Superdiffusion of energetic particles at shocks: A fractional diffusion and Lévy flight model of spatial transport

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ABSTRACT

Context. The observed power laws in space and time profiles of energetic particles in the heliosphere can be the result of an underlying superdiffusive transport behavior. Such anomalous, non-Gaussian transport regimes can arise, for example, as a consequence of intermittent structures in the solar wind. Non-diffusive transport regimes may also play a critical role in other astrophysical environments such as supernova remnant shocks.

Aims. To clarify the role of superdiffusion in the transport of particles near shocks, we study the solutions of a fractional diffusion-advection equation to investigate this issue. A fractional generalization of the Laplace operator, the Riesz derivative, provides a model of superdiffusive propagation.

Methods. We obtained numerical solutions to the fractional transport equation by means of pseudo-particle trajectories solving the associated stochastic differential equation driven by a symmetric, stable Lévy motion.

Results. The expected power law profiles of particles upstream of the plasma shock, where particles are injected, can be reproduced with this approach. The method provides a full, time-dependent solution of the fractional diffusion-advection equation.

Conclusions. The developed models enable a quantitative comparison to energetic particle properties based on a comprehensive, superdiffusive transport equation and allow for an application in a number of scenarios in astrophysics and space science.

Key words. acceleration of particles – plasmas – shock waves – Sun: heliosphere

1. Introduction

There is growing evidence from in situ observations of energetic particles near heliospheric shocks that the transport behavior of these particles can deviate from standard Gaussian diffusion. This includes observed power law profiles of energetic particles upstream of interplanetary shocks (Perri & Zimbardo 2007, 2008; Giacalone 2012; Perri et al. 2022), in solar energetic particle events (Trotta & Zimbardo 2011), and at the solar wind termination shock (Perri & Zimbardo 2009, 2012). This is different from Galactic cosmic ray propagation, where anomalous transport has not been explored systematically (see, e.g., Becker Tjus & Merten 2020, for a review).

Full-orbit particle simulations in weak, isotropic, random-phase Kolmogorov turbulence added to a uniform background field show different propagation regimes depending on the particles’ reduced gyro radius (Reichherzer et al. 2020, 2022a,b). All of them show either ballistic or Gaussian diffusive behavior with varying energy dependencies. However, from particle tracing in dynamo-generated fields (Shukurov et al. 2017) or synthetic turbulence models (Pucci et al. 2016) extended phases of non-diffusive behavior are observed. Zimbardo et al. (2006) found that in a generalized turbulence geometry, in a quasislab scenario, parallel particle transport can show prolonged superdiffusive transport, while the perpendicular propagation remains subdiffusive. Most studies to date only consider random-phase synthetic turbulence without accurate structure formation due to a lack of phase correlations (e.g., Dundovic et al. 2020; Mertsch 2020). Similarly, Magnetohydrodynamic (MHD) simulations (e.g., Beresnyak 2019), while capturing in principle the full dynamics of turbulence, exhibit only a very short and often spectrally distorted inertial range due to a lack of resolution. Next to such fundamental studies, it is thus important to also develop effective models that go beyond the paradigm of diffusive transport and that can be compared to simulation results and observations.

Anomalous diffusion encompasses both so-called subdiffusion and superdiffusion, which, to the first order, are characterized by a nonlinear temporal evolution of the particle mean-square displacement; that is, the second moment of the particle distribution. This is expressed quantitatively as

\[ \langle (\Delta x)^2 \rangle = \kappa_\eta \zeta t^\zeta, \]

with \( \zeta \neq 1 \) being the anomalous diffusion index and \( \kappa_\eta \) a generalized diffusion constant, where one can distinguish the three diffusion regimes depending on \( \zeta \), namely subdiffusion \((0 < \zeta < 1)\), normal or Brownian or Gaussian diffusion \((\zeta = 1)\), and superdiffusion \((1 < \zeta < 2)\).

For a comprehensive overview of anomalous diffusion processes and their description, including the necessity of a generalized central limit theorem (e.g., Bouchaud & Georges 1990), see Metzler & Klafter (2000) and Metzler & Klafter (2004). More recent reviews of nonclassical transport processes in laboratory, heliophysical, and astrophysical plasmas have been presented by Perrone et al. (2013), Zimbardo et al. (2015), and Perri et al. (2022).

In the context of energetic particle transport, a number of theoretical studies have been performed, which have investigated...
the physical foundations and consequences of anomalous transport. Duffy et al. (1995) and Kirk et al. (1996) studied subdiffusive shock acceleration, assuming long-range coherent fields at perpendicular shocks. This way, a more rapid acceleration may be possible. Ragot & Kirk (1997) investigated the anomalous diffusion of electrons in a galaxy cluster. Köta & Jokipii (2000) analyzed the classical concept of effective subdiffusion due to compound perpendicular diffusion. The general result is a non-Markovian perpendicular transport process with an anomalous diffusion index, $\zeta = 1/2$, resulting from the superposition of field-line wandering and parallel transport. Webb et al. (2006) introduced (amongst other approaches) a fractional Fokker–Planck model for the compound perpendicular diffusion of cosmic rays. Shalchi & Kourakis (2007), Shalchi et al. (2007) and le Roux et al. (2010) extended the compound diffusion model to nonlinear theories. The relation between intermittent heliospheric turbulence caused by magnetic flux-tube structures and anomalous transport has been explored more systematically in recent years (e.g., Alouani-Bibi & le Roux 2014; Malandraki et al. 2019; le Roux & Zank 2021; le Roux 2022). Anomalous transport has also been studied in connection with: high-energy cosmic rays in galactic super- \#s (Barghouty & Schnee 2012); superdiffusive acceleration in the heliospheric termination shock, supernova shocks, and galaxy cluster shocks (Perri & Zimbardo 2012; Perri et al. 2016; Zimbardo & Perri 2017), yielding harder energy spectra than predicted from standard theory; and galactic propagation (Buonocore & Sen 2021; Hu et al. 2022). The acceleration model has been augmented by the consideration of Lévy walks in the studies by Zimbardo & Perri (2013) and Prete et al. (2019).

In the present paper, we aim to explicitly formulate and solve a fractional transport equation for particles near a plasma shock wave that can describe superdiffusive transport. In the following section, we introduce the transport equation based on fractional derivatives and describe the solution method based on the equivalent stochastic differential equation (SDE) with a Lévy symmetric process for the superdiffusive term. In Sect. 3, we explore the solutions to the equation. Finally, in Sect. 4 we discuss the results and future prospects for our approach.

2. Anomalous transport with fractional equations and Lévy flights

In general, subdiffusive behavior corresponds to a time-fractional diffusion equation, while superdiffusion can be described by a space-fractional diffusion operator (e.g., Chukbar 1995; Zimbardo et al. 2017). In the present work, we concentrate on superdiffusion, and thus consider a symmetric, fractional diffusion operator. This approach is complementary to a propagator approach employing asymptotic properties (e.g., Perri & Zimbardo 2009). Solving a fractional transport equation allows for the detailed computation of the distribution function of energetic particles in dependence of the phase-space coordinates and a subsequent extension to higher dimensions in future studies.

A general form of the fractional Fokker-Planck equation (FFPE) for the distribution function, $f(x,t)$, at position $x$ and time $t$, with a prescribed potential, $V(x)$, and a generalized, spatially constant diffusion coefficient, $\kappa_0$, is given by (Metzler & Klafter 2000; Magdziarz & Weron 2007)

$$\frac{\partial f(x,t)}{\partial t} = \alpha D_t^{1-\alpha} \left[ \frac{\partial}{\partial x} V''(x) + \kappa_0 \nabla^\alpha \right] f(x,t),$$  \hspace{1cm} (2)

with the Riemann–Liouville fractional derivative defined as

$$\alpha D_t^{1-\beta} f(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \int_0^t (t-s)^{\beta-1} f(s) \, ds,$$ \hspace{1cm} (3)

and the Riesz derivative (Podlubny 1998; Metzler & Klafter 2004) given by

$$\nabla^\alpha f(x) = -\frac{1}{2 \cos(\alpha \pi/2)} (-\mu D_t^\alpha + x D_x^\alpha) f(x),$$ \hspace{1cm} (4)

From dimensional analysis, the diffusion coefficient in Eq. (2) is related to the one in Eq. (1) via $\kappa_0 = \kappa_{\zeta} / \zeta$, with $\alpha = 2 / \zeta$, so that the dimension of $\kappa_0$ is (length)$^\alpha$(time)$^{1-\alpha}$ and that of $\kappa_{\zeta}$ is (length)$^2$(time)$^\zeta$ (Metzler & Klafter 2000; Fichtner et al. 2014). In the special case of Gaussian diffusion – that is, $\zeta = 1$ and $\alpha = 2$ – one has $\kappa_0 = \kappa_{\zeta} \equiv \kappa$.

The FFPE can also be derived from a generalized master equation (Metzler et al. 1999). In general, the Riesz derivative can be regarded as the proper fractional generalization of the Laplace operator, due to its symmetry properties and its Fourier transform characterization, $\mathcal{F}[\nabla^\alpha f(x)] = -|k|^\alpha \mathcal{F}[f(k)]$. In the following, we consider this equation in the usual way:

$$\hat{f}(k, t) = \mathcal{F}[f(x, t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x, t) \exp(-ikx) \, dx,$$ \hspace{1cm} (5)

2.1. Shock transport model

We employed a simplified model of an interplanetary or astrophysical nonrelativistic shock and the transport of energetic particles in its vicinity. The shock is assumed to be one-dimensional and planar. The considerable complication of an oblique or spherical shock and the introduction of further dimensions was left to future studies, because here we were primarily interested in the effect of superdiffusive particle propagation close to the shock. Thus, also ignoring all momentum-dependent and subdiffusive effects, the time-fractional Riemann–Liouville derivative in the FFPE gives just the identity operator (setting $\beta = 1$). For the space-fractional index, $\alpha$, we allowed for values between 1 and 2. Using further a potential function giving the usual advective term and introducing a delta-functional, time-homogeneous, mono-energetic source of the particles at the origin, the FFPE was written explicitly for only one dimension using the common partial derivative notation for the Riesz derivative. The background velocity, $a$, and the (fractional) diffusion coefficient, $\kappa_0$, were assumed to be constant.

In the following, we consider this equation as the basis of our modeling and we introduce nondimensional units. In a helio-physical scenario, this means for example that the length is given in AU, $L_0 = 1$ AU $= 1.49 \times 10^{13}$ cm, and the velocity in solar wind speed units, $a_0 = 400 \text{ km s}^{-1}$ (for our form of the diffusion advection equation, this velocity has to be negative to allow for a solar wind flowing from negative to positive x values). This gives the normalization time as the time it takes the solar wind to cross 1 AU, as $t_0 = 372500 \approx 4.3$ days. The diffusion coefficient is thus measured in units of $\kappa_0 = 5.96 \times 10^{20}$ cm$^2$/s.
In the limit of Gaussian diffusion ($\alpha = 2$), Eq. (6) reduces to the usual diffusion-advection equation
\[
\frac{\partial}{\partial t} f(x, t) = a \frac{\partial}{\partial x} f(x, t) + \kappa \frac{\partial^2}{\partial x^2} f(x, t) + C_0 \delta(x),
\]
which of the steady state solution is readily given as
\[
f(x, \infty) = C_0 \frac{1}{|a|} \exp \left( -\frac{a}{\kappa} x \right) + C_1,
\]
for $x < 0$, and $f(x, \infty) = C_0 |a|^{-1} + C_1$ for $x > 0$. Here, $C_0$ and $C_1$ are arbitrary integration constants that can be utilized to model the source strength and a constant particle background. This choice implies that $a < 0$ is equivalent to an advection in a positive $x$ direction (i.e., opposite to the choice made in Litvinenko & Effenberger 2014 and Stern et al. 2014).

### 2.2. Fourier series solution

Stern et al. (2014) have found a Fourier series representation of the fundamental solution to the fractional diffusion equation (i.e., Eq. (6) without a source and advection term) on a finite periodic domain of length $L$:
\[
f(x, t) = \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{L} \cos \left( \frac{n\pi x}{L} \right) \cdot \exp \left( -\left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} t \right).
\]
This series representation coincides with the exact solution given in terms of Fox’s H-function in the limit $L \to \infty$ (Mainardi et al. 2005). To include the advection contribution, this Fourier series solution was modified as
\[
f(x, t) = \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{L} \cos \left( \frac{n\pi (x + at)}{L} \right) \cdot \exp \left( -\left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} t \right).
\]
Finally, to obtain the full solution to Eq. (6) including the source, the previous expression can be integrated with respect to time, which yields
\[
f(x, t) = \int_0^t \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{L} \cos \left( \frac{n\pi (x + at')}{L} \right) \cdot \exp \left( -\left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} t' \right) \, dt' + \sum_{n=1}^{\infty} \left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} \kappa_\alpha L \cos \left( \frac{n\pi x}{L} \right) - \frac{n\pi}{\kappa \kappa_t} \sin \left( \frac{n\pi x}{L} \right) - \frac{n\pi}{\kappa \kappa_t} \kappa_\alpha L \cos \left( \frac{n\pi (x + at)}{L} \right) \exp \left( -\left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} t \right) + \frac{n\pi}{\kappa \kappa_t} \sin \left( \frac{n\pi (x + at)}{L} \right) \exp \left( -\left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} \left( \frac{1}{2} + \frac{n\pi}{\kappa \kappa_t} \right) \right) \right)
\]
\[
= \left( \frac{n\pi}{\kappa \kappa_t} \right)^{\alpha} \left( \kappa_\alpha L \left( L^2 + n^2 \pi^2 a^2 \right) \right).
\]

The convergence of this series depends on the period, $L$, with $f(x, t) = f(x + 2L, t)$ and consequently it does not converge on the exact solution for $t \to \infty$, since the periodic domains begin to overlap. In practice, $L$ has to be chosen to be sufficiently large to prevent these influences. We used this solution to validate our Lévy flight model.

### 2.3. Numerical method based on a stochastic differential equation

We used a modified version of CRPropa3.2 (Alves Batista et al. 2022) based on the equivalence between a certain class of SDEs and the Fokker–Planck equation (Gardiner 2009). The standard SDE methods have become increasingly popular in the energetic particle modeling context due to their simplicity and scalability with modern computer architecture (see, e.g., Kopp et al. 2012 for an account of numerical techniques). They have been applied successfully to, for example, solar energetic particle transport (Dröge et al. 2010) and cosmic ray modulation problems (Strauss et al. 2011), as well as high-energy cosmic ray transport, including acceleration in the case of Gaussian diffusion (Achterberg & Schure 2011; Aerdker et al. 2024). Strauss & Effenberger (2017) give an extended review of the relevant methods and applications.

For the full FFPE (Eq. (2)), an SDE-type method has been developed by Magdziarz & Weron (2007) that resembles a competition between subdiffusion and Lévy flights. In this approach, the subdiffusive aspect is represented by a time rescaling of the stochastic process, called subordination, while the space-fractional aspect of Lévy flights is modeled by an $\alpha$-stable
random distribution, which deviates from the Gaussian exponential behavior. Some more details on particle transport modeling with this approach are also described in Effenberger (2014). Since in our model we are only considering superdiffusion, the main aspect is the generalization to $\alpha$-stable distributions. This gives the following SDE,

$$\mathrm{d}X(t) = -a \mathrm{d}t + \sqrt{2\kappa_0^\alpha} \mathrm{d}L_\alpha(t),$$

(12)

where $\mathrm{d}L_\alpha(t)$ is a Lévy $\alpha$-stable distribution that fulfills the Fourier transform characteristic

$$\mathcal{F}\{e^{i\xi L_\alpha(t)}\} = e^{-|\xi|^\alpha}.\quad(13)$$

In each simulation time step, $\Delta t$, a random number, $\eta_\alpha$, is drawn from the Lévy $\alpha$-stable distribution and the pseudo-particle position is updated based on an advective step, $-a\Delta t$, and a diffusive step, $\sqrt{2\kappa_0^\alpha} \eta_\alpha \Delta t^{1/\alpha}$. The random number generation is based on the Chambers–Mallows–Stuck method (Chambers et al. 1976), which is explained in more detail in Appendix A.

Figure 1 shows a comparison between pseudo-particle orbits driven by a Wiener process ($\alpha = 2$) and a Lévy motion ($\alpha = 1.7$) for a simple SDE with $a = 0$ and $\kappa_0 = 1$ obtained with CRPropa3.2. All pseudo-particles are injected at $t = 0$ and followed for a unit time interval. The trajectories show the prominent Lévy flight behavior of the pseudo-particles, giving occasional large spatial jumps that result from the heavy tails in the random distribution (compare to Fig. A.1). These jumps become more pronounced and frequent for smaller values of $\alpha$. It has to be emphasized at this point that these trajectories represent only samples of the stochastic process and not real particle trajectories. Their jumps thus do not violate any finite particle speed effects. Still, the mean squared displacement of the Lévy distribution diverges (Metzler & Klafter 2000). This is a potential issue for the spatial transport of massive particles, which could be solved by replacing Lévy flights with Lévy walks (Prete et al. 2019, 2021). Lévy walks are spatiotemporal-coupled and long jumps get a time penalty, which recovers a well-defined mean squared displacement (see Metzler & Klafter 2000, 2004 and references therein).

The solution to the fractional diffusion-advection (Eq. (6)) is given by the conditional probability of the random process described by the SDE (Eq. (12)). In practice, the distribution function, $f(x, t)$, is obtained from an ensemble average over a large number of pseudo-particle trajectories sampling the stochastic process. We only consider the time-forward approach here, although in principle one can also integrate trajectories backward in time from the phase-space point of interest (see again Kopp et al. 2012 for details on the difference between forward and backward methods in the classical SDE context).

To model the chosen source term in the transport equation, pseudo-particles need to be continuously injected at the origin. The structure of CRPropa3.2, however, does not allow for an intrinsic description of such continuous injection. Instead, the pseudo-particles position in phase-space are stored at times $T_i = i\Delta T$ during the simulation. In the later analysis, the distribution function, $f(x, T)$, at time $T$ can be constructed by summing over all distributions, $f(x, T_i)$, weighted by the time interval, $\Delta T$. The time resolution, $\Delta T$, is sufficient, if the distribution function does not change too much during the time interval, $\Delta T$. For more details, we refer to Merten et al. (2017, 2018), Aerdker et al. (2024).

3. Results for the transport of shock particles

To validate our solution methods presented in the previous chapter, we compared them in the framework of our shock model. We considered the time evolution of the spatial structure of the distribution function for Gaussian diffusion and for a case of superdiffusion with $\alpha = 1.7$, which is similar to the values estimated in the data analysis of Perri & Zimbardo (2009) for the termination shock.

The result at three different times for Gaussian diffusion is given in the left panel of Fig. 2 in a semi-logarithmic plot. The solid lines giving the Fourier series solution (as is presented in Sect. 2.2) coincide very well with the symbols, which show the numerical solution to the SDE (Eq. (12)) calculated with a total

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1 Note, that $\Delta t$ and $\Delta T$ do not necessarily need to be the same.
number of $10^4$ pseudo-particles released at the origin. In case of Gaussian diffusion, the steady state is reached at $t = 100$ for $x < 30$.

The resulting distributions in the superdiffusive case (Fig. 2, right panel) show an equally good agreement between both solution methods. In particular, the upstream solution shows some prominent deviation from the exponential behavior of the classical result. A peaked structure at the origin is formed as well. This is visible more clearly in Fig. 3, where the peak is shown for different values of $\alpha$.

We find that for different values of $\kappa_\text{ff}$ (keeping the velocity at unity) the upstream Fourier series solution has different turning distances, as is illustrated in Fig. 4. It can be seen that close to the shock the solution resembles more the behavior of a Gaussian diffusion model before turning into a power law for larger distances. This enables us to estimate possible values for the superdiffusion coefficient, $\kappa_\text{ff}$, from the turnover into the power law in comparison with data (Perri et al. 2015). In Fig. 4, the upstream solution of the distribution is compared for different values of the diffusion coefficient, $\kappa_\text{ff}$, for Gaussian diffusion and superdiffusive transport. It can clearly be seen that the turnover into the power laws depends on the (super)diffusion coefficient.

This behavior is also clearly visible in Fig. 5, where we investigate the transition to the power law as dependent on $\alpha$. It can be seen that the break scale is dependent on alpha, while the Gaussian diffusion case of $\alpha = 2$ remains an exponential rollover that is matched closely by the other cases in the vicinity of the shock.

4. Conclusions

We have presented a stochastic solution method based on Lévy flights for the fractional diffusion-advection equation for cosmic ray transport. With the modified CRPropa3.2 code\(^2\), we are able to reproduce time-dependent spatial profiles of the full solution of the space-fractional equation as ensemble averages of pseudo-particles. Our numerical solutions are validated by semi-analytic Fourier series solutions of the space-fractional equation.

\(^2\) We used the CRPropa3.2 version with the git hash 83a85e54, which will become publicly available in the future.

A particular feature of these solutions is a pronounced peak in the spatial profile near the shock position. Figure 3 illustrates this and the dependence of the feature on the superdiffusivity parameter, $\alpha$. This may provide a further observational constraint on superdiffusion models and can also be of relevance to the effective acceleration spectra in a superdiffusion scenario. While the acceleration of particles itself was beyond the scope of our present study, we aim to investigate the superdiffusive acceleration of particles in a comprehensive SDE model in future work.

We emphasize the advantages of the SDE approach in such a context. For superdiffusive shock acceleration, a second ordinary differential equation can be solved along with Eq. (12), describing the energy gain of pseudo-particles due to the divergence of the advective background flow, $a(x)$, at the shock (for more details on simulating diffusive shock acceleration with SDEs, see e.g., Achterberg & Schure 2011 and Aerdker et al. 2024).

An additional advantage of the SDE approach is that three-dimensional superdiffusive transport could also be modeled.
by a system of SDEs, analogously to Eq. (12). This would enable the study of spherical shocks and, potentially, galactic transport.

In summary, a quantitative model of superdiffusive energetic particle transport, such as the one presented in this paper, enables us to bridge our theoretical understanding of anomalous diffusion with specific questions related to the modeling of observed energetic particle profiles. Combined with new insights from full-orbit simulations in turbulence, extensions to higher dimensions, and the inclusion of particle acceleration, future work should give us a more complete picture of anomalous diffusion in space plasmas.

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Appendix A: Random number generation

In the modified version of CRPropa3.2, a random number, $\eta_\alpha$, is drawn from the $\alpha$-stable Lévy distribution for each time step, $\Delta t$, and for every pseudo-particle to get the diffusive step $\Delta x_{\text{diff}} = \sqrt{2} (\kappa_\alpha \Delta t)^{1/\alpha} \eta_\alpha$. In Figure A.1 pseudo-random numbers (PRNs) drawn from an $\alpha$-stable Lévy distribution with $\alpha = 1.5$ and $\alpha = 2$ are shown. Compared to the Gaussian distribution, the $\alpha$-stable Lévy distribution with $\alpha \neq 2$ has enhanced tails and an enhanced center.

The algorithm from Chambers et al. (1976) (see also Magdziarz & Weron 2007) was used to generate the PRNs:

$$\eta_\alpha = \frac{1}{\sqrt{2}} \frac{\sin(\alpha U)}{[\cos(U)]^{1/\alpha}} \left[ \frac{\cos((1-\alpha)U)}{V} \right]^{(1-\alpha)/\alpha}. \quad (A.1)$$

Here, $U = \pi(\xi - 0.5)$ and $V = -\log(\xi)$, and $\xi$ is a random number uniformly distributed on $(0, 1)$.

![Figure A.1. Histogram (200 bins) of $10^7$ PRNs drawn from a symmetric $\alpha$-stable distribution with $\alpha = 1.5$ and $\alpha = 2$.](image)

Appendix B: Delta injection

The power law asymptotics of the superdiffusive process can already be studied in the simpler case of pure fractional diffusion,

$$\frac{\partial f}{\partial t} = \kappa_\alpha \nabla^{\alpha} f, \quad f(x, 0) = \delta(x). \quad (B.1)$$

Figure B.1 shows the solution of the distribution function, $f$, decaying over time as compared to the Fourier series approximation (Equation 9) with excellent agreement, giving further validation of our method.

![Figure B.1. Fourier series (dash-dotted line) and SDE (dots) solution of the distribution function for the delta injection of pseudo-particles at $t = 0$.](image)