Field-level Lyman-\(\alpha\) forest modeling in redshift space via augmented nonlocal Fluctuating Gunn-Peterson Approximation

F. Sinigaglia\(^{1,2,3,4}\), F.-S. Kitaura\(^{1,2}\), K. Nagamine\(^{5,6,7}\), Y. Oku\(^{5}\), and A. Balaguera-Antolínez\(^{1,2}\)

1 Instituto de Astrofísica de Canarias, Calle Via Láctea s/n, 38205 La Laguna, Tenerife, Spain
e-mail: f.sinigaglia@iac.es

2 Departamento de Astrofísica, Universidad de La Laguna, 38206 La Laguna, Tenerife, Spain

3 Department of Physics and Astronomy, Università degli Studi di Padova, Vicolo dell’Osservatorio 3, 35122 Padova, Italy
e-mail: francesco.sinigaglia@phd.unipd.it

4 INAF – Osservatorio Astronomico di Padova, Vicolo dell’Osservatorio 5, 35122 Padova, Italy

5 Theoretical Astrophysics, Department of Earth and Space Science, Graduate School of Science, Osaka University,
1-1 Machikaneyama, Toyonaka, Osaka 560-0043, Japan

6 Kavli-IPMU (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8583, Japan

7 Department of Physics & Astronomy, University of Nevada, Las Vegas, 4505 S. Maryland Pkwy, Las Vegas, NV 89154-4002, USA

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ABSTRACT

Context. Devising fast and accurate methods of predicting the Lyman-\(\alpha\) forest at the field level, avoiding the computational burden of running large-volume cosmological hydrodynamic simulations, is of fundamental importance to quickly generate the massive set of simulations needed by the state-of-the-art galaxy and Ly\(\alpha\) forest spectroscopic surveys.

Aims. We present an improved analytical model to predict the Ly\(\alpha\) forest at the field level in redshift space from the dark matter field, expanding upon the widely used Fluctuating Gunn-Peterson Approximation (FGPA). Instead of assuming a unique universal relation over the whole considered cosmic volume, we introduce a dependence on the cosmic web environment (knots, filaments, sheets, and voids) in the model, thereby effectively accounting for nonlocal bias. Furthermore, we include a detailed treatment of velocity bias in the redshift space distortion modeling, allowing the velocity bias to be cosmic-web-dependent.

Methods. We first mapped the dark matter field from real to redshift space through a particle-based relation including velocity bias, the redshift space distortion modeling, allowing the velocity bias to be cosmic-web-dependent.

Results. We find evidence for a significant difference between the same model parameters in different environments, suggesting that for the investigated setup the simple standard FGPA is not able to adequately predict the Ly\(\alpha\) forest in the different cosmic web regimes. We reproduce the summary statistics of the reference cosmological hydrodynamic simulation that we use for comparison, yielding an accurate mean transmitted flux, probability distribution function, 3D power spectrum, and bispectrum. In particular, we achieve maximum deviation and average deviation accuracy in the Ly\(\alpha\) forest 3D power spectrum of \(\sim 3\%\) and \(\sim 0.1\%\) up to \(k \sim 0.4\hMpc\), and \(-5\%\) and \(\sim 1.8\%\) up to \(k \sim 1.4\hMpc\).

Conclusions. Our new model outperforms previous analytical efforts to predict the Ly\(\alpha\) forest at the field level in all the probed summary statistics, and has the potential to become instrumental in the generation of fast accurate mocks for covariance matrices estimation in the context of current and forthcoming Ly\(\alpha\) forest surveys.

Key words. large scale structure of Universe – dark energy – quasars: emission lines – surveys – methods: numerical

1. Introduction

The neutral hydrogen (H\(\text{I}\)) absorption features imprinted in quasar spectra, also known as the Lyman-\(\alpha\) forest (Ly\(\alpha\) forest), represent promising cosmological probes over the current and next decades. The Ly\(\alpha\) forest traces the density field continuously along the line of sight, and the high number density of quasar sight lines per square degree delivered by state-of-the-art cosmological redshift surveys will allow building accurate three-dimensional maps of the Universe at \(z \gtrsim 1.8\), in other words, at distances that are currently beyond the reach of wide-field galaxy redshift surveys. In the wait of next-generation astronomical facilities such as the Square Kilometre Array Observatory\(^{1}\) and the Extremely Large Telescope\(^{2}\), the Ly\(\alpha\) forest is anticipated to play a pivotal role in refining the constraints on cosmological models and advancing our comprehension of the high-redshift Universe.

To maximize the scientific return and optimize the extraction of cosmological information from the unprecedented incoming Ly\(\alpha\) forest datasets provided by surveys as DESI (Levi et al. 2013), Euclid (Amendola et al. 2018), WEAVE-QSO (Pieri et al. 2016), and Subaru-PFS (Takada et al. 2014), it is extremely important to develop accurate analytical models and numerical tools to efficiently interpret and analyze Ly\(\alpha\) forest observables.

\(^{1}\) SKAO: https://www.skao.int

\(^{2}\) ELT: https://elt.eso.org
In particular, a primary goal is to formalize an effective model of the bias relation linking the Ly\(\alpha\) forest to the dark matter field. In fact, an effective bias mapping allows the prediction of the Ly\(\alpha\) forest from the dark matter field in a fast and accurate way, enabling the generation of massive sets of Ly\(\alpha\) forest mock lightcones. Mock catalogs have become the standard tool used in large-scale cosmological surveys to robustly address uncertainties on cosmological parameters, to test pipelines and commissioning tools, and to assess the feasibility of studies targeting new observables, among others. Furthermore, knowing an effective bias description makes it possible to forward-model the Ly\(\alpha\) forest at the field level and iteratively reconstruct the Ly\(\alpha\) forest (see e.g., Horowitz et al. 2019; Porqueres et al. 2020) and its baryon acoustic oscillations (BAOs), improving on methods of de-evolving cosmic structures back in time with some approximation for the displacement field (e.g., Eisenstein et al. 2007, and later works implementing higher-order Lagrangian perturbation theory refinements).

In addition, the dark matter field – Ly\(\alpha\) forest connection has found an interesting application in tomographic studies as well, such as in the LATIS survey (Newman et al. 2020), in the CLAMATO survey (Lee et al. 2018; Horowitz et al. 2022b), and in the PSF Galaxy Evolution Survey (Greene et al. 2022).

Pioneering studies (Hui & Gnedin 1997; Rauch 1998; Croft et al. 1998; Weinberg et al. 1999) proposed to model the Ly\(\alpha\) forest using a deterministic scaling relation mapping – known as Fluctuating Gunn-Peterson Approximation (FGPA) – between the dark matter density and the H I optical depth. While useful, fairly accurate on large linear scales, and fast to compute, the FGPA has been shown to lose accuracy and not adequately model the summary statistics of the Ly\(\alpha\) forest in the weakly nonlinear regime \((k > 0.1 \, h \, \text{Mpc}^{-1})\) (e.g., Sinigaglia et al. 2022). A plethora of works employed the FGPA to forward-model the Ly\(\alpha\) forest from dark matter density fields obtained through N-body simulations (Meiksin & White 2001; Viel et al. 2002; Slosar et al. 2009; White et al. 2010; Roa et al. 2013; Lee et al. 2014), approximated gravity solvers (Horowitz et al. 2019), and Gaussian or lognormal field (Le Goff et al. 2011; Font-Ribera et al. 2012; Farr et al. 2020).

In particular, the L\(\text{Y}^{\text{ALPHA}}\)-COLORe method (Farr et al. 2020), applying the FGPA to lognormal fields, with free parameters constrained so as to match the position and width of the acoustic peak in the Ly\(\alpha\) forest two-point correlation function and the amplitude of the 1D Ly\(\alpha\) forest power spectrum, was adopted to produce the mock lightcones used for the BAO measurements of the Ly\(\alpha\) forest from the final eBOSS data release (SDSS DR16; du Mas des Bourboux et al. 2020).

On the other hand, more sophisticated iterative methods of modeling the Ly\(\alpha\) forest, employing different strategies, have been proposed, such as for example LYMAS (Peirani et al. 2014, 2022) and the Iteratively-Matched Statistics (Ims, Sorini et al. 2016). These techniques have been shown to yield promising results, although they still feature deviations of 5–20% in the 3D power spectrum. Moreover, these techniques were not applied to approximated gravity solvers, but rely on full N-body simulations. Therefore, they are not able to overcome the computational burden of running massive sets of simulations.

With the flourishing of machine learning methods, and the extraordinary attention that this field is receiving in astrophysics, the generation of AI\(^3\)-accelerated simulated cosmological volumes is witnessing a rapid expansion.

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3 Artificial Intelligence.
\[ m_{\text{gas}} = 1.57 \times 10^9 \text{ h}^{-1} M_\odot \] for gas particles. The gravitational softening length was set to \( \epsilon_g = 16 \text{ h}^{-1} \text{kpc} \) (comoving), but we allowed the baryonic smoothing length to become as small as \( 0.1 \epsilon_g \). This means that the minimum baryonic smoothing at \( z = 2 \) is about physical 533 \text{ h}^{-1} \text{pc}. The star formation and supernova feedback were treated as described in Shimizu et al. (2019). The code also contains important refinements, such as the density-independent formulation of SPH and the time-step limiter (Saitoh & Makino 2009, 2013; Hopkins et al. 2013).

The main baryonic processes that shape the evolution of the gas are photo-heating, photo-ionization under the UV background radiation (Haardt & Madau 2012), and radiative cooling. All of these processes are accounted for and solved by the Grackle library\(^4\) (Smith et al. 2017), which determines the chemistry for atomic (H, D and He) and molecular (H\(_2\) and HD) species. The chemical enrichment from supernovae is also treated with the CEL1b chemical evolution library by Saitoh (2017). The initial conditions are generated at redshift \( z = 99 \) using MUSIC2 (Hahn et al. 2021), with cosmological parameters taken from Planck Collaboration VI (2020).

In this work we used the output at \( z = 2 \) (for which the computation times amount to \( 1.44 \times 10^5 \) CPU hours), reading both gas and dark matter properties.

To compute the Ly\(_\alpha\) forest flux field \( F = \exp(-\tau)^5 \), we first obtained the HI optical depth, \( \tau \), by means of a line-of-sight integration as follows (Nagamine et al. 2021):

\[
\tau = \frac{\pi e^2}{m_e c^2} \sum_j f(x-x_j) m_{\text{HI}}(x_j) \Delta l,
\]

where \( e, m_e, c, n_{\text{HI}}, f, x_j, \) and \( \Delta l \) denote, respectively, the electron charge, electron mass, speed of light in vacuum, \( \text{HI} \) number density, absorption oscillator strength, line-of-sight coordinate of the \( j \)th cell, and the physical cell size. The Voigt-line profile, \( \phi(x) \), in Eq. (1) was provided by the fitting formula of Tasitsiomi (2006). Where necessary, relevant quantities (e.g., \( \text{HI} \) number density) were previously interpolated on the mesh according to the SPH kernel of the simulation. The coordinates, \( x_j \), of cells along the line of sight refer to the outcome of the interpolation of particles based either on their positions in real space, \( r_j \), or redshift space, \( s_j = r_j + v_\text{los}/aH \), where \( v_\text{los} \), \( a \), and \( H \) are the \( j \)th particle velocity components along the line of sight, the scale factor, and the Hubble parameter at \( z = 2 \), respectively.

We interpolated the dark matter and the Ly\(_\alpha\) forest fields in real and in redshift space onto a 256\(^3\)-cell cubic mesh using a CIC mass assignment scheme (Hockney & Eastwood 1981). Such a resolution corresponds to a physical cell-volume of \( \partial V \sim (1.95 \text{ h}^{-1} \text{Mpc})^3 \) and a Nyquist frequency of \( k_{\text{Nyq}} \sim 1.6 \text{ h} \text{Mpc}^{-1} \).

It is worth noting that the simulations utilized for the analyses of the Ly\(_\alpha\) forest in Momose et al. (2021), Nagamine et al. (2021), the exploration of the cosmological HI distribution in Ando et al. (2019), and the examination of bias in cosmological gas tracers in Sinigaglia et al. (2021, 2022) were all executed using the same code.

3. Cosmic web classification and nonlocal bias

The cosmic web (see e.g., Bond et al. 1996) arises as a result of gravitational instability and the formation and growth of cosmic structures from tiny perturbations of the primordial matter density field in the early Universe. While several different methods have been proposed in the literature to mathematically define the cosmic web (see Cautun et al. 2014; Libeskind et al. 2017, for a summary), we focus here on the following procedure.

To quantitatively describe the large-scale matter distribution and split it into different cosmic web environments, Hahn et al. (2007) proposed a classification scheme based on the signature of the eigenvalues of the gravitational tidal field tensor,

\[
\mathbf{T}_j(r) = \partial_i \partial_j \phi(r),
\]

where \( \phi \) denotes the gravitational potential and \( r \) stands for Eulerian coordinates. Considering the equations of motion in comoving coordinates

\[
r = -\nabla \phi(r)
\]

for a test particle subject to the gravitational potential, \( \phi \), and assuming \( \nabla \phi(r) = 0 \) at the center of mass of dark matter haloes (i.e., there is a local minimum), one can linearize the equation of motions and realize that the dynamics close to local extrema of the gravitational potential in the linear regime is ruled by the three eigenvalues \( \lambda_i \) (\( i = 1, 2, 3 \)) of \( \mathbf{T}_j \) (we refer the reader to Hahn et al. 2007, for more details on the calculations). In a close analogy to the Zel’dovich approximation (Zel’dovich 1970), and sorting the \( \lambda_i \) in decreasing order such that \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \), Hahn et al. (2007) defined a region at coordinates \( r \) to belong to:

- a knot, if \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \);
- a filament, if \( \lambda_1, \lambda_2 \geq 0, \lambda_3 < 0 \);
- a sheet, if \( \lambda_1 \geq 0, \lambda_2, \lambda_3 < 0 \);
- a void, if \( \lambda_1, \lambda_2, \lambda_3 < 0 \).

At this point, one can drop the assumption of local extrema at the center of mass of the dark matter halo and generalize the cosmic web classification to any point in the density field. This implies the emergence of a constant additive acceleration term to the linearized equations of motion, which however does not affect the role played by the tidal tensor in linear order.

Elaborating on this work, Forero-Romero et al. (2009) proposed to relax the threshold for classifying the cosmic web, based on simple dynamical arguments and realizing that raising the threshold from \( \lambda_0 = 0 \) to \( \lambda_0 = 0.1 \) provides a classification that better matches the visual appearance of the cosmic web.

The T-web classification has been widely adopted to investigate the properties of dark matter haloes, galaxies, and intergalactic gas in the cosmic web (Hahn et al. 2007; Yang et al. 2017; Martizzi et al. 2019; Sinigaglia et al. 2021), to develop fast, accurate bias models to generate haloes and galaxy mock catalogs (e.g., Zhao et al. 2015; Balaguer-Antolínez et al. 2018, 2019, 2023; Pellejero-Ibáñez et al. 2020; Kitaura et al. 2022), and to perform observational cosmological analysis (e.g., Lee & White 2016; Krolewski et al. 2017; Horowitz et al. 2019), among other things.

The T-web classification has also been shown to provide a direct connection between the phenomenology of the cosmic web and the mathematical description of halo bias relying on Eulerian perturbation theory (see e.g., McDonald & Roy 2009). In fact, discriminating between different T-web environments corresponds to considering a perturbative bias expansion up to the third order, including both local and nonlocal long-range bias terms. In this sense, following Kitaura et al. (2022), we denote the three invariants of the tidal field tensor as:

- \( I_1 = \lambda_1 + \lambda_2 + \lambda_3 \);
- \( I_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \);
- \( I_3 = \lambda_1 \lambda_2 \lambda_3 \).

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\(^4\) https://grackle.readthedocs.io/

\(^5\) The notation \( F = \exp(-\tau) \) for the Ly\(_\alpha\) forest flux field actually refers to the transmitted flux \( F/F_\odot \), in other words to the quasar spectrum normalized to its continuum \( F_\odot \). We will omit hereafter the normalization and denote the transmitted flux just as \( F \) unless specified otherwise.
These three items represent an alternative formulation of the perturbative bias expansion up to the third order (Kitaura et al. 2022), but they can also be straightforwardly linked to the phenomenological T-web classification as follows:

- knots: $I_1 > 0 & I_2 > 0 & I_1 > A_1$;
- filaments: $I_1 < 0 < I_2 < 0 \parallel I_3 < 0 < I_2 > 0 \& \lambda_3 < I_1 < \lambda_3 - \lambda_3 A_3 A_1$;
- sheets: $I_1 > 0 & I_2 < 0 \parallel I_3 < 0 & I_2 > 0 \& \lambda_1 - \lambda_1 A_1 A_3 > I_3 < I_1$;
- voids: $I_1 > 0 & I_2 > 0 \& I_3 < \lambda_1$.

Eventually, one can generalize this description by relaxing the value of the eigenvalues threshold and replacing zero with any other arbitrary value.

From an intuitive point of view, this means that T-web models the full perturbative expansion up to the third order adopting a low-resolution binning, that is, just four categories identified as cosmic web types (see Kitaura et al. 2022, and references therein). While using the invariants, $I_i$, of $T_{ij}$ has been shown to provide a more accurate description of nonlocal bias and of anisotropic clustering than T-web, the T-web classification has the advantage of being much less likely to incur overfitting.

In this work, we applied the T-web classification to the modeling of the Lyα forest in order to introduce nonlocal contributions within the FGPA prescription. In particular, as is described in more detail in Sect. 4, we made the models for both RSDs and the FGPA dependent on the T-web by fitting the parameters of such models separately for each cosmic web type. Also, we extracted the cosmic web classification both in real and in redshift space. Table 1 reports the volume filling factor of each cosmic web type for the real space dark matter field, and the redshift space dark matter field obtained by applying the RSD description, including velocity bias dependent on the real space T-web with parameters described in the upper part of Table 2 (see Sect. 4.2 for the details about the procedure), and used to compute our final nonlocal FGPA model.

<table>
<thead>
<tr>
<th></th>
<th>Real space (%)</th>
<th>Redshift space (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knots</td>
<td>2.1</td>
<td>2.7</td>
</tr>
<tr>
<td>Filaments</td>
<td>21.4</td>
<td>21.9</td>
</tr>
<tr>
<td>Sheets</td>
<td>49.2</td>
<td>48.8</td>
</tr>
<tr>
<td>Voids</td>
<td>27.3</td>
<td>26.6</td>
</tr>
</tbody>
</table>

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4.1. Standard FGPA

A popular way to compute a fast proxy for the Lyα forest consists of the FGPA, which assumes an equilibrium between optically thin photoionization and collisional recombination of the intergalactic H I. In particular, because the majority (>90%) in volume and >50% in mass, Lukić et al. 2015) of the gas probed by the Lyα forest found in regions with mildly nonlinear density contrasts is diffuse and not shock-heated, the gas density, $\rho_{\text{gas}}$, and temperature, $T_{\text{gas}}$, can be linked through the power law relation

$$T_{\text{gas}} = T_0 (\rho_{\text{gas}}/\bar{\rho}_{\text{gas}})^\gamma$$

where $\bar{\rho}$ is the mean gas density and $T_0$ and $\gamma$ depend on the reionization history and on the spectral slope of the UV background model. These vary commonly within the ranges $4000 \mathring{\text{K}} \leq T_0 \leq 10^5 \mathring{\text{K}}$ and $0.5 \leq \gamma \leq 0.6$ (see e.g., Hui & Gnedin 1997). Assuming a photo-ionization equilibrium, one can express the Lyα forest optical depth, $\tau$, as a function of the gas density as

$$\tau = A (1 + \delta_{\text{dm}})^{\alpha}. \quad (5)$$

This expression represents the FGPA, as $\tau$ is described as a field that fluctuates with the underlying gas distribution. While $\alpha = 2 - 0.7\gamma \sim 1.6$ (see e.g., Weinberg et al. 1999; Viel et al. 2002; Seljak 2012; Rorai et al. 2013; Lukić et al. 2015; Cieplak & Slosar 2016; Horowitz et al. 2019), $A$ is a normalization constant that depends on redshift and on the details of the hydrodynamics (e.g., Weinberg et al. 1999). Given that in the cool, low-density regions, dark matter and gas densities display a very high cross-correlation (see e.g., Sinigaglia et al. 2021), and given that the FGPA constitutes a useful tool to be applied to the dark matter field to obtain Lyα forest predictions without solving the equations of hydrodynamics, $\delta_{\text{gas}}$ can be replaced with $\delta_{\text{dm}}$ in Eq. (5), so that the FGPA is applied directly to the dark matter field:

$$\tau = A (1 + \delta_{\text{dm}})^{\alpha}. \quad (6)$$

The FGPA has been shown to represent a good approximation for very high-resolution full N-body simulations (e.g., Sorini et al. 2016; Kooistra et al. 2022), in other words when representing density fields on regular grids with cell size $l \sim (100\mathring{\text{kpc}}) h^{-1}$, where particle positions and velocities are known with high accuracy thanks to the exact solution of collisionless and fluid dynamics for dark matter and gas particles or cells, respectively. However, this is not the case if either a coarser grid resolution ($\geq1-2\mathring{\text{Mpc}} h^{-1}$) and/or approximated gravity solvers are to be adopted, as in the case of the massive generation of large-volume Lyα forest boxes encompassing all the universe up to $z \sim 4$ (e.g., Farr et al. 2020). In particular, the summary statistics predicted by the FGPA have been shown to depart from the corresponding hydrodynamic flux field statistics obtained with the full hydro computation already at cell resolutions of $l \sim 0.8\mathring{\text{Mpc}} h^{-1}$, both in real and in redshift space (Sinigaglia et al. 2022).

In this work, we computed the FGPA based on the cubic mesh used to CIC-interpolate the simulation particles, that is, with physical cell size $l \sim 1.95\mathring{\text{Mpc}} h^{-1}$. To motivate the choice of a coarse grid, we notice that such a resolution implies that one needs to resort to a grid with $N = 5120^3$ cells to represent a simulation box of volume $V = (10\mathring{\text{Gpc}} h^{-1})^3$, covering a full-sky cosmic realization at $0 \leq z \leq 3.8$, to cover the relevant volume needed for Lyα forest studies. This implies an already quite
To put ourselves in realistic observational conditions, we computed the FGPA in redshift space. To this end, we displaced particles from real to redshift space via the mapping (Kaiser 1987; Hamilton et al. 1998)

\[
s_i = r_i + \frac{(\theta_i \cdot \hat{r}_i)}{aH},
\]

(7)

where \(s_i\) and \(r_i\) are the Eulerian comoving coordinates of the \(i\)th particle in redshift space and real space, \(\hat{r}_i = r_i/|r_i|\), \(v_i\) is its velocity, and \(a\) and \(H\) are the scale factor and the Hubble parameter at the redshift of interest, respectively. Under the assumption of plane-parallel approximation that we adopt, and therefore neglecting the lightcone geometry, Eq. (7) simplifies to only one scalar component along the line of sight.

In this setup, we computed the “standard” FGPA by setting \(\alpha = 1.6\) and determining the parameter \(A\) as the value that matches the normalization of the Lyα forest 3D power spectrum on large scales. In this case, we find \(A = 0.27\).

4.2. RSD modeling

While the mapping from real to redshift space presented in Eq. (7) holds exactly for dark matter particles, it does not account for any velocity bias contribution in the Lyα forest. Therefore, following Sinigaglia et al. (2022), we generalized such a model as follows.

We start by considering the dark matter field in real space. For each cell, \(i\), we assigned a set of \(N\) fictitious particles, with the Eulerian position, \(r_i\), coinciding with the center of the cell. Applying an inverse nearest grid point (INGP) scheme, each particle, \(j\), inside a cell, \(i\), was assigned a mass \(M_j = \rho_i V_i/N\), where \(\rho_i\) and \(V_i\) are the dark matter density and volume of the cell, respectively. The \(j\)th particle was then displaced from real to redshift space using a modified version of Eq. (7) accounting for the velocity bias:

\[
s_j = r_j + b_v \frac{(\theta_{dm,j} \cdot \hat{r}_i)}{aH},
\]

(8)

where \(s_j\) and \(r_j\) are the redshift space and real space Eulerian comoving coordinates of the \(j\)th particle, \(\theta_{dm,j}\) is the modeled velocity of the particle, and \(b_v\) is a velocity bias factor. The particle velocity, \(\theta_{dm,j}\), was modeled as the sum of two components, \(\theta_{dm,j} = \theta_{dm,j}^{\text{coh}} + \theta_{dm,j}^{\text{disp}}\). Here, \(\theta_{dm,j}^{\text{coh}} = \theta_{dm,j}^{\text{disp}}\) corresponds to the velocity field at cell \(i\), interpolated on the mesh with the same mass assignment scheme as the density field, and models the large-scale coherent flows. We draw the attention of the reader to the fact that the velocity bias, \(b_v\), directly multiplies the coherent flows component of the velocity field. On the other hand, \(\theta_{dm,j}^{\text{disp}}\) is built as a velocity dispersion term, randomly sampled from a Gaussian with zero mean and a variance proportional to the value of the density field inside the cell, \(i\), in real space through a power law: \(\theta_{dm,j}^{\text{disp}} \sim N(\mu = 0, \sigma = B (1 + \beta^2))\), with \(B\) and \(\beta\) free parameters (Heß et al. 2013; Kitaura et al. 2014). This latter component models the quasi-virialized velocity dispersion motions, which are smoothed out by the interpolation of the velocity field on the mesh.

6 We adopt the nomenclature “standard” in contrast to the nonlocal FGPA version presented in the following sections.

To determine the optimal values for the RSDs and FGPA normalization parameters, we jointly maximized the cross-correlation coefficient between the FGPA prediction and the Lyα forest field flux from the reference simulation and matched the normalization of the Lyα forest 3D power spectrum at large scales. After fixing \(\alpha = 1.6\) as a fiducial value, we find the choice \(A = 0.7\), \(b_v = -0.8\), \(b_v = 1.5\), and \(\beta = 2.0\) to fulfill the aforementioned constraints.

4.3. Nonlocal bias contribution

Looking at Eq. (6), one easily realizes that the standard FGPA prescription provides a way to compute the Lyα forest optical depth relying exclusively on the local dark matter density, and neglecting any nonlocal contribution. While the simplicity of the FGPA model is appealing as it allows one to compute the Lyα forest with just a few parameters, it attempts to model distinct cosmic web environments with the same physical model, and therefore fails to capture the intrinsic diversity of physical conditions. As a major contribution of this paper, we propose to make the FGPA cosmic-web-dependent, to relax the exponent \(\alpha\) and let it vary, and to determine the sets of coefficients that best describe each cosmic web type. We point out that we are considering at this stage the dark matter field in redshift space as modeled in Eq. (8), and we therefore extract the T-web classification directly in redshift space.

Analogously, we also make the RSD model in Eq. (8) dependent on the cosmic web environment, thereby passing from the three free parameters \(\{b_v, B, \beta\}\), to 12 parameters \((3 \times 4 = 12)\) cosmic web environments). However, in order to keep the number of free parameters as small as possible, we notice that the virial theorem allows us to write \(v \propto \sqrt{\rho}\), and therefore we fix \(\beta = 0.5\). Moreover, peculiar velocity is closer to the linear or quasi-linear regime in sheets and voids, and hence one can expect the velocity dispersion component to be negligible there, allowing us to set \(B_x = B_{\beta} = 0\). These two tricks allow us to reduce the number of free parameters from twelve to six.

We notice that, in contrast to the cosmic web classification discussed above, here the T-web is extracted in real space, as the T-web itself is needed to perform the mapping from real to redshift space.

In summary, to introduce the nonlocal dependence on the cosmic web in the FGPA we:

1. extracted the T-web from the dark matter in real space;
2. determined the distinct parameters \(\{b_v, B, \beta\}\) depending on the real space T-web;
3. mapped particles from real to redshift space, adopting the cosmic-web-dependent version of Eq. (8);
4. extracted the T-web from the dark matter in redshift space;
5. used the redshift space T-web to model nonlocalities in the FGPA by making the free parameters \(\{A, \alpha\}\) dependent on the cosmic web category each cell belongs to.

4.4. Threshold bias

It is well known from cosmological hydrodynamic simulations that the Lyα forest transmitted flux one-point probability distribution function (PDF) is starkly bimodal, featuring two sharp peaks around \(F = 0\) and \(F = 1\), and displaying a much lower occurrence of cells with intermediate values \(0.1 \lesssim F \lesssim 0.9\) (e.g., Nagamine et al. 2021, and references therein). These

7 We hereafter adopt the following subscript notation: kn = knots, fl = filaments, sh = sheets, vd = voids.

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characteristics can mainly be attributed to the exponential mapping $F = \exp(-\tau)$, which regularizes the domain of $\tau$, mapping it in the interval $[0, 1]$, truncating low values of $\tau$ to $F \sim 1$, and rapidly saturating high values of $\tau$ to $F \sim 0$. The bimodal nature of the Lya forest PDF, together with the high nonlinearity induced by the exponential mapping, makes it particularly hard to capture such behavior and accurately predict fluxes, which can suffer rapid variations within the domain, even in neighboring cells.

To improve on the FGPA model regarding this aspect, we adopted the perspective of introducing a threshold bias (Kaiser 1984; Bardeen et al. 1986; Sheth et al. 2001; Mo & White 2002; Kitaura et al. 2014; Neyrinck et al. 2014; Vakili et al. 2017), which in state-of-the-art models of halo bias is used to suppress the formation in overdense regions.

In analogy with the formalism of the PATCHY method (Kitaura et al. 2014, 2015, 2016a,b), Vakili et al. (2017) and expanding upon it, we updated Eq. (6) by introducing two multiplicative exponential terms,

$$
\tau = A(1 + \delta)^{\alpha} \exp \left( -\frac{\delta}{\delta_1^0} \right) \exp \left( \frac{\delta}{\delta_2^0} \right),
$$

(9)

where the term $\exp \left( -\delta/\delta_1^0 \right)$ represents a threshold bias term acting as a cutoff, while $\exp \left( \delta/\delta_2^0 \right)$ represents an inverse threshold bias acting as a boost. Both terms have a negligible effect when $\delta \ll \delta^*$, whereas they acquire importance when $\delta \gg \delta^*$. Here we did not use the step function used in Vakili et al. (2017), and $\delta_1^0$ and $\delta_2^0$ were left free. Also, the parameters for RSD were left unchanged ($b_v = 0.8$, $b_h = 1.5$, $\beta = 2.0$). We again fixed $\alpha = 1.6$, and found the best-fitting parameter values to be $A = 0.26$, $\delta_1^0 = 0.81$, and $\delta_2^0 = 0.49$.

On top of this modification, following Sect. 4.3, we made the four free parameters of the model depend on the T-web classification, in other words, $[A, \alpha, \delta_1^0, \delta_2^0]$, with $i = [\text{knots, filaments, sheets, voids}]$. In this way, we modeled the FGPA differently depending on distinct cosmic web environments, thereby effectively including nonlocal bias.

So far, the model has $22 = 6 + 16$ free parameters (6 for RSD as described in Sect. 4.2, $16 = 4 \times 4$ for the augmented nonlocal FGPA prescription). We describe the procedure we adopted to determine such parameters and our findings in Sect. 5 and Table 2, respectively.

4.5. Stochasticity

As the final ingredient, we added a stochastic component, $\epsilon$, to the model expressed in Eq. (9) and the subsequent nonlocal modification:

$$
\tau = A_i(1 + \delta)^{\alpha_i} \exp \left( -\frac{\delta}{\delta_1^0_i} \right) \exp \left( \frac{\delta}{\delta_2^0_i} \right) + \epsilon_i,
$$

(10)

where $\epsilon_i = f_i N_i$ is the noise term in the $j$th cell for $i = [\text{knots, filaments, sheets, voids}]$, and $f_i$ is a normalization constant. While $f_i$ is a parameter controlling the amplitude of the stochastic term, $N_i$ is sampled from a negative binomial distribution (see Kitaura et al. 2014; Vakili et al. 2017), parameterized as

$$
P(N|n, p) = \frac{\Gamma(N + n)}{N!\Gamma(n)} p^n(1 - p)^N,
$$

(11)

where $N$ is the sampled term and corresponds to the number of failures in the standard negative binomial distribution definition, $n$ stands for the number of successes, $N + n$ is the total number of trials, and $p$ is the success probability.

In particular, as will be commented on in more detail in Sect. 5, the latest nonlocal modification of Eq. (9) without stochasticity yields predictions that lack power toward small scales and display suppression of small-scale structures, especially in underdense environments. To this end, we decided to model the stochastic term only in voids, leaving the improved nonlocal FGPA model deterministic in knots, sheets, and voids.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{v,kn}$</td>
<td>$-0.9$</td>
<td>Coh. flows velocity bias in knots</td>
</tr>
<tr>
<td>$b_{v,f}$</td>
<td>$-0.8$</td>
<td>Coh. flows velocity bias in filaments</td>
</tr>
<tr>
<td>$b_{v,sh}$</td>
<td>$-1.0$</td>
<td>Coh. flows velocity bias in sheets</td>
</tr>
<tr>
<td>$b_{v,v}$</td>
<td>$-0.5$</td>
<td>Coh. flows velocity bias in voids</td>
</tr>
<tr>
<td>$B_{kn}$</td>
<td>$3.0$</td>
<td>Q.v.m. velocity bias in knots</td>
</tr>
<tr>
<td>$B_{f}$</td>
<td>$2.0$</td>
<td>Q.v.m. velocity bias in filaments</td>
</tr>
<tr>
<td>$B_{sh}$</td>
<td>$0.0$</td>
<td>Q.v.m. velocity bias in sheets</td>
</tr>
<tr>
<td>$b_{v,0}$</td>
<td>$0.0$</td>
<td>Q.v.m. velocity bias in voids</td>
</tr>
<tr>
<td>$\beta_{kn}$</td>
<td>$0.5$</td>
<td>Q.v.m. exponent in knots</td>
</tr>
<tr>
<td>$\beta_{f}$</td>
<td>$0.5$</td>
<td>Q.v.m. exponent in filaments</td>
</tr>
<tr>
<td>$\beta_{sh}$</td>
<td>$-0.6$</td>
<td>Q.v.m. exponent in sheets</td>
</tr>
<tr>
<td>$\beta_{v,0}$</td>
<td>$-0.7$</td>
<td>Q.v.m. exponent in voids</td>
</tr>
</tbody>
</table>

**Notes.** The left column reports the symbol used throughout the paper for each parameter, the central column reports the value and the associated uncertainties, and the right column provides a brief description of what the parameter stands for.

4.7. Mean bias

<table>
<thead>
<tr>
<th>Mean bias</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{kn}$</td>
<td>$0.23^{+0.05}_{-0.06}$</td>
<td>Normalization in knots</td>
</tr>
<tr>
<td>$A_{f}$</td>
<td>$0.11^{+0.03}_{-0.02}$</td>
<td>Normalization in filaments</td>
</tr>
<tr>
<td>$A_{sh}$</td>
<td>$0.20^{+0.04}_{-0.09}$</td>
<td>Normalization in sheets</td>
</tr>
<tr>
<td>$A_{vd}$</td>
<td>$0.15^{+0.36}_{-0.12}$</td>
<td>Normalization in voids</td>
</tr>
<tr>
<td>$\alpha_{kn}$</td>
<td>$4.03^{+0.66}_{-0.16}$</td>
<td>Power law exponent in knots</td>
</tr>
<tr>
<td>$\alpha_{f}$</td>
<td>$3.97^{+0.69}_{-0.67}$</td>
<td>Power law exponent in filaments</td>
</tr>
<tr>
<td>$\alpha_{sh}$</td>
<td>$2.21^{+0.71}_{-0.67}$</td>
<td>Power law exponent in sheets</td>
</tr>
<tr>
<td>$\alpha_{vd}$</td>
<td>$2.03^{+0.72}_{-0.68}$</td>
<td>Power law exponent in voids</td>
</tr>
<tr>
<td>$\delta_{1,kn}$</td>
<td>$1.36^{+0.55}_{-0.53}$</td>
<td>Exponential cutoff scale in knots</td>
</tr>
<tr>
<td>$\delta_{1,f}$</td>
<td>$0.93^{+0.51}_{-0.50}$</td>
<td>Exponential cutoff scale in filaments</td>
</tr>
<tr>
<td>$\delta_{1,sh}$</td>
<td>$0.85^{+0.42}_{-0.45}$</td>
<td>Exponential cutoff scale in sheets</td>
</tr>
<tr>
<td>$\delta_{1,vd}$</td>
<td>$1.01^{+0.39}_{-0.68}$</td>
<td>Exponential cutoff scale in voids</td>
</tr>
<tr>
<td>$\delta_{2,kn}$</td>
<td>$0.41^{+0.48}_{-0.37}$</td>
<td>Exponential boost scale in knots</td>
</tr>
<tr>
<td>$\delta_{2,f}$</td>
<td>$0.17^{+0.51}_{-0.11}$</td>
<td>Exponential boost scale in filaments</td>
</tr>
<tr>
<td>$\delta_{2,sh}$</td>
<td>$0.60^{+0.42}_{-0.39}$</td>
<td>Exponential boost scale in sheets</td>
</tr>
<tr>
<td>$\delta_{2,vd}$</td>
<td>$0.25^{+0.47}_{-0.18}$</td>
<td>Exponential boost scale in voids</td>
</tr>
</tbody>
</table>

**Stochasticity**

<table>
<thead>
<tr>
<th>Stochastic</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\epsilon}$</td>
<td>$1.38^{+0.38}_{-0.27}$</td>
<td>Normalization of stochastic term</td>
</tr>
<tr>
<td>$n$</td>
<td>$0.11^{+0.34}_{-0.08}$</td>
<td>Number of successes</td>
</tr>
<tr>
<td>$p$</td>
<td>$0.58^{+0.34}_{-0.28}$</td>
<td>Success probability ($0 &lt; p &lt; 1$)</td>
</tr>
</tbody>
</table>

Notes. The left column reports the symbol used throughout the paper for each parameter, the central column reports the value and the associated uncertainties, and the right column provides a brief description of what the parameter stands for.

8 While other parameterizations are possible, we stick to the one used in this work and reported in the numpy (Harris et al. 2020) documentation, for the sake of the reproducibility of results.

9 $n$ is in principle an integer number, but its definition can be generalized to reals, as is done in this work.
to limit the loss of cross-correlation. This implies $f_e = 0$ in knots, filaments, and sheets. As is shown later in the paper, we find this choice to be sufficient to accomplish the desired goal.

We can develop an intuition about the application of discrete tracer statistics to the Ly$\alpha$ forest in voids as follows. From a mathematical point of view, we start by noticing that we can in principle sample the noise term from any arbitrary continuous distribution. In fact, we are effectively restricting our study to the second moment of the distribution. Starting from this, we can then wonder how to model the variance of the distribution. One possibility would be to assume a Poisson variance, or go beyond it, as is done in this paper by choosing the negative binomial. Therefore, our model is statistically robust. This stochastic component can then be interpreted as follows. Assuming that H I is found in sparse clouds in voids at high redshift, then we can treat them as point sources and model them in a similar way as haloes and galaxies. We will test the validity of this assumption in future work. For the moment, we argue that this effective model works well and achieves the goal it was designed for (see Sect. 5 for the results).

The negative binomial distribution can be interpreted as an overdispersed alternative to the Poisson distribution, whereby one can allow the mean and variance to be different. This aspect can turn out to be useful in cases when two events have a positively correlated occurrence (i.e., the two events have a positive covariance term and are not independent), and this causes a larger variance than in the case in which the events are independent. In the presented parametrization, $n$ controls the deviation from Poissonity, making the distribution converge to the Poisson distribution for large $n$ and causing an overdispersion for small $n$. While a thorough investigation of the deviation from Poissonity for the Ly$\alpha$ forest goes beyond the scope of this paper, it is sensible and convenient to allow the distribution from which we sample the stochastic term to feature a non-Poissonian variance including a correlation term (see e.g., Peebles 1980, for a detailed treatment applied to galaxies).

5. Analysis, results and discussion

In this section, we describe the procedure we adopted to determine the optimal parameters for our model, and present our findings.

5.1. Fit of model parameters

Given the model developed in Sects. 4.2 and 4.3, we fit the coefficients of our model as follows.

We first determined the RSD model parameters by finding the values that maximize the cross-correlation between the predicted and reference Ly$\alpha$ forest field (see the upper part of Table 2). In particular, we started with an initial guess based on the prior knowledge we had, that is, we initialized the parameters $b_{ij} = -0.8$, and $B_i = 1$ for $i = \{\text{knots, filaments, sheets, voids}\}$. As anticipated in Sect. 4.2, we fixed $B_i = 0.5$ for all the cosmic web types, and set $B_{\delta} = B_{ed} = 0$. We then varied the parameters around the initial guess, one cosmic web at a time, applied the standard FGPA prediction, and monitored the variation in the cross-correlation between our prediction and the reference Ly$\alpha$ forest field from the simulation. We noticed that at this point we were still applying the standard FGPA, with the same parameters for all cells, because we had not yet fit the parameters for the nonlocal FGPA model. In order to be able to predict flux with the nonlocal FGPA at this point, one should perform an end-to-end automatic extraction of the parameters, going from the real space dark matter field to the redshift space nonlocal FGPA Ly$\alpha$ forest flux. However, this would imply a computationally expensive optimization procedure, involving the interpolation of particles to the grid at each step. Therefore, we limited ourselves to a simpler manual determination that, while in principle suboptimal, turned out to be very instructive and allowed us to develop a good understanding of how different choices of model parameters impact the cross-correlation at different scales.

Afterward, we determined the parameters for the modified nonlocal FGPA model. Here, we resorted to automatic parameter estimation. In particular, we determined the optimal values for our parameters by sampling their posterior distribution through the affine-invariant emcee Markov chain Monte Carlo (MCMC) sampler (Goodman & Weare 2010; Foreman-Mackey et al. 2013).

We proceeded as follows. We separately fit the Ly$\alpha$ forest flux PDFs $\rho(F)$ of knots, filaments, sheets, and voids, in other words performing a separate fit for each cosmic web type. In this way, each MCMC chain samples a separate distribution, $p(\theta_i|\text{data}) \propto p(\text{ref}|\theta_i)p(\theta_i)$, of the parameters, $\theta_i = \{A_i, \alpha, \delta_1^\ast, \delta_2^\ast\}$, for $i = \{\text{knots, filaments, sheets, voids}\}$, where “ref” denotes the reference simulation Ly$\alpha$ forest PDF $\rho_{\text{ref}}(F)$.

We assumed a Gaussian likelihood for $\rho_{\text{ref}}(F)$:

$$P(\rho(F)|\theta) = \prod_F \frac{1}{\sqrt{2\pi\sigma_F^2}} \exp\left(-\frac{(\rho_{\text{ref}}(F) - \rho_{\text{mock}}(F))^2}{2\sigma_F^2}\right),$$

where $\rho_{\text{ref}}(F)$ and $\rho_{\text{mock}}(F)$ are the flux PDFs of the reference and predicted Ly$\alpha$ forest fields, and $\sigma_F$ corresponds to the number of cells containing a flux value inside the same bin of the PDF as $F$. Furthermore, we chose the following flat priors for the model parameters: $0 < A_i < 2$, $1 < \alpha_i < 5$, $0 < \delta_{\text{1j}} < 1.5$, $0 < \delta_{\text{2j}} < 1.5$, $\forall i$. After running the chain with 32 walkers for 2000 iterations, we computed the autocorrelation length, $\tau_{\text{cld}}$, using the built-in emcee function, and found that in all cases $\tau_{\text{cld}} \sim 100$ iterations. Therefore, we conservatively discarded the first 500 iterations, that is, $\sim 5$ times the chain autocorrelation length. As an example, we show in Fig. 1 the resulting posterior distribution of the $[A, \alpha, \delta_1^\ast, \delta_2^\ast]$ free parameters obtained by fitting the Ly$\alpha$ forest PDF in knots. One clearly notices that there is a tight correlation between $\delta_1^\ast$ and $\delta_2^\ast$.

Eventually, we computed the median of the resulting posterior distribution, and the 16th and 84th percentiles intervals as associated uncertainties.

Up to this point, the model does not include the stochastic noise term, and the predicted $P(k)$ suffers from a lack of power toward small scales with respect to the reference $P(k)$ (see Sect. 5.4 for details). Therefore, the model parameters determined thus far correspond to the nonlocal FGPA without stochasticity. As described in Sect. 4.5, we included a further additive noise term, and the predicted $P(k)$ was sampled from a negative binomial distribution. To determine the best parameters for the binomial distribution $\theta' = \{f_e, n, p\}$, we first manually varied the parameters until we reached a combination of $\theta'$ that enhances the small-scale power enough and provides a good fit of the power spectrum. (see Sect. 5.4 for details). Eventually, we refit the parameters altogether, including the random component and using as an initial guess the optimal parameter values previously found. Furthermore, we adopted the following upper and lower bounds as priors on $\theta'$: $1 < f_e < 2$, $0 < n < 1$, $0 < p < 1$.

We report the final estimated parameter values and their uncertainty in Table 2. We find clear differences in the best-fitting values of the same parameter in different cosmic web environments. This supports our finding that including nonlocal
Fig. 1. Posterior distributions of the \( \{A, \alpha, \delta^*_1, \delta^*_2\} \) free parameters, obtained from the partial fit of the \( \text{Ly}\alpha \) forest PDF in knots.

5.2. Visual inspection of slices through the simulation box

Figure 2 shows a comparison of slices through the simulation box, obtained by averaging two contiguous slices 1.95\,h^{-1}\,Mpc thick parallel to the (x,z) plane, displaying redshift space distortions along the z axis. A visual inspection between the prediction from the standard FGPA (bottom right) and the reference cosmological simulation (top left) immediately reveals that the former fails to adequately render the redshift space cosmic structures, displaying less pronounced redshift space distortions. When introducing a velocity bias correction to the standard FGPA (bottom left), redshift space distortions appear to model more appropriately the observed elongation of structures in the cosmic web; however, it turns out not to capture the saturation at \( F \sim 1 \) in the underdense regions, which is also reflected in the probability distribution function (see also Fig. 3). We again stress that the free parameters for the FGPA model have been chosen in this case in order to match the large-scale amplitude of the power spectrum. It would also be possible to find the parameters terms is playing an informative role in the model about missing physical dependencies in the standard FGPA.

As a final note, we stress that only the fit of parameters for the negative binomial distribution governing the random sampling of the stochastic term was determined by considering deviations between the reference and predicted power spectrum. In fact, in the fit of the parameters of the nonlocal FGPA prescription, only the PDFs were taken into account. In this sense, the fact that this bias model reproduces the power spectrum and the bispectrum with such good accuracy constitutes a reassuring sanity check.
that best fit the PDF, but this yields a severe bias ($\gtrsim 30\%$) in the large-scale predicted power spectrum amplitude. Eventually, our nonlocal FGPA model$^{10}$ (top right) visually resembles the reference simulation to a good degree of approximation. In particular, it succeeds both in reproducing the redshift space structure of the cosmic web and in properly predicting the right values of flux in all the different density regimes. Without the stochasticity, underdense regions in our predicted flux would appear emptier than they actually are. This aspect is reflected in a lack of power toward small scales, as will be commented on in more detail in Sect. 5.4. The addition of the noise term, $\epsilon$, only in voids, is meant to compensate for the lack of substructures. Therefore, we do not expect those small-scale structures to visually match the reference simulation in position. Rather, in a global sense, with the addition of the noise term, voids appear to be more filled with substructures than in the case in which we do not model stochasticity.

5.3. Probability distribution function

Figure 3 shows a comparison between the Ly$\alpha$ forest PDF from the reference simulation (solid red line) and the prediction from a

---

$^{10}$ We hereafter omit to write that the nonlocal model also includes velocity and threshold bias.
standard FGPA (dashed yellow line), from an FGPA with velocity bias (dotted purple line), from an FGPA with velocity and threshold bias (dash-dotted black line), and from our nonlocal FGPA with and without the addition of stochastic fluctuations in voids (dash-dotted blue and dashed green lines, respectively).

The sharp bimodality of the Ly\(\alpha\) forest PDF makes it particularly non-trivial to correctly predict both flux regimes, yielding the correct shape and average flux. While the standard FGPA model fails to reproduce the height of both peaks, the FGPA with velocity (and threshold) bias prescriptions achieve a better prediction toward \(F \sim 0\), but a worse result at \(F \sim 1\), as was already commented on in Sect. 5.2. In particular, the FGPA model with threshold bias achieves a better prediction of the PDF with respect to the FGPA model with only velocity bias, but neither of the two is able to properly account for the low-density peak. The nonlocal FGPA, however, succeeds in reproducing both peaks, and qualitatively also achieves the best result among the studied cases in the intermediate flux regime, \(0.1 \leq F \leq 0.9\).

We stress that our final model, including stochasticity, with model parameters fitted to reproduce the PDFs of knots, filaments, sheets, and voids separately, yields a predicted average flux of \(\bar{F}_{\text{pred}} \sim 0.75\), in great agreement with the value from the simulation, \(\bar{F}_{\text{ref}} \sim 0.76\), even though the model has not been explicitly calibrated to reproduce such a quantity.

We point the reader’s attention to the fact that the difference between the nonlocal FGPA (dashed green line) and the FGPA model with velocity and threshold bias (dash-dotted black line) consists in making the model sensitive to nonlocal terms. It turns out to be clear that the nonlocal dependencies play an important role in improving the prediction of the Ly\(\alpha\) forest PDF.

### 5.4. Power spectrum and cross-correlation

In Fig. 4 we present a comparison between Ly\(\alpha\) forest 3D power spectra, \(P(k)\), in the top panel, power spectrum ratios, \(P_{\text{mock}}(k)/P_{\text{ref}}(k)\), in the mid panel, and cross-correlation coefficients, \(C(k)\), between the reference Ly\(\alpha\) forest and the predictions from the different tested models. In each panel, the investigated reference simulation summary statistics are shown as a solid red line, while the prediction by the standard FGPA is displayed as a dashed yellow line, the FGPA with velocity bias as a dotted purple line, the FGPA with velocity and threshold bias as a dash-dotted black line, the nonlocal FGPA as a dashed green line, and the nonlocal FGPA with stochasticity as a dash-dotted blue line. In the mid panel, the gray shaded areas stand for 1%, 2%, 5%, and 10% deviations, from darker to lighter. In the bottom panel, the gray shaded areas stand for 5% and 10% deviations, from darker to lighter.
coefficients, \(C(k)\), in the bottom panel. We display the reference simulation summary statistics as a solid red line, while we plot the predicted Ly\(\alpha\) forest flux summary statistics as a dashed yellow line in the case of the standard FGPA, as a dotted purple line in the case of the FGPA with velocity bias, as a dash-dotted black line in the case of the FGPA with velocity and threshold bias, as a dashed green line in the case of nonlocal FGPA, and as a dash-dotted blue line the nonlocal FGPA with stochasticity. The top and bottom panels clearly highlight that the standard FGPA (dashed yellow line) and its augmented versions including velocity bias (dotted purple line) and velocity and threshold bias (dash-dotted black line) rapidly lose power toward small scales, reaching ratios \(P_{\text{mock}}(k)/P_{\text{ref}}(k) < 20\%\), \(P_{\text{mock}}(k)/P_{\text{ref}}(k) < 50\%\), and \(P_{\text{mock}}(k)/P_{\text{ref}}(k) < 60\%\), respectively, at the Nyquist frequency \(k_{\text{Nyq}} \sim 1.6 \, h\, \text{Mpc}^{-1}\). As already anticipated, the \(P(k)\) predicted by the nonlocal FGPA model with stochasticity ensures a good fit of the target power spectrum, featuring a maximum deviation of \(-3\%\) and average deviations of \(-0.1\%\) up to \(k \sim 0.4 \, h\, \text{Mpc}^{-1}\), a maximum deviation of \(-5\%\) and average deviations of \(-1.8\%\) up to \(k \sim 1.4 \, h\, \text{Mpc}^{-1}\), and a maximum deviation of \(-8\%\) and average deviations of \(-0.8\%\) up to the Nyquist frequency \(k \sim 1.6 \, h\, \text{Mpc}^{-1}\).

An inspection of the bottom panel of Fig. 4, instead, reveals that the standard FGPA (dashed yellow line) delivers a lower \(C(k)\) with respect to all the other cases that incorporate the modeling of velocity bias. In fact, looking at Fig. 2, it can be appreciated by eye how the standard FGPA (top right) does not model properly the enhancement of cosmic structure elongation along the line of sight clearly seen in the reference simulation (top left). In this sense, an adequate treatment of velocity bias ensures a better cross-correlation at all scales. The nonlocal FGPA (dashed green line) displays a larger \(C(k)\) than the standard FGPA (dashed yellow line) and the local FGPA with velocity bias (dotted purple line) and then the local FGPA with velocity bias (dotted purple line) at all scales, from a 1% gain in \(C(k)\) on large scales up to a \(-3\%\) gain in \(C(k)\) at \(k \sim 1.0 \, h\, \text{Mpc}^{-1}\). Eventually, the nonlocal FGPA with stochasticity (dash-dotted blue line) features a slightly lower \(C(k)\) than the nonlocal FGPA (dashed green line) up to \(k \sim 0.5 \, h\, \text{Mpc}^{-1}\), while it starts to depart from the latter at larger \(k\), lacking a \(-6\%\) \(C(k)\) at \(k \sim 1.0 \, h\, \text{Mpc}^{-1}\). This is expected by construction, due to the addition of a random component, which has the effect of lowering the cross-correlation. However, the loss of \(C(k)\) is not dramatic, and the nonlocal FPGA model with stochasticity (dash-dotted blue line) still improves the local FGPA with velocity bias \(C(k)\) (dotted purple line) up to \(k \sim 0.8 \, h\, \text{Mpc}^{-1}\). Remarkably, the FPGA model with velocity and threshold bias is found to provide the
highest $C(k)$ across all scales, even though it is very similar to the prediction by the nonlocal FGPA model.

Here again, the difference between the nonlocal FGPA (dashed green line) and the FGPA model with velocity and threshold bias (dash-dotted black line) consists of the inclusion of the nonlocal dependence on the cosmic web. The remarkable difference (≥20% gain in the accuracy of the predicted $P(k)$ at the Nyquist frequency) makes clear the improvement accounted for by the nonlocal model.

5.5. Bispectrum

To address the capability of the model to reproduce higher-order statistics, we also assessed the accuracy of the predicted bispectrum. We report in Fig. 5 the reduced bispectrum, $Q(\theta_{12})$, as a function of the subtended angle, $\theta_{12}$, for four different triangular configurations: (i) $k_1 = 0.1$, $k = 0.2\, h\, \text{Mpc}^{-1}$ (top left), (ii) $k_1 = 0.2$, $k = 0.2\, h\, \text{Mpc}^{-1}$ (top right), (iii) $k_1 = 0.3$, $k = 0.5\, h\, \text{Mpc}^{-1}$ (bottom left), and (iv) $k_1 = 0.5$, $k = 0.5\, h\, \text{Mpc}^{-1}$ (bottom right). The plot displays the bispectrum of the reference simulation (solid red line), compared to the predictions from the standard FGPA (dashed yellow line), the FGPA with velocity bias (dotted purple line), the FGPA with velocity and threshold bias (dashed-dotted black line), the nonlocal FGPA (dashed green line), and the nonlocal FGPA with stochasticity (dash-dotted blue line). The latter (i.e., our preferred model) reproduces the target bispectrum with reasonable accuracy for all the probed triangular configurations, while the others start to feature significant deviations at $k \geq 0.2\, h\, \text{Mpc}^{-1}$. In particular, the nonlocal FGPA model with stochasticity reproduces not only the overall shape of the reference bispectrum, but also some peculiar features, such as the position and amplitude of the peak around $\theta \sim 2.3$ in the $k_1 = k_2 = 0.5\, h\, \text{Mpc}^{-1}$ configuration, as well as the position (although not very much the amplitude) of the peak around $\theta \sim 2.1$ in the $k_1 = 0.3$, $k_2 = 0.5\, h\, \text{Mpc}^{-1}$ configuration.

6. Lightcone generation

In this section we illustrate how our method can be generalized to other redshifts and applied to generate Ly$\alpha$ forest lightcones.

We start by noticing that we have the following redshifts available from the simulation: $z = 1.8$, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, and 4.0.

First, we show that the method also generalizes at other redshifts. To do so, we re-calibrated the parameters of our model at $z = 1.8$ and 2.2. Figure 6 shows the resulting Ly$\alpha$ forest power spectra ratios at $z = 1.8$ (dotted purple line) and at $z = 2.2$ (dashed green line). The achieved accuracy is very similar to that shown in Fig. 4 (mid panel) at $z = 2$. Subsequently we demonstrate that, once the calibration is performed for some redshift snapshots, we can interpolate the parameters for the redshifts in between and ensure a continuous smooth redshift evolution. In particular, we linearly interpolated the parameters obtained at $z = 1.8$ and $z = 2.2$ and predicted the Ly$\alpha$ forest field at $z = 2$. We show the resulting $P(k)$ at $z = 2$ in Fig. 6 (dash-dotted blue line), noticing that again the achieved accuracy is very similar to the one we get via direct calibration at $z = 2$.

Afterward, we also re-calibrated the model parameters for the remaining redshift snapshots and generated a lightcone box as follows. We considered the redshift range $\Delta z = [1.8, 3.8]$. The comoving radial distance subtended by this redshift interval is $l \sim 1500\, h^{-1}\, \text{Mpc}$. Therefore, after having mapped the dark matter fields at all redshift from real to redshift space, we replicated the same simulation box three times in the redshift direction.

![Fig. 6. Comparison of 3D Ly$\alpha$ forest power spectrum ratios, $P_{\text{mock}}(k)/P_{\text{ref}}(k)$, between the predicted and reference power spectra at different redshifts. Results at $z = 1.8$ (dotted purple line) and $z = 2.2$ (dashed green line) are obtained via independent calibrations. Instead, the $z = 2$ result (dash-dotted blue line) corresponds to the Ly$\alpha$ forest power spectrum that we get by applying the nonlocal FGPA model with parameters linearly interpolated at $z = 2$ from calibrations at $z = 1.8$ and $z = 2.2$. The gray shaded areas stand for 1%, 2%, 5%, and 10% deviations, from darker to lighter.]

Then, we connected the redshift space dark matter field snapshots transitioning from one to the following at the average redshift between the two. Even though this implies a sharp transition in the dark matter field, we painted the Ly$\alpha$ forest on the lightcone by interpolating the parameters of the model with a cubic spline interpolation scheme at each redshift slice. This ensured a smooth redshift evolution. We show a slice through the final lightcone box in Fig. 7. One can clearly visualize how the Ly$\alpha$ forest changes from $z = 1.8$ to $z = 3.8$, with voids becoming bigger and emptier toward the low-redshift end.

In a forthcoming publication, we will present full-sky Ly$\alpha$ forest mocks by generating dark matter fields with lightcone geometry and smooth evolution in redshift already in the dark matter field (Sinigaglia et al., in prep.).

7. Potential future improvements

While the presented model, as anticipated, turns out to be sufficient to fulfill the purpose it was designed for, there are further potential improvements that we leave to be explored in future works.

In this work, we include nonlocal dependencies in the bias formulation only through the T-web. Such dependencies are known to have a long-range effect (e.g., McDonald & Roy 2009), and we are hence neglecting short-range terms, the effect of which kicks in when it comes to modeling the nonlinear clustering toward small scales. Short-range nonlocal terms can be constructed as scalars from the curvature tensor $\delta_{ij} = \partial_i \partial_j \delta$, such as its Laplacian $\nabla^2 \delta$ in Eulerian (e.g., Peacock & Heavens 1985; McDonald & Roy 2009; Werner & Porciani 2020; Kitaura et al. 2022) or Lagrangian space (e.g., Zennaro et al. 2022), which characterizes the shape of the local maxima of the density field (Peacock & Heavens 1985). More generally, one can build short-range nonlocal terms as scalars computed from arbitrarily higher-order derivatives of the density field, such as $\delta_i \partial_j \partial_k \delta$ (e.g., Kitaura et al. 2022). Alternatively, in analogy with the T-web description based on the invariants of the tidal field tensor
(see also Sect. 3), one can work in a framework relying on the invariants of the curvature tensor. This latter description has been proven to encode crucial information to model the bias of baryon density fields (e.g., Sinigaglia et al. 2021, 2022). Since we do not model such a family of dependencies here, we stress that we may be missing some relevant piece of physical information. However, we also stress that computing such terms requires an accurate enough modeling of small scales, which is not guaranteed here. Therefore, by introducing short-range nonlocal terms we may face the risk of introducing a noisy component, to the detriment of the final accuracy. Moreover, in the computationally cheapest way of modeling such dependencies, we would be extracting the analogous to the T-web based on \( \partial_i \), which would imply a non-negligible number of additional free parameters in our model.

In connection to the previous point, we note that Kitaura et al. (2022) show that the I-web model encodes a larger amount of information on the clustering of biased tracers with respect to the T-web, at the expense of a larger number of parameters (the number of bins used to described the invariants), and therefore is at a greater risk of incurring overfitting. Moreover, while describing the nonlocal quantities by explicitly using the invariants of the tidal tensor is possible (e.g., Pellejero-Ibañez et al. 2020), it implies devising a suitable functional form for each additional term used in the bias, which is in principle nontrivial.

Another point that can be improved consists of the way we modeled the low-density regions, identified as cosmic voids according to the T-web classification. In fact, in order to compensate for the lack of substructures, we randomly sampled a noise component from a negative binomial distribution. At this point we noticed that this could be avoided, or its impact alleviated, by improving the modeling of the deterministic component. One easy modification could consist of binning the density distribution in voids into an arbitrary number of intervals, and identifying distinct parameters for such distinct density bins. However, this would again introduce a larger number of free parameters in our model.

Eventually, if a noise term is to be included as is done in this work, we note that we have randomly sampled the stochastic component from a negative binomial distribution, for the reasons previously stated, but this is not the only possible solution. In fact, one may find that using a different distribution may be more convenient.

We leave all these potential improvements to be investigated in future publications, or in applications adopting this model.

8. Conclusions

This work presents a significantly augmented version of the widely used FGPA to predict the Ly\( \alpha \) forest in redshift space, as observed from current and forthcoming spectroscopic galaxy surveys. This new model relies on explicit modeling of velocity bias and redshift space distortions and of a modification to the FGPA introducing a cutoff and a boosting scale, with free parameters (determined based on both heuristic and physical arguments, as well as on an efficient MCMC scheme) made dependent on the cosmic web environment as described by the T-web classification (knots, filaments, sheets, and voids, Hahn et al. 2007). In this sense, we label our model a “nonlocal” FGPA. In fact, the Ly\( \alpha \) forest flux in each cell is here made dependent on the content of the surrounding cells and the geometry of the gravitational field, and the model effectively incorporates nonlocal bias information up to third-order in Eulerian perturbative bias expansion (see Kitaura et al. 2022). In addition, since we find that such a model fails to reproduce the small-scale clustering in the underdense regions, we add a further stochastic term only in voids, randomly sampled from a negative binomial distribution. This random component has the effect of enhancing the small-scale power, and hence of improving the fit of the Ly\( \alpha \) forest forest 3D power spectrum toward small scales.

We predicted Ly\( \alpha \) forest fluxes with a standard FGPA, our preferred nonlocal FGPA model with stochasticity, and three other intermediate models in between, on a mesh with \( V = (500 h^{-1} \text{ Mpc})^3 \) volume and \( N_c = 256^3 \) cells, with physical cell resolution \( l \sim 1.95 h^{-1} \text{ Mpc} \), trying to reproduce a full cosmological hydrodynamic N-body/SPH simulation (spanning the same volume) and run with \( N = 1024^3 \) particles. We assessed the accuracy of the prediction of such models by comparing the results regarding the mean transmitted flux, \( \bar{F} \), the PDF, the power spectrum, and the bispectrum, with the analogous summary statistics computed from the reference simulation.

The augmented nonlocal FGPA with stochasticity improves upon the standard FGPA model in all the investigated summary statistics. In fact, the nonlocal FGPA accurately reproduces the mean transmitted flux, \( \bar{F} \), the flux PDF, the power spectrum, and the bispectrum. In particular, our model yields a mean transmitted flux, \( \bar{F}_{\text{red}} \approx 0.75 \), in excellent agreement with the value \( \bar{F}_{\text{red}} \approx 0.76 \), and a power spectrum featuring a maximum deviation of \( \sim 3\% \) and average deviations of \( \sim 0.1\% \) up to \( k \sim 0.4 h \text{ Mpc}^{-1} \), a maximum deviation of \( \sim 5\% \) and average deviations of \( \sim 1.8\% \) up to \( k \sim 1.4 h \text{ Mpc}^{-1} \), and a maximum deviation of \( \sim 8\% \) and average deviations of \( \sim 0.8\% \) up to the Nyquist frequency \( k \sim 1.6 h \text{ Mpc}^{-1} \). The predicted PDF and the
bispectrum clearly outperform the results from any other model tested in this paper as well.

Compared to other schemes based on machine learning (e.g., Harrington et al. 2022; Horowitz et al. 2022a; Sinigaglia et al. 2022) and other methods based on iterative calibrations (e.g., Sorini et al. 2016; Peirani et al. 2022), our model has the appeal of being a purely analytical method, and therefore it is fast to compute, less prone to the overfitting problem, and can be straightforwardly generalized to larger or smaller volumes, different mesh resolutions, and at different redshift snapshots. This method can even ensure a continuous smooth treatment of redshift evolution via interpolation of the fitting coefficients in between the redshift snapshots used for calibration, avoiding discontinuity problems at the edge of distinct contiguous redshift shells.

We point out that the resolution (1.95 h^{-1} Mpc physical cell size) considered in this work is not sufficient to provide realistic Lyα forest high-resolution spectra with correct modeling of the 1D power spectrum, needed to perform studies of small-scale (k \sim 5–10 h^{-1} Mpc^{-1}) physics, such as bounds on the mass of warm dark matter particles (Viel et al. 2005, 2013) and of neutrinos (Palanque-Delabrouille et al. 2015) and constraints on the thermal state of the intergalactic medium (Garzilli et al. 2017, and references therein), among others. While the application of the nonlocal FGPA to high-resolution density fields is possible, one should bear in mind that resolving sub-Mpc scales on a 3D mesh makes it computationally very expensive to realize large-volume cosmological boxes. Therefore, in future works, we will explore novel techniques based on Bayesian inference aimed at upsampling the resolution of 1D Lyα forest skewers extracted from low-resolution Lyα forest flux boxes generated following the procedure presented in this work.

We plan to adopt this novel scheme to generate Lyα forest mock lightcones for the DESI survey, in combination with accurate approximated gravity solvers correcting for shell-crossing (e.g., Kitaura & Hess 2013; Tosone et al. 2021; Kitaura et al. 2023). In particular, we aim to use a recent fast structure formation model based on iterative Eulerian Lagrangian perturbation theory (eALPT, Kitaura et al. 2023), which has been shown to reproduce the clustering of N-body simulations with percent accuracy deep into the nonlinear regime. Furthermore, the application of the nonlocal FGPA model to approximated dark matter fields generated through the ALPT or eALPT approximations can potentially represent a new avenue in Bayesian density field reconstruction studies using the Lyα forest. We will investigate this point in future publications.

In conclusion, we argue that this novel nonlocal FGPA scheme represents a significant step forward with respect to previous analytical efforts, and could become instrumental in the generation of fast accurate mocks for covariance matrices estimation in the context of current and forthcoming Lyα forest surveys.

The code implementing the model presented in this work has been made publicly available. \footnote{https://github.com/francescosinigaglia/nl-fgpa}

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