Higher order ionospheric effects in testing gravitational redshift by space frequency signal transfer

Pengfei Zhang\textsuperscript{1}, Wen-Bin Shen\textsuperscript{1,2}, Chenxiang Wang\textsuperscript{1}, Ziyu Shen\textsuperscript{3}, Chenghui Cai\textsuperscript{1}, Xiao Sun\textsuperscript{1}, Wei Xu\textsuperscript{1}, Rui Xu\textsuperscript{1}, Abdelrahim Ruby\textsuperscript{1,2,4}, and Mostafa Ashry\textsuperscript{1,2,5}

\textsuperscript{1} Time and Frequency Observation Center, School of Geodesy and Geomatics, Wuhan University, 129 Luoyu Road, Wuhan 430079, PR China
e-mail: wbshen@sgg.whu.edu.cn
\textsuperscript{2} State Key Laboratory of Information Engineering in Surveying, Mapping and Remote Sensing, Wuhan University, 129 Luoyu Road, Wuhan 430079, PR China
\textsuperscript{3} School of Resource, Environmental Science and Engineering, Hubei University of Science and Technology, 88 Xianning Avenue, Xianning 437100, Hubei Province, PR China
\textsuperscript{4} Geomatics Engineering Department, Faculty of Engineering at Shoubra, 108 Shoubra street, Benha University, Cairo 11629, Egypt
\textsuperscript{5} Civil Engineering Department, Faculty of Engineering, Misr Aswan Agricultural Road, EL MAHATTA, Minia University, Minia 61111, Egypt

Received 2 November 2021 / Accepted 31 July 2023

ABSTRACT

\textbf{Context.} When a microwave passes through the ionosphere, it produces ionospheric refraction and path bending, leading to changes in frequency and reducing the accuracy of frequency transmission. Currently, the Atomic Clock Ensemble in Space (ACES, 2023) and China Space Station (CSS, 2022) carry atomic clocks with a long-term stability of $10^{-16}$ and $10^{-18}$. The accuracy of the frequency comparison and gravitational redshift (GRS) test matches the corresponding order of magnitude.

\textbf{Aims.} Based on ground-space frequency links and considering the frequency shift caused by the higher order terms of the ionosphere, the gravitational redshift (GRS) test could be achieved at a higher level of accuracy.

\textbf{Methods.} We formulated a higher order ionospheric frequency shift model and analyzed the ionosphere effects on the one-way frequency transfer, as well as the dual- and tri-frequency combination methods, for frequency transfer between a space station (ACES or CSS) and a ground-based station.

\textbf{Results.} The analysis shows that for one-way frequency transfer, the second-order ionospheric frequency shift is about $10^{-15}$, $10^{-17}$, and $10^{-18}$ for the S-, Ku-, and Ka-bands, respectively. The second- and third-order ionospheric frequency shifts were eliminated using the dual-frequency combination method for CSS frequency transfer. When using the tri-frequency combination method for frequency transfer, the second ionospheric frequency shifts are about $10^{-16}$ to $10^{-17}$ for ACES and $10^{-19}$ for CSS, while the third-order frequency shifts are smaller than $10^{-19}$ for two missions.

\textbf{Conclusions.} Concerning the current atomic clock’s accuracy and microwave link frequencies for ACES and CSS missions, the second-order ionospheric frequency shift needs to be considered and eliminated, but the third-order term does not need to be considered. To get the accuracy of the GRS test to reach $10^{-8}$ to $10^{-9}$, we can use the dual- or tri-frequency combination method. Our study also shows that even for the mm accuracy level requirement, the third-order ionospheric frequency shift can be neglected.

\textbf{Key words.} relativistic processes – gravitation – space vehicles: instruments – atmospheric effects – time

1. Introduction

According to Einstein’s theory of general relativity (GR), a clock at a higher gravity potential (geopotential) runs faster (i.e., it has a higher vibrational frequency) than a clock at a lower geopotential (Einstein 1915; Weinberg 1972). The frequency shift can be measured by comparing identical precise clocks’ vibration frequencies at different heights via the remote frequency signal transfer technique. Hence, comparisons of microwave frequencies between two remote atomic clocks can determine the gravitational potential (GP; Shen et al. 1993) and test gravitational redshift (GRS; Snider 1972; Vessot & Levine 1979; Vessot et al. 1980; Pound & Snider 1965).

Due to the increasing requirements for high precision of time and frequency measurements, such as tests of GRS (Shen et al. 2017; Sun et al. 2021), fundamental quantum physics (Delva et al. 2017), gravity potential, orthometric height determination (Cai et al. 2020) and redefinition of the second by optical clocks (Riehle 2015), high-precision atomic clocks have developed very quickly, especially in the case of optical atomic clocks (OACs). The stability of atomic clocks had been greatly improved to $7 \times 10^{-19}$ (Oelker et al. 2019; McGrew et al. 2018) and the technique of OACs down-conversion for microwave frequencies has since reached an instability of $10^{-18}$ (Nakamura et al. 2020). All of these technologies continue to promote research on the applications of remote time and frequency transmission.

In particular, international space atomic clocks projects have aimed to carry atomic clocks of the highest precision. Among them, Atomic Clock Ensemble in Space (ACES) onboard the international space station (ISS) led by the European Space Agency (ESA) will be equipped with atomic clocks with the long-term stability of $10^{-16}$ (Cacciapuoti & Salomon 2011; Duchayne et al. 2009; Meynadier et al. 2018). The China Space Station (CSS) will payload the OACs with the long-term stability of $10^{-18}$ (Guo 2021; Wang 2021). Both missions will use microwave links (MWLs) for remote time and frequency transfer.
frequency comparison. By implementing these space atomic clock projects, scientists may measure the frequency at a higher accuracy level and develop applications to determine the GP.

Many scholars have carried out experiments or provided theoretical formulations to test GRS. In 1976, based on the GP-A experiment, GRS was tested at the accuracy level of $7 \times 10^{-5}$ (Vessot et al. 1980). Then, scientists developed the time and frequency transfer equations of $c^{-3}$ (Blanchet et al. 2001) and $c^{-6}$ (Linet & Teyssandier 2002), with accuracy levels higher than $10^{-16}$ and $10^{-18}$, respectively. However, they did not consider the influence of the troposphere and ionosphere. Later, studies established the higher order ionosphere model on time transfer. By the simulation experiments, they showed that if the clock’s long-term stability is limited to $10^{-16}$, the high-order ionospheric effects can be ignored for the $Ku$-band (Duchayne 2008; Meynadier et al. 2018). Recently, by focusing on frequency links between the space-ground station, a tri-frequency combination approach was used to test GRS. The simulation results show that based on ACES onboard the ISS, an accuracy level of at least $2 \times 10^{-6}$ could be achieved (Sun et al. 2021). However, since it is limited by the measurement accuracy of the atomic clocks, until now the frequency transfer model only considered the first-order ionospheric frequency shift (Shen et al. 2017; Sun et al. 2021).

Compared with the time transfer to test GRS, frequency transfer offers some advantages, such as reducing the influence of phase ambiguities and getting the instant geopotential. At present, many scholars have made great efforts in investigating frequency transmission. The newest free-space optical frequency transfer simulation results show the instability of frequency transfer in free-space will achieve $4 \times 10^{-18}$ at 3000 s (Shen et al. 2021a). The significant challenges to improve the precision is how to cancel or greatly reduce the Doppler effects, atmospheric (troposphere and ionosphere) effects, and other effects related to the time and frequency transfer (Vessot & Levine 1979; Pound & Snider 1965; Shen et al. 2017; Sun et al. 2021). In particular, the ionospheric effects, which are relevant to the signal frequencies, should be eliminated. Until now, the higher order ionospheric frequency shifts have not been modeled or analyzed. However, to achieve the geopotential with an equivalent height of 1 cm, such studies need to consider the higher order ionospheric frequency shifts (Shen et al. 2017; Kopeikin et al. 2015).

In this work, we study the effects of higher order ionospheric terms on frequency transfer and construct a model of the impact of the ionosphere on frequency. We investigate how to obtain the parameters such as total electron content (TEC) and geomagnetic field in the model. By analyzing the magnitude of the higher order ionospheric effects on frequency shift, we may decide whether it should be considered in future frequency transfer experiments when the long-term stability of atomic clocks becomes $10^{-18}$.

The paper is organized as follows. Section 2 establishes the ionospheric frequency shift model and the calculation method of parameters in the model. In Sect. 3, we formulate the ionospheric frequency shift models of one-way frequency transmission, dual-frequency transmission, and tri-frequency combination technique for frequency transfer, respectively. By eliminating the first-order term of the ionospheric frequency shift and calculating the coefficients of the residual second- and third-order terms, the effects of the second- and third-order terms in different frequency transfer methods can be evaluated. In Sect. 4, we use the ISS orbits and the ionosphere model IRI-2016 to assess the influence of higher order ionospheric terms on different frequency transfer methods. In Sect. 5, we discuss relevant issues related to higher order ionospheric effects on frequency transfer and draw our main conclusions.

2. Formulation

2.1. Ionospheric frequency shift formula

It is widely known that when a microwave signal transfers from the ground to a craft, the Doppler frequency shift $\Delta f$ is caused by the time variation of the phase path (Jacobs & Watanabe 1966; Bennett 1968; Namazov et al. 1975) as follows:

$$
\Delta f = -f \frac{dP}{c} \frac{d}{dt},
$$

where $c$ is the velocity of light in vacuum, $f$ is the proper frequency, and the phase path, $P$, in the atmosphere can be calculated by the following formula (Jacobs & Watanabe 1966; Bennett 1968; Davies 1965):

$$
P = \int L n \cos \beta dl,
$$

where $n$ is the refractive index of the troposphere or ionosphere, $L$ is the signal’s propagation path through the troposphere or ionosphere, and $\beta$ is the angle between the microwave normal and the propagation direction. If we assume that the atmosphere is isotropic and consequently, the index of refraction does not change with direction, then $\beta = 0$ (Davies 1965).

In this study, we focus on the ionospheric frequency shift $\Delta f_{ion}$, caused by the ionosphere, expressed as (Shen et al. 2017):

$$
\Delta f_{ion} = -f \frac{d}{dt} \int L n_i dl,
$$

and after the ionospheric refractive index $n_i$ (or simplified noted as $n$ in the following text) is determined, the ionospheric frequency shift can be deduced.

2.2. Refractive index of ionosphere

The ionosphere is a dispersive medium to microwave signals. When a microwave signal propagates through the ionosphere, it is affected by the electron density and the geomagnetic field strength and direction, and the signal’s frequency will change. The refractive index of the ionosphere must be specified to determine the effects. The phase ionospheric refractive index can be derived based on the Appleton-Hartree equation (Alizadeh et al. 2013; Appleton 1932; Bassiri & Hajji 1993):

$$
n^2 = 1 - \frac{X}{1 - \frac{1}{4} Y \sin^2 \theta + \frac{1}{4} \left( Y^4 \sin^4 \theta + Y^2 \cos^2 \theta (1 - X^2) \right)^{1/2}},
$$

where

$$
X = \frac{N e^2}{4 \pi^2 \varepsilon_0 m f^2}, \quad Y = \frac{e B_0}{2 \pi m f},
$$

where $e = 1.6022 \times 10^{-19}$ C, and $m = 9.1094 \times 10^{-31}$ kg. The ionosphere refractive index of the carrier is as same as that of...
the phase (Duchayne 2008). Accurate to the order of \( f^{-4} \), it can be deduced via Eq. (4) (Hoque & Jakowski 2007, 2008a, 2012):

\[
\begin{align*}
N_e(l) & = \frac{7527 \times c}{2f^3} N_i B_0 \cos \theta - \frac{812.3 \times N_0^2}{f^4} - \frac{1.58 \times 10^{22}}{f^3} N_i B_0^2 (1 + \cos^2 \theta), \\
\end{align*}
\]

(5)

where \( f \) is the frequency of the signal, \( N_e \) is the ionospheric electron density, \( B_0 \) is the strength of the geomagnetic field, \( \theta \) is the angle between the wave propagation direction and the geomagnetic field vector. When the microwave is left-hand circularly polarized, the sign is positive (\( + \)), whereas if the microwave is right-hand circularly polarized, the sign is negative (\( - \)) (Hoque & Jakowski 2007; Hartmann & Leitinger 1984). The last \( f^{-4} \) term can be neglected here because it is two orders of magnitude smaller than another \( f^{-2} \) term.

From Eq. (5), we see that the first-order refractive index is affected by the ionospheric electron density. The Earth’s magnetic field and interactions with the ionosphere electron density affect the second- and third-order terms. In order to precisely consider the frequency shift in frequency transfer, the higher order terms of ionospheric refraction should be taken into account.

Substituting Eq. (5) into Eq. (3), the absolute ionospheric frequency shift can be represented as:

\[
\begin{align*}
\Delta f_{\text{ion}} &= 40.3 \frac{1}{c f^2} \int L_i N_e d l - \frac{7527}{2f^2} \int L_i N_i B_0 \cos \theta d l + \frac{812.3}{c f^3} \int L_i N_0^2 d l, \\
\end{align*}
\]

(6)

where \( f^{-i} (i = 1, 2, 3) \) denotes the \( i \)th-order of ionospheric frequency shift. The first-order \( \int N_e d l \) is the slant total electron content (STEC) along with the microwave propagation and it is expressed in total electron content (TEC) unit (TECU), where 1 TECU = \( 10^{16} \) electrons m\(^{-2}\). The second-order term includes not only the STEC but also the strength of the geomagnetic field \( B_0 \) with unit Tesla (1 T = \( 10^4 \) nT), which is considered as constant throughout propagation (Hoque & Jakowski 2007), and we take an average value \( B_0 \cos \theta \) for the magnetic field component \( B_0 \cos \theta \) (see details in Sect. 2.3). The positive sign (+) corresponds to the right-hand circularly polarized waves and the negative sign (−) corresponds to the left-hand one. From the shape parameter \( \eta = \int N_0^2 d l / (N_m \ast \text{STEC}) \) (Hartmann & Leitinger 1984; Brunner & Gu 1991), the integral of the third term can be represented as (Hartmann & Leitinger 1984; Brunner & Gu 1991):

\[
\begin{align*}
\int L_i N_0^2 d l &= \eta N_m \int L_i N_e d l, \\
\end{align*}
\]

(7)

We found that \( \eta \) hardly varies with elevation, and at elevation 7.5° the value of \( \eta \) is 0.6635, and at elevation 90°, it is 0.6498. For a Chapman layer, the value of \( \eta \) is 0.6577 (Hoque & Jakowski 2008a; Hartmann & Leitinger 1984; Brunner & Gu 1991).

We use \( s \) to represent STEC, namely:

\[
\begin{align*}
s &= \int L_i N_e d l, \\
\end{align*}
\]

(8)

and substituting the average (equivalent) value \( B_0 \cos \theta \) and Eq. (7) into Eq. (6), the ionospheric frequency shift can be obtained:

\[
\begin{align*}
\Delta f_{\text{ion}} &= 40.3 \frac{1}{c f^2} \int L_i N_e d l - \frac{7527}{2f^2} \int L_i B_0 \cos \theta d l + \frac{812.3}{c f^3} \int N_0^2 d l. \\
\end{align*}
\]

(9)

From Eq. (9), we can find that each order of the ionospheric frequency shift is related to the rate of STEC along the slant path and the second-order term \( f^{-2} \) is associated with the geomagnetic field.

2.3. Calculation of relevant quantities

To calculate the first-, second-, and third-order terms of the ionospheric frequency shift, it is necessary to calculate the STEC. Due to the lack of actual observational data, we cannot use the two signals to detect the STEC (Brunner & Gu 1991). The empirical formula was chosen to solve the vertical TEC (VTEC; Liu et al. 2018b) and then use the mapping function to calculate the STEC. The commonly used models for calculating electron density include Klobuchar Model (Klobuchar 1987), Nequick model (Angrisano et al. 2013), IRI (International Reference Ionosphere) model (Bilitza et al. 2017) and the NTCM (Neustrelitz TEC model) model (Hoque & Jakowski 2015; Jakowski et al. 1998). In this study, we chose the IRI model to calculate the electron density and VTEC.

The IRI model results from international cooperation sponsored by the Committee on Space Research (COSPAR) and the International Union of Radio Science (URSI). It has been recognized as an international standard for the ionosphere since 1999 (Liu et al. 2019). The model was based on all available ground and space data sources. When new data become available, older data sources are thoroughly evaluated and exploited, and then the model is revised by these new results. It describes the electron density, electron temperature, ion temperature, ion composition, and several additional parameters in the altitude range from 50 to 2000 km. When inputting coordinates and time, the model can calculate the VTEC of any integral height. The model has several versions, such as IRI-2001, IRI-2007, IRI-2012; the latest version is IRI-2016. The peak electron density \( N_m \) \( (F_2 \) layer has the maximum electron density \( N_m, F_2 \)), VTEC and the corresponding height \( H_m, F_2 \) can be calculated by the IRI-2016 model. These parameters will be used in solving the ionospheric frequency shift.

In addition, electron density presents a certain periodic change with the solar activity cycle, and the 10.7 cm solar radio flux (F10.7) index is usually used to describe the solar activity (Tariku 2020). Figure 1 shows the variation of the F10.7 index according to the data obtained from the Space Environment Prediction Center\(^1\). The F10.7 index reached its peak in 2002 and 2014 respectively, with a cycle of about 11 yr.

It is necessary to project the VTEC calculated by IRI-2016 to the slant direction and establish the corresponding ionospheric mapping function model to obtain the STEC between the space station and the ground station. As shown in Fig. 2, we assume a thin ionosphere layer where STEC and VTEC are compressed in it. The line between the ISS/CSS and ground station intersects with the thin layer at a point is called the ionospheric pierce

\(^1\) http://www.sepc.ac.cn/SCF_chn.php
The EH is generally expressed by the height of the center of ionospheric equivalent height (EH) of interest in this study, where the expression of $F(z)$ is given by (Angrisano et al. 2013):

$$F(z) = \frac{1}{1 - \left(\frac{R}{H}\right)^2 \sin(z)},$$

where $R$ is the radius of the Earth, $H$ is the compression height of the ionosphere.

The height of the ACES/CSS orbits above the ground is about 400 ~ 450 km. Hence, the compression height is the ionospheric equivalent height (EH) of interest in this study, somewhere between 60 km and the space station orbit height. The EH is generally expressed by the height of the center of mass of the electron density (Liu et al. 2018a), which is quite approximate. Referring to Fig. 3a, here we provide a more rigorous expression. Based on the electron density distribution, the EH can be expressed as (Liu et al. 2018a; Wolfson 2009):

$$H_{EH} = \frac{\int_{H_b}^{H_t} N_e dh}{\int_{H_b}^{H_t} N_e dh},$$

where $H_b$ and $H_t$ are the heights of the bottom and top spheres of the ionosphere of interest, respectively, $h$ is the height of the integral element $dh$ (see Fig. 2).

The electron density presents a certain periodic change with the solar activity cycle. Here, we selected the high solar activity period, June 2014, and the low solar activity period, June 2021, and we used the IRI-2016 model to draw the curve of ionospheric electron density with the change of height. Figure 3a shows that ionospheric density increases dramatically during high solar activity years. In 2014, the peak ionospheric density was about $6.72 \times 10^{11}$ electrons m$^{-3}$ at 12:00 am and $4.26 \times 10^{11}$ electrons m$^{-3}$ at 0:00 am. By comparison, in 2021, the peak ionospheric density is about $3.93 \times 10^{11}$ electrons m$^{-3}$ at 12:00 am, and $2.82 \times 10^{11}$ electrons m$^{-3}$ at 0:00 am. The electron density in 2014 (blue line and red line) is much bigger than that in 2021 (green line and purple line) at the same daily time. Since the distribution of electron density changes with time, the calculated EH changes with time either. When the spacecraft passes over the ground station, we should calculate the EH for every circle. As shown in Fig. 3b, we see that the EH for ISS/CSS is lower than that for GNSS by about 80 km on average. In this study, we calculate the EH based on the temporal distribution of electron density; when the space station is visible for the ground station as the green area in Fig. 2, we calculate the compression height every second and select the average value as the EH.

The IPP coordinate must be determined to calculate the VTEC. Figure 2 shows the relationship between the IPP coordinate and the ground receiver coordinate, which can be expressed as (Klobuchar 1987; Odijk 2002):

$$\psi_{IPP} = \arcsin(\varphi_R \cos \psi + \cos \varphi_R \sin \psi \cos \alpha),$$

where $\varphi_R$ and $\lambda_R$ are respectively the geographical latitude and longitude of the receiver in radians, $\alpha$ is the azimuth angle of the satellite at the ground station, $\psi$ is the geocentric angle of the receiver and the IPP, expressed as (Klobuchar 1987; Odijk 2002):

$$\psi = z - \arcsin\left(\frac{R}{R + H} \sin(z)\right).$$

Taking the IPP coordinate into the IRI-2016 model to calculate the value of VTEC and combining Eqs. (10) and (11), the STEC can be calculated.

The ST EC rate ($ds/dt$) has been defined as the change of STEC between two measurement epochs. In Fig. 2, at the time, $t_i$, the slant TEC is $s_i$ and at the next epoch, $t_{i+1}$, the slant TEC is $s_{i+1}$, and the STEC rate can be expressed as (Hoque & Jakowski 2010):

$$\frac{ds}{dt} = \frac{s_{i+1} - s_i}{t_{i+1} - t_i},$$

where $s_i$ denotes STEC at time $t_i$. 

---

**Fig. 1.** Annual variation of 10.7 cm solar radio flux ($F(10.7)$) index. The blue curve is the average of the $F10.7$ index one month; the curve is the smooth data of $F10.7$ index, the dots are the scattered values of the monthly $F10.7$ index.

**Fig. 2.** Supposing the TEC is compressed into an Ionospheric thin layer, the International Space Station (ISS)/Chinese Space Station (CSS) emitter to the receiver on Earth connections intersect at the Ionospheric thin layer, which point is called Ionospheric Piercing Point (IPP). VTEC is the TEC in the vertical direction at IPP and STEC is the slant TEC from ISS/CSS to the Earth. $H_b$ and $H_t$ are the heights from the Earth’s surface to the bottom and top spheres of the ionosphere layer of interest. The green area between the left-side point IPP and the right-side one is visible from the ground station to the space station.
To calculate the effect of the second-order term of the ionospheric frequency shift, we need to calculate the average strength of the geomagnetic field. We choose the International Geomagnetic Reference Field (IGRF) model to calculate the geomagnetic field. The IGRF model is a series of models describing the large-scale internal part of the Earth’s magnetic field and predicting its annual rate of change (known as a secular variation) for 5 yr beyond the date of issue. The thirteenth revision of IGRF updated two new spherical harmonic main field models for epochs 2015.0 and 2020.0 and a model of the predicted secular variation for the interval 2020.0 to 2025.0 by 15 candidates worldwide and was released in December 2019 (Alken et al. 2021). The IGRF model provides the geomagnetic field elements northward and eastward components of horizontal intensity, vertical downward direction intensity, and total geomagnetic field intensity.

Based on a previous study, for a fixed link, we find that changes of \( B_0 \cos \theta \) are not significant for scale and peak layer heights within the ranges mentioned (Hoque & Jakowski 2008b). We select the value of \( B_0 \cos \theta \) at IPP as the equivalent value. Once the space station and ground station coordinates are determined, we can calculate them by the following formula (Duchayne 2008; Hoque & Jakowski 2007, 2008a):

\[
\overline{B_0 \cos \theta} = B_{\text{ipp}} \cdot k_{\text{ipp}},
\]

where \( B_{\text{ipp}} \) is the geomagnetic field vector at IPP, \( k_{\text{ipp}} \) is the unit vector of the microwave propagation direction at IPP.

### 3. Ionospheric frequency shift effect on frequency transfer

In the ACES/CSS mission, the microwave frequencies and polarization directions are different, so we need different methods to process the ionospheric frequency shift. Namely, we use one-way frequency transfer, dual-frequency combination, and tri-frequency combination frequency transfer to eliminate various errors. The residual ionospheric frequency shift models accurate to different orders are formulated.

3.1. Ionospheric frequency shift in one-way frequency transfer

For one-way frequency transfer between two clocks, it is required to determine the photons frequency ratio \( f_A/f_B \), where \( f_B \) is the proper frequency of the photon measured on the ground B and \( f_A \) is the proper frequency measured on satellite A (Blanchet et al. 2001; Linet & Teyssandier 2002; Shen et al. 2016, 2017), and the ratio can be expressed as:

\[
\frac{f_A}{f_B} = f_B + \frac{\Delta f_{\text{dop}} + \Delta f_{\text{rel}} + \Delta f_{\text{ion}} + \Delta f_{\text{tide}}}{f_B} + \Delta f_{\text{ref}}, \quad (17)
\]

where \( \Delta f_{\text{dop}} \) is the Doppler frequency shift, \( \Delta f_{\text{rel}} \) is the relativistic frequency shift, \( \Delta f_{\text{ion}} \) is the tropospheric frequency shift, \( \Delta f_{\text{ion}} \) is the ionospheric frequency shift, \( \Delta f_{\text{tide}} \) is the bending effect on frequency and \( \Delta f_{\text{ref}} \) is the tide effect on frequency.

For one-way frequency transfer, we must construct a correction model for each error and estimate the correction parameters in Eq. (17). In this study, we only consider the ionospheric frequency shift \( \Delta f_{\text{ion}} / f_B \). Combining it with Eq. (9), the relative ionospheric frequency shift can be estimated as:

\[
\Delta f_{\text{ion}}^1/f = 40.3 \frac{1}{c f^2} \frac{ds}{dt}, \quad (18)
\]

\[
\Delta f_{\text{ion}}^2/f = \frac{7527}{2 f^3} \overline{B_0 \cos \theta} \frac{ds}{dt}, \quad (19)
\]

\[
\Delta f_{\text{ion}}^3/f = \frac{812.3}{c f^3} \eta N_m \frac{ds}{dt}. \quad (20)
\]

The coefficient in Eqs. (18)–(20) have the same meaning as those in Eq. (9), the superscripts 1, 2, and 3 represent the first-, second-, and third-order terms of the relative ionospheric frequency shifts, respectively.

3.2. Ionospheric frequency shift in dual-frequency combination method frequency transfer

In the CSS mission, a microwave signal with the frequency \( f_A \) emitted from the CSS antenna at point A and epoch \( t_A \), and...
the ground station B emitted a microwave signal with the frequency \( f_B \) at the epoch \( t_B \). At the epoch \( t_B' \), the ground station moved to \( B' \) and received a \( f_B' \) signal and at the epoch \( t_B'' \), the CSS antenna received a \( f_B'' \) signal at the point \( A' \), and these signals are recorded by their respective devices (see Fig. 4). Our study only considers the difference between emitting frequency and receiving frequency, the difference of \( T_{AB} \) or \( T'_{AB} \leq 1 \mu \text{s} \) (Meynadier et al. 2018), where \( T_{AB} = t_B - t_A \) and \( T'_{AB} = t_B' - t_A' \) can be ignored.

The CSS mission has two frequency signals with the same frequency of 30.4 GHz and different polarization directions. We can get the GRS by subtracting the two MWLs, which is called the dual-frequency combination method. In the dual-frequency combination method, we have one up-link \( f_B' / f_B \) and one down-link \( f_A' / f_A \) and the frequency ratio of emitting and receiving signals are expressed as:

\[
f_A' / f_A = \frac{f_A - \Delta f_A^{\text{rel}} + \Delta f_A^{\text{ion}} + \Delta f_A^{\text{others}}}{f_A}.
\]

\[
f_B' / f_B = \frac{f_B + \Delta f_B^{\text{rel}} + \Delta f_B^{\text{ion}} + \Delta f_B^{\text{others}}}{f_B}.
\]

From Eq. (19), this pair of links have the same second-order ionospheric frequency shift but opposite in sign.

As \( T_{AB} \leq 12 \mu \text{s} \), the time of microwaves travel between spacecraft and ground station is about 1 ms (Meynadier et al. 2018). In such a short time, when the signals pass through the atmosphere, the effects of the troposphere and ionosphere are similar for up- and down-links, but the GRS equals in magnitude and opposite in sign. By subtracting Eq. (22) from Eq. (21), the GRS effects can be expressed as:

\[
\frac{\Delta f_{\text{rel}}}{f_0} = \frac{1}{2} \left( \frac{f_B}{f_B'} - \frac{f_A}{f_A'} - \frac{\Delta f_B^{\text{ion}} - \Delta f_A^{\text{ion}}}{f_B} \right).
\]

Let \( I_1 \), \( I_2 \) and \( I_3 \) denote the coefficients of the first-, second-, and third-order terms, and then by combining Eqs. (18)–(20), the ionospheric frequency shift can be simplified as:

\[
\Delta f_A^{\text{ion}} / f_A = \frac{I_1}{f_A'} + \frac{I_2}{f_A''} + \frac{I_3}{f_A'''}
\]

\[
\Delta f_B^{\text{ion}} / f_B = \frac{I_1}{f_B'} + \frac{I_2}{f_B''} + \frac{I_3}{f_B'''}
\]

\[
\Delta f = f_0 - f_0' - \frac{f_0'' - f_0}{2}.
\]

For the up- and down-links, the \( \cos \theta \) or \( I_2 \) values have different signs. When we substitute Eqs. (24) and (25) with Eq. (23), the ionospheric frequency shifts are eliminated.

### 3.3. Ionospheric frequency shift in tri-frequency combination method for frequency transfer

In the ACES mission, there are only three different frequency signals with one up-link \( f_1 \) and two down-links \( f_2 \) and \( f_3 \). The signals transfer is shown in Fig. 5: one up-link signal with a frequency, \( f_1 \), emitted from point \( B \) at the time, \( t_1 \), and received at point \( A \) at the time, \( t_1' \), with the frequency \( f_1' \), immediately two down-link signals with frequencies \( f_2 \) and \( f_3 \) emitted from point \( B' \) at the times \( t_2 \) and \( t_3 \), and received at point \( A' \) at times of \( t'_2 \) and \( t'_3 \) with frequencies of \( f_2' \) and \( f_3' \).

There are different ways to test GRS: time comparison and frequency comparison. Meynadier et al. (2018) focused on time transfer and they used the code and carrier data to determine the time series of clock differences and then combined the two types of observations to calculate the ionospheric delay. The signals transfer of ACES is very similar to the GP-A experiment mission, so we considered choosing the tri-frequency combination method (Vessot & Levine 1979; Pound & Smider 1965; Shen et al. 2017) for the frequency transfer. In the GP-A experiment, the procedures are similar to the above description. The emitting signal frequency at the ground is \( f_0 \), after it arrives at the rocket and is transpondered to the ground, the received signal frequency at the ground is \( f_0'' \) (go-return link1 and link2); at the same time, the rocket emits a new signal with the frequency \( f_0 \) and the received signal’s frequency at the ground is denoted as \( f_0' \) (one-way link3). The tri-frequency combination model (Vessot & Levine 1979; Vessot et al. 1980; Shen et al. 2016, 2017; Kleppner et al. 1970) is expressed as:

\[
\Delta f = f_0' - f_0 - \frac{f_0'' - f_0}{2}.
\]

In the GP-A experiment setup, since the frequencies of all three microwave links are almost same, the frequency separation is required to prevent regeneration, which is realized by using rational multiples (\( P/Q = 76/49, N/M = 240/221 \), and
of the maser output frequency. By the frequency combination, the ionospheric frequency shift from \( f^{-1} \) term was eliminated by choosing the uplink and downlink frequency according to the relationship \( P/Q = \sqrt{2}(R/S) \) [1 + \((N/M)^2\)]^{1/2} = 0 (Vessot & Levine 1979; Vessot et al. 1980). Setting rational multiples into the equation, the ionospheric frequency shifts can be controlled to a magnitude smaller than \( 2 \times 10^{-15} \), which satisfies the accuracy requirement of being lower than \( 5 \times 10^{-15} \).

The frequency comparison has the following advantages: first, frequency comparisons need not consider phase ambiguity, which must be considered in time transfer. Second, frequency comparison may test GRS in a short time interval, while time comparison must accumulate data to solve the time-changing rate to test GRS. In the ACES/CSS mission, since the conditions are different, to eliminate the first-order ionospheric effect we use the modified tri-frequency model given by Sun et al. (2021) or Shen et al. (2021b). Sun et al. (2021) modified tri-frequency combination method for frequency transfer, and the one-way frequency transfer can be expressed as an equation (see Appendix A):

\[
\frac{f_1}{f_2} = \frac{m_1}{m_2} \times \hat{A}_{rel}, \tag{27}
\]

where \( \hat{A}_{rel} \) is the relativity effects which include the gravitational red-shift, transverse Doppler frequency shift and Shapiro effect, and \( m_1/m_2 \) expressed as (Sun et al. 2021):

\[
\frac{m_1}{m_2} = 1 + \frac{f_2}{f_1} - 1 \left( \frac{f_2^2}{f_1^2} - 1 \right) \left( \frac{f_2}{f_1} \right)^2 \times 1 - \frac{f_2}{f_1}.
\tag{28}
\]

The parameters in Eqs. (27) and (28) have the same notations as in Sun et al. (2021). This model can eliminate the first-order ionospheric frequency shift effect on the frequency transfer. Considering the higher order term ionospheric frequency shifts, Eq. (28) can be expressed as (for more details, see Appendix A):

\[
\frac{m_1}{m_2} = 1 + \left( \frac{f_2}{f_1} - 1 \right) \left( \frac{f_2^2}{f_1^2} - 1 \right) \left( \frac{f_2}{f_1} - 1 \right) \times \frac{1 - f_2}{f_1} \left( \frac{1 - f_2}{f_1} \right)^2 \times 1 - \frac{f_2}{f_1}.
\tag{29}
\]

where the term \( \pm \frac{f_2}{f_1} \) takes the positive and negative signs for the left-hand and right-hand circularly polarized microwaves, respectively.

By combining Eqs. (27) and (29), we get the coefficients of the second- and third-order ionospheric frequency shifts, expressed as:

\[
C_2 = \hat{A}_{rel} \times \left[ \frac{f_2}{f_1} - 1 \right] \times \left( \frac{f_2^2}{f_1^2} - 1 \right) \times \left( \frac{f_2}{f_1} - 1 \right) \times \left( 1 - \frac{f_2}{f_1} \right), \tag{30}
\]

\[
C_3 = \hat{A}_{rel} \times \left[ \frac{f_2^2}{f_1^2} - 1 \right] \times \left( \frac{f_2^2}{f_1^2} - 1 \right) \times \left( 1 - \frac{f_2}{f_1} \right), \tag{31}
\]

4. Evaluation of the influence of high order ionospheric terms on frequency transfer

To study the ionospheric effects on frequency transmission, we must know the different frequency payloads by ACES and CSS. As shown in Fig. 6, there are one up-link and two down-links in the ACES. The Ku-band with the frequencies 13.475 GHz (up) and 14.703 GHz (down) and S-band with the frequency 2.248 GHz (down), respectively. In CSS, there are four frequencies, two up-links are K-band and Ka-band with the frequencies 26.8 GHz and 30.4 GHz, two down-links are K-band and Ka-band with the frequencies 20.8 GHz and 30.4 GHz. In addition, the polarization directions of the frequency payload by the CSS are different.

Table 1 shows the signals’ frequencies and their polarization directions. The CSS up-links polarization direction is left-hand circularly polarized. Down-links of CSS and all ACES links are right-hand circularly polarized. In the CSS, up-link2 and down-link2 have the same working frequency and different polarization directions. This setting will reduce the difficulty of data processing and make the data processing different from the ACES.

As the orbit altitude of both the CSS and the ISS is around 400 ~ 450 km, in this study, we chose the orbit data of the ISS when calculating the ionospheric frequency shifts of the CSS. The ISS orbit data can be downloaded from CELESTRAK\(^2\).

Our experiment selected Observatoire de Paris (OP) as the ground station and the WGS84 ellipsoid as the reference ellipsoid. In Sect. 3.1, we established the model of ionospheric frequency shift, with the relevant parameters shown in Table 2.

\(^2\) http://www.celestrak.com
Table 1. Atomic Clock Ensemble in Space (ACES) and Chinese Space Station (CSS) frequencies and their polarization direction.

<table>
<thead>
<tr>
<th>Mission</th>
<th>Links</th>
<th>Frequency (GHz)</th>
<th>Polarization direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSS</td>
<td>up-link1</td>
<td>26.8</td>
<td>Left-hand circularly polarized</td>
</tr>
<tr>
<td></td>
<td>up-link2</td>
<td>30.4</td>
<td>Left-hand circularly polarized</td>
</tr>
<tr>
<td></td>
<td>down-link1</td>
<td>20.8</td>
<td>Right-hand circularly polarized</td>
</tr>
<tr>
<td></td>
<td>down-link2</td>
<td>30.4</td>
<td>Right-hand circularly polarized</td>
</tr>
<tr>
<td>ACES</td>
<td>down-link1</td>
<td>14.703</td>
<td>Right-hand circularly polarized</td>
</tr>
<tr>
<td></td>
<td>down-link2</td>
<td>2.248</td>
<td>Right-hand circularly polarized</td>
</tr>
</tbody>
</table>

Table 2. Relevant parameters in the experiment.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Latitude of OP</td>
<td>48.835, 954° N</td>
</tr>
<tr>
<td>Longitude of OP</td>
<td>2236, 422° E</td>
</tr>
<tr>
<td>Height of OP</td>
<td>124.2 m</td>
</tr>
<tr>
<td>Earth Radius R</td>
<td>6, 378, 137 m</td>
</tr>
<tr>
<td>Earth flatness ε²</td>
<td>0.006, 694</td>
</tr>
<tr>
<td>Light speed in vacuum c</td>
<td>299, 792, 458 m s⁻¹</td>
</tr>
<tr>
<td>Peak ionospheric height</td>
<td>Hₑ, F₁ (IRI - 2016)</td>
</tr>
<tr>
<td>Peak ionospheric density Nₑ</td>
<td>Nₑ, F₁ (IRI - 2016)</td>
</tr>
<tr>
<td>Cut-off elevation angle</td>
<td>15°</td>
</tr>
</tbody>
</table>

4.1. Calculation of STEC rate and $B₀ \cos θ$

We calculated the STEC rate and ionospheric frequency shifts according to the method described in Sect. 2. First, we determined the orbits of ACES/ISS and the ground station coordinate. Second, we calculated the EH when the ISS/CSS is visible to the ground station each cycle. Then, we got the VTEC in the IPP and took into the slant path of the spacecraft and the ground station by the mapping function. At last, we calculated the STEC rate of the elevation angles are $θ = 0°$ to $360°$ and the elevation angles are $0°$, $15°$, $30°$, $45°$, $75°$, $90°$. Figures 7a,b show the values of $B₀ \cos θ$ for frequencies $14.703$ GHz and changes gradually with the increase of elevation angle. Comparing Figs. 7d,f, we see that $B₀ \cos θ$ has the same value but the opposite sign for up- and down-links.

The absolute value of $B₀ \cos θ$ ranges from $0$ to $5 \times 10^{-5}$ T and changes significantly with the azimuth angle and elevation angle. We use the value of $B₀ \cos θ$ at the IPP with a one-second sampling interval to evaluate the second-order ionosphere frequency shift.

4.2. Ionospheric frequency shifts

We selected three frequencies of ACES mission and four frequencies of CSS (as shown in Table 1) to estimate the different orders of the ionospheric frequency shifts for one-way frequency transfer. Other relevant parameters in the experiment are shown in Table 2.

Here, we plot the absolute value distribution of different-order ionospheric frequency shifts in one orbital period by calculating the experiment data. Figure 8 shows the relative frequency shifts for ACES and CSS. The symbols $f_{ion1}$, $f_{ion2}$, and $f_{ion3}$ denote the absolute values of first-, second-, and third-order frequency shifts calculated by Eqs. (18)–(20). Respectively, we see that ionospheric frequency shifts become smaller with the space station closer to the ground station. Ionospheric frequency shifts decrease with the increase of the frequency. Figure 8a shows the relative frequency shifts and that of the order is approximately $10^{-15}$ for the frequency of $2.248$ GHz. Figures 8b,c show (respectively) the second-order frequency shifts with magnitudes being $10^{-15}$ for frequencies $14.7$ GHz and $14.7$ GHz. Figures 8d–f show, for the CSS, that the magnitudes of the second-order ionospheric frequency shifts are about $10^{-18}$ for the three frequency signals and that of the third-order frequency shifts is smaller than $10^{-20}$.

The accuracies of frequency transfer models for $c^{-3}$ and $c^{-4}$ are respectively about $5 \times 10^{-15}$ (Blanchet et al. 2001) and $10^{-19}$ (Linet & Teyssandier 2002). When we consider the model accurate to $c^{-3}$, the ionospheric frequency shifts bigger than $10^{-17}$ should be corrected. If the accuracy requirement is $c^{-4}$, the ionospheric frequency shifts bigger than $10^{-19}$ should be corrected. According to characteristics of ACES and CSS missions, they will payload the atomic clocks with long-term stabilities of $3 \times 10^{-16}$ (Cacciapuoti & Salomon 2011; Duchayne et al. 2009; Meynadier et al. 2018) and $8 \times 10^{-18}$ (Wang 2021; Guo 2021). The third-order frequency shift is beyond the measurement accuracy of the atomic clocks, which can be ignored. In the ACES mission, the second-order terms of the $S$-band should be considered. The second-order ionospheric frequency shifts of the $K$-band and $Ka$-band must be eliminated or corrected for the CSS mission.
Table 3. Maximum value of ionospheric frequency shift of different frequencies.

<table>
<thead>
<tr>
<th>Method</th>
<th>Mission</th>
<th>Frequency (GHz)</th>
<th>1st order term</th>
<th>2nd order term</th>
<th>3rd order term</th>
</tr>
</thead>
<tbody>
<tr>
<td>One-way frequency</td>
<td>ACES</td>
<td>2.248</td>
<td>$2.189 \times 10^{-11}$</td>
<td>$7.272 \times 10^{-15}$</td>
<td>$2.396 \times 10^{-17}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>13.475</td>
<td>$6.093 \times 10^{-13}$</td>
<td>$3.376 \times 10^{-17}$</td>
<td>$1.856 \times 10^{-20}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>14.703</td>
<td>$5.118 \times 10^{-13}$</td>
<td>$2.599 \times 10^{-17}$</td>
<td>$1.309 \times 10^{-20}$</td>
</tr>
<tr>
<td></td>
<td>CSS</td>
<td>20.8</td>
<td>$2.557 \times 10^{-13}$</td>
<td>$9.180 \times 10^{-18}$</td>
<td>$3.268 \times 10^{-21}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>26.8</td>
<td>$1.540 \times 10^{-13}$</td>
<td>$4.292 \times 10^{-18}$</td>
<td>$1.187 \times 10^{-21}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30.4</td>
<td>$1.197 \times 10^{-13}$</td>
<td>$2.940 \times 10^{-18}$</td>
<td>$7.163 \times 10^{-22}$</td>
</tr>
<tr>
<td>Dual-frequency</td>
<td>CSS</td>
<td>30.4</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>combination</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tri-frequency</td>
<td>ACES</td>
<td>$f_2 = 14.703$</td>
<td>–</td>
<td>$9.280 \times 10^{-17}$</td>
<td>$1.037 \times 10^{-19}$</td>
</tr>
<tr>
<td>combination</td>
<td>CSS</td>
<td>$f_2 = 20.8$</td>
<td>–</td>
<td>$2.231 \times 10^{-19}$</td>
<td>$1.745 \times 10^{-22}$</td>
</tr>
</tbody>
</table>

Notes. The symbol “–” means the ionospheric frequency shift has been eliminated.

Fig. 8. Ionospheric frequency shifts of one-way frequency transfer for ACES (a–c) and CSS (d–f). $f_\text{ion1}$, $f_\text{ion2}$, and $f_\text{ion3}$ denote the first-, second- and third-order frequency shifts of different frequency signals.

The CSS mission carried four microwave frequency signals, two of which have the same frequency and opposite polarization directions (see Table 1). We can use the dual-frequency combination method to make the frequency comparison and test the GRS to eliminate ionospheric frequency shifts (see Table 3). When we use the tri-frequency combination technique to transfer the frequency signals in ACES or CSS (as introduced in Sect. 3.3), we need to consider the effects of the second-order ionospheric frequency shift. We estimated the second-order ionospheric frequency shift coefficient value by Eq. (30), that is, $-3.5707$ for ACES and $-0.0243$ for CSS. By combining these coefficients with the values of frequency shifts with proper frequencies 14.703 GHz and 20.8 GHz (as shown in Table 3), the modified tri-frequency combination method data show that the second-order frequency shifts are about $9.280 \times 10^{-17}$ for ACES and $2.231 \times 10^{-19}$ for CSS.

In the same space environment, ionospheric frequency shift decreases with the increase of the frequency of microwave signals (see Table 3) and the CSS provides the higher frequency signals that can effectively reduce the impact of the ionosphere.

As the analysis given in Sect. 2.3, we know that electron density has a link to solar activity. It influences the TEC and ionospheric frequency shift. We plotted the second-order ionospheric frequency shifts of frequencies 2.248 GHz and 20.8 GHz in different solar activity years. Figure 9 shows the second-order ionospheric frequency shifts have similar distribution for the high solar activity year 2014 and low solar activity year 2021. In the lower left-hand corner of Figs. 9a,b, we enlarge the figure of $0 \sim 100$ s, and the red dotted lines are $10^{-14}$ and $10^{-17}$ are for frequencies of 2.248 GHz and 20.8 GHz, respectively. It shows that the magnitudes of the ionospheric frequency shifts are about $10^{-14}$ in 2014 and $10^{-15}$ in 2021 for frequency 2.248 GHz, and
Following on the development of time and frequency science and technology, especially OACs developments and the successful launch of the CSS, here we propose an approach that considers higher order ionospheric frequency contributions, which could achieve a higher order accuracy of testing GRS, based on a higher order term model of ionospheric frequency shift and different frequency transfer methods.

**References**


Alken, P., Thebault, E., Beggan, C., et al. 2021, Earth Planets Space, 73, 1


Einstein, A. 1915, in Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften, 844

Guo, Y. 2021, in *The 12th China Satellite Navigation Conference, CSNC 2021*


Hoque, M. M., & Jakowski, N. 2007, *J. Geodesy*, 81, 259


Hoque, M. M., & Jakowski, N. 2008b, *GPS Solutions*, 12, 87


Liu, T., Zhang, B., Yuan, Y., & Li, M. 2018b, *J. Geodesy*, 92, 1


Riehle, F. 2015, *Comptes Rendus Physique*, 16, 506


Taraka, Y. A. 2020, *AJ*, 160, 185


Wolfson, R. 2009, *Essential University (Physics Volume 2)*, 2 (Pearson Education (India))
Appendix A: Derivation of the modified tri-frequency combination formula

In the tri-frequency combination method for frequency transfer (Vessot & Levine 1979; Vessot et al. 1980; Shen et al. 2017; Sun et al. 2021), the propagation paths of the two down-link signals \( f_2' / f_3' \) and \( f_2' / f_3' \) are similar (see Fig. 5). Based on Equations 14-17 of Sun et al. (2021), dividing the \( f_2' / f_3' \) by \( f_2' / f_3' \), we obtain the expression:

\[
\frac{f_2'}{f_3'} = 1 + \frac{\Delta f_{2\text{ion}}(f_2')}{f_2'} - \frac{\Delta f_{3\text{ion}}(f_2')}{f_3'} + \left(1 - \frac{f_2'}{f_2'}\right) \left[\delta f_{2\text{ref}} + \frac{40.3 N_{\text{AB}} \cdot \nu_A}{c f_2^2}\right], \tag{A.1}
\]

where \( \delta f_{2\text{ref}} \) is the frequency shift caused by the refraction \( N_{\text{AB}} \) is the unit vector from point A to point B and \( \nu_A \) is the velocity vector at point A. From Equation 9, the two downlink signals’ ionospheric frequency shifts are:

\[
\Delta f_{2\text{ion}}(f_2') = \frac{40.3 \, ds}{c f_2^2} + \frac{7527}{2 f_2^2} B_0 \cos \theta \frac{ds}{dt} + \frac{812.3 \, N_m \, ds}{c f_2^3}, \tag{A.2}
\]

Substituting Equation A.2 into Equation A.1, we obtain:

\[
\frac{f_2'}{f_3'} = 1 + \frac{40.3 \, ds}{c f_2^2} + \frac{7527}{2 f_2^2} B_0 \cos \theta \frac{ds}{dt} + \frac{812.3 \, N_m \, ds}{c f_2^3} \right) + \left(1 - \frac{f_2'}{f_2'}\right) \left[\delta f_{2\text{ref}} + \frac{40.3 N_{\text{AB}} \cdot \nu_A}{c f_2^2}\right]. \tag{A.3}
\]

By rearranging the relevant terms, Equation A.3 can be written as:

\[
\frac{f_2'}{f_3'} = 1 + \frac{40.3 \, ds}{c f_2^2} + \left(1 - \frac{f_2'}{f_2'}\right) \left[\delta f_{2\text{ref}} + \frac{40.3 N_{\text{AB}} \cdot \nu_A}{c f_2^2}\right]. \tag{A.4}
\]

Thus, we can define:

\[
\delta f_2 = \left(\frac{f_2'}{f_3'}\right) \delta f_1. \tag{A.6}
\]

from Sun et al. (2021)’s study, since \( \delta f_{2\text{ref}} \) is inversely proportional to the square of the frequency \( f_2' \), we get \( \delta f_2 = \kappa / f_2' \), where \( \kappa \) is a constant, namely \( \delta f_2 f_2' = \delta f_2 f_2' \). Hence, we have

\[
\delta f_2 = \left(f_2' / f_3'\right) \delta f_1. \tag{A.6}
\]

Equation (A.4) can be written as:

\[
\frac{f_2'}{f_3'} = 1 + \left(1 - \frac{f_2'}{f_2'}\right) \delta f_2 + \frac{812.3 \, N_m \, ds}{c f_2^3} \cdot \bigg(1 - \frac{f_2'}{f_2'}\bigg). \tag{A.7}
\]

Similarly, for the up- and down-links, we know \( \theta_u = 180^\circ - \theta_e \) and \( \cos \theta_u = -\cos \theta_e = -\cos \theta \), the up- and down-links have the different sign, so:

\[
\frac{f_1'}{f_2'} = \left[1 + \left(1 - \frac{f_1'}{f_2'}\right) \delta f_1 + \frac{812.3 \, N_m \, ds}{c f_2^3} \right] \cdot \bigg(1 - \frac{f_1'}{f_2'}\bigg), \tag{A.8}
\]

which includes the \( f_1 \) frequency shift \( \delta f_1 \), namely, \( \delta f_1 / f_1' \) is expressed by \( \delta f_1 \). We may also express \( f_1' / f_2' \) by \( f_2 \) frequency shift \( \delta f_2 \) by following procedure. We multiply \( f_1' / f_2' \) by \( f_3' / f_2' \) frequency shift \( \delta f_2 \) (they are equal to 1) with the second, third, and fourth terms of square brackets in Equation A.8, respectively, obtaining the expression:

\[
\frac{f_1'}{f_2'} = \left[1 + \left(1 - \frac{f_1'}{f_2'}\right) \delta f_1 + \frac{812.3 \, N_m \, ds}{c f_2^3} \right] \cdot \bigg(1 - \frac{f_1'}{f_2'}\bigg), \tag{A.9}
\]

and after rearranging, we get

\[
\frac{f_1'}{f_2'} = \left[1 + \left(1 - \frac{f_1'}{f_2'}\right) \delta f_1 + \frac{812.3 \, N_m \, ds}{c f_2^3} \right] \cdot \bigg(1 - \frac{f_1'}{f_2'}\bigg). \tag{A.10}
\]

Substituting Equation A.6 into Equation A.10, we get

\[
\frac{f_1'}{f_2'} = \left[1 + \left(1 - \frac{f_1'}{f_2'}\right) \delta f_2 + \frac{812.3 \, N_m \, ds}{c f_2^3} \right] \cdot \bigg(1 - \frac{f_1'}{f_2'}\bigg). \tag{A.11}
\]

where \( \delta f_2 \) can be obtained from Equation A.7:

\[
\delta f_2 = \left(\frac{f_2'}{f_3'}\right) \left[1 - \left(1 - \frac{f_2'}{f_3'}\right) \frac{7527}{2 f_2^2} B_0 \cos \theta \frac{ds}{dt} \right] \cdot \bigg(1 - \frac{f_2'}{f_3'}\bigg). \tag{A.12}
\]

Substituting Equation A.12 into Equation A.11, we obtain Equation 29, namely:

\[
\frac{m_1}{m_2} = 1 + \left(1 - \frac{f_2'}{f_3'}\right) \left[1 - \left(1 - \frac{f_2'}{f_3'}\right) \frac{7527}{2 f_2^2} B_0 \cos \theta \frac{ds}{dt} \right] \cdot \bigg(1 - \frac{f_2'}{f_3'}\bigg). \tag{A.13}
\]

From Equations A.8, A.11, and A.13, we can estimate the magnitude of the higher order ionospheric frequency shifts.