Eigenbases for a force-constrained position control of adaptive shell mirrors

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ABSTRACT

Context. The deployment of meter-scale (hitherto pre-focal) adaptive deformable mirrors finds some prominent examples in the leading ground-based visible to near-infrared facilities (e.g. the Very Large Telescope (VLT), the Large Binocular Telescope (LBT), or the Magellan Telescope) and is being adopted by several others (e.g. the Multiple Mirror Telescope (MMT) or Subaru). Furthermore, two out of the three giant segmented-mirror telescopes now under design will feature them. In all these cases, the proprietary technology is based on voice-coils and is limited in force, stroke, and velocity.

Aims. Because of the nature of their purpose, that is, adaptive wave-front correction, any kind of optimality relies on the control of a subset of principal wave-front components or eigenmodes, for short, a basis of functions in a mathematical sense. Here we provide algorithmic procedures for generating such eigenbases, also called Karhunen-Loève (KL) modes, that integrate force limitations in their definitions whilst maintaining standard orthonormality, statistical independence, and deformable mirror span.

Methods. The double-diagonalisation method was revisited to build KL modes ranked by the force applied on the actuators.

Results. We analysed this new KL basis for von Kármán turbulence statistics and present the fitting error and the distribution of their definitions whilst maintaining standard orthonormality, statistical independence, and deformable mirror span.

Key words. instrumentation: adaptive optics – telescopes – methods: analytical

1. Introduction

Adaptive shell (hitherto pre-focal) deformable mirrors (DM) [Salinari et al. 1994] bring adaptive optics (AO) into the core of the optical path of a telescope. Unlike classical post-focal instrument-proprietary AO, these adaptive telescopes promise to reduce the number of reflecting surfaces leading to improved near-IR (NIR) observations, larger corrected fields, more compact optical designs, and so on.

Notable examples are the deformable secondary at the Multiple Mirror Telescope (MMT) [Brusa et al., 2003], the Very Large Telescope Adaptive Optics Facility (VLT-AOF) [Arsenault et al., 2006], and the adaptive secondaries of the Large Binocular Telescope (LBT) [Riccardi et al., 2010] and Magellan telescopes [Close et al., 2010], all based on the Microgate technology. Future giant segmented-mirror telescopes such as the European Extremely Large Telescope (ELT) and the Giant Magellan Telescope (GMT) will feature adaptive secondaries based on voice-coil technology. As we show below, the developments made here will therefore cater to any of these systems.

The shells surface is deformed by controlling actuator positions recurring to an inner closed loop at frame rates that are orders of magnitude higher than typical AO frame-rates, using position sensors (e.g. capacitive sensors). These dynamic details are unimportant for the analysis that ensues because from an AO point of view, they occur much beyond the bandwidths relevant for our analysis. However, the budget of local forces applied to the location of the actuators along the z-axis orthogonal to the surface is limited to a few Newton. Other limitations in terms of stroke and velocity also apply, but they fall outside of the modal basis optimisation that we intend to outline here because i) they concern amplitude, which cannot be encoded in a basis, and ii) they concern velocity, which is a time dimension parallel to the space dimension covered here.

Classical DMs are controlled by minimising one quantity (either voltages or currents) that directly drive the local displacement of the DM actuators. Unlike the latter, the fast inner position-control loop in adaptive shells means that their response is not passive, but active. The commands sent to the DM thus achieve a different nature: they become z position setpoints that are kept in place by the inner control loop. Despite this, the local deformations (whether passive or active) can still be represented by a steady-state influence function that describes the bi-dimensional optical-path-difference (OPD) deformation in response to a unitary command.

We present an evolution of the double-diagonalization method (Gendron (1995)) that is employed to build a modal basis for wave-front (WF) reconstruction in AO. Our purpose is to revisit the generation of control bases that conserve key properties. The first property is that the bases correspond to the principal components of the atmospheric turbulence. The second property is that they are statistically independent. The third property is that they are orthonormal in phase space with respect to the primary aperture. The fourth property is that they are fully defined in the space spanned by the deformable mirror. To these properties, the control of thin adaptive shells adds constraints in terms of force, which we factor in the generation of the control basis itself whilst keeping the above properties intact.

Without loss of generality, the presentation that follows is illustrated for (and motivated by) the case of the European Ex-
tremely Large Telescope (ELT) pre-focal quaternary deformable mirror (the M4) \citep{Vernet2019}. We consider an interface to the DM where the command vector is the vector of actuator mechanical position setpoints.

To do this, we revisit the Karhunen-Loève (KL) modal basis generation through double-diagonalisation in §3 and then expand it to force minimisation in §4.

2. KL modal expansion

The KL modal expansion of space- and time-indexed processes is one of the most frequently used statistical techniques for the feature extraction of continuous stochastic processes. In its discrete formulation, it is simply empirical orthogonal function (EOF) analysis or principal component analysis (PCA) \citep{Fontanella2012}. Hence, both KL expansion and EOF are useful for efficiently reducing the dimension of large degree-of-freedom data-sets. In its simplest description, it is the solution of the covariance eigenvalue equation.

We outline the steps for synthesising (but also to analysing) a basis set of functions (or modes) to describe the wavefront on a given telescope pupil, over which the modal expansion writes

\[ \phi(r) = b^T \mathcal{B}(r), \]

where \( b^T \) is the transpose vector of modal coefficients, and \( \mathcal{B}(r) \) are the basis functions (modes) indexed by the bi-dimensional variable \( r \). It is useful to define an inner product between two bi-dimensional functions \( f_1(r) \) and \( f_2(r) \),

\[ f_1(r) \ast f_2(r) = \frac{1}{\Omega} \int_{\Omega} f_1(r)f_2(r)dr, \]

where the binary aperture function \( \Omega(r) \) is zero-one valued for points outside or inside the pupil, respectively. The KL modes are eigenfunctions and therefore respect two properties: The first one is the orthonormality

\[ \mathcal{B}_i(r) \ast \mathcal{B}_j(r) = \delta_{ij}, \]

and the second one the statistical independence

\[ \langle b_i b_j \rangle = \langle b_i^2 \rangle \delta_{ij}, \]

where \( \delta_{ij} \) is the Kronecker symbol, and \( \langle \cdot \rangle \) represents the ensemble average. From these equations, we obtain the two-dimensional KL integral equation

\[ \langle \phi(r)\phi(r) \rangle \ast \mathcal{B}_i(r) = \langle b_i^2 \rangle \mathcal{B}_i(r). \]

2.1. Eigendecomposition of \( \langle \phi(r)\phi(r) \rangle \)

The KL modes can instead be computed numerically for any pupil type, form, or shape, with or without central obstruction. In this case, we compute the eigenvectors of the phase covariance function whose columns express the KL modes on the chosen underlying basis.

Adopting a discrete, point-wise description of the phase (also called zonal expansion), we determine the phase covariance matrix \( C_\phi = \langle \phi(r)\phi(r) \rangle = \langle \phi^2 \rangle \) by summing over all the layers that were sampled by the values of the covariance function \( C_i(r) \) at \( i, j \in \Omega \). We compute \( \rho = \| r_i - r_j \| \), which is shorthand notation for a square matrix of the distances between all the intercepts within the considered aperture (whose shape can be any desired shape),

\[ C_\phi[i, j] = \sum_j C_i(r) = | r_i - r_j | \quad r_i, r_j \in \Omega, \]

where

\[ C_i(r) = \begin{cases} \frac{L_0}{\rho_0(l)}^{5/3} \Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{5}{6} \right) \left( \frac{24}{5} \right)^{5/6} \left( \frac{2\pi \rho}{L_0} \right)^{5/6} K_{5/6} \left( \frac{2\pi \rho}{L_0} \right) & \text{if } \rho > 0, \\ 0 & \text{otherwise} \end{cases} \]

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An example is provided in Fig. [1]. The figure depicts commonly accepted KL modes such as slightly bent tilt modes and a family of Zernike-resembling modes computed from the eigendecomposition of the piston-removed spatial phase covariance matrix shown in Fig. [2].

Alternative methods that care to provide an orthonormal basis on arbitrary pupil shapes use for example Gram-Schmidt minimisation \citep{Upton2004}, but they fall short of a basis retaining all the key features for AO control (listed in the introduction).

![First 25 KL modes](image_url)

Furthermore, it is convenient to express the basis functions within the space spanned by the deformable-mirror (DM) influence functions (IFs). This step is necessary to build a basis that is relevant for AO control. It leads not to one, but to two eigenvalue-determination problems, as is explained next.
3. Original double-diagonalisation method

The developments covered here were first presented in Gendron (1995) with the purpose of realising the so-called optimal modal gain integrator (OMGI), a technique whereby a set of (spatial) control modes are optimally weighted to obtain the lowest wavefront residual. Some mathematical aspects presented in Lai et al. (2000) were used as the starting point of our work. The method is called double-diagonalisation because it requires two steps involving the calculation of eigenmodes of some quantity in some space.

The concept behind the generation of one such KL basis is the following: The generation of the principal components of the atmosphere (stochastic process) is expressed as a linear combination of the DM principal components (geometric steady-state modes).

Mathematically, we seek the basis \( B \) of modes that respects

\[
\begin{align*}
B^{-1}CB^T &= \Lambda \quad \text{(diagonal matrix)} \\
B^TAb &= I,
\end{align*}
\]

where \( C \) is a statistical covariance matrix (from turbulence), and \( \Lambda \) a geometric cross-product matrix of DM IFs (sometimes also called geometric covariance matrix) to ensure orthonormality between the modes. Both are symmetric and positive definite.

3.1. Definition of DM related variables

The vector space spanned by the DM is given by the steady-state IFs representing the most complete basis of the DM. The OPD is the shape of the wave front after reflection by the deformable mirror. Any valid OPD deformation of the DM can be decomposed uniquely in the IF space,

\[
\phi(\rho) = \sum_i p_i h_i(\rho),
\]

where \( p_i \) is the \( i \)th actuator command setpoint, and \( h_i \) is the \( i \)th bi-dimensional influence function.

From now on and for compactness, we write the equations in the form of matrix operations whenever possible (the dot operator representing the matrix multiplication), and when we define a matrix, we append the explicit dimensions as follows:

\[
M[\text{dim}_a, \text{dim}_b].
\]

Hence the matrix representation of Eq. (10) becomes

\[
\phi = H \cdot p.
\]

where \( \phi[n_p] \) is the vector of the \( n_p \) valid OPD numerical points within the pupil, and \( p[n_{act}] \) is the vector of the actuator positions (throughout this paper, the position of an actuator denotes the mechanical displacement of the shell surface in the centre of the actuator along the \( z \)-axis). We consider in the \( n_{act} \) set all actuators, regardless of whether within or outside of the pupil.

\[
H[n_p, n_{act}]
\]

is the matrix that collects the IFs as vectors on the valid pupil points in its columns.

We define for later use a handy geometric IFs cross-product matrix \( \Delta \), which also appears in Eq. (9), and is given by

\[
\Delta = H^\dagger \cdot H.
\]

The two variables \( \phi \) and \( p \) are expressed in meters.

In the classical method, it is common practice to select a number of valid actuators \( n_{act} \) defined by some reasonable criterion. In this case, the influence matrix is limited to the selected actuators (\( H[n_p, n_{act}] \)), and \( \Delta[n_{act}, n_{act}] \) is assumed to be invertible.

To any vector \( p \) corresponds a vector of forces \( f[n_{act}] \) that is expressed as a function of a stiffness matrix \( K[n_{act}, n_{act}] \) (expressed in newton-meter),

\[
f = K \cdot p.
\]

In the case of M4, the matrix \( K \) is a block diagonal matrix in which each block is equal to the stiffness matrix for a shell (all six shells are carbon copies of each other). This is common knowledge, and the method developed here is general enough to cover cases of non-segmented shells.

The least-squares solution for the best fit in terms of DM positions \( \phi_{\text{fit}} \) to an input OPD \( \phi \) can be written in a general form with the Tikhonov regularisation matrix \( V \) :

\[
\phi_{\text{fit}} = (\Delta + V^T \cdot V)^{-1} H^T \cdot \phi.
\]

In the classical method of a least-squares fit with actuator selection, in Eq. (13) \( V = 0 \), and \( \Delta \) is invertible.

3.2. Specific modes and seed basis

The first step in the definition of a useful modal basis for control in AO is to define specific modes (SpM) that explicitly appear in the reconstruction.

We define \( n_{spm} \), the number of SpMs, and \( \phi_{spm}[n_p, n_{spm}] \), the exact pupil-bounded OPD shape to fit. We use the notation of Ferreira et al. (2018) to write the best fit (in positions space) \( \tau[n_{spm}, n_{spm}] \) to the OPD of SpMs \( \phi_{spm} \),

\[
\tau = \Delta^{-1} \cdot H^\dagger \cdot \phi_{spm}.
\]

This defines the SpM. Its complementary space \( H_\perp \) on which we build the orthonormal seed basis \( SB[n_{act}, n_{act} - n_{spm}] \) is defined by subtracting the projection of each IF on the SpM,

\[
H_\perp = H - (H\tau)(H\tau)^T H.
\]
which can be expressed in terms of a generator $\mathbf{G}[n_{sel}, n_{act}]$ of modified IFs,
\[
\mathbf{H}_\perp = \mathbf{H} \cdot \mathbf{G},
\]
where $\mathbf{G}$ is given by Ferreira et al. [2018]
\[
\mathbf{G} = \mathbf{I} - \mathbf{T} \cdot \mathbf{A} \cdot [\mathbf{T} \cdot \mathbf{A}]^{-1} \cdot \mathbf{T} \cdot \mathbf{A}
\]
and is readily obtained by plugging Eq. (15) into Eq. (16) and further developing terms. The seed basis is defined by
\[
\mathbf{S}_B = \mathbf{G} \cdot \mathbf{D} \cdot \sqrt{n_p \Lambda^{-1}},
\]
where $\mathbf{D}$ and $\Lambda$ are the eigenvectors and diagonal matrix, respectively, of the eigenvalues of the cross product of $\mathbf{H}_\perp$.
\[
\mathbf{D} \cdot \Lambda \cdot \mathbf{D}^T = \text{svd} \left( \mathbf{G}^T \cdot \mathbf{A} \cdot \mathbf{G} \right),
\]
where we use some abuse of notation $\text{svd}(\cdot)$ to represent the singular-value decomposition (SVD), which formally provides three matrices, the product of which is the one shown.

We note that from Eq. 20, $\mathbf{D}$ is a collection of eigenvectors expressed in components of $\mathbf{G}$ (modified IFs). Hence, to obtain $\mathbf{S}_B$, expressed in actuator space (original IFs), a change of basis from $\mathbf{G}$ to actuators is performed by multiplication by $\mathbf{G}$ in front of Eq. 19. Then, using the definition and unitary nature of $\mathbf{D}$, it is straightforward to show that $\mathbf{S} \mathbf{B}$ is an orthonormal basis and therefore verifies $(\mathbf{H} \mathbf{S})^T \mathbf{H} \mathbf{S} \mathbf{B} = \mathbf{I}$. We caution that this way of computing DM eigenmodes uses a particular property (invertibility) of the cross product $\mathbf{A}$ and will no longer hold when force minimisation is considered. This is explained in section 4.2.

3.3. KL modes

The two properties of KLs we are interested in are statistical independence for turbulence correction and orthonormality in OPD space. We define the covariance matrix $\mathbf{C}_p$ as the covariance of the best-fit positions ($\mathbf{V} = 0$) for atmospheric turbulence (Kolmogorov or von Karman) statistics,
\[
\mathbf{C}_p = \langle \mathbf{p}_{\text{fit}} \cdot \mathbf{p}_{\text{fit}}^T \rangle.
\]
The ensemble average is done over independent realisations of turbulence phase $\phi$. The matrix $\mathbf{C}_p$ has been derived in Gendron (1995) and Lai et al. (2000), and we rewrite the derivation in a slightly different way in Appendix A. The different writing supports the generalisation to a force-optimised basis.

The KL modes are the eigenmodes $\mathbf{A}$ of $\mathbf{C}_p$ expressed in the SB in order to satisfy the two properties. The matrix $\mathbf{K}_L[n_{act}, n_{act} - n_{spms}]$ represents the actuators position setpoints encapsulating the KL shapes,
\[
\mathbf{K}_L = \mathbf{S}_B \cdot \mathbf{A},
\]
where $\mathbf{A}$ is obtained from the SVD of $\mathbf{C}_p$ after a change of basis to SB,
\[
\mathbf{A} \cdot \mathbf{S} \cdot \mathbf{A}^T = \text{svd} \left( \mathbf{S}_B^{-1} \cdot \mathbf{C}_p \cdot \mathbf{S}_B^{-1} \right).
\]

Working versions used in classical double-diagonalisation are publicly available in Matlab and IDL (and maybe other codes). An example is OOMAO (Conan & Correia 2014).

The diagonal matrix $\mathbf{S}$ gives the spectrum, root mean square (RMS) of the OPD, of the turbulent phase projected on the KLs, and we have
\[
\mathbf{S} = \mathbf{K}_L^{-1} \cdot \mathbf{C}_p \cdot \mathbf{K}_L^{-T}.
\]

3.3.1. Complete basis

The complete basis is the concatenation of the special modes and the KL modes,
\[
\mathbf{B} = [\tau | \mathbf{K}_L].
\]
This matrix is commonly called in adaptive-optics jargon the modes-to-commands (MC) matrix and it is all that is needed to perform AO modal control.

4. Double diagonalisation and force minimisation

In the preceding section, only the selected actuators $n_{act}$ inside or very near the pupil were considered. In this case, $\Delta$ is invertible, and an orthonormal basis can always be found. In most of the AO systems, this is sufficient to define the command of the DM (an extrapolation at border actuators is sometimes applied).

For adaptive shell mirrors, the situation is different, and the DM with its actuator grid may be significantly larger than the pupil. For example, the ELT M4 presents an overall shell size of 2.40 m, while the optical pupil (for the centre of the field) has a size of 2.25 m. The diameter is 7% larger and represents about three rings of actuators (see Fig. 3).

In the particular case of M4, we need to control outer actuators with little or no influence on the phase in the pupil, and this despite the extrapolation imposed on the outermost actuators that follows from the classical methods. We consider now $\mathbf{H}[n_{act}, n_{act}]$ defined for all actuators. Concretely, this means that $\Delta$ is no longer invertible (because it is ill conditioned), and hence, non-independent sets of modes can be obtained. Some computations such as inversions become pseudo-inversions, and diagonalisations involve the selection of valid eigenvectors. As for the method in this section, we always ensured that we sought an orthonormal basis (for the SpMs, the seed basis, or the KLs), not only to satisfy the original properties, but also as a guarantee that the resulting bases are independent.

For instance, for a basis $\mathbf{B}$ expressed in positions space, we define the orthonormality criterion $C_O$ as
\[
C_O = \max \left( \frac{1}{n_p} \mathbf{B}^T \mathbf{A} \mathbf{B} - 1 \right) < < 1,
\]
which we consider to be satisfied when it is of the order of the expected computer precision. A similar criterion is applied to check the diagonality of the statistical covariance matrix.

The concept of the method that follows is to compute the SpMs, seed basis, and KLs in force space (instead of position). However, we found that SpMs such as often-cases tilt modes appear associated with the lowest values of the stiffness matrix. In other words, low-order modes (or equivalently, low spatial-frequency modes) require the least force (lower by orders of magnitude when compared to higher-order modes). In the implementations made leading to the results of this paper, we found that expressing the basis in terms of force setpoints leads to numerical instabilities when projecting on low-order SpMs (e.g. tip and tilt). For this reason, we proceeded in three steps: The first step consists of finding the best OPD fit (expressed in positions setpoints) to SpMs by weighted least-squares projection on the space spanned by the DM IFs. The second step is determining the seed basis by diagonalisation in force-like space of the sub-space orthogonal to the SpMs. Then, express the resulting seed basis in position space. The third step is computing the KL basis as in the classical method (in position space), but with an adaptation to take the ill-conditioned $\Delta$ into account.
4.1. Specific modes: WF fitting with force regularisation

One natural way to determine the SpMs is to compute the positions using a regression with Tikhonov regularisation using the forces as a penalty term. We used Eq. 14 with $\Gamma \neq 0$ and proportional to the stiffness matrix $K$,

$$\Gamma = \sqrt{\alpha K}.$$  

(27)

This is equivalent to minimising

$$\arg \min_{\phi} ||H \cdot \phi - p||^2 + \alpha ||K \cdot \phi||^2.$$  

(28)

The solution minimises the fitting error while keeping the forces sufficiently low. $\alpha$ is the only parameter that is tuned. All actuators are considered valid. We can rewrite Eq. 15 and express $\tau_\alpha$, the best fit of the SpMs minimising forces, as

$$\tau_\alpha = \left( \Lambda + \alpha K^T \cdot K \right)^{-1} \cdot H^T \cdot \phi_{spm},$$  

(29)

where $\alpha$ needs to be adjusted until the two error sources in Eq. 28 reach acceptable values. Typical values in the case of M4 are $\alpha = 10^{-15}$ for low-order modes (piston, tip-tilt, differential piston modes, etc.) and must be adapted case by case. We suggest in any case to make a sanity check of the positions and forces obtained. We also realised that cropping the influence function to save memory may affect the optimum $\alpha$ values significantly.

4.2. First diagonalisation: A seed basis from expressing OPD as a function of force

The classical way of computing the first diagonalisation is equivalent to analysing OPD as a function of DM actuator position setpoint amplitudes. The DM eigenmodes thus obtained are ranked in position setpoint amplitudes.

We now instead compute the diagonalisation of IFs expressed as a function of force (or a quantity close to force). To do this, we performed the following change of basis:

$$\phi = H \cdot F_\beta \cdot f.$$  

(30)

where $F_\beta$ is related to the inverse of the stiffness matrix $K$, allowing the expression of the OPD as a function of force setpoints applied on the actuators instead of position setpoints,

$$F_\beta = \left( K + 1/\beta \cdot I \right)^{-1}.$$  

(31)

This is equivalent to expressing the DM IFs not as a function of actuator position setpoint, but force setpoints,

$$H_\beta = H \cdot F_\beta.$$  

(32)

For $\beta = \infty$, the change of basis is simply from IFs as a function of positions ($H$) to IFs as a function of forces ($H_\beta$). We can verify that

$$H \cdot p = H_\beta (\beta = \infty) \cdot f.$$  

(33)

The parameter $\beta$ has been introduced to damp large positions of actuators at the outer edge of the shells that occur under some conditions when the shell is completely free outside the pupil. In this sense, $H_\beta$ are the IFs as a function of a quantity that is slightly modified forces that keep track of the position information. A typical value is $\beta = 10^{-3}$. This value has to be tuned as a function of the overall stroke budget of the DM. A high value of $\beta$ would just leave the IFs unchanged (modulo a multiplicative factor), while intermediate non-small values may give rise to a non-invertible basis $F_\beta$.

We define $G_\beta$ and $\Delta_\beta$ by simply replacing $H$ by $H_\beta$ in Eq. 12 and in Eq. 18 The generator $G_\beta$ forces in write space

$$G_\beta = I - F_\beta^{-1} \cdot \tau_\alpha \left( \tau_\alpha^T \cdot F_\beta^{-T} \cdot \Delta_\beta^{-1} \cdot \Delta_\beta^{-1} \cdot \tau_\alpha^T \cdot F_\beta^{-T} \cdot \Delta_\beta \right)^{-1} \cdot \tau_\alpha^T \cdot F_\beta^{-T}.$$  

(34)

If we followed the classical method, we would rewrite Eq. 20 and compute the DM eigenmodes by making the SVD of the IFs cross product. While it is easy to show that the eigenvectors of a matrix ($H$) are also eigenvectors of the cross product ($H^T \cdot H$), the inverse proposition is generally not true, and in particular, in our case, the matrix $\Delta_\beta$ is highly ill-conditioned. We therefore compute the new DM eigenmodes by computing the SVD of the (modified) IFs directly. This is a much larger matrix than the $\Delta_\beta$, but it is much better conditioned.

To compute the seed basis, we perform the SVD of the modified IFs with the same intention as in Eq. 20 but without going through the cross-product step,

$$U_F \cdot S_F \cdot V_F^T = \text{svd} \left( H_F \cdot G_\beta \right).$$  

(35)

The final form of the seed basis is equal to the normalised eigenmodes with the change of basis $F_\beta$ to return to position space,

$$S_{B_\beta} = F_\beta \cdot G_\beta \cdot V_F \cdot S_F^{-1} \cdot \sqrt{\mu}.$$  

(36)

In contrast to Eq. 19 a number of non-zero eigenmodes must be selected in the pseudo-inversion of $S_F$. The number of modes $nm$ are chosen in a reasonable way, such that the basis is orthogonal (hence criterion of Eq. 26 is satisfied). This reasonable number is in general very close to the optimum number of selected actuators of the classical method, so $nm \approx n_{act} - n_{spm}$.

The seed basis computed in this section is ready to be used for the next step of the method for KL computation. The properties of this basis, satisfying orthogonality in OPD space and force minimisation with ranking like the stiffness modes, may also be used for other purposes and to generate other types of basis. In particular, other conditions, such as continuity conditions (e.g. minioning), are possible if the $K$ matrix is adapted or other terms are added to Eq. 51.

4.3. Second diagonalisation: KL modes

The next step is to compute the covariance matrix in the seed basis $S_{B_\beta}$. Because $\Delta$ is not invertible, we cannot use the equivalent of Eq. 24 (which in turn relies on Eq. 4). We define the cross product of the $S_{B_\beta}$ basis (which is diagonal in this particular case, but we keep it for generality),

$$\Delta_{S_{B_\beta}} = S_{B_\beta}^{-T} \cdot \Delta \cdot S_{B_\beta}.$$  

(37)

When we write the vector $m$ of the modal coefficients in the SB explicitly, we obtain

$$m = \Delta^{-1}_{S_{B_\beta}} \cdot (H \cdot S_{B_\beta})^T \cdot \phi.$$  

(38)

After a development similar to Eq. 2.28 of Gendron (1995), we obtain the expression of the covariance matrix in the basis $S_{B_\beta}$,

$$C_F = \left( m \cdot m^T \right) = \Delta^{-1}_{S_{B_\beta}} \cdot S_{B_\beta}^{-T} \cdot C_h \cdot S_{B_\beta} \cdot \Delta_{S_{B_\beta}}^{-T}.$$  

(39)
Plugging Eq.\,[37]\ into Eq.\,[39]\ we can easily find Eq.\,[23]\ as we intended for the cases when $\Delta^{-1}$ exists.

In the case of an orthonormal basis, this equation becomes

$$C_{\mathcal{F}} = \frac{1}{n_p} s \bar{S}_{\mathcal{F}}^T \cdot C_h \cdot \bar{S}_{\mathcal{F}}.$$  \hspace{1cm} (40)

As in section 3.3, we diagonalise the covariance matrix and express the KL basis in DM position setpoint space,

$$A_{\mathcal{F}} \cdot S \cdot A_{\mathcal{F}}^T = \text{svd} (C_{\mathcal{F}}),$$  \hspace{1cm} (41)

from which the KL modes are obtained as

$$KL_{\mathcal{F}} = \bar{S}_{\mathcal{F}} \cdot A_{\mathcal{F}}.$$  \hspace{1cm} (42)

4.4. Complete basis

The final force-optimised orthonormal $\mathcal{F}$ is obtained by concatenating the SpMs and the KL modes,

$$\mathcal{B}_{\mathcal{F}} = [\tau_s \mid KL_{\mathcal{F}}].$$  \hspace{1cm} (43)

5. Assessing the efficiency and suitability of any basis

We aim to analyse the performance in terms of fitting error, actuator position excursion, and forces applied to actuators for a Von Karman turbulence with a Fried parameter $r_0 = 0.1$ and an outer scale $L_0 = 30$.

5.1. Fitting error

We rewrite Eq.\,[39]\ in its general form for any basis $\mathcal{M}$ (not necessarily orthonormal),

$$C_m = \langle m \cdot m^T \rangle = \Delta_m^{-1} \cdot M^T \cdot C_h \cdot M \cdot \Delta_m^{-T}. $$  \hspace{1cm} (44)

We use this equation to evaluate the following bases: The first basis is the classical KL basis with optimised positions. The second basis is the KL basis with optimised forces (two cases: $\beta = 0$ and $\beta = 10^{-5}$). The third basis is the seed basis $SB$ corresponding to the DM eigenmodes minimising positions. The fourth basis $SB_{\beta}$ is composed of the DM eigenmodes minimising forces ($\beta = 10^{-5}$). The fifth basis is the stiffness modal basis composed of the eigenmodes of the $K$ matrix reduced to a selection of 4300 actuators, with forces set to zero for the outer actuators.

In the following, we assume that the piston is always the first mode of the basis. The total RMS error $\text{RMS}_{\text{urb}}$ contained in turbulence is computed by integrating the numerical 2D map $W_{i,k}$ created from Eq.\,[37]\ and this was used to compute the modes,

$$\text{RMS}_{\text{urb}} = \sqrt{\int W_{i,k}(\kappa) d^2 \kappa}. $$  \hspace{1cm} (45)

We define the RMS of the sum of the projection of turbulence on the modes $\mathcal{M}$ from mode $a$ to $b$,

$$\text{RMS}_{a:b} = \sqrt{\frac{1}{n_p} \sum_{i=a}^{b} \sum_{j=a}^{b} (C_m \odot \Delta_m)_{ij}.} $$  \hspace{1cm} (46)

The fitting error corresponding to an effective number $q$ of corrected modes is given by

$$\text{RMS}_{\text{eff},q} = \sqrt{\text{RMS}_{\text{urb}}^2 - \text{RMS}_{0,q}^2}. $$ \hspace{1cm} (47)

5.2. Distribution of positions and forces

For adaptive shell mirrors, the two quantities of interest for ensuring the good functioning and integrity of the system are the actuator mechanical positions and the forces applied to them. The control system will apply maximum threshold values on these two quantities. With the analytical development presented here, we can statistically evaluate the variance of these quantities in the hypothesis of a perfect fit of the turbulence by the modes. A more thorough estimation including the wave-front sensor and the WF reconstruction algorithm will require a full end-to-end simulation and goes beyond the scope of this paper. Section 6 presents some examples of an optimisation of modal bases with respect to positions and forces.

The statistical covariance of positions for a given number of modes $q$ is computed from the modal covariance $C_{m,1:q}$ and the modal basis $M_{1:q}$, restricted to $q$ modes,

$$\langle pp^T \rangle_q = M_{1:q} \cdot C_{m,1:q} \cdot M_{1:q}^T.$$ \hspace{1cm} (48)

The RMS of the actuator positions is given by

$$\text{RMS}_{\text{POS},q} = \sqrt{\text{diag} \left( \langle pp^T \rangle_q \right)}.$$ \hspace{1cm} (49)

The RMS of Forces applied to actuators is given by

$$\text{RMS}_{\text{FOR},q} = \sqrt{\text{diag} \left( \langle K (pp^T) K \rangle_q \right)}.$$ \hspace{1cm} (50)

6. Application to the European Extremely Large Telescope wavefront control

6.1. European Extremely Large Telescope pre-focal deformable mirror

The ELT M4 features a total of 5352 actuators, 4866 of which are available for instrument-led adaptive control. The outer 486 are kept for telescope-only control and are used to maintain overall stability. Figure 7 provides the geometry and positions of the (inner and outer) actuators overlaid on the primary aperture (M1) of the telescope, whose segmented nature is not explicitly depicted here.

In this section, we evaluate the performance of a DM based on the IFs and the stiffness matrix $K$ in a semi-analytical way.

6.2. Nominal modal basis

We call nominal basis the modal basis that is obtained by applying the method in Sect. 4 to the 5352 actuators that are used as a set of independent degrees of freedom.

6.2.1. Fitting efficiency

One of the properties expected from a modal basis is the efficiency to correct atmospheric turbulence with a minimum number of modes. This is important for instance when using OGMI (Gendron 1995). We plot in Fig. 8 the efficiency $\text{RMS}_{\text{eff},q}$ for different bases. This shows that the choice of a basis matters:

The best basis is the KL basis because by definition, the KLS are statically independent. The worst basis is the minimum position DM eigenbasis. The reason for this certainly is that no regularisation was used to derive this basis. The minimum-force eigenbasis differs from the KLS only in the case of low-order correction. Otherwise, it is as efficient as KLS for more that 1000 corrected modes. The reason is that force-minimisation is similar to
a Kolmogorov regularisation where high spatial frequencies are damped.

One important question that arises for KLs is whether there is a penalty from force minimisation rather than the classical position minimisation. Figure 4 compares the fitting efficiency of the classical minimum position KL basis to the minimum force KL basis ($\beta = 10^{-5}$ and $\beta = \infty$). The three cases perform equally well. The difference is a larger fitting error of a few nanometers at most for the position-optimised basis. Moreover, the damping parameter $\beta$ has a negligible impact on the fitting error. This shows that minimising for forces does not impact the fitting performance.

6.2.2. Statistics on position and force

The safety checks on shell mirrors usually impose maximum values for actuator celerity, position amplitude, and force amplitude at any given time. Typical values for M4 are a few tens of microns for maximum positions stroke and a maximum force of about 1.5 Newton. A full analysis of the effect of these limits would require studying a full stroke budget and is reserved for a later work. In this section, we compute the RMS of positions and forces based on the hypothesis that phase screens following Van Karman statistics are projected on the KL modes. We analyse the effect of the algorithm choice on the distribution of the statistics. With this tool, we can compare bases. We qualify a basis as valid when both position and force RMS are taking reasonable values.

Fig. 6 compares the RMS of positions for 4000 corrected modes and for the classical case (no force constraint = minimised position) and for two $\beta$ settings for the force-optimised case. Six groups of actuators are visible as identified in Fig. 3, each with two zones: there is a rather flat part identical to all cases, which corresponds to the correction of the phase in the pupil. The shape of the curve corresponding to actuators outside the pupil differs and is influenced by the parameters of the algorithm.

Positions clearly tend to zero in the zone for the classical basis (no force constraint). For the force-optimised case without damping ($\beta = \infty$), the positions tend to have large and outstanding amplitudes outside the pupil. When we varied the damping parameter ($\beta = 10^{-5}$ in Fig. 5), a compromise in positions was possible and we were able to limit the excursion of positions while having little influence on the force distribution, as shown in the next figure.

Figure 7 shows that the case without a force constraint (hence when positions are forced to be small) generates very strong forces on actuators outside the pupil. For the force-optimised cases, the forces are well contained and are weaker outside than inside the pupil. The green and orange curves are almost the same and demonstrate that it is interesting to apply the damping factor $\beta$ to limit the position excursions while preserving optimum force distribution. These results give an idea of the trade-off that can be made for the optimal control in positions and forces of an adaptive shell mirror.

To illustrate the dependence of the force on spatial frequency, we show in Fig. 8 the maximum RMS of the force as a function of the number of corrected modes for the force-optimised KL basis with $\beta = 10^{-5}$. This plot shows that up to about 700 modes, the forces are quite weak. In a regime up to 4200 modes, the dependence is almost linear with the number of modes. After 4200 corrected modes, the forces increase abruptly.
Fitting error in function of modes corrected

Fig. 5. Fitting error for classical KLs and force-optimised KLs. The green and orange curves show the fitting error difference with respect to the blue curve.

Maximum RMS force for nominal basis

Fig. 8. Maximum force RMS as a function of the number of corrected modes for the force-optimised KL basis with $\beta = 10^{-5}$.

Fig. 6. Position distribution for force-optimised KLs with different $\beta$ parameters. No force-constraint stands for original double-diagonalisation method with position minimisation. The assumption for the number of corrected modes is 4000.

Fig. 7. Force distribution for force-optimised KLs with different $\beta$ parameter and no force constraint. The assumption for the number of corrected modes is 4000.

6.3.1. Zero-force policy on outer actuators

The zero-force policy on the outer M4 actuators consists of the following: The outer positions are adjusted such that the forces applied on them are rigorously zero while the inner actuators do not move. We compute the forces as

$$
\begin{pmatrix}
    f'_i \\
    f'_o
\end{pmatrix} =
\begin{pmatrix}
    K_{ii} & K_{io} \\
    K_{oi} & K_{oo}
\end{pmatrix}
\begin{pmatrix}
    p_i \\
    p_o
\end{pmatrix},
$$

(51)

where the forces and positions are explicitly decomposed into inner and outer actuators. The outer actuators are minimised by applying a zero-force to them, that is, $f'_o = 0$. Solving for $f'_i$, that is, the forces to apply to the inner actuators, we find

$$
f'_i = (K_{ii} - K_{io}K_{oo}^{-1}K_{oi}) p_i
$$

(52)

and

$$
p'_o = -K_{oo}^{-1}K_{oi} p_i
$$

(53)

for outer actuator setpoints, ensuring that a zero-force policy is followed. From Eq. (53), we can write the minimisation matrix $M$ as

$$
M = \begin{pmatrix}
    I \\
    -K_{oo}^{-1}K_{oi}
\end{pmatrix}.
$$

(54)

6.3.2. Modal basis computation with minimising

To account for the effect of the outer actuators on the OPD in the pupil, we define a modified influence matrix $H^m[n_{act}, n_i]$, a reduced stiffness matrix $K^m[n_i, n_i]$, and a covariance matrix $C^m_{ii}[n_i, n_i]$ with the following definitions:

$$
H^m = H \cdot M \quad K^m_i = (K \cdot M)_i \quad C^m_{ii} = M^T \cdot C_h \cdot M.
$$

(55)

In the case of the zero-force policy in Eq. (54), we can retrieve the result of Riccardi (2021) for the reduced stiffness matrix.

$$
K^m = K_{ii} - K_{io}K_{oo}^{-1}K_{oi}.
$$

(56)

The modal basis with minimising is obtained with the same calculations as in Sect. 4 but we replace $H$ by $H_m$, $K$ by $K_m$, and $C_h$ by $C^m_{ii}$ to obtain the full basis (Eq. 43), which we call $B^m_{ii}[n_i, n_{modes}]$. It is expressed in the 4866 inner actuators and can be used for the control.
For the statistical analysis of the performance of the basis, we must express the basis on all actuators and execute the calculations of Sect. 5.

\[ B_{F}^{\text{mod}} = M \cdot B_{F}^{m}. \]  
(57)

We apply this zero-force policy and repeat the calculations of the basis and the statistical analysis. The damping parameter need to be slightly adjusted \( (\beta = 2.10^{-5}) \) for the zero-force policy case. Figure 9 presents the force distribution with the zero-force policy basis compared to the nominal one. The force distributions are globally very similar, except when the forces are null by design. The difference is more visible for the positions, as shown in Fig. 10. The excursion of outer actuators for the zero-force policy is higher by about 15%, but still with values below the mean value over the pupil. This shows that in the case of the selection of 4866 actuators of M4, the zero-force policy does not lead to any significant deterioration of forces and positions statistics.

6.3.3. Alternative: A posteriori modification of the nominal basis

Although the inner actuator setpoints are not affected by the zero-force policy, strictly speaking, the same is not true with the OPD induced by outer actuators. This justifies that in the general case, the method in Sect. 6.3.2 is used.

However, in the particular case of M4, the OPD induced by outer actuators is rather small. In this case, we can compute the nominal basis and alter it by applying the minioning, as in the following equation:

\[ B_{F}^{\text{mod}} = M \cdot B_{F}[i, \cdot]. \]  
(58)

Application of Eq. 58 mostly preserves the position and force statistics and only slightly affects the mode orthornormality \( (C_0 \approx 0.01) \). This means that we can alternatively compute the KL basis (from Sec. 4) on 5352 actuators without enforcing any specific policy) without any minioning whatsoever and apply it to the basis a posteriori. The effect on positions and forces is shown in Fig. 11 and Fig. 12. We can verify that the forces indeed reach zero on the outer actuators. A slight increase of forces on inner actuators near the border is visible in detail. The positions are barely changed, which demonstrates that the nominal basis is very close to a zero-force policy.

Fig. 9. Force distribution for force-optimised KLs with zero-force policy compared to the nominal case. The assumption for number of corrected modes is 4000.

Fig. 10. Position distribution for force-optimised KLs with zero-force policy compared to the nominal case. The assumption for the number of corrected modes is 4000.

Fig. 11. Position distribution for a nominal basis alteration with zero-force policy compared to the nominal basis. The assumption for the number of corrected modes is 4000. There are 4866 inner actuators.

7. Conclusion

This paper provides the mathematical foundations underpinning the generation of a set of bi-dimensional deformation modes suitable for the optimal control of force-constrained adaptive shell mirrors using the KL formalism. The latter was revisited using the double-diagonalisation method to compute the atmospheric turbulence (stochastic process) eigenfunctions expressed as a linear combination of the DM principal components (geometric force-dependent mechanical deformations). The use of force instead of actuator position control is the novelty we introduced. We developed the means for generating the bases orthogonal to special modes (e.g. tip-tilt) and the subtleties of how to create them to minimise forces and avoid large positions.

The analysis conducted here confirmed that the new basis minimises force, as was expected by construction, shown in Fig. 4. Although the bi-dimensional modal deformations may be different from the classical calculation, the fitting error is by no
Fig. 12. Force distribution for a nominal basis alteration with zero-force policy compared to the nominal basis. The assumption for the number of corrected modes is 4000. There are 4866 inner actuators.

means degraded (Fig. 5). We also showed how to balance force and mechanical excursion (i.e. position) by introducing regularisation. This acts as a damping parameter to maintain both force and position at reasonable values.

The formalism we used is suitable for vetting any control basis an instrument wishes to use in terms of correction efficiency, force, and mechanical excursion. This is an important contribution leading to the assessment of acceptable control bases when another than the one developed here is to be adopted.

Furthermore, our developments are capable of handling different force policies on actuators, as is commonly done with adaptive shell mirrors. This is so as long as they can be expressed as linear penalties in either position or force. We illustrated the application to a first-generation instrument on the ELT in the absence of any constraint policy or with a zero-force policy (Fig. 10). The assessment of the impact of different policies now under consideration for the ELT-M4 will be the subject of a forthcoming paper.

References


**Appendix A: Covariance matrix \( C_p \) expressed in actuator position setpoint space**

Eqs [21] and [23] show how the covariance matrix is used in the classical method. The method will be slightly different in the force-minimisation case, where \( \Delta \) is non-invertible (when all actuators are selected).

The atmospheric statistics is taken into account by computing the covariance matrix of the projection \( h \) of the IFs on turbulent phase screens \( \phi \), like in \( \text{Lai et al.} \ (2000) \).

\[
\mathbf{h} = \mathbf{H}^T \cdot \mathbf{\phi}.
\]  
(A.1)

The covariance matrix is

\[
\mathbf{C}_h = \langle \mathbf{h} \mathbf{h}^T \rangle = \mathbf{H}^T \cdot \langle \mathbf{\phi} \mathbf{\phi}^T \rangle \cdot \mathbf{H} = \mathbf{H}^T \cdot \mathbf{C}_\phi \cdot \mathbf{H},
\]  
(A.2)

where \( \mathbf{C}_\phi[n_p, n_p] \) is the covariance matrix of the turbulent phase.

For the classical method, the step from \( \mathbf{C}_h \) to \( \mathbf{C}_p \), that is, from projection coefficients to DM position setpoints, is performed in a double integration in the space spanned by the DM IFs, \( \mathbf{C}_p \) defined in \( \text{Lai et al.} \ (2000) \).

The expression of \( \mathbf{C}_p \) is derived here. From \( \text{Eq.} [14] \) (with \( \Gamma = 0 \) when \( \Delta \) is invertible), we have

\[
\mathbf{p} = \Delta^{-1} \cdot \mathbf{h},
\]  
(A.3)

and using \( \text{Eq.} [21] \) we obtain

\[
\mathbf{C}_p = \Delta^{-1} \cdot \mathbf{C}_h \cdot \Delta^{-T}.
\]  
(A.4)

Plugging \( \text{Eq.} [A.2] \) into \( \text{Eq.} [A.4] \) effectively means computing

\[
\mathbf{C}_p = \langle \mathbf{p}_{\text{fit}} \mathbf{p}_{\text{fit}}^T \rangle = \mathbf{H}^T \langle \mathbf{\phi} \mathbf{\phi}^T \rangle \mathbf{H}^{-1},
\]  
(A.5)

that is, the least-squares projection of the phase-covariance function in the space spanned by the DM IFs.

Using the Wiener-Khinchine theorem, we can compute the covariance function more efficiently in the Fourier space by noting

\[
\langle h_i h_j \rangle = \int \mathbf{W}_d(\mathbf{\kappa}) h_i^* h_j \mathbf{d}^2 \mathbf{\kappa},
\]  
(A.6)

where \( \mathbf{W}_d \) is the 2D phase spatial power-spectral density indexed by \( \mathbf{\kappa} = (\kappa_x, \kappa_y) \), and \( ^* \) denotes Fourier-transformed variables. This property is reformulated in \( \text{Appendix B} \) as a set of computer-efficient matrix operations.

**Appendix B: Optimised algorithm for computing \( \mathbf{C}_h \) for von Karman turbulence**

In the expression of \( \mathbf{C}_h \), the covariance matrix \( \mathbf{C}_\phi[n_p, n_p] \) appears. For an ELT with a typical IF resolution of 5 cm, we have \( n_p \approx 5 \times 10^5 \). This makes the matrix \( \mathbf{C}_\phi \) impractical to use. One solution of this problem was proposed in \( \text{Lai et al.} \ (2000) \), where each element of \( \mathbf{C}_h \) was computed in a double integration in the direct space using the relation between covariance matrix and the structure function. We used a method in the Fourier space that was proposed by \( \text{Gendron} \ (1995) \). We present this method and show how it can be written in a matrix form that can be parallelised efficiently.

The Fourier transform (FT) of the covariance matrix of phase points \( \mathbf{C}_h \) is equal to the FT of the power spectrum of phase points (Wiener-Khinchine theorem),

\[
\hat{\mathbf{C}}_h(\mathbf{\kappa}) = \mathbf{W}_d(\mathbf{\kappa}) = 0.023 r_0^{-5/3} \left( \frac{k^2 + \frac{1}{L_0^2}}{\kappa^2} \right)^{-11/6},
\]  
(B.1)

where \( \kappa \) is the 2D vector of spatial frequencies and its module \( \kappa = |\kappa| \).

We define \( n_f \) as the number of points in the Fourier domain. For a Nyquist sampling, we have \( n_f = (2 \times \text{dim})^2 \) at least, where \( \text{dim} \) is the linear dimension of the smallest array containing a pupil.

We define \( f_{\text{max}} = 0.5 \text{dim/diameter} \) as the maximum spatial frequency that can be computed numerically. \( \mathbf{W}_d(\mathbf{\kappa}) \) is a bi-dimensional function.

With the goal of expressing the math in matrix form, we define a vector \( \mathbf{W}[n_f] \) obtained from reshaping the matrix \( \mathbf{W}_d(\mathbf{\kappa}) \) in a vector.

In the same way, we define \( \mathbf{H}[n_f, n_{\text{act}}] \) as the matrix form of the collection of vectors describing the Fourier transform of the IFs. We define also the matrix \( \mathbf{D}[n_f, n_{\text{act}}] \) whose columns are the replication of \( n_{\text{act}} \) times the vector \( \mathbf{W} \) multiplied by some normalisation factor,

\[
\mathbf{D}_L = (2 \kappa_{\text{max}})^2 \mathbf{W}.
\]  
(B.2)

After some developments using the Parseval theorem (see e.g. \( \text{Ellerbroek} \ (2002) \), we can rewrite \( \text{Eq.} [A.2] \) using the expressions in the Fourier domain (where \( \circ \) is the element-wise matrix multiplication, and \( \mathbf{M}^H \) is the hermitian transpose of a matrix),

\[
\mathbf{C}_h = \hat{\mathbf{H}}(\cdot) \circ (\mathbf{D} \circ \hat{\mathbf{H}})(\cdot),
\]  
(B.3)

where \( (\cdot) \) represents matrix derasterisation into a vector format. This matrix equation can easily be parallelised by block and by taking advantage of the fact that it is symmetric.

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