A modified Milne-Eddington approximation for a qualitative interpretation of chromospheric spectral lines

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ABSTRACT

Context. The Milne-Eddington approximation provides an analytic and simple solution to the radiative transfer equation. It can be easily implemented in inversion codes used to fit spectro-polarimetric observations and infer average values of the magnetic field vector and the line-of-sight velocity of the solar plasma. However, in principle, it is restricted to spectral lines that are formed under local thermodynamic conditions, namely, photospheric and optically thin lines.

Aims. We show that a simple modification to the Milne-Eddington approximation is sufficient to infer relevant physical parameters from spectral lines that deviate from local thermodynamic equilibrium, such as those typically observed in the solar chromosphere.

Methods. We modified the Milne-Eddington approximation by including several exponential terms in the source function to reproduce the prototypical shape of chromospheric spectral lines. To check the validity of such an approximation, we first studied the influence of these new terms on the profile shape by means of the response functions. Then we tested the performance of an inversion code including the modification against the presence of noise. The approximation was also tested with realistic spectral lines generated with the RH numerical radiative transfer code. Finally, we confronted the code with synthetic profiles generated from magnetohydrodynamic simulations carried out with the Bifrost code. For the various tests, we focused on the vector magnetic field and the line-of-sight velocity. We compared our results with the weak-field approximation and center of gravity technique as well.

Results. The response function corresponding to the new terms in the source function have no trade-offs with the response to the different components of the magnetic field vector and line-of-sight velocity. This allows us to perform a robust inference of the physical parameters from the interpretation of spectral line shapes. The strategy has been successfully applied to synthetic chromospheric Stokes profiles generated with both standard models and realistic magnetohydrodynamic (MHD) simulations. The magnetic field vector and velocity can be successfully recovered with the modified Milne-Eddington approximation.

Conclusions. Milne-Eddington model atmospheres that include exponential terms are not new to the solar community but have been overlooked for quite some time. We show that our modification to the Milne-Eddington approximation succeeds in reproducing the profile shape of two chromospheric spectral lines, namely, the Mg I b2 line and the Ca II b at 854.2 nm. The results obtained with this approach are in good agreement with the results obtained from the weak field approximation (for magnetic field) and the center of gravity (for velocity). However, the Milne-Eddington approximation possesses a great advantage over classical methods since it is not limited to weak magnetic fields or to a restricted range of velocities.

Key words. Sun: chromosphere – Sun: magnetic fields – polarization – radiative transfer

1. Introduction

The Milne-Eddington (ME) approximation is a useful method to interpret spectral line radiation under local thermodynamic equilibrium (LTE) and to recover relevant physical information of the solar plasma and magnetic field. It is the solution to the Radiative Transfer Equation (RTE) under the conditions of a plane parallel atmosphere, constant magnetic and velocity fields, and a source function with a linear dependency on the opacity. With respect to the Sun, these conditions are typically met in the photosphere and the approximation works properly in this region. Higher up in the chromosphere, the physical conditions start to depart from the LTE and therefore they are no longer compatible with the assumptions that lead to the ME approximation. Departures from LTE conditions lead to non-typical spectral line shapes (i.e., non Gaussian-like) and the assumption of a linear source function proposed in the ME approximation is not good enough to reproduce them.

Lites et al. (1988) presented a modification of the ME approximation proposing a non-linear expression for the source function based on the solution of the non LTE (NLTE) transfer problem for a standard two-level atom. The new approximation consists of adding various exponential terms with an explicit dependency on the optical depth to the ME linear source function. The above-mentioned authors showed that the modified ME model with a single exponential term was sufficient to reproduce the prototypical Mg I b triplet. They also suggested that the new source function was similar to the original one.

To test the validity of the new approximation, the authors took a modified semi-empirical umbral model (Maltby et al. 1986; Lites et al. 1987) and considered a constant magnetic field to synthesize the Mg I b triplet. Then they performed inversions applying different initial model configurations to test the validity of the new approach (i.e., with one and two exponential terms in the source function) or using either fixed or free thermodynamic parameters. Two remarkable conclusions were drawn from their
study: firstly, only one exponential term in the source function was sufficient to reproduce the Mg I b2 profile shape. Secondly, to successfully retrieve the original model parameters, the inversion should include the four components of the Stokes vector (i.e., the polarization signals) with varying weights among them (i.e., weights of 0.01 for Stokes $I$ and 0.1 for Stokes $Q$, $U$, and $V$). The authors argued that inverting the full Stokes vector with the mentioned weights helped to properly fit the four Stokes profiles simultaneously because the trade-offs between the thermodynamic parameters become less relevant. They also showed that there is a correlation between the sensitivity to the magnetic field strength and the line-to-continuum absorption coefficient in the weak field regime. It is important to remark that the authors focused on the line core only and they obtained slightly underestimated field strengths for inclinations around 45°.

Based on the idea of Lites et al. (1988), the main goal of this work is to ascertain whether it is possible to systematically reproduce the shape of chromospheric spectral line profiles using this modified version of the Milne-Eddington approximation (mME, from now on, i.e., ME approximation with a non linear source function). In Sect. 2, we present the mathematical description of the mME approximation and briefly describe the weak field approximation (WFA) as well as the center of gravity (CoG) method. In Sect. 3, we test the validity of the approximation and its applicability as a fast inverson on synthetic profiles of the Mg I b2 line generated from a FALC model (Fontenla et al. 1993). In Sect. 4, we apply the mME approximation to synthetic profiles generated from a radiative MHD simulation and compare the retrieved magnetic field and LOS velocity with those obtained from the WFA and CoG, respectively, as well as with the original stratifications in the simulation. Finally, in Sect. 5, we discuss the results, along with the capabilities and the shortcomings of the mME approximation.

2. Methodology

2.1. Modified Milne-Eddington approximation

The Milne-Eddington approximation to the RTE makes the assumptions that: the atmosphere is plane-parallel and semi-infinite; the physical quantities $B$, $\theta$, $\chi$, and $\Delta\theta$ (magnetic field strength, inclination, azimuth, and Doppler width, respectively) are depth-independent. Consequently, the magnetic field vector, the line-of-sight (LOS) velocity, and the absorption and anomalous dispersion profiles are also independent of the optical depth, $\tau$. Nonetheless, the source function, $S$, depends linearly on $\tau$ as follows:

$$S(\tau) = S_0 + S_1 \tau. \quad (1)$$

Under these assumptions, an analytical solution for the RTE can be found (see, e.g., Unno 1956; Rachkovsky 1962, 1967; Landolfi & Landi Degl’Innocenti 1982; Orozco Suárez & Del Toro Iniesta 2007). In total, a ME model can be fully described with just nine parameters: three for the magnetic field vector, one for the LOS velocity, and five for the so-called thermodynamic parameters, that is, $S_0$, $S_1$, the line-to-continuum absorption coefficient, $\rho_0$, the Doppler width, $\Delta\theta$, and the damping parameter, $a$. Therefore, the ME approximation does not provide information about gradients with the height of the physical parameters and cannot reproduce asymmetries of the line profiles. Nonetheless, it is a much faster method than typical inversion codes that solve the RTE under LTE assumptions (for a review see del Toro Iniesta & Ruiz Cobo 2016), which makes it an ideal method for performing statistical estimates of the magnetic field in the solar photosphere. However, in order to be applicable in chromospheric spectral lines, the ME approximation requires some modifications.

The chromosphere is the region of the solar atmosphere where the temperature rises and magnetic pressure starts to dominate over the gas pressure. This leads to a departure from LTE, so to model chromospheric lines, it is necessary to solve the RTE in NLTE conditions. The ME approximation can be modified in a way so that the source function reproduces the temperature rise in the chromosphere. In practice, this is done by adding depth-dependent exponential terms in the linear source function, as follows:

$$S(\tau) = S_0 + S_1 \tau + \sum_{i=1}^{N} A_i e^{-\alpha_i \tau} \approx S_0 + S_1 \tau + A_1 e^{-\alpha_1 \tau} - A_2 e^{-\alpha_2 \tau}, \quad (2)$$

where the new source function has two new parameters, $A$ and $\alpha$, per exponential. To solve the inverse problem and to preserve the uniqueness and stability of the solution, a low number of exponentials ought to be used. Fortunately, setting $N = 2$ provides two new terms and four additional free parameters that successfully reproduce the prototypical shape of chromospheric spectral lines. To maintain consistency with Landi Degl’Innocenti & Landolfi (2004), in Eq. (2), we changed the sign in the second exponential. In fact, this new source function yields to new analytical expressions for the Stokes parameters (Lites et al. 1988; Landi Degl’Innocenti & Landolfi 2004):

$$I = [S_1 + S_1 K^{-1} + A_1 (1 - \alpha_1 (\alpha_1 + K)^{-1})] + A_2 (1 - (1 + \alpha_2) (\alpha_2 + K)^{-1}) U, \quad (3)$$

where $I$ is the Stokes vector whose components are $(I, Q, U, V)^t$, $K$ is the propagation matrix, $U$ is the unit vector $(1, 0, 0, 0)^t$, and $I$ is the identity matrix.

The role of the two new exponentials is as follows: one of them (e.g., $i = 1$) is meant to reproduce possible emission features in the line core (through $A_1$) and to make the line wider along the wings ($\alpha_1$). The other one (e.g., $i = 2$) narrows the shape of the profiles around the line core ($\alpha_2$) and increases the line-to-continuum contrast ($A_2$). This expression can be reduced to the Unno-Rachkovsky solution when $A_1 = A_2 = 0$ (i.e., the classical ME approximation for a linear source function). In the case of the mME model the intensity of the nearby continuum is:

$$I_c = S_0 + S_1 + \frac{A_1}{1 + \alpha_1}. \quad (4)$$

and since the line core opacity at the continuum is negligible, $k_\lambda \rightarrow 0$, the propagation matrix can thus be replaced by the identity matrix, $K \rightarrow I$. It is worth to mention that, if $k_\lambda \rightarrow \infty$, the intensity of the line core can be written as $I_{core} = S_0 + A_1 - A_2$. Hence, the mME produce emission in the line core when $A_1 - A_2 < 0$, that is, when $A_2 < A_1$.

Figure 1 displays a synthetic Stokes $I$ profile corresponding to the Mg I b2 line at 517.3 nm, generated from realistic MHD simulations (see Sect. 4). The plot compares the fits resulting from both the ME and the mME approximations. The ME fit resembles a typical Voigt profile and cannot reproduce the synthetic profile. In contrast, the mME approach, with only two
exponential terms, is able to reproduce both: the narrow line core and the wide wings. The difference in the quality of the fits between the blue and red wings of the line illustrates the fact that the synthetic profile is not symmetric while the mME approach provides strictly symmetric Stokes I profiles. Table 1 summarizes the results obtained for the different free parameters from both inversions. Unlike Lites et al. (1988) who used only the line core, we use a wider spectral window to include the full Mg I b2 line. The reason is that they used a data set that corresponds to a sunspot, where large magnetic field gradients are expected. In our case, the data set represents a portion of the quiet Sun, where the gradients of the magnetic field with height are milder than in sunspots, and therefore, we can use the full spectral range. This is the reason why they concluded that only one exponential term was enough to reproduce the shape of the line. As we will show later by means of response functions, two exponential terms in the source function are needed to successfully reproduce not just the core, but the full line profile as well.

2.2. Response Functions in the mME case

One of the strengths of the ME approximation is that it is possible to derive response functions analytically. The response functions give information about how the Stokes profiles change due to the variation of a given physical parameter. They are defined as the partial derivatives of the Stokes profiles with respect to one of the physical parameters in the model (Orozco Suárez & Del Toro Iniesta 2007). These authors have shown a very important property of ME model atmospheres through the interpretation of the corresponding RFs: The sensitivity of ME Stokes profiles to perturbations in the magnetic field is uncorrelated with the perturbations in the line-of-sight (LOS) velocity and with the thermodynamic parameters (del Toro Iniesta et al. 2010; del Toro Iniesta & Ruiz Cobo 2016). This fact makes the ME approach robust when determining those physical parameters. However, there is much more uncertainty in the determination of the thermodynamic parameters since the corresponding response functions show clear trade-offs between them.

As explained before, the mME approximation with \( N = 2 \) has four new thermodynamic parameters. Hence, describing a mME model requires a total of thirteen free parameters, nine of which are meant just to reproduce the shape of the profiles. The new mME response functions with respect to the new four free parameters can also be calculated analytically. The analytical response functions to the first exponential parameters \( A_1 \) and \( a_1 \) are:

\[
\frac{\partial I}{\partial A_1} = 1 - a_1 d_{00}(a_1); \quad \frac{\partial I}{\partial a_1} = -a_1 d_{00}[1 - a_1 d_{00}(a_1)], \tag{5}
\]

\[
\frac{\partial Q}{\partial A_1} = -a_1 d_{01}(a_1); \quad \frac{\partial Q}{\partial a_1} = A_1 a_1 d_{02}^2(a_1), \tag{6}
\]

\[
\frac{\partial U}{\partial A_1} = -a_1 d_{02}(a_1); \quad \frac{\partial U}{\partial a_1} = A_1 a_1 d_{02}^2(a_1), \tag{7}
\]

\[
\frac{\partial V}{\partial A_1} = -a_1 d_{03}(a_1); \quad \frac{\partial V}{\partial a_1} = A_1 a_1 d_{03}^2(a_1), \tag{8}
\]

and to the second exponential parameters \( A_2 \) and \( a_2 \):

\[
\frac{\partial I}{\partial A_2} = (1 + a_2) d_{00}(a_2) - 1; \quad \frac{\partial I}{\partial a_2} = A_2 d_{00}(a_2)[1 - (1 + a_2) d_{00}(a_2)], \tag{9}
\]

\[
\frac{\partial Q}{\partial A_2} = (1 + a_2) d_{01}(a_2) - 1; \quad \frac{\partial Q}{\partial a_2} = -A_2(1 + a_2) d_{02}^2(a_2), \tag{10}
\]

\[
\frac{\partial U}{\partial A_2} = (1 + a_2) d_{02}(a_2) - 1; \quad \frac{\partial U}{\partial a_2} = -A_2(1 + a_2) d_{02}^2(a_2), \tag{11}
\]

\[
\frac{\partial V}{\partial A_2} = (1 + a_2) d_{03}(a_2) - 1; \quad \frac{\partial V}{\partial a_2} = -A_2(1 + a_2) d_{03}^2(a_2), \tag{12}
\]

where \( d_{00} \) are the elements of the inverse matrix defined by \( D(\alpha) = (\alpha + K)^{-1} \) which are given by

\[
d_{00}(\alpha) = \Delta^{-1}(\alpha)(1 + \alpha + k_1)(1 + \alpha + k_2) + f_i^2 + f_v^2 + f_f^2, \]

\[
d_{01}(\alpha) = -\Delta^{-1}(\alpha)[(1 + \alpha + k_1)^2 k_2 + (1 + \alpha + k_1)(k_3 f_v - k_v f_3)] + f_0(k_4 f_3 + k_4 f_v + k_v f_4), \]

\[
d_{02}(\alpha) = -\Delta^{-1}(\alpha)[(1 + \alpha + k_1)^2 k_3 + (1 + \alpha + k_1)(k_4 f_v - k_v f_4)] + f_0(k_4 f_3 + k_4 f_v + k_v f_4), \]

\[
d_{03}(\alpha) = -\Delta^{-1}(\alpha)[(1 + \alpha + k_1)^2 k_4 + (1 + \alpha + k_1)(k_4 f_v - k_v f_4)] + f_0(k_4 f_3 + k_4 f_v + k_v f_4). \tag{13}
\]

According to Landi Degl’Innocenti & Landolfi (2004). The term \( \Delta(\alpha) \) denotes the determinant of the matrix \( D \).

\[
\Delta(\alpha) = (1 + \alpha + k_1)^4 + (1 + \alpha + k_1)^2(f_f^2 + f_v^2 + f_f^2) - k_i^2 - k_i, \]

\[
- k_1^2 - k_1 - k_2^2 - k_2 - k_3^2 - k_3 - k_4^2 - k_4 - k_5^2 - k_5. \tag{14}
\]

In the mME model atmosphere there are more chances for trade-offs between the thermodynamic parameters and the vector magnetic field and LOS velocity. Fortunately, the sensitivity of the Stokes profiles to these new parameters show no trade-offs with the magnetic field vector or the LOS velocity. Figure 2 shows the analytical Stokes I response functions to the four parameters describing the exponential terms. The shape of the
response functions shows how these parameters modify the profile shape to accommodate the prototypical chromospheric line profile shapes, that is, the possible emission features in the line core and the wider wings and narrower line core. By comparing these response functions with the ones presented in Orozco Suárez & Del Toro Iniesta (2007) we can easily check that there are no trade-offs between the thermodynamic parameters and the vector magnetic field and the velocity. Although there is a strong trade-off between the different thermodynamic parameters as in the ME case.

For the tests described in this paper, we have numerically implemented the mME model in the LMpyMilne (LMfit Inversion in a Milne-Eddington atmosphere) code\(^2\) and in the MILOS\(^3\) code (Orozco Suárez & Del Toro Iniesta 2007). In particular, we added two exponential terms to the source function and used the new analytical solution to the RTE – Eq. (3). Both codes are based on the Levenberg-Marquardt algorithm for minimizing the residuals of the Stokes profiles in order to fit them. During the inversion process, all the parameters are defined to be always positive except for \(A_1, A_2\), and \(v_{\text{los}}\).

### 2.3. Weak field approximation method

We apply the weak field approximation (WFA) to our data and compare the results with our method. When the Zeeman splitting in the spectral line is much smaller than the Doppler width of the line (\(\Delta \lambda_B \ll \Delta \lambda_D\)), we can perform a perturbative analysis of the RTE (Landi Degl’Innocenti & Landi Degl’Innocenti 1973; Jeffries et al. 1973; Landi Degl’Innocenti & Landolfi 2004; Centeno 2018) and deduce some properties of the solutions to the RTE without solving it formally. The first order approximation leads to an expression that relates the circular polarization (Stokes \(V\)) with the derivative of the intensity (Stokes \(I\)):

\[
V(\lambda) = -\Delta \lambda_B \tilde{g} \cos \theta \frac{\partial I(\lambda)}{\partial \lambda},
\]

where \(\tilde{g}\) the effective Landé factor of the transition. The contribution from the magnetic field to this expression comes from the LOS component by means of \(\Delta \lambda_B = \kappa \lambda \xi_0 B\) (where \(\kappa = 4.6686 \times 10^{-13} \text{G}^{-1} \text{Å}^{-1}\) and \(\cos \theta\), with \(\theta\) the inclination of the magnetic field and \(\lambda_0\) the rest wavelength of the spectral line of interest.

The linear polarization profiles, \(Q\) and \(U\), are given by the second order approximation by considering the azimuth as constant and equal to zero, which leads to:

\[
Q(\lambda) = -\frac{1}{4} \Delta \lambda_B^2 \tilde{G} \sin^2 \theta \frac{\partial^2 I(\lambda)}{\partial \lambda^2},
\]

where \(\tilde{G}\) is the transversal Landé factor, defined as:

\[
\tilde{G} = \tilde{g}^2 - \delta
\]

with

\[
\delta = \frac{1}{80} (g_u - g_l)^2 [16 J_u (J_u + 1) + J_l (J_l + 1)] - 7 J_u (J_u + 1) - J_l (J_l + 1)^2 - 4],
\]

where \(J_u\) and \(J_l\) being the total angular momentum of the upper and lower level, respectively, and \(g_u\) and \(g_l\) are Landé factors of these levels. Strictly speaking, the WFA approximation shown in Eq. (16) is only valid at the line center (Landi Degl’Innocenti & Landolfi 2004). However, since our goal is to make a one-to-one comparison with the mME model, we decided to apply the WFA as such to the whole line profile.

From the boundary conditions it can be deduced that \(U(\lambda)/Q(\lambda) = \tan 2\lambda\) for \(Q(\lambda) \neq 0\), so this expression is valid in the reference frame where \(U(\lambda) = 0\). Using these expressions, we can infer the longitudinal and transverse magnetic field components by performing a minimization of the residuals between the observed circular and linear polarization signals and the derivative of the intensity. The minimization can be obtained analytically and leads to the following expression for the longitudinal component of the magnetic field with respect to the line of sight (LOS):

\[
B_\parallel = -\sum V_i \left( \frac{\partial I}{\partial \lambda} \right)_i C_i \sum \left( \frac{\partial I}{\partial \lambda} \right)^2_i,
\]

where \(C_i = k_2 \lambda \xi_0^2 g_l\). Likewise, using Eq. (16) and following the same procedure than before, but now relating linear polarization signals (i.e., \(Q\) and \(U\)) with the second derivative of Stokes \(I\), we obtain an expression for the so-called transverse magnetic field component with respect to the LOS:

\[
B_\perp = \frac{\sum \left( \frac{\partial I}{\partial \lambda} \right)_i}{C_2 \sum \left( \frac{\partial I}{\partial \lambda} \right)^2_i},
\]

where \(C_2\) is a constant defined as \(C_2 = k_2 \lambda \xi_0^2 G/4\) and \(L_i = \sqrt{Q_i^2 + U_i^2}\) is the total linear polarization, which in our reference frame is given only by Stokes \(Q\) since \(U = 0\).

### 2.4. Center of gravity method

Following Uitenbroek (2003) and references therein (e.g., Semel 1967), the LOS velocity can be determined as

\[
v_{\text{CoG}} = \frac{c (\lambda_0 - \lambda_{\text{CoG}})}{\lambda_0},
\]
with $\lambda_{\text{CoG}}$ defined as

$$\lambda_{\text{CoG}} = \frac{\int \lambda (I_c - I)d\lambda}{\int (I_c - I)d\lambda},$$

(22)

where $I_c$ is the continuum intensity; $I$ the intensity at a given wavelength; $\lambda$ $\lambda_{\text{CoG}}$ the center of gravity wavelength; and $c$ the speed of light.

3. Validity of the mME model

To test the validity of the mME approximation in chromospheric spectral lines, we considered different scenarios. We first checked the convergence and uniqueness of the solution using mME profiles (Sect. 3.1). Then we generated realistic Stokes parameters with the RH NLTE synthesis code (Uitenbroek 2001, 2003). We performed the synthesis in a wavelength range that goes from −400 to +400 mÅ around the line core, with a sampling of 10 mÅ. We did not take into account PRD effects because they are negligible for Mg I b2, so it was performed under CRD conditions (Quintero Noda et al. 2018) (Sect. 3.2).

3.1. Convergence and uniqueness of the solution against noise

Although the mME approximation can be written down analytically, the extraction of the physical quantities from the observed Stokes profiles requires the use of inversion techniques, that is, an automatic procedure that minimizes the squared differences between a synthetic Stokes profile (associated to a given model atmosphere) and the observed one. The absence of trade-offs between the different model parameters provides confidence for the determination of the different physical quantities. However, the inversion algorithm can also introduce uncertainties into the process. For instance, the algorithm might not be robust enough due to the large number of parameters to be fit or due to the noise in the observations. Hence, to check the robustness of the inversion code, a set of synthesized Stokes profiles emerging from 100,000 mME model atmospheres with added random noise at the level of $10^{-3} I_c$, $0.5 \times 10^{-3} I_c$, and $10^{-4} I_c$ were generated, assuming a random model atmosphere. The model was built with a uniform random distribution of vector magnetic fields (with strength from 0 to 1500 G and inclination and azimuth from 0° to 180°) and LOS velocities between $-2$ km s$^{-1}$ to $+2$ km s$^{-1}$. Both the magnetic field vector and the LOS velocity are constant with height. The thermodynamic parameters are allowed to vary randomly around 20% of the initial value. The initial model parameters are: $S_0 = 0.06$, $S_1 = 0.86$, $y_0 = 900$, $\Delta I_D = 0.56$, $a = 0.03$, $A_1 = 0.74$, $a_1 = 11.42$, $A_2 = 0.76$, and $a_2 = 25.58$. All the tests were done with the Mg I b2 line at 517.3 nm so the thermodynamic parameters correspond to those that best reproduce this spectral line. The number of spectral samples across the line was 100 with a sampling of 0.3 pm. In the inversion, we allowed a maximum of 100 iterations and, contrary to Lites et al. (1988), we did not need to apply different weights. We think variable weights, as they established, are really necessary as soon as the Stokes profiles show asymmetries. The averaged time of the inversion of those random pixels is ∼0.6 s per pixel.

The inversion results can be seen in Fig. 3. The root mean squared (rms) errors depend on the amount of noise applied to the simulated profiles. In particular, for magnetic fields stronger than 200 G, the rms errors of the magnetic field strength are ∼37, 19, and 4 G for $10^{-3} I_c$, $0.5 \times 10^{-3} I_c$, and $10^{-4} I_c$ noise levels, respectively. For weaker fields, the rms values are ∼80, 53, and 18 G, although the most important contribution to the error comes from a clear deviation of the mean value for the inferred field strengths. The field strength is always overestimated by more than 100 G when the field is smaller than 200 G for a noise
level of $10^{-3} I_c$. The deviation is almost negligible as soon as the Stokes profiles are affected by an rms noise of $10^{-4} I_c$. For the field inclination, the noise increases monotonically as the field strength decreases, being on average $6.5^\circ$, $4.2^\circ$, and $1.4^\circ$ for $10^{-3} I_c$, $0.5 \times 10^{-3} I_c$, and $10^{-4} I_c$ noise levels, respectively, and for fields stronger than $200$ G. The azimuth is the most affected parameter by the noise, with rms values always above $20^\circ$.

### 3.2. Comparison with standard chromospheric models

In this subsection, we test the robustness of the mME approximation against realistic Stokes profiles. First, we combined different magnetic field configurations with a FALC model atmosphere (Fontenla et al. 1993) and generated a set of Stokes profiles with the RH code. In the models, the magnetic field strength spans from $100$ G to $1500$ G in steps of $100$ G, while the field inclination is set to $0^\circ$, $45^\circ$, and $90^\circ$. In all models, the magnetic field azimuth is constant and set to zero degrees. For the tests, we concentrated on the Mg I b2 line at $517.2$ nm. We performed $100$ mME inversions for each of the magnetic field configurations mentioned above. We use the same initialization each time but adding a varying random noise signal of $0.5 \times 10^{-3} I_c$ to the synthetic Stokes profiles. We also applied the WFA in different field regimes: weak, intermediate, and strong fields.

Figure 4 shows the mean magnetic field strength that results from the $100$ fits for each model using the mME approximation (top panel) and the WFA (bottom panel) – both displayed versus the input magnetic field strength values. In the mME case, the inferred field strengths are slightly overestimated for horizontal magnetic fields ($90^\circ$) and the deviations to the input values, in terms of relative errors, reach up to $13\%$ for the weakest fields and about $6.7\%$ for the strongest ones; while for intermediate inclinations ($45^\circ$) and longitudinal fields ($0^\circ$) the resultant magnetic fields are slightly underestimated. This behavior was already pointed out by Lites et al. (1988). However, for $45^\circ$ inclinations, they obtained errors of about $9\%$ after including the full Stokes vector in the inversion; whereas in our case, the inferred values are underestimated up to $3\%$ but such relative errors do not show a clear dependency on the magnetic field strength. The smaller error values of the retrieved magnetic field strengths may be simply because we consider a larger spectral window than Lites et al. (1988). It is important to remark that the standard deviations calculated from the $100$ inversions are very small (of the order of $1$ G), which means that the deviations from the identity line are not due to poor fits.

The bottom panel in Fig. 4 shows the limitations of the WFA when it is applied to realistic Stokes profiles. It can be seen how the inferred fields significantly deviate from the input fields at inclinations of $0^\circ$ and $45^\circ$ for strengths above $1000$ G. This is because the WFA is valid for magnetic fields where the Zeeman splitting is smaller than the Doppler width of the line. Therefore, the stronger the fields, the larger the errors are. Anyhow, for vertical fields, the deviation of the WFA is slightly smaller than in the mME approximation. However, it can also be noticed that errors get significantly larger as the magnetic field inclination increases, in part because the transverse component of the field is based on a second order approximation, unlike the longitudinal component of the magnetic field.

It can be argued that the mME approximation in general provides better results than the WFA for different magnetic field strengths and inclinations. It has limitations, however, as the deviations may be as large as $200$ G for horizontal fields.

### 3.3. Source function determination

Lites et al. (1988) pointed out that the mME source function they obtained was quite similar to the original one, which might open the possibility to infer temperatures from the mME source functions. However, we have found that there are clear discrepancies between both source functions (see Fig. 5). The reason is that the mME approximation considers a constant propagation matrix ($K$) with optical depth, which would certainly not be enough for reproducing chromospheric profile shapes. The mME uses the exponential terms of the source function to reproduce the line shapes discarding the physics of the line formation, thus, the mME source function deviates from the real one. Nonetheless, the approximation can successfully reproduce the line profile shapes, provided they are symmetric.

Figure 5 shows a comparison between the source function inferred with the mME model for two different magnetic field strengths and the original source function computed by the RH synthesis code. The mME source function is understood as a thermodynamic parameter and, as discussed in the previous section, it does not show trade-offs with the parameters describing the vector magnetic field. Hence, the two results are quite similar except for a small deviation at around $\log \tau = -1.0$. However, the source function is quite different – apart from the fact that all of them tend to decrease towards higher atmospheric...
layers, for example, there are clear humps around $\log \tau = -1.2$ in the mME source functions, while the source function shows the hump at around $\log \tau = -2.5$ with a much smaller amplitude.

4. Comparison against realistic MHD models

We use the 3D radiative magneto-hydrodynamic simulation of a portion of the enhanced network run performed with the Bifrost code (Gudiksen et al. 2011) by Carlsson et al. (2016). The physical size of the original box is $24\text{ Mm} \times 24\text{ Mm} \times 16.8\text{ Mm}$ with a $48\text{ km}$ sampling in the horizontal domain and a $19\text{ km}$ sampling in the vertical one. The model goes from $2.4\text{ Mm}$ below the photosphere (upper convection zone) to about $14.4\text{ Mm}$ above it, including the chromosphere and part of the corona.

The magnetic field in the simulation is a bipolar structure seen at photospheric heights as two clusters of magnetic field concentrations that have similar strength but display opposite polarities. Here, we focus on a $100 \times 100$ pixels region of positive polarity. The area contains pixels with both strong and weak magnetic field concentrations, so that the mME approximation can be tested for a wide range of magnetic field strengths. From these simulations, Quintero Noda et al. (2018) synthesized the Mg I b triplet (b1, b2, and b4) and the Ca II 854.2 nm line with a wavelength sampling of 10 mÅ using the RH code. Here, we use those synthetic profiles in the cropped region.

We inferred the line-of-sight velocity and the magnetic field vector from the synthetic profiles using the mME approximation. The mME inversion has been performed in a pixel by pixel basis, obtaining for each of the pixels: the magnetic field strength ($B$), inclination ($\gamma$), azimuth ($\phi$), the line-of-sight velocity ($v_{\text{LOS}}$), the line-to-continuum absorption coefficient ratio ($\eta_0$), the damping coefficient ($\alpha$), the Doppler width ($\Delta v_D$), and the six parameters describing the mME source function (given in Sect. 2.1). The inversion was repeated three times: first, the initial model was initialized randomly; after the first inversion, we calculated the mean value of each of the inferred model parameters and set them as the initial model for a second inversion; finally, the process was repeated using the results of the second inversion. The first tests showed that if the thermodynamic parameters in the input model were very different from those in the original model, the code does not converge towards the correct solution for a significant number of pixels. This is the reason for repeating the inversion three times. With the spatial averaged values as input model in the second and third inversion, we help the code to approach to the correct solution for all pixels. The procedure helps us to reach a better convergence – at the cost of computing time, however. It is important to remark that these tests were carried out without taking into account the influence of noise sources coming from either the measurements process (i.e., telescope diffraction, limited sampling) or photon noise. This is beyond the scope of this paper and will be treated elsewhere.

The results of the inversion will be compared with the original model atmosphere but on an optical depth scale. This model was obtained during the spectral synthesis (Quintero Noda et al. 2018). For completeness, the results obtained with mME code will also be compared with those obtained from the weak field approximation (WFA – Sect. 2.3) for the magnetic field strength and inclination and from the CoG technique (Sect. 2.4) for the LOS velocity.

4.1. Magnetic field strength

Figure 6 shows the maps of the magnetic field strength from the simulations at $\log \tau_{500} = -3.0$ and $\log \tau_{500} = -5.0$, optical depths that give the minimum rms when compared to the mME approximation applied to the Mg I b2 line and to the Ca II 854.2 line, respectively. In terms of geometrical heights, these optical depths correspond to a wide range of kilometers whose averages correspond to $\sim 460 \text{ km}$ and $\sim 880 \text{ km}$, respectively. As expected, the Ca II 854.2 provides information from higher layers than the Mg I b2 line. The figure also shows the maps of the field strength obtained from the mME inversion and from the WFA in the two lines. In the mME, magnetic field strength is one of the inverted physical parameters. In WFA we have to use transversal and longitudinal components – Eqs. (19) and (20) – to compute it with $B = \sqrt{B_\parallel^2 + B_\perp^2}$. These components have been obtained by applying the approximation to the whole spectral range.

The resemblance among the mME and WFA in the context of the simulations is quite significant but looking into the details, there are several mismatches that can be appreciated. For instance, the inferred magnetic field maps show slightly stronger fields within magnetic field concentrations for both methods in the Mg I b2 case, while in the Ca II 854.2, it only happens for
Fig. 6. Maps of the magnetic field strength in the region of interest from the Bifrost simulation (left panels), from the mME approximation (central panels), and from the WFA (right panels); and for the Mg I b2 line (top) and for the Ca II 854.2 nm (bottom). The maps in the left panels are taken at the estimated optical depths according to the mME inversion results ($\log \tau_{500} = -3.0$ for Mg I b2 and $\log \tau_{500} = -5.0$ for Ca II).

Likewise, the magnetic field inferred by the mME method and corresponding to the Ca line shows stronger fields than the WFA. Indeed, the rms are slightly smaller for the WFA than for the mME in this particular case.

Figure 7 displays 2D histograms comparing the magnetic field strengths resulting from the mME, WFA, and the Bifrost simulations. The dispersion is rather large for both the mME and the WFA when compared to Bifrost, which just highlights the fact that the inferences cannot be ascribed to single heights but the locations where the lines are sensitive to the magnetic field encompass a wide range of geometrical heights. The histograms in the bottom panels show a good correlation between mME and WFA for the Mg I b2 line; while for the Ca II 854.2 line the mME provides stronger fields than the WFA. This can also be seen in Fig. 6, where the WFA map shows weaker fields than the mME map, mainly in the regions with the strongest field concentrations.

It is important to mention that the radiative transfer problem of the mME is a much stronger approximation than that of the WFA. This latter is limited to regions where the magnetic field is weak enough, or in detail, whenever the Zeeman splitting is much smaller than the Doppler broadening of the line. The good resemblance between the inferred magnetic field maps shown here indeed suggests that the WFA is adequate for having a glance at the longitudinal magnetic fields for chromospheric lines in weak field areas. However, we ought to keep in mind that the WFA will fail in those regions where the approximation does not hold (i.e., whenever Stokes V is no longer proportional to the longitudinal component of the field). This can happen because of the underlying physics, such as strong field areas or instrumental effects, as with the broadening of the line that induce linear deviations and thus an erroneous inference for the longitudinal field.
Fig. 8. Magnetic field inclination (γ) maps of the region chosen for this study (in degrees). First row represents inclination corresponding to the Mg I b2 line. Left: Bifrost γ at Mg I b2 formation height; central: inverted γ using mME; right: inverted γ using WFA. Second row contains inclination maps corresponding to Ca II 854.2 line. Left: Bifrost γ at Ca II 854.2 formation height; central: inverted γ using mME; right: inverted γ using WFA. Formation heights have been estimated choosing those whose residuals were minimum with the mME inverted B.

4.2. Magnetic field inclination

The results for the magnetic field inclination are shown in Fig. 8. As for the magnetic field strength case, the figure shows the inclinations from the simulations taken at the same optical depths as in the field strength case, as well as those inferred with the mME approximation and the WFA, for both lines of interest. As we mention above, the magnetic field inclination (γ) is one of the physical quantities that we invert with the mME model atmosphere. However, for the WFA case (as we did with the magnetic field strength), the inclination has to be obtained from \( B_\parallel \) and \( B_\perp \), using Eqs. (19) and (20) as:

\[
\gamma = \arctan \left( \frac{B_\perp}{B_\parallel} \right). \quad (23)
\]

The magnetic structures appear to be more compact in the maps of the Mg I b2 line than in those of the Ca II line. This is due to the opening of the magnetic field and the increase in the inclination with height. The Mg I b2 show fields that are more vertical simply because this spectral line is sampling deeper layers.

Figure 9 shows histograms of mME inclination versus Bifrost’s values for both lines. In both cases we can see a clear correlation between mME and the original atmospheres. There is a discrepancy for inclinations around 90° (i.e., horizontal fields) in both lines. This is due to the fact that the majority of these horizontal fields are associated with weak fields in regions where the vertical stratifications change rather dramatically, hence, giving rise to strong asymmetries in Stokes \( Q \) and \( U \). This is the reason why the mME approximation fails in reproducing those profiles.

4.3. Line-of-sight velocities

Figure 10 shows LOS velocity maps for Mg I b2 and Ca II 854.2 from the simulations, from the mME inversion, and from the CoG technique applied to the synthetic profiles and also applied to the best fit profiles resulting from the mME inversion. The velocity maps for the Mg I b2 line from mME show a clear granulation pattern, typical of the photosphere, which is not seen at \( \log \tau_{500} = -3.0 \). In this case, the CoG applied to the synthetic profiles provides a more similar velocity map to Bifrost, while

Fig. 9. Normalized 2D inclination (in degrees) histogram for Mg I b2 line (left panel) and for Ca II 854.2 line (right panel). We represent mME inverted γ vs. Bifrost γ at the line formation height.
the CoG applied to the mME best fits also shows the granulation pattern as the mME inversions. The large differences might be due to large asymmetries in the Stokes I profiles that are not taken into account by mME.

For Ca II, the velocity maps provide a more chromospheric appearance. The larger structures seem to be recovered well, but at small scales, there are clear differences between mME and the original maps. The results from the CoG also seem to be consistent with the original velocity map. The main difference is the presence of some small patches with strong blue-shifts that are not seen in the simulations.

Finally, it can also be seen that the CoG applied to the best-fit mME profiles retrieve the inferred LoS velocity with the mME inversions quite well, in both the Mg I b2 and the Ca II 854.2 nm lines, while the CoG applied to the synthetic profiles always display some differences with respect to mME. This result, although obvious, highlights the influence of spectral line asymmetries in the determination of the CoG wavelength within the line, asymmetries that are not taken into account by mME.

5. Conclusions

In this paper we have presented a modification to the classical Milne-Eddington approximation that allows for the interpretation of solar chromospheric lines, such as the Mg I b2 and the Ca II 854.2 nm lines. The mME approximation was already tested by Lites et al. (1988) using realistic chromospheric models; however, in our opinion, this was not done to great extent. Here, we have recycled the mME approximation since it allows for a faster and simpler interpretation of chromospheric spectropolarimetric observations against those provided by full non-LTE inversions. The conclusions listed here are valid for both the Mg I b2 and the Ca II 854.2 nm lines. We have deepened in the usefulness and reliability of the mME approximation by first analyzing the so-called response functions under the mME approach. The analysis of the analytical mME response functions suggests that trade-offs between the response functions to the vector magnetic field, the line-of-sight velocity, and the thermodynamic parameters are negligible. The latter is a necessary condition for an effective interpretation of chromospheric Stokes profiles under the mME approach since it allows for inversion codes to disentangle the magnetic and velocity field information from the thermodynamic data. We have also shown that, at least, two exponential terms in the source function are needed for successfully reproducing chromospheric profile shapes unlike Lites et al. (1988), who concluded that only one exponential was sufficient. Next, we tested its validity by applying it to synthetic profiles generated from a FALC model atmosphere, including different magnetic field configurations. From this analysis we conclude that:

1. The mME approximation is able to reproduce chromospheric Stokes profiles emerging from standard atmospheric models and to determine the vector magnetic field rather accurately. The inferred mME model shows slight deviations in the determination of the field strength for inclination values at around 90°.
2. Although the mME source function shows similar behavior with optical depth as those computed from the original atmospheric model (i.e., it generally decreases monotonically with optical depth), it does present clear deviations. This fact was not put forward in the work by Lites et al. (1988). In particular, the mME source function presents clear humps that cannot be found in the original one. The humps in the mME source function allow for the profile shape to be fit but carry negligible physical meaning.

After testing the mME approximation against the standard FALC model, we confronted it with Stokes profiles generated from realistic MHD simulations (Carlsson et al. 2016; Quintero Noda et al. 2018). For the sake of comparison, we also used the WFA and the CoG techniques to compare the mME inversion results with simpler techniques. The results show that even though both the mME and WFA give similar results for the magnetic field strength and inclination, the mME interprets the data in a more comprehensive manner. For instance, the mME provides direct information about magnetic field strength, inclination, or velocity fields – unlike the WFA, which only provides independent measurements for the longitudinal and transverse components of the field vector. Another advantage of the mME approach is
that it is not restricted to weak magnetic fields – again, unlike the WFA.

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