Dynamical essence of the eccentric von Zeipel–Lidov–Kozai effect in restricted hierarchical planetary systems

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ABSTRACT

Aims. The eccentric von Zeipel–Lidov–Kozai (ZLK) effect is widely used to explain dynamical phenomena in a variety of astrophysical systems. The purpose of this work is to clarify the dynamical essence of the eccentric ZLK effect by constructing an inherent connection between this effect and the dynamics of secular resonance in restricted hierarchical planetary systems.

Methods. Dynamical structures of apsidal resonance were studied analytically by means of perturbative treatments. The resonant model was formulated by averaging the Hamiltonian (up to octupole order) over rotating ZLK cycles, producing an additional motion integral. The phase portraits under the resonant model can be used to analyse dynamical structures, including resonant centres, dynamical separatrices, and islands of libration.

Results. By analysing phase portraits, five branches of libration centres and eight libration zones are found in eccentricity–inclination space. The analytical results of the libration zone and the numerical distributions of the resonant orbit agree very well, indicating that the resonant model for apsidal resonances is valid and applicable. Additionally, we found that in the test-particle limit, the distributions of flipping orbits are dominated by the apsidal resonances that are centred at an inclination of \(i = 90^\circ\).

Conclusions. The eccentric ZLK effect is dynamically equivalent to the effect of apsidal resonance in restricted hierarchical planetary systems. The dynamical response of the eccentric ZLK effect (or of the effect of apsidal resonance) is to significantly excite the eccentricities and/or inclinations of test particles in the very long-term evolution.

Key words. methods: analytical – planets and satellites: dynamical evolution and stability – celestial mechanics

1. Introduction

Hierarchical three-body configurations are common in various astrophysical systems, ranging from satellite and planet scales to supermassive black holes (Naoz 2016). Under the test-particle approximation, this configuration reduces to the so-called restricted hierarchical three-body problem, where the test particle moves around the central body under the gravitational perturbation from the perturber. For a situation in which the perturber moves on a circular orbit, Kozai (1962) and Lidov (1962) studied the long-term dynamics of test particles and found that a resonance occurs between the longitude of pericentre \(\sigma\) and the longitude of ascending node \(\Omega\) when the inclination is greater than \(39.2^\circ\). In the long-term evolution, coupled oscillations between eccentricity and inclination are called standard Kozai–Lidov oscillations (or Kozai–Lidov effect; Lithwick & Naoz 2011). About the same issue, Ito & Ohtsuka (2019) pointed out that Von Zeipel (1910) performed a similar analysis; thus they suggested to refer to this mechanism as the von Zeipel–Lidov–Kozai (ZLK) effect.

Under the circular assumption (for the orbit of perturber), the vertical angular momentum \(H = \sqrt{1 - e^2} \cos i\) is conserved in the long-term evolution, showing that the orbits of test particle cannot flip between prograde and retrograde. This means that the standard ZLK effect cannot lead to the phenomenon of orbit flips. However, the situation is different when the circular assumption is relaxed. In this context, the Hamiltonian needs to be formulated up to the octupole order in the semi-major axis ratio. Under the octupole-level approximation, the vertical angular momentum is no longer constant (Naoz 2016). In particular, long-timescale modulations of the Kozai–Lidov cycle can force the variation of vertical angular momentum \(H\), leading to striking features including flipping between prograde and retrograde, extremely high eccentricities, and chaotic behaviours (Katz et al. 2011). This mechanism can be referred to as the eccentric ZLK effect (Ito & Ohtsuka 2019). It has been used to interpret various dynamical phenomena in astrophysical systems (Libert & Tsiganis 2009; Shevchenko 2016; Naoz 2016). In the test-particle limit, Antognini (2015) investigated the timescales of ZLK oscillations at quadrupole- and octupole-order dynamical models.

Naoz et al. (2011) found that planetary orbits can flip between prograde and retrograde with respect to the invariant plane of the system due to the eccentric ZLK effect. They proposed a possible clue for forming hot Jupiters on retrograde orbits by combining the eccentric ZLK effect with tidal friction at the late stage. To analytically understand the eccentric ZLK effect, Katz et al. (2011) performed an average for the secular equations of motion over the period of the Kozai–Lidov cycle and found a new constant of motion. In particular, they derived an analytical criterion for orbit flips. At the same time, Lithwick & Naoz (2011) performed a numerical investigation of the eccentric ZLK effect. They numerically mapped the initial conditions for which
flipping orbits occur for various values of $\epsilon$ ($\epsilon$ stands for the contribution of the octupole-order Hamiltonian). By using surfaces of section and Lyapunov exponents, Li et al. (2014a) studied the chaotic and quasi-periodic evolutions caused by the eccentric ZLK effect. For the same topic, Li et al. (2014b) classified flipping orbits into two types: the low-eccentricity, high-inclination (LeHi) case, and the high-eccentricity, low-inclination (HeLi) case. They pointed out that the first type of flipping orbit is governed by the joint effect of quadrupole-order and octupole-order resonances, and the second type of flipping orbit is dominated by octupole-order resonances (Li et al. 2014a,b). Recently, Sidorenko (2018) interpreted the eccentric ZLK effect working in the LeHi space as a resonant phenomenon. In a recent work (Lei 2022), a systematic study was performed for the dynamics of orbit flips caused by eccentric ZLK effect through three approaches: Poincaré surfaces of the section, dynamical system theory (periodic orbit and invariant manifold), and perturbative treatments. Through these studies, the dynamical essence of flipping orbits is very clear: flipping orbits are a kind of quasi-periodic (or resonant) trajectory around stable, polar, periodic orbits (Sidorenko 2018; Lei 2022).

However, the dynamical essence of the eccentric ZLK effect is not clear. We know that the eccentric ZLK effect is the dynamical response under the octupole-level Hamiltonian model. This means that there must be a certain correspondence between the secular dynamics and the eccentric ZLK effect. Based on this consideration, the purpose of this work is twofold. The first purpose is to analytically explore the dynamical structures of secular resonances (apsidal resonances) under the octupole-level approximation by means of the perturbative treatments developed by Henrard & Lemaître (1986) and Henrard (1990). This theory was adopted by Sidorenko (2018) to study the same topic. The second purpose is to construct the dynamical connection between the eccentric ZLK effect and apsidal resonances at the octupole-level approximation in restricted hierarchical planetary systems. Our results show that (a) the webs of apsidal resonance constitute basic backbones imbedded in phase space that govern the very long-term dynamics of particles, (b) in the test-particle limit, the eccentric ZLK oscillations are attributed to the effect of apsidal resonance, and (c) only apsidal resonances with centres at $i = 90^\circ$ may cause orbit flips.

Throughout this study, the inherent connection between the eccentric ZLK effect and apsidal resonances becomes clear: from the point of view of dynamics, the eccentric ZLK effect is equivalent to the effect of apsidal resonance under the octupole-level approximation in restricted hierarchical planetary systems. The dynamical consequence of the eccentric ZLK effect (or the effect of apsidal resonance) is to significantly excite eccentricities and/or inclinations of test particles in the long-term evolution. In particular, the behaviour of orbit flip is just one kind of dynamical response due to the eccentric ZLK effect (or the effect of apsidal resonance). In this sense, the present work can be considered as an extension of the resonant interpretation of the eccentric ZLK effect (Sidorenko 2018).

The remaining part of this work is organised as follows. In Sect. 2, the Hamiltonian model is briefly introduced under the test-particle and octupole-order approximation. In Sect. 3, the fundamental frequencies and nominal location of apsidal resonance are identified under the quadrupole-order Hamiltonian flow. Resonant model for apsidal resonances is formulated in Sect. 4 by means of first-order perturbation theory. Results including the dynamical structures, libration zones, and applications are presented in Sect. 5. Finally, the conclusions of this work are summarised in Sect. 6.

2. Hamiltonian model

We investigated secular resonances for an inner test particle moving around a central star under gravitational perturbation from a distant planet. This dynamical model is called a restricted hierarchical planetary system. It is widely adopted as the basic dynamical model for studying secular dynamics in various astrophysical systems (Lithwick & Naoz 2011; Li et al. 2014a,b; Sidorenko 2018; Luo et al. 2016; Lei et al. 2018; Lei 2019, 2021a,b, 2022; Kaiz et al. 2011; Antognini 2015). The mass of the central star is denoted by $m_0$ and the mass of the perturber is denoted by $m_p$. In the test-particle limit, the orbit of the perturber around the central star is unchanged, while the test particle moves around the central star on a perturbed Keplerian orbit. Under this hierarchical configuration, the invariant plane of the system is coincident with the orbit of the perturber.

For convenience, we introduce a right-handed inertial reference frame, with the origin at the central star, the $x$–$y$ plane at the invariant plane (i.e. the perturber’s orbit), the $x$-axis along the eccentricity vector of the perturber’s orbit, and the $z$-axis parallel to the vector of the total angular momentum. In this coordinate system, the orbits of the test particle (perturber) are described by the semi-major axis $a(\alpha_p)$, the eccentricity $e(\epsilon_p)$, the inclination $i(\iota_p)$, the longitude of the ascending node $\Omega(\Omega_p)$, the argument of pericentre $\omega(\omega_p)$, and the mean anomaly $M(M_p)$. For both prograde and retrograde configurations, the longitude of pericentre and mean anomaly can be defined in a general manner (Shevchenko 2016)

$$\omega = \Omega + \text{sign}(\cos \iota)\omega, \quad \lambda = M + \omega$$

for the test particle and

$$\omega_p = \Omega_p + \text{sign}(\cos \iota_p)\omega_p, \quad \lambda_p = M_p + \omega_p$$

for the perturber. Here, sign($x$) is a sign function of $x$ and $\epsilon$ is equal to 1.0 when $x$ is greater than zero and $\epsilon$ is equal to $-1.0$ when $x$ is smaller than zero. In the chosen reference frame, $\iota_p = 0$ and $\iota_p = 0$ (this is due to the setting that the $x$-axis is along the eccentricity vector of the perturber orbit). If not stated otherwise, we adopt the variables with subscript $p$ for the perturber and variables without subscripts for the test particle.

In the long-term evolution, the short-period terms arising in the Hamiltonian can be filtered out by means of double-averaging techniques over the orbital periods of the test particle and the perturber (Ford et al. 2000; Naoz et al. 2013; Naoz 2016; Luo et al. 2016; Shevchenko 2016; Lei et al. 2018; Lei 2019). This process of phase averaging is called secular approximation (Naoz 2016). Due to the hierarchical configuration, the semi-major axis ratio $\alpha = \frac{a_p}{a}$ is a small parameter, leading to the fact that the Hamiltonian can be truncated at a certain order in the semi-major axis ratio. The Hamiltonian truncated at the second order corresponds to the quadrupole-level approximation, and the Hamiltonian truncated at the third order corresponds to the octupole-level approximation.

The (normalised) double-averaged Hamiltonian, up to the octupole order in the semi-major axis ratio, can be written as (Lithwick & Naoz 2011; Naoz 2016)

$$\mathcal{H} = -\left(\mathcal{F}_\text{quad} + \epsilon\mathcal{F}_\text{oct}\right),$$

where the coefficient $\epsilon$, measuring the significance of the octupole-order contribution, is a small parameter, given by

$$\epsilon = \frac{a_p}{a} \frac{\epsilon_p}{1 - \epsilon_p^2}.$$
showing that when the semi-major axis ratio $a = \frac{a_2}{a_0}$ or the eccentricity of the perturber $e_p$ is higher, the contribution of the octupole-order term is greater. The quadrupole-order term is given by

$$F_{\text{quad}} = -\frac{1}{2} e^2 + \frac{3}{2} e^2 \cos^2 i + \frac{5}{2} e^2 \left(1 - \cos^2 i\right) \cos (2\omega)$$

and the octupole-order term is given by

$$F_{\text{oct}} = \frac{5}{16} \left( e + \frac{3}{4} e^2 \right) \times \left[ \left( 1 - 11 \cos i - 5 \cos^2 i + 15 \cos^3 i \right) \cos (\omega - \Omega) + \left( 1 + 11 \cos i - 5 \cos^2 i - 15 \cos^3 i \right) \cos (\omega + \Omega) \right] - \frac{175}{64} e^3 \left( 1 - \cos i - \cos^2 i + \cos^3 i \right) \cos (3\omega - \Omega) + \left( 1 + \cos i - \cos^2 i - \cos^3 i \right) \cos (3\omega + \Omega) .$$

The double-averaged Hamiltonian up to an arbitrary order in $\epsilon$ can be found in Laskar & Boué (2010) and Lei (2021a).

In order to formulate the Hamiltonian model, a set of (normalised) Delaunay variables are introduced as follows (Lithwick & Naoz 2011):

$$g = \omega, \quad G = \sqrt{1 - e^2},$$

$$h = \Omega, \quad H = G \cos i.$$

In terms of Delaunay’s variables, the Hamiltonian can be further expressed as follows:

$$\mathcal{H}(g, h, G, H) = \mathcal{H}_2(g, G, H) + \mathcal{H}_1(g, h, G, H) = -F_{\text{quad}}(g, G, H) - \epsilon F_{\text{oct}}(g, h, G, H),$$

which determines a dynamical model with two degrees of freedom. The canonical Hamiltonian relations lead to the equations of motion as follows (Morbidelli 2002):

$$\frac{dg}{dt} = \frac{\partial \mathcal{H}}{\partial G}, \quad \frac{dG}{dt} = -\frac{\partial \mathcal{H}}{\partial g},$$

$$\frac{dh}{dt} = \frac{\partial \mathcal{H}}{\partial H}, \quad \frac{dH}{dt} = -\frac{\partial \mathcal{H}}{\partial h} .$$

The Hamiltonian given by Eq. (2) holds the following symmetries (Sidorenko 2018):

$$\mathcal{H}(g, h, G, H) = \mathcal{H}(2\pi - g, h, G, -H) = \mathcal{H}(g, 2\pi - h, G, -H),$$

which implies that the solution curves under the Hamiltonian flow are symmetric with respect to $H = 0$ (i.e. $i = 90^\circ$).

In this section, we identify the fundamental frequencies under the quadrupole-order dynamical model. Based on this, we determine the nominal location of the apsidal resonance.

The quadrupole-order Hamiltonian $\mathcal{H}_2$ is very simple and can be written as

$$\mathcal{H}_2 = \frac{1}{2} \left( 1 - G^2 - \frac{H^2}{G^2} - \frac{3}{2} \left( \frac{1}{G^2} - 1 \right) H^2 \right) ,$$

where the angular coordinate $h$ is absent from $\mathcal{H}_2$, indicating that the $z$-component of the angular momentum $H$ is conserved under the quadrupole-order model. The conserved quantity $H$ can be specified by the critical inclination $i_c$ (when the eccentricity is assumed as zero) of the manner (Kozai 1962)

$$H = \sqrt{1 - e^2} \cos i = \cos i_c .$$

In the following discussions, we often use $i_c$ to stand for $H$. The dynamical model determined by $\mathcal{H}_2$ is of one degree of freedom. With given $H$ (or $i_c$), the solution curves under the Hamiltonian flow of $\mathcal{H}_2$ are usually called the ZLK cycles, including the librating ZLK cycles and the rotating ZLK cycles. Rotating and librating ZLK cycles are divided by means of a dynamical separatrix in the phase space.

The global structures in the phase space can be explored by analysing phase portraits that correspond to level curves of Hamiltonian with given motion integral. As an example, the case of $|H| = 0.5$ (corresponding to the critical inclination at $i_c = 60^\circ$ or $i_c = 120^\circ$) is considered, and the associated phase portrait is presented in Fig. 1. In this case, the ZLK resonance may occur. At the ZLK centre, the eccentricity and inclination should satisfy (Kozai 1962)

$$\cos^2 i = \frac{3}{5} \left( 1 - e^2 \right) .$$
In Fig. 1, the black dot stands for the position of the ZLK centre, at which the Hamiltonian takes its maximum. The dynamical separatrix, shown as a red line, divides the rotating ZLK cycles from the librating ZLK cycles. It is known that the separatrix corresponds to the level curve of a Hamiltonian passing through \( G = 1 \) (corresponding to \( e = 0 \)). Substituting \( G = 1 \) into the quadrupole-order Hamiltonian \( \mathcal{H}_2 \) given by Eq. (4), it is not difficult to obtain the expression of the separatrix (Lei 2021b)

\[ \mathcal{H}_2 + H^2 = 0. \]

For the current example (the motion integral is taken as \( |H| = 0.5 \)), the Hamiltonian of the separatrix is equal to \( \mathcal{H}_2 = -0.25 \). From the phase portrait, we observe that in the phase space, the region with \( \mathcal{H}_2 > -0.25 \) is of ZLK libration and the region with \( \mathcal{H}_2 < -0.25 \) is of ZLK circulation.

Inside the regions filled with ZLK librating cycles, the dynamics is dominated by the quadrupole-order resonance (i.e. ZLK resonance). Only in regions filled with ZLK rotating cycles does the octupole-order resonance play an important role in governing the very long-term behaviours of particles (Li et al. 2014a). Figure 1 further shows that the ZLK cycles starting from \( 2g = 0 \) are of circulation. In the following, we focus on the regions that are filled with rotating ZLK cycles, where the octupole-order resonances may appear and dominate the long-term dynamics. Without loss of generality, we fix the initial condition at \( 2g = 0 \) for rotating ZLK cycles (Katz et al. 2011; Li et al. 2014a; Sidorenko 2018; Lithwick & Naoz 2011) and then determine the fundamental frequencies in the parameter space spanned by orbit elements \((e, i)\) or conserved quantities \((\mathcal{H}_2, H)\).

In order to study the dynamics of secular resonance by means of perturbative treatments (Henrard & Lemaitre 1986; Henrard 1990), we introduce the following action–angle variables under the quadrupole-order Hamiltonian flow,

\[
g^* = g - \rho_g (g^*, G^*, H^*), \quad G^* = \frac{1}{2\pi} \int_0^{2\pi} G dg, \tag{5}
\]

\[
h^* = h - \rho_h (g^*, G^*, H^*), \quad H^* = H,
\]

which is canonical with the generating function

\[
S_1 (g, h, G^*, H^*) = h H^* + \int G (\mathcal{H}_2 (G^*, H^*), g, H^*) \, dg.
\]

In Eq. (5), \( \rho_g \) and \( \rho_h \) are periodic functions with the same period of the rotating ZLK cycle. The variable \( G^* \) is called Arnold action, which stands for the phase-space area bounded by the ZLK cycle (divided by \( 2\pi \)).

Under the quadrupole-order dynamical model, we denote the period of \( g \) as \( T_g \) and the period of \( h \) as \( T_h \). Thus, the linear functions of \( g^* \) and \( h^* \) can be expressed as

\[
g^* = \frac{2\pi}{T_g} t, \quad h^* = h^0 + \frac{2\pi}{T_h} t,
\]

where \( h^0 \) is a constant under the quadrupole-order Hamiltonian flow.

In Fig. 2, a rotating ZLK cycle in the phase space \((g, G)\) is shown in the left panel, and the relation between the Arnold action \( G^* \) and \( G(0) \) is plotted in the right panel. In the left panel, the location of \( G^* \) for the particular example is marked in red. Evidently, \( G^* \) is a constant under the quadrupole-order Hamiltonian flow.

In practice, we produce the ZLK cycle as well as the action \( G^* \) by integrating the following differential equations over one period of ZLK cycle (i.e. \( T_g \)) under the quadrupole-order Hamiltonian flow,

\[
\dot{g} = \frac{\partial \mathcal{H}_2}{\partial G}, \quad \dot{G} = -\frac{\partial \mathcal{H}_2}{\partial g}, \quad W = G \frac{\partial \mathcal{H}_2}{\partial G}.
\]

At the initial instant, \( g^* = 0 \) and \( h^* = h^0 \) for rotating ZLK cycles.

The ZLK cycle \( G(g) \) and the Arnold action \( G^* \) can alternatively be produced by means of elliptic integrals, as presented by Sidorenko (2018).

According to the generating function, we can obtain an alternative expression for the new set of angles \((g^*, h^*)\) as follows:

\[
g^* = \frac{\partial S_1}{\partial G^*}, \quad h^* = \frac{\partial S_1}{\partial H^*}.
\]
Thus, we can obtain the expressions for computing periodic functions $\rho_\sigma = g - g^*$ and $\rho_h = h - h^*$ as

$$
\rho_\sigma(g^*, G^*, H^*) = \int \frac{\partial H_2}{\partial G} \dd t - \frac{\partial}{\partial G} \int G(H_2(G^*, H^*) , g, H^*) \dd q,
$$

$$
\rho_h(g^*, G^*, H^*) = - \frac{\partial}{\partial H} \int G(H_2(G^*, H^*) , g, H^*) \dd q,
$$

which indicate that $\rho_\sigma = g - g^*$ and $\rho_h = h - h^*$ are equal to zero when $g^* = 0$ or $g^* = 2\pi$ (Henrard 1990). This means that the old and new sets of angles are coincident at the initial instant and at one period of ZLK cycle.

For the example shown in the left panel of Fig. 2, the time histories of $(g, h)$ and $(g^*, h^*)$ are shown in the right panel of Fig. 3. The differences between the old and new sets of angles $\rho_\sigma$ and $\rho_h = h - h^*$ are absent from the Hamiltonian $H_2$. In turn indicates that $G^*$ and $H^*$ are conserved quantities along the ZLK cycle. The fundamental frequencies under the quadrupole-order dynamical model are identified by

$$
g^* = \frac{\partial H_2(G^*, H^*)}{\partial G^*}, \quad h^* = \frac{\partial H_2(G^*, H^*)}{\partial H^*},
$$

which determines the periods of $g$ and $h$ as

$$
T_g = \frac{2\pi}{g^*}, \quad T_h = \frac{2\pi}{h^*}.
$$

Based on the fundamental frequencies, it is possible for us to determine the nominal location of secular resonance by the following resonant condition:

$$
k_1 g^* + k_2 h^* = 0,
$$

where $k_1, k_2 \in \mathbb{Z}$.

As for the octupole-order resonance, the critical argument is given by

$$
\sigma = h^* + \text{sign}(H^*)g^*.
$$

In Fig. 4, the solution of $\dot{\sigma} = h^* + \text{sign}(H^*)g^* = 0$ is distributed in $(i, e)$ space in the left panel and in $(H, H_2^*)$ space in the right panel. In the left panel, the eccentricity and inclination are evaluated when the angle $g$ is equal to zero. This is because we fixed $2g = 0$ as initial conditions of rotating ZLK cycles. In the right panel, the eccentricity and inclination are evaluated when the angle $g$ is equal to zero. This is because we fixed $2g = 0$ as initial conditions of rotating ZLK cycles. In the right panel, the shaded region shows the ZLK circulation, the upper boundary of circulation region is given by $H_2 + H^2 = 0$, and the bottom boundary is given by $H_2 + H^2 + 2 = 0$ (Lei 2021b). In both panels, the distribution of the nominal location of the resonant centre is symmetric with respect to $i = 90^\circ$ (or $H = 0$).

The left panel of Fig. 4 shows that there are three branches of resonant centres in the considered parameter space: one branch corresponds to polar orbits at arbitrary eccentricities, and the other two branches occupy the low-eccentricity space. The latter two branches are called asymmetric families of the resonant centre. The right panel of 4 shows that these two asymmetric branches are close to the upper boundary of the circulation region represented by $H_2 + H^2 = 0$ (this boundary corresponds to the ZLK separatrix, as discussed in Fig. 1).

Next, we discuss the essence of the resonance with the critical argument of $\sigma = h^* + \text{sign}(H^*)g^*$. According to the general definition of the longitude of pericentre $\sigma^*$ for prograde and retrograde orbit configurations (Shevchenko 2016),

$$
\sigma^* = \Omega^* + \text{sign}(\cos i^*)\omega^*,
$$

the critical argument $\sigma$ is equal to the longitude of pericentre $\sigma^*$. Because of the choice of the reference frame, it is known that the longitude of pericentre of the perturber orbit is fixed at zero, that is, $\sigma_p = 0$ and $\sigma_0 = 0$. As a result, we can further write the critical argument as

$$
\sigma = h^* + \text{sign}(H^*)g^* = \sigma^* - \sigma_p,
$$

which means that in essence, the resonances arising in Fig. 4 are the so-called apsidal resonances with the critical argument of $\sigma = \sigma^* - \sigma_p$. For simplicity, we denote the critical argument of the apsidal resonance as $\sigma = h^* + \text{sign}(H^*)g^*$ in the test-particle limit. Based on the set of Delaunay variables $(g, h, G, H)$, Sidorenko (2018) defined the critical argument as $\sigma = h + \text{sign}(H)g$, which was also adopted by Lei (2022). Additionally, Katz et al. (2011) introduced the longitude
σ transformation is introduced: 

\[ S = \sigma \]

which is canonical, with the generating function 

\[ \sigma = h + \text{sign}(H)g \]

\( h^* + \text{sign}(H^*)g^* = 0 \)

The magnitude of the canonical variables \( A \) can be filtered out by means of an averaging technique, which corresponds to the lowest-order perturbation method (Naoz 2016). To this end, we can formulate the resonant Hamiltonian by performing a further average for the Hamiltonian over one period of \( \sigma_2 \).

\[ \mathcal{H}^* = \mathcal{H}_2 = -H^2 \]

\[ \mathcal{H}_2 = -H^2 + \text{sign}(H^*)g^* = 0 \]

Under the new set of canonical variables \( (\sigma_1, \sigma_2, \Sigma_1, \text{and } \Sigma_2) \), the Hamiltonian can be further written as

\[ \mathcal{H} = \mathcal{H}_2 = \Sigma_1^2 + \Sigma_2^2 = H^2 + \text{sign}(H)g \]

\[ \mathcal{H}(\sigma_1, \sigma_2, \Sigma_1, \Sigma_2) = \mathcal{H}_2 = \mathcal{H}_3(\sigma_1, \sigma_2, \Sigma_1, \Sigma_2) \]

It is difficult to obtain the explicit expression of Eq. (12). In practice, we computed the Hamiltonian \( \mathcal{H} \) numerically when the set of variables \( (\sigma_1, \sigma_2, \Sigma_1, \text{and } \Sigma_2) \) was given.

At the resonant centre shown in Fig. 4, it holds

\[ \sigma_1 = \frac{\partial H_2}{\partial \Sigma_1} = 0. \]

As the octupole-order Hamiltonian \( H_3 \) is small compared to the quadrupole-order Hamiltonian \( H_2 \), we can obtain

\[ \sigma_1 = \frac{\partial H_3}{\partial \Sigma_1} \]

\[ \sigma_2 = \frac{\partial H_3}{\partial \Sigma_2} \]

\[ \sigma_2 \sim O(e) \]

which shows that under the perturbation of the octupole-order interaction, apsidal resonances may also take place.

When the test particle is inside an apsidal resonance, the resonant angle \( \sigma_1 \) becomes a short-period variable and the angle \( \sigma_2 \) is a short-period variable. This is a separable Hamiltonian system (Henrard 1990). Thus, the terms involving \( \sigma_2 \) in the Hamiltonian yield short-period influences. They can therefore be filtered out by means of an averaging technique, which corresponds to the lowest-order perturbation method (Naoz 2016). To this end, we can formulate the resonant Hamiltonian by performing a further average for the Hamiltonian over one period of \( \sigma_2 \).

\[ \mathcal{H}^* = \int_0^{2\pi} \mathcal{H}(\sigma_1, \sigma_2, \Sigma_1, \Sigma_2) \ d\sigma_2 \]

\[ \Sigma_2 = G^* - |H^*| = 1 - |\cos i_\ast| \]

\[ \Sigma_2 = G^* - |H^*| = 1 - |\cos i_\ast| \]

Because of the definition \( \sigma_2 = g^* \), we can understand that this average is performed over one period of a rotating ZLK cycle under the quadrupole-order Hamiltonian flow (Sidorenko 2018; Katz et al. 2011). Under the dynamical model determined by Eq. (13), the angle \( \sigma_2 \) becomes a cyclic coordinate, so that its action \( \Sigma_2 \) becomes a constant of motion (or motion integral). The resulting resonant model determined by Eq. (13) has one degree of freedom, depending on the motion integral \( \Sigma_2 \).

When the eccentricity is assumed to be zero, the motion integral \( \Sigma_2 \) can be specified by a critical inclination \( i_\ast \):

\[ \Sigma_2 = G^* - |H^*| = 1 - |\cos i_\ast| \]
The critical inclination $i_*$ corresponds to the minimum inclination in prograde space and to the maximum inclination in retrograde space. Clearly, $i_*$ and $\pi - i_*$ stand for the same motion integral.

5. Results

In the previous section, the resonant Hamiltonian for apsidal resonances was formulated as $H^\ast (\sigma_1, \Sigma_1, \Sigma_2)$, where $\Sigma_2$ is the motion integral of the resonant model. The global dynamics of apsidal resonance in phase space can be revealed by phase portraits. In this section, we produce phase portraits and then analyse the phase portraits to estimate the resonant width (Lei 2021a, 2022; Lei & Li 2021). We finally construct a connection between the libration zones of the apsidal resonance and the numerical distributions of the resonant orbit (or flipping orbit).

5.1. Dynamical structures of the apsidal resonance

In Fig. 5, the (pseudo-) phase portraits are shown in the $(\sigma_1, i_0)$ space for the motion integral specified by the critical inclination at $i_*=30^\circ(150^\circ)$, $45^\circ(135^\circ)$, $50^\circ(130^\circ)$, and $65^\circ(115^\circ)$. $i_0$ given on the $y$-axis corresponds to the inclination when the angle $g$ is equal to zero. Dynamical separatrices are marked as red lines. The resonant width, denoted by $\Delta l_0$, measures the maximum size of the island of libration.

When the critical inclination is at $i_* = 30^\circ(150^\circ)$ (see the top left panel of Fig. 5), the structure arising in the phase portrait is like a pendulum: there is one resonant centre and one saddle point. The resonant centre is located at $(\sigma_1 = 180^\circ, i_0 = 90^\circ)$ and the saddle point is located at $(\sigma_1 = 0^\circ, i_0 = 90^\circ)$. The single island of libration is bounded by the dynamical separatrix shown by the red line. As for this low-inclination case, there is another island of libration arising in low-eccentricity space, which is shown in Fig. 6.

When the critical inclination is increased up to $i_* = 45^\circ(135^\circ)$ (see the top right panel of Fig. 5), the dynamical structures become complex. In total, there are three islands of libration: one is centred at $(\sigma_1 = 0^\circ, i_0 = 90^\circ)$, and the other two islands are centred at $(\sigma_1 = 180^\circ, i_0 = 90^\circ)$; the latter two islands of libration are symmetric with respect to $i_0 = 90^\circ$. There are three separatrices (shown by the red lines), stemming from three saddle points. These separatrices provide boundaries for three isolated islands of libration in phase space.

When the critical inclination is up to $i_* = 50^\circ(130^\circ)$ (see the bottom left panel of Fig. 5), there are also three resonant centres and three saddle points. However, it is different from the case of $i_* = 45^\circ(135^\circ)$ because there are only two separatrices: an inner separatrix, and an outer separatrix. The inner separatrix bounds two asymmetric islands of libration, and the outer separatrix bounds the island centred at $(\sigma_1 = 0^\circ, i_0 = 90^\circ)$ and the island centred at $(\sigma_1 = 180^\circ, i_0 = 90^\circ)$. The region bounded by the inner and outer separatrices is also a libration region. The resonant trajectory inside this region holds the maximum variation of the inclination up to $\sim 80^\circ$.

When the critical inclination is at $i_* = 65^\circ(115^\circ)$ (see the bottom right panel of Fig. 5), the dynamical structure becomes pendulum-like again: there is a single island of libration, centred at $(\sigma_1 = 180^\circ, i_0 = 90^\circ)$. Along the resonant trajectory inside...
this island, the maximum variation of the inclination can reach $\sim 50^\circ$.

The phase portraits shown by Fig. 5 indicate that the inclinations of test particles can be effectively excited following the trajectories inside the islands of libration. This significant variation of the orbit orientation is due to the effect of apsidal resonance in the octupole-order Hamiltonian model.

For the low-inclination cases, two examples are presented in Fig. 6 for the cases of $i_0 = 20^\circ (160^\circ)$ and $i_0 = 30^\circ (150^\circ)$. Different from Fig. 5, the phase portraits are plotted in $(\sigma_1, e_0)$ space. Here $e_0$ corresponds to the eccentricity when the angle $g$ is equal to zero. In addition to the islands shown in Fig. 5 (see the top left panel), a new island arises in low-eccentricity space and is centred at $\sigma_1 = 0$. The dynamical separatrices are shown as red lines. In these two plots, the resonant width is measured by $\Delta e_0$. Following the trajectories inside the islands of libration, the eccentricities of the test particles can be excited through the effect of apsidal resonance.

### 5.2. Libration zones of the apsidal resonance

By analysing phase portraits at different levels of motion integral $\Sigma_2$, it is possible for us to identify the location of resonant centre and boundaries of libration zones. The main results of this work are reported in Fig. 7, where the resonant centres are marked as black dots and the libration zones are shown as shaded areas with different colours. The boundaries of each libration zone are provided by the dynamical separatrices evaluated at the angle of the corresponding resonant centre (see Figs. 5 and 6 for representative phase portraits). The level curves of the motion integral $\Sigma_2$ are plotted as dashed lines as background. The resonant width is measured along the isolines of the motion integral.

Figure 7 shows that the dynamical structures arising in the $(e, i)$ space are symmetric with respect to the line of $i = 90^\circ$. In total, there are five branches of the libration centres: one branch located on the polar line, two branches occupying the low-eccentricity and low-inclination space, and the remaining twobranches located in the space in which the eccentricities change from $\sim 0$ to $\sim 0.4$. In addition, there are eight libration zones, denoted by numbers from 1 to 8. The branches of the resonant centres in zones 1 and 2 are absent in the quadrupole-level dynamical model (see Fig. 4 for the nominal location of apsidal resonance). This means that these two branches are present due to the effect of the octupole-order Hamiltonian alone. Except for zones 1 and 2, all the other branches of libration centres are consistent with those shown in Fig. 4.

In libration zone 1 (see Fig. 6 for the representative phase portraits), the bottom boundary is located at $e = 0$, and the resonant width decreases first and then increases with inclination $i$. Inside this zone, Funk et al. (2011) found an interesting dynamical region around $i \sim 35^\circ$, where the low-eccentricity orbits are long-term stable, meaning that an inclined ($\sim 35^\circ$) quasi-circular Earth-mass companion can survive in the habitable zone of an extrasolar system. Libert & Delsate (2012) explained that the long-term stable dynamics at an inclination of $\sim 35^\circ$ is due to the secular resonance associated with $\sigma = \omega - \Omega$. In the low-eccentricity space with $i < 39^\circ$, Lei (2021a) studied the long-term dynamics by means of Lie series transformation. They reported that the long-term stability inside this zone is governed by the apsidal resonance with an argument of $\sigma = \omega + \Omega$, which is consistent with the result of the present work.

Libration zone 1 has a symmetric zone in the retrograde space, denoted by 2. In this zone, the bottom boundary of the libration is located at $e = 0$, and the resonant width decreases first and then increases with $\pi - i$. Inside zones 1 and 2, the effect of the apsidal resonance is to excite the eccentricities. However, the inclination has little variation during the long-term evolution. The low-eccentricity islands of libration therefore cannot be found in the phase portrait plotted in $(\sigma_1, i_0)$ space (see the first panel of Fig. 5).

Libration zone 3 appears when the critical inclination is higher than $\sim 59^\circ$ and lower than $\sim 121^\circ$ (see the bottom right panel of Fig. 5 for the representative phase portrait). The eccentricity of this zone ranges from zero to $\sim 0.35$. The effect of the apsidal resonance inside this zone is to exchange the eccentricity and inclination of the test particle. The resonant trajectories inside this zone could flip from prograde to retrograde and back again.

There are two subregions in zone 4: one zone in the prograde space, and the other one in the retrograde space. According to the phase portrait shown in the bottom left panel of Fig. 5, this zone of libration holds centres at $i = 90^\circ$ and is bounded by the inner and outer separatrices. The effect of apsidal resonance inside this zone is to significantly change the eccentricity and inclination of test particles. In addition, all the trajectories inside this zone could realise flips between prograde and retrograde.

Zones 5 and 6 are symmetric with respect to $i = 90^\circ$. The top-right panel of Fig. 5 shows the representative phase portrait.
Fig. 7. Resonant centres of apsidal resonance distributed in the inclination–eccentricity space (black dots) and libration zones obtained by analysing phase portraits (shaded areas). The level curves of the motion integral $\Sigma_2$ are shown as background and the resonant width is measured along the isoline of $\Sigma_2$. There are eight libration zones, denoted by numbers from 1 to 8. Evidently, the distribution of libration zones is symmetric with respect to the line of $i = 90^\circ$.

Fig. 8. Analytical results for the resonant regions for apsidal resonances centred at $\sigma_1, c = \pi$ (left panel) and the associated numerical results for the distribution of the apsidal resonance (right panel). For the numerical results, the initial condition is $g_0 = 0$ and $h_0 = \pi$ (corresponding to $\sigma_1, c = \pi$ at the initial instant). In the left panel, level curves of the motion integral $\Sigma_2$ are presented as background. In the right panel, blue dots stand for resonant trajectories inside islands centred at $i = 90^\circ$, and red dots show resonant trajectories inside asymmetric islands of libration. The difference arising in the top space is due to the fact that the analytical results are restrained by level curves of the motion integral $\Sigma_2$.

For these two zones, the line of $i = 90^\circ$ provides one boundary, so that the resonant trajectories are restrained in either prograde or retrograde space. This means that the trajectories inside these two zones cannot flip from prograde to retrograde or vice versa.

The phase portraits of last two zones 7 and 8 are shown in the top left and top right panels of Fig. 5. Zone 7 is located in the intermediate-eccentricity region, and zone 8 is located in the high-eccentricity region. Both of them hold resonant centres at $i = 90^\circ$. As a result, all the resonant trajectories inside these two zones can flip from prograde to retrograde and back again.

Figures 8 and 9 provide comparisons between analytical and numerical results for the libration zones of the apsidal resonance. In particular, Fig. 8 corresponds to apsidal resonances centred at $\sigma_1, c = \pi$, and Fig. 9 corresponds to those centred at $\sigma_1, c = 0$. For analytical results, libration zones 3, 4, 5, 6 and 8 are shown in the left panel of Fig. 8 and libration zones 1, 2 and 7 are presented in the left panel of Fig. 9. To be consistent, the initial conditions of numerical results are assumed as $g_0 = \pi$ and $h_0 = 0$ corresponding to the resonant centre at $\sigma_1, c = \pi$, and they are assumed as $g_0 = 0$ and $h_0 = 0$, corresponding to
the resonant centre at $\sigma_{1,c} = 0$. To produce numerical results, the equations of motion represented by Eq. (3) are numerically integrated over 500 units of dimensionless time. The orbit is recorded as a librating trajectory if the maximum variation of critical argument $\sigma = h + \text{sign}(H)g$ is smaller than $2\pi$ during the considered integration period. This means that the critical argument is librating during the integration period. The numerical distributions of the apsidal resonance are shown in the right panels of Figs. 8 and 9. As expected, good agreement can be found between analytical and numerical results, indicating that the resonant Hamiltonian formulated in the previous section is valid and can be applied to explore the dynamics of the apsidal resonance under the octupole-level approximation.

5.3. Application to orbit flips

According to the results given in Fig. 7, it is known that the resonant trajectories with resonant centres at $i = 90^\circ$ can flip from prograde to retrograde and back again. In Fig. 10, analytical results of libration zones causing orbit flips are compared with the numerical distribution of flipping orbits under the octupole-level Hamiltonian model. Four libration zones cause orbit flips (zones 3, 4, 7, and 8). Inside zones 3, 4, and 8, the resonant centres are at $\sigma_{1,c} = \pi$, and inside zone 7, the resonant centres are at $\sigma_{1,c} = 0$. Analytical results for the libration zones causing orbit flips are compared with the distribution of apsidal resonance (right panel). For the numerical results, the initial condition is $g_0 = 0$ and $h_0 = 0$ (corresponding to $\sigma_{1,c} = 0$ at the initial instant). In the left panel, level curves of the motion integral $\Sigma_9$ are presented as background. In the right panel, blue dots stand for resonant trajectories inside islands centred at $i = 90^\circ$, and red dots show resonant trajectories occurring in low-eccentricity, low-inclination spaces.

Fig. 9. Analytical results for the resonant regions for apsidal resonances centred at $\sigma_{1,c} = 0$ (left panel) and the associated numerical results for the distribution of apsidal resonance (right panel). For the numerical results, the initial condition is $g_0 = 0$ and $h_0 = 0$ (corresponding to $\sigma_{1,c} = 0$ at the initial instant). In the left panel, level curves of the motion integral $\Sigma_2$ are presented as background. In the right panel, blue dots stand for resonant trajectories inside islands centred at $i = 90^\circ$, and red dots show resonant trajectories occurring in low-eccentricity, low-inclination spaces.

Fig. 10. Analytical results of libration zones for apsidal resonances centred at $\sigma_{1,c} = 0$ (left panel) and the numerical results of flipping regions (right panel). The left panel shows four libration zones causing orbit flips (zones 3, 4, 7, and 8). Apsidal resonances with centres at $i = 90^\circ$ cause orbit flips. The right panel shows three distinct regions of orbit flips, located in low-eccentricity, intermediate-eccentricity, and high-eccentricity space.
the high-eccentricity region of flipping orbits corresponds to libration zone 8, where the critical argument $\sigma = h + \text{sign}(H)g$ librates around $\pi$.

The comparison made in Fig. 10 shows an excellent agreement between the analytical and numerical results. The results imply that the dynamics of orbit flips can be well understood with the help of dynamical structures of the apsidal resonance.

6. Conclusions

We studied the dynamics of apsidal resonance by means of perturbation treatments under the octupole-level approximation in restricted hierarchical planetary systems. The Hamiltonian function is composed of the quadrupole-order term $\mathcal{H}_2$ and the octupole-order term $\mathcal{H}_3$. From the point of view of perturbative treatments, the quadrupole-order Hamiltonian is considered as the kernel function, and the octupole-order Hamiltonian plays the role of perturbation to the quadrupole-order dynamics. By introducing the action-angle variables (a kind of canonical transformation), the quadrupole-order Hamiltonian can be converted to be independent of angular coordinates, leading to the fact that the action variables become conserved quantities under the quadrupole-order Hamiltonian flow. The transformed secular resonances with critical argument of $\sigma = h^* + \text{sign}(H^*)g^*$ occur in the considered parameter space. Secular resonances with critical argument of $\sigma = h^* + \text{sign}(H^*)g^*$ can be equivalently expressed as $\sigma = \sigma^* - \sigma_0$, which corresponds to apsidal resonances.

To study the dynamics of apsidal resonance, a canonical transformation was introduced. After transformation, it becomes a typical separable Hamiltonian model, so that we can apply first-order perturbation theory to formulate the resonant Hamiltonian model by averaging the Hamiltonian over rotating ZLK cycles. Application of first-order perturbation theory gives rise to a new constant of motion, and the resulting resonant Hamiltonian model has one degree of freedom. Phase portraits (level curves of a new constant of motion) illustrate the global dynamical structures of apsidal resonance. In particular, the location of the resonance centre, the saddle points, the dynamical separatrix between circulating and librating regions, as well as islands of libration can be determined from the resonant model.

Our main results are reported in Fig. 7. We conclude that (a) dynamical structures are symmetric with respect to $i = 90^\circ$, (b) there are five branches of libration centres, and (c) there are eight libration zones. The comparisons between analytical and numerical results for libration zones of apsidal resonance shows that they agree well.

Islands of libration centred at $i = 90^\circ$ can cause orbit flips. Thus, libration zones with resonant centres at $i = 90^\circ$ correspond to flipping regions in phase space. To validate this point, the analytical results of libration zones were compared with numerical distributions of flipping orbits. A perfect correspondence can be found between the analytical and numerical results.

Through this study, we can conclude that from the point of view of dynamics, the eccentric ZLK effect is equivalent to the effect of apsidal resonance at the octupole-level approximation in restricted hierarchical planetary systems. The dynamical response of the eccentric ZLK effect (or the effect of apsidal resonance) is to significantly excite eccentricities and/or inclinations (even flipping) of test particles in the very long-term evolution.

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