Determination of the escape velocity of the Milky Way using a halo sample selected based on proper motion

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ABSTRACT

Context. The Gaia mission has provided the largest catalogue ever of sources with tangential velocity information. However, it is difficult to use this catalogue for dynamical studies because most of the stars lack line-of-sight velocity measurements. Recently, we presented a selection of $\lesssim 10^4$ halo stars with accurate distances that were selected based on their photometry and proper motions.

Aims. Using this sample, we model the tail of the velocity distribution in the stellar halo locally and as a function of distance. Our goal is to measure the escape velocity, and based on this, to constrain the mass of our Galaxy.

Methods. We fitted the tail of the velocity distribution with a power-law distribution, a commonly used approach that has long been established. For the first time, we used tangential velocities that were accurately measured for an unprecedented number of halo stars to estimate the escape velocity.

Results. In the solar neighbourhood, we obtain a very precise estimate of the escape velocity, which is $497^{+8}_{-8}\text{ km s}^{-1}$. This estimate is most likely biased low, our best guess is by 10%. As a result, the true escape velocity is most likely closer to 550 km s$^{-1}$. The escape velocity directly constrains the total mass of the Milky Way. To find the best-fitting halo mass and concentration parameter, we adjusted the dark (spherical Navarro-Frenk-White) halo of a realistic Milky Way potential while keeping the circular velocity at the solar radius fixed at $v_c(R_0) = 232.8\text{ km s}^{-1}$. The resulting halo parameters are $M_{200}^{105} = 1.11^{+0.08}_{-0.07} \times 10^{12} M_\odot$, and the concentration parameter is $c^{105} = 11.8^{+0.3}_{-0.2}$, where we use the explicit notation to indicate that they are corrected for the 10% bias. The slope of the escape velocity with galactocentric distance is as expected in the inner Galaxy based on Milky Way models. Curiously, we find a disagreement beyond the solar radius where the estimated escape velocity is higher than at the solar radius. This result is likely an effect of a change in the shape of the velocity distribution and could be related to the presence of velocity clumps. A tentative analysis of the escape velocity as a function of $(R, z)$ shows that the slope is shallower than expected for a spherical halo when standard values are used for the characteristic parameters describing the galactic disc.

Key words. Galaxy: kinematics and dynamics – Galaxy: structure – Galaxy: fundamental parameters

1. Introduction

Numerous studies have attempted to measure the mass of the Milky Way, but it has been notoriously difficult to obtain precise and model-independent constraints. Most works now agree that the mass of the Milky Way dark matter halo is $10^{12} M_\odot$, within a factor of 2 (see Fig. 7 of Callingham et al. 2019, for a recent compilation). The kinematics of globular clusters, dwarf galaxies, and halo stars have often been used in these studies (Kochanek 1996; Xue et al. 2008; Watkins et al. 2010; Deason et al. 2012; Fragione & Loeb 2017; Posti & Helmi 2019; Callingham et al. 2019; Fritz et al. 2020). The timing argument and the properties of debris streams such as those from the Sagittarius dwarf galaxy (e.g., Dierickx & Loeb 2017; Zaritsky et al. 2020) have provided additional but still similar constraints. In this work, we aim to derive a very precise estimate of the escape velocity near the Sun and hence, under further assumptions, of the mass of the Milky Way.

The escape velocity is the maximum velocity that stars can have while still being bound to the Galaxy. In principle, the single fastest moving bound star places a lower limit on the escape velocity. However, in practice, individual stars might be affected by large measurement uncertainties or they might be outliers (such as escapees). A more robust approach is to fit the velocity distribution as a whole, as was put forward by Leonard & Tremaine (1990, hereafter LT90), who described the tail of the velocity distribution with a power law.

Several works have used the LT90 method in the past. For example, Smith et al. (2007) and Pfiff et al. (2014b), hereafter S07 and P14, estimated the escape velocity locally to lie in the range of $[500–600]\text{ km s}^{-1}$, using only radial velocity information from RAVE (Steinmetz et al. 2006). The analysis of 2017 (2017, hereafter W17) supports these values and these authors also show that the escape velocity drops to $\sim 300\text{ km s}^{-1}$ at a distance of 50 kpc. The advent of full phase-space information with Gaia DR2 has not led to a reduction in the estimated range for the escape velocity in the solar neighbourhood: It is still $[500–640]\text{ km s}^{-1}$ (Monari et al. 2018; Deason et al. 2019, hereafter M18 and D19), a result that can largely be attributed to the different underlying assumptions used by the authors.

In this paper, we use a sample of halo stars with only tangential velocities from Gaia DR2 to infer the escape velocity applying also the LT90 method. This sample comprises orders of magnitude more halo stars than any other sample used before. Samples making use of only tangential velocities have not been popular for this type of studies in the past because of the large...
uncertainties in the velocities, particularly induced by the distance uncertainties. The lack of (accurate) proper motion measurements for large numbers of stars was even more dramatic. However, Gaia DR2, containing about ~200× more stars with proper motions than radial velocities, makes this type of study feasible now. We proceed in this work as follows. We describe the data we used and their properties in Sect. 2 and the methods in Sect. 3. In Sect. 4 we test the method for determining the escape velocity using mock data and cosmological simulations. In Sects. 5 and 6 we present our results in the solar neighbourhood and as a function of galactocentric distance, respectively. In Sect. 7 we use the local escape velocity to derive an estimate of the mass of the Milky Way dark halo and to identify likely unbound stars. In Sect. 8 we present our conclusions.

2. Data

The determination of the escape velocity is contingent upon having a sample of halo stars with high-quality measurements and large velocity amplitudes. Most of the data used in this work are provided by the Gaia mission (Gaia Collaboration 2016, 2018). We mainly use the sample of halo stars that was selected and analysed in Koppelman & Helmi (2021, KH21 hereafter). This sample comprises ~10^7 main-sequence (MS) halo stars, and we refer to it as the reduced proper motion or the 5D sample hereafter. Additionally, we make use of a set of nearby halo stars with full phase-space information.

2.1. Velocity information

To transform the observed motions (proper motions and radial velocities) into space velocities, we proceeded as follows. We computed the tangential velocity of a star by combining the proper motion and its distance as

\[ v_j = 4.74057 \text{ km s}^{-1} \left( \frac{\mu_j}{\text{mas yr}^{-1}} \right) \left( \frac{d}{\text{kpc}} \right) \]

where \( j = (\ell, b) \). These velocities were then corrected for the solar motion using the values for the motion of the Sun with respect to the local standard of rest (LSR) given by Schönrich et al. (2010) and the motion of the LSR given by McMillan (2017); they are \((U_\odot, V_\odot, W_\odot) = (11.1, 12.24, 7.25) \text{ km s}^{-1}\) and \(V_{\text{LSR}} = 232.8 \text{ km s}^{-1}\). The transformation to correct the tangential velocities is

\[ v_j' = v_j + v_{\rho,0} \]

where \( v_{\ell,0} \) and \( v_{b,0} \) are defined as

\[ v_{\ell,0} = -U_\odot \sin \ell + (V_\odot + V_{\text{LSR}}) \cos \ell, \]

\[ v_{b,0} = W_\odot \cos b - \sin b \cdot (U_\odot \cos \ell + (V_\odot + V_{\text{LSR}}) \sin \ell). \]

Finally, the tangential velocity in the Galactic frame of rest as observed from the Sun is calculated as

\[ v_\rho = \sqrt{(v_\ell + v_{\rho,0})^2 + (v_b + v_{b,0})^2}. \]

Similarly, the line-of-sight velocity can be corrected for the solar reflex motion using \( v'_{\text{los}} = v_{\text{los}} + v_{\text{los},0} \), where \( v_{\text{los},0} \) is \( W_\odot \sin b + \cos b \cdot (U_\odot \cos \ell + (V_\odot + V_{\text{LSR}}) \sin \ell). \)

To derive space velocities, we used the following expressions:

\[ v_x = v'_{\text{los}} \cos \ell \cos b - v_y \sin \ell - v_y' \cos b \sin b, \]

\[ v_y = v'_{\text{los}} \sin \ell \cos b + v_y' \cos \ell - v_y' \sin \ell \sin b, \]

\[ v_z = v'_{\text{los}} \sin b + v_b' \cos b. \]

To transform the coordinates into a galactocentric frame, we placed the Sun at \( X = -8.2 \text{ kpc} \) (McMillan 2017). We used this value for the distance to the Galactic centre because it is consistent with the McMillan (2017) potential that we employ later, and the same is true for the LSR velocity. We note, however, that the McMillan values agree well with the more recent determination of the distance to the Galactic centre by the GRAVITY Collaboration (2018) and with the circular velocity at the position of the Sun by Eilers et al. (2019).

To isolate a halo sample using the Gaia DR2 data, we considered stars with velocity vectors that deviate more than 250 km s\(^{-1}\) from the velocity vector of the LSR (i.e. the velocity vector of a typical disc star), namely \( |v - v_{\text{LSR}}| \geq 250 \text{ km s}^{-1}\). This type of selection is known as a “Toomre” selection.

In APOGEE we did not use the line-of-sight velocity information was available, we used Eq. (6) and set \( v_\rho = 0\). In that case, we refer to the velocity vector as \((v_x, v_y, v_z)\) to stress that these are not the true Cartesian velocities. For this set of stars, which constitute the majority of our sample, we used an adapted Toomre selection to isolate a halo sample, namely \( |v - v_{\text{LSR}}| \geq 250 \text{ km s}^{-1}\).

2.2. Sample with full phase-space information

In the solar neighbourhood, we used a sample of stars with full phase-space information from Gaia that is known as the 6D or the radial velocity spectrometer (RVS) sample (Katz et al. 2019). We extended this dataset by adding sources with radial velocities observed by APOGEE (Wilson et al. 2010; Abolfathi et al. 2018), LAMOST (Cui et al. 2012), and RAVE (Kunder et al. 2017), see Sect. 2 of Koppelman et al. (2019) for a full description of this catalogue. The cross-matches with APOGEE and RAVE were obtained from the Gaia archive (Marrese et al. 2019).

For this sample and in line with M18 and D19, we used the quality cuts described in Marchetti et al. (2019), namely:

- astrometric_gof_al < 3,
- astrometric_excess_noise_sig ≤ 2,
- -0.23 ≤ mean_varpi_factor_al ≤ 0.32,
- visibility_periods_used > 8,
- rv_nb_transits > 5,

and also imposed the following quality criteria:

- ruwe < 1.4,
- parallax_over_error > 5.

For the additional spectroscopic data we used the same quality cuts, with the exception of the criterion on rv_nb_transits. Additionally, we used survey-specific quality constraints. For APOGEE we used:

- SNR > 20,
- STARFLAG == 0,
- abs(SYNTHVEL10_AVG - OBSVEL10_AVG) < 50,
- NVISITS > 2,

for RAVE.

- eHRV < 10,
- Algo_Conv_K! = 1,
- SNR_K > 20,

and for LAMOST,
- snr > 20,
- snrg > 20.

Several studies have reported that the sources in the RVS sample, and bright sources in general, contain a parallax offset...
of \( \sim 0.05 \) mas (see Schönrich et al. 2019; Leung & Bovy 2019; Zinn et al. 2019; Chan & Bovy 2020). Therefore we corrected the parallaxes in the 6D sample for an offset of \( \sim 0.054 \) mas as estimated by Schönrich et al. (2019). Following these authors, we increased the parallax uncertainties by 0.006 mas to account for the uncertainties in the offset and by 0.043 mas to account for the RMS in the offset reported by Lindegren et al. (2018), both of which were added in quadrature.

Nonetheless, to mitigate the effects of the parallax offset, we only considered sources within 2 kpc. As explained earlier, we selected halo stars as those with \( |\varpi - n_{\text{SR}}| > 250 \) km s\(^{-1}\). Finally, we removed the star with \textit{Gaia} DR2 source Id 5932173855446728064 because its radial velocity reported in \textit{Gaia} DR2 is known to be incorrect (Boubert et al. 2019). The final sample comprises 2067 high-quality stars, of which 495 are from the \textit{Gaia} RVS sample, 10 from APOGEE, one from RAVE, and 1561 from LAMOST.

Since the spectroscopic surveys add a considerable number of stars, mostly from LAMOST, we verified that they do not bias our results. They are fully consistent with using only \textit{Gaia} RVS sources. The stars from the spectroscopic surveys do not dominate the determination of \( v_{\text{MS}} \) because they have larger uncertainties in general. However, they do help in closing the confidence contours, as we show in Sect. 5.

2.3. Reduced proper motion sample

For the complete description of the reduced proper motion (RPM) sample, we refer to the KH21 paper. Here we summarise the details that are relevant for this paper. By virtue of the selection method, the RPM sample comprises only MS stars. The relatively linear colour-magnitude relation for these types of stars can be used to calculate photometric distances with typical uncertainties of 7%.

In KH21 we have introduced several quality cuts. Here we prune the sample even further. Firstly, we targeted the purest set of halo stars by imposing the following cut on velocities \( |\varpi - n_{\text{SR}}| > 250 \) km s\(^{-1}\) (see Sect. 2.1). Then we selected stars with tangential velocities higher than \( v_t > 250 \) km s\(^{-1}\). Furthermore, we isolated stars that are least affected by extinction, that is, we considered only those with \( A_G < 0.2 \). Next, we selected stars in the colour range where the photometric distances have the smallest uncertainty: \( 0.50 < G - G_{Bp} < 0.715 \). The blue limit here is stricter than in KH21 to remove any possible contamination from the MS turn-off. Finally, we selected stars at high latitudes to remove contamination from the disc: \(|b| > 20^\circ\).

Some stars in the RPM sample have less precise photometric distance than trigonometric distance (i.e. parallax from \textit{Gaia}). Furthermore, some stars may have been excluded because they did not satisfy the last three quality cuts described above, even though their trigonometric parallaxes are accurate. Therefore we returned these stars to the sample. We also replaced the photometric distances with trigonometric distances for stars with \( \text{parallax\_over\_error} > 10 \) if the latter has a smaller uncertainty than the first, and we only considered stars with parallaxes \( > 0.5 \) mas.

As mentioned above, the trigonometric parallaxes from \textit{Gaia} DR2 are known to contain a zero-point offset that has a complex dependence on other observational parameters (e.g., the colour and magnitude of the stars). Because most of the stars in the 5D sample (without radial velocities) are fainter than those in the 6D sample, we corrected their parallaxes with a different offset. Following Lindegren et al. (2018), we used a value of \( \sim 0.029 \) mas for the parallax offset and increased the parallax uncertainties by 0.043 mas (the uncertainties were added in quadrature) to account for variations in the offset.

Within 1 kpc, about 90% of the distances stem from the \textit{Gaia} parallaxes, and at 2 kpc, this percentage drops to \( \sim 50\% \). The final pruned sample comprises 197449 sources, 18236 of which have \textit{Gaia} parallaxes.

2.4. Inspection of the RPM sample

Figure 1 shows the spatial distribution of the stars in the sample. The maps are coloured by the logarithm of the counts per bin. The quality cuts described in the previous section affect the spatial distribution of the stars, most notably, by removing low-latitude stars. The overdensity in the solar neighbourhood (centre of the figure) is caused by the addition of sources with accurate parallaxes. The median heliocentric distance of the sample is 3.6 kpc.

In Fig. 2 we show the tail of the tangential velocity distribution as a function of galactocentric distance by slicing the sample into uniformly spaced overlapping bins, ranging from 4–12 kpc, of 1 kpc width, which is larger than the typical uncertainty in the distances. A visual inspection reveals only small variations in the distributions. These clearly resemble a power law (as anticipated), but with a slight tendency to become more exponential with distance from the Galactic centre.

We propagated the uncertainties in the tangential velocities, which we denote as \( \sigma_t \), using the standard uncertainty propagation approximation. This approximation uses a Taylor expansion to linearise the coordinate transformations, which is a common practice in the literature. We started the propagation from the measurement uncertainties of \textit{Gaia} DR2, where the uncertainty of the distance is derived from the parallax uncertainty for the trigonometric sample. For the RPM sample, we derived distance uncertainties separately because this sample contains photometric distances (see Sect. 3.3. of KH21 for more information). Typically, the photometric distance uncertainties are at the 7% level.

The distribution of the relative uncertainties in the tangential velocity \( v_t \) is shown in Fig. 3 for the photometric and trigonometric distances separately. On average, the tangential
velocities derived from the trigonometric parallaxes are slightly more accurate than those based on the photometric distances. This is a selection effect because only sources with very accurate parallaxes are included in our sample. When the uncertainties are propagated in the velocities, we find that the uncertainty distribution for sources with photometric distances peaks at 7% and has a median of 8%. For the trigonometric distances, the distribution in the velocity uncertainties peaks at ~5%. The distribution of \( \sigma_t \) has a tail towards higher uncertainties because of proper motions uncertainties, and there is only a small dependence on magnitude at the faint end (i.e. for \( G \gtrsim 20 \)).

**3. Methods**

**3.1. Determining \( v_{\text{esc}} \)**

As described earlier, we used the LT90 method to determine the escape velocity, denoted hereafter as \( v_{\text{esc}} \). The motivation of this method is that the tail of the velocity distribution can be described by a power law, and \( v_{\text{esc}} \) is the velocity at which the probability of finding a star reaches zero. Although we closely followed Sections Iia and Iic from LT90 and adopted their notation, the formalism we used reveals some differences.

As just stated, the probability of finding a star in a local volume with a velocity in the range \( (v, v + dv) \) is described close to the escape velocity as a power law,

\[
 f(v|v_{\text{esc}}, k) = \begin{cases} 
 A(v_{\text{esc}} - v)^k, & v_{\text{cut}} < v < v_{\text{esc}}, \\
 0, & v \geq v_{\text{esc}}, 
\end{cases}
\]

where \( k \) is the exponent, \( v_{\text{esc}} \) is the escape velocity, and \( v_{\text{cut}} \) is a threshold velocity below which the distribution is poorly represented by a power law. It is important to set \( v_{\text{cut}} \) accordingly such that only the tail of the distribution is fit. The normalisation constant is defined as \( A = \frac{1}{\int_{v_{\text{cut}}}^{v_{\text{esc}}} f(v|v_{\text{esc}}, k) \, dv} \).

The expression in Eq. (7) describes the distribution of velocities corrected for the solar motion (including peculiar and LSR), for example at the location of the Sun. If \( f dv \) is the probability of finding a star with velocity \( v \) in the range \( (v, v + dv) \), this implies that some distribution \( g(v) \) exists such that \( \int_{v_{\text{cut}}}^{v_{\text{esc}}} g(v) \, dv = f(v) \, dv \) under the assumption that the velocity distribution is isotropic.

We now wish to obtain the probability distribution for the tangential velocity (i.e. \( f_t(v, v_t|v_{\text{esc}}, k) \)). This can be derived from the joint distribution \( f_{t}(v, v_t|v_{\text{esc}}, k) \), which gives the probability of...
finding a star with a given line-of-sight velocity and tangential velocity as \( f_{\nu}(v, v_1|\nu, k)\). By performing a transformation of variables

\[
f_{\nu}(v, v_1|\nu, k) = \int g(\theta|v, k) \delta(v - \nu - \hat{n} \delta(v_1 - |\nu \times \hat{n}|)) \, d\theta,
\]

where \( \hat{n} \) is a unit vector along the line of sight. To express the distribution function in terms of \( v_1 \), alone, we integrated over the line-of-sight component (and over angle),

\[
f_{\nu}(v_1|\nu, k) = \frac{1}{2\pi} \int g(\theta|v, k) \delta(v_1 - |\nu \times \hat{n}|) \, d\theta.
\]

The distribution \( f_{\nu}dv \), gives the probability of finding a tangential velocity \( v_1 \) in the range \( (v_1, v_1 + dv_1) \).

Evaluating this integral in spherical coordinates, with \( \hat{n} \) aligned with the z-axis (implicitly assuming the stars are distributed isotropically), we obtain

\[
f_{\nu}(v_1|\nu, k) = \int f(\nu) \delta(v - \nu \sin \theta) \nu^2 \sin \theta \, dv \, d\theta,
\]

which, by changing the order of integration and the substitution of \( u = v - \nu \sin \theta \), reduces to

\[
f_{\nu}(v_1|\nu, k) = -\int_{v_1}^{v_{\text{esc}}} f(u) \left[ \nu^2 - v^2 \right]^{-\frac{1}{2}} \, dv.
\]

The resulting integral for \( f(\nu) \) given by Eq. (7) can be solved with Mathematica (and depends on the regularised hypergeometric \( zF_1 \) function). When the Taylor series expansion of \( v_1 \to v_{\text{esc}} \) is evaluated for the integral, we obtain

\[
f_{\nu}(v_1|v_{\text{esc}}, k_i) \propto \begin{cases} 
(v_{\text{esc}} - v_1)^{k_i+\frac{3}{2}}, & v_{\text{cut}} < v_1 < v_{\text{esc}}, \\
0, & v_1 \geq v_{\text{esc}},
\end{cases}
\]

which is the expression found by LT90. It can be normalised by multiplying with the constant \( A_i = \frac{\nu_{\text{esc}} + 1.5}{v_{\text{esc}} - v_{\text{cut}}} \), which is derived from the requirement that \( \int_0^{v_{\text{esc}}} f_{\nu}(v_1|v_{\text{esc}}, k_i) \, dv_1 = 1 \), and where we have replaced \( k \) with \( k_i \) for clarity. The reason for this is that only in the case of \( v_1 \to v_{\text{esc}} \) are the two power-law indices of Eqs. (7) and (12) related, and \( k_i = k \). It is thus best to think of \( f(\nu|v_{\text{esc}}, k_i) \) in Eq. (12) simply as a power-law description of the tangential velocity tail, an approximation that is supported by Fig. 2. We show in Sect. 4.1 that it is in general not quite true that \( k_i = k \) for the \( v_{\text{esc}} \) values that are typically considered in the literature. In what follows, we therefore reserve the notation \( k_i \) for the power-law exponent of \( f_\nu \), use \( k \) for the exponent using the distribution from Eq. (7), and use \( k_\nu \) to indicate the exponent for a sample using only line-of-sight velocities (e.g., when comparing to values in the literature).

So far, we have assumed that the velocities \( (v, v_1) \) are the true velocities. However, in reality, we are dealing with ‘observed’ velocities, which are a combination of the true velocity and some unknown uncertainty. In this section we use \( v' \) to indicate the observed tangential velocity and reserve \( v_1 \) for the true value. To account for the uncertainty, we smoothed the velocities by convolving them with an error distribution \( \epsilon(v_1 - v'_1, \sigma_\epsilon) \), where \( \sigma_\epsilon \) is the uncertainty in \( v'_1 \). When we assume that \( v_1 \) and \( \nu \) have Gaussian errors, then the distribution \( \epsilon(v_1 - v'_1, \sigma_\epsilon) \) follows the Beckmann distribution\(^1\) (i.e. it is non-Gaussian). However, when it is evaluated far away from the origin \( (v'_1/\sigma_\epsilon \gg 1) \), this distribution is well approximated by a Gaussian. This gives us another reason to choose a sufficiently large \( v_{\text{cut}} \). In what follows, we therefore approximate \( \epsilon(v_1 - v'_1, \sigma_\epsilon) \) by a Gaussian \( f_G(v_1 - v'_1, \sigma_\epsilon) \). The convolution of the power law from Eq. (12) and the Gaussian is given by

\[
C(v'_1, \sigma_\epsilon, \theta) = \int v_{\text{esc}}^{v_{\text{cut}}} f_\nu(v_1|v_{\text{esc}}, k_i) f_G(v_1 - v'_1, \sigma_\epsilon) \, dv_1,
\]

where \( \theta = (v_{\text{esc}}, k, v_{\text{cut}}) \).

We note that we took \( v_{\text{esc}} \) as the lower boundary and not zero, as in Eq. (17) of LT90, because the velocity distribution below \( v_{\text{esc}} \) is poorly described by a power law. In this range, it is better described by a different distribution function \( f'(v_1) \) for which the convolution over the range \( 0 < v < v_{\text{cut}} \) would take the form

\[
C'(v'_1, \sigma, v_{\text{cut}}) = \int v_{\text{esc}}^{v_{\text{cut}}} f'(v_1) \epsilon(v_1 - v'_1, \sigma) \, dv_1,
\]

which neither depends on \( v_{\text{esc}} \) nor on \( k \). As we show below, we may thus ignore this part of the velocity distribution. This also means that we also ignored stars with an apparent \( v'_1 \) below \( v_{\text{esc}} \), but with a finite probability of having a true \( v_1 \) above it. We show in Sect. 4 that these assumptions do not affect the ability of the method to infer \( v_{\text{esc}} \).

By normalising Eq. (13), we find \( P(v'_1, \sigma_\epsilon|\theta) \), the probability of finding a star with \( v'_1 \) in the range \( (v'_1, v'_1 + dv'_1) \),

\[
P(v'_1, \sigma_\epsilon|\theta) = \frac{C(v'_1, \sigma_\epsilon, \theta)}{\int_{v'_1}^{v_{\text{esc}}} C(v'_1, \sigma_\epsilon, \theta) \, dv'_1}.
\]

By definition, because both \( f_\nu \) and \( f_G \) are normalised, the integral in the denominator is unity because the area under a convolution is \( \int f(\theta) \, d\theta = \int f(\theta) \, d\theta \). The resulting likelihood function is given by

\[
L = \prod_{i=1}^{n} P(v'_i, \sigma_\epsilon|\theta).
\]

The probability distribution of the model parameters \( v_{\text{esc}} \) and \( k \) is found by using Bayes’ theorem,

\[
P(\theta|v'_i, \sigma_\epsilon) \propto P(v_{\text{esc}}|P(k)) \prod_{i=1}^{n} P(v'_i, \sigma_\epsilon|\theta),
\]

where \( P(v_{\text{esc}}) \) and \( P(k) \) are priors for \( v_{\text{esc}} \) and \( k \). For numerical reasons, the logarithm of the probability is evaluated, which also allows us to ignore the normalisation because that is constant and independent of the model parameters.

The procedure that is outlined above implicitly makes the following assumptions. Firstly, the tail of the velocity distribution is populated up to the escape velocity and, secondly, the tail of the velocity distribution is smooth. Furthermore, we assume that there are no unbound stars in our sample and that there is no contamination from a rotating (disc-like) population. The latter would break the isotropy on the sky.

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\(^1\) The Beckmann distribution is the most general form of the distribution \( \rho(r) \) of parameter \( r = \sqrt{x^2 + y^2} \), where \( x \) and \( y \) drawn from a bivariate Gaussian distribution (see https://reference.wolfram.com/language/ref/BeckmannDistribution.html). The distribution can generally only be expressed in integral form (e.g., Eq. (31) of Beckmann 1962), but takes an explicit form for specific cases. For example, when \( x \) and \( y \) are independent and drawn from a standard normal distribution, \( \rho(r) \) takes the form of the Rice distribution. The more general case of \( \rho(r) \), in which \( x \) and \( y \) are drawn from an uncorrelated bivariate Gaussian distribution, is known as the non-central chi distribution.
Perhaps the most problematic assumption is the first. There is no guarantee that the velocity distribution locally, or at any other location in the Milky Way, extends up to the escape velocity. Most likely, it is truncated at some lower value. As a result, the LT90 method may easily underestimate the true \( v_{\text{esc}} \). For example, cosmological simulations show velocity distributions that are truncated at 90% of \( v_{\text{esc}} \) (e.g., S07). The exact location of the truncation depends on the assembly history of the galaxy, and quite possibly, also on the resolution of the simulation. We quantify the truncation of the velocity distribution using mock data in Sect. 4. We stress that because of this truncation, whatever value we derive for \( v_{\text{esc}} \) it most likely is a lower limit.

The second assumption has recently been tested by Grand et al. (2019), who found that clustering in the velocity distribution biases the estimation of \( v_{\text{esc}} \), and can result in both under- and overestimates. Nonetheless, these authors showed that the estimated \( v_{\text{esc}} \) is typically underestimated by 7%. To emphasise the importance of this bias: A difference of 7% in the escape velocity results in a 21% bias in the estimated mass.

It seems unlikely that our sample contains many unbound stars because hyper-velocity stars are typically young stars ejected from the Galactic centre and not old stars in the halo (e.g., Brown 2015; Boubert et al. 2018). Furthermore, the velocity distributions shown in Fig. 2 are relatively smooth, suggesting the presence of a single population dominated by main-sequence halo stars. Nonetheless, it would be interesting to spectroscopically follow up stars near the escape velocity. In Sect. 7.3 we revisited possible outliers in the solar neighbourhood.

### 3.2. Adopting a prior on \( v_{\text{esc}} \) and \( k_t \)

In line with the literature, we assumed a simple prior on \( v_{\text{esc}} \) of the form \( P(\log(v_{\text{esc}})) \propto 1/v_{\text{esc}} \). For \( k \) (we use the notation \( k \) here, understanding that it only compares to \( k_t \) and \( k_f \) in the limiting case) there is some debate in the literature on what to assume, and because \( v_{\text{esc}} \) and \( k \) are highly degenerate (see next section), the prior assumed might bias the resulting \( v_{\text{esc}} \). For example, the M18 and D19 estimates of \( v_{\text{esc}} \) differ by \( \pm 50 \text{ km s}^{-1} \) mainly because different ranges were considered for \( k \). Attempting to estimate \( v_{\text{esc}} \) and \( k \) simultaneously is only possible with a large sample with very small uncertainties. For example, LT90 estimated that a sample of >200 stars with high-quality radial velocities above \( v_{\text{cut}} \) is necessary to estimate both values simultaneously.

LT90 argued that \( k \) values should be in the range \([0.5–2.5]\) because this brackets \( k = 1.5 \), which is the value expected for a system that has undergone violent relaxation (Aguilar & White 1986; Jaffe 1987; Tremaine 1987). S07 have compared stellar halos in cosmological simulations of Milky Way-like galaxies and found a range of \([2.7–4.7]\) to be more appropriate. P14, building on more recent simulations like this, produced this range to \([2.3–3.7]\), which is also the range used by M18. D19 updated the criteria for finding Milky Way analogues based on recent discoveries regarding the merger history of the Milky Way (Belokurov et al. 2018; Helmi et al. 2018). When using cosmological simulations, the range \([1.0–2.5]\) was found to be more favoured. Using a sample of blue horizontal branch (BHB) stars, K giants, and main-sequence turnoff stars from the Sloan Digital Sky Survey (SDSS) with only line-of-sight velocities, W17 determine both \( k \) and \( v_{\text{esc}} \) simultaneously. They report a value for \( k \) of 4 ± 1 for the local stars.

The above paragraph shows that no consensus has been reached on the value of \( k \) for the Milky Way. To complicate matters, the ranges mentioned above were determined for the stellar halo at the position of the Sun (in the simulations). It is not clear whether \( k \) remains constant as a function of distance to the Galactic centre. In this work, we mainly rely on the estimate of \( k \) at the location of the Sun. This is where our sample contains many stars with reliable parallax information that are approximately isotropically distributed on the sky. For this local sample of stars, we calculated the marginalised posterior distribution for \( k \). We applied this posterior as a prior to other distance bins in which we estimate \( v_{\text{esc}} \). In doing so, we assumed that \( k_t \) does not vary (much) over the distance range that we probe, which is also justified by the analysis we carry out in Sect. 4.2.

### 4. Validating the method

Before applying the method to the data, we attempted to establish the accuracy of the method, the sample size required to estimate both \( v_{\text{esc}} \) and \( k_t \) at the same time, and the effect of the cut-off parameter \( v_{\text{cut}} \). We first consider mock data and then apply the method to cosmological simulations.

#### 4.1. Tests with mock data

The mock data were drawn from an idealised power-law distribution. We sampled velocities according to the distribution given by Eq. (12) assuming \( v_{\text{esc}} = 550 \text{ km s}^{-1} \) and \( k_t = 2.3 \). We convolved the resulting distribution with realistic uncertainties drawn from the distribution of uncertainties (i.e. that shown in Fig. 3) for the sample of photometric distances.

The left panel of Fig. 4 shows the results of applying the formalism described in Sect. 3 to this dataset for three different sample sizes (see annotation) above \( v_{\text{esc}} = 250 \text{ km s}^{-1} \). The contours mark the 1, 2, and 3σ levels estimated by the level where the probability has dropped to 61%, 14%, and 1% of the probability of the most likely parameter combination. The true \( v_{\text{esc}} \) and \( k_t \) of the parent distribution are indicated with a red marker. Decreasing the number of stars (from 10 000 to 500) results in higher uncertainties in the estimates of the \( v_{\text{esc}} \) and \( k_t \) parameters. A sample with \( \sim 10^4 \) stars is sufficiently large to determine both \( v_{\text{esc}} \) and \( k_t \) at the same time, given the amplitude of the velocity uncertainties.
We also tested a procedure drawing the mock distribution of \( v_t \) starting from a parent distribution of full 3D velocities (e.g., starting from Eq. (7) rather than from Eq. (12)). The 3D velocities were then transformed into \( v_t \) velocities by artificially setting one component to zero, assuming that the velocities are distributed isotropically. Arguably, the resulting distribution of mock \( v_t \) more closely describes the observed distribution of \( v_t \) than the one drawn directly from Eq. (12). The right panel of Fig. 4 shows that the resulting values for \( \nu_{\text{esc}} \) are close to the input value. However, the values for \( k_t \) are slightly overestimated. This overestimate arises from the difference between \( k \) and \( k_t \) for low \( \nu_{\text{cut}} \) and was already anticipated in Sect. 3.1.

To emphasise this behaviour, we show in Fig. 5 the behaviour of the artificial \( v_t \) distribution (drawn without uncertainties) compared to the expected distribution of \( k_t \) \( \rightarrow k \) if \( v_t \) \( \rightarrow \nu_{\text{esc}} \) (i.e. Eq. (12)). The two distributions are only equivalent for cut-off velocities \( \nu_{\text{cut}} \) very close to the escape velocity. Although this does not invalidate our approach at all because \( \nu_{\text{esc}} \) is robustly determined without any biases, we nonetheless have to be cautious when we compare the value of \( k \) obtained using tangential velocities. Similar considerations are in order when the LT90 method is applied to a sample of line-of-sight velocities only.

### 4.2. Tests on Aurigaia Milky Way-like halos

We now test the method on two halos from the Auriga suite of mock Gaia catalogues (Grand et al. 2019). We explore here whether the tail of the velocity distribution is well described by a power law, the effect of velocity clumps, the behaviour of \( k_t \) as a function of distance, and the power of the method given the typical uncertainties in the tangential velocity in our sample.

The Aurigaia catalogues have been generated from the Auriga suite of Milky Way-like galaxies (Grand et al. 2017), which is a suite of high-resolution, zoomed-in re-simulations based on galaxies extracted from the EAGLE simulations (Schaye et al. 2014). The mock catalogues that we analysed correspond to halos 6 and 27, have the bar at 30 degrees orientation, and were generated with the SNAPDRAGONS code (Hunt et al. 2015). We refer to these simulations as Au-06 and Au-27. These specific halos are chosen somewhat at random, although Au-06 is the most similar to the Milky Way (based on halo spin, Grand et al. 2018). The halo of Au-06 has a similar mass as the Milky Way (i.e. \( M_{200} \approx 10^{12} M_\odot \)), whereas that of Au-27 is slightly more massive: \( M_{200} \approx 1.7 \times 10^{12} M_\odot \) (Grand et al. 2017).

Both halos are mildly prograde (\( \sim 30\text{–}70 \text{ km s}^{-1} \)), as measured by the mean rotational velocity of accreted stars with heliocentric distances smaller than 1 kpc.

Because the original Auriga simulations do not have the resolution of Gaia DR2 (~10° stars), the SNAPDRAGONS code was used to artificially increase the number of objects, whereby simulated stellar particles are split into multiple ‘stars’. This leads to artificial enhancement of the clustering of stars in phase-space, which can lead to biases in the determination of the escape velocity. Therefore we only used unique stellar particles by filtering all duplicates using the true Coordinates and HVelocities parameters in the Aurigaia catalogue.

#### 4.2.1. High-velocity tail in Aurigaia

Figure 6 shows the velocity of the fastest moving stars relative to the escape velocity and as a function of distance. We see that the fastest star typically moves at 90–100% of the true \( \nu_{\text{esc}} \) throughout the range of galactocentric distances probed. The escape velocity was calculated as the velocity needed to reach \( r \rightarrow \infty \) for the potential given in the Aurigaia catalogue (parameter: GravPotential), and using

\[
\nu_{\text{esc}}(r \rightarrow \infty) = \sqrt{2|\Phi(r) - \Phi(\infty)|} = \sqrt{2|\Phi(r)|}. \tag{18}
\]

In Fig. 6 the black markers correspond to the fastest star, and the grey markers indicate the location of the tenth fastest star and provide an idea of the steepness of the velocity tail.

Interestingly, for the halo of Au-27, the velocity distribution is truncated close to the escape velocity around a galactocentric radius of ~15 kpc. In the inner regions, particularly for Au-6 but for Au-27 to some extent as well, the difference between the fastest and the tenth fastest moving star typically shows less scatter, indicating that there are many stars near the truncation of the velocity distribution.

#### 4.2.2. Determining the escape velocity in Aurigaia

We followed a similar procedure as for the data to select stars with high tangential velocities from the Aurigaia halos. Firstly, the tangential velocities were convolved with uncertainties drawn from the ‘observed’ uncertainty distribution shown in Fig. 3 (and as in Sect. 4.1). We then selected stars that have

\[
\frac{v_{\text{max}}}{\nu_{\text{esc}}}
\]
\( v_t > 200 \text{ km s}^{-1} \). Next, we artificially set the line-of-sight velocities to zero and selected stars with \( |\vec{v} - \vec{v}_{\text{LSR}}| > 250 \text{ km s}^{-1} \) (as in Sect. 2.3). Although the Aurigaia catalogues do not exactly represent the Milky Way, these velocity cuts serve to remove the thin disc and (most of) the thick disc present in the simulations.

We then determined \( v_{\text{esc}} \) and \( k_t \) in concentric shells of 1 kpc in width centred on the galaxy centre, with radii ranging from 2–21 kpc. For both Auriga halos, the cut-off velocity was set at \( v_{\text{cut}} = 250 \text{ km s}^{-1} \). This is well below the escape velocity in all the distance bins we probed. We also tested a heuristic procedure to determine \( v_{\text{esc}} \) by taking the maximum of 250 km s\(^{-1}\) and the velocity of the 10,000th fastest moving star (20,000th for Au-27). For bins with a large number of stars, this pushes the cut-off to higher values.

We note that for the Au-27 halo, the top 20,000 stars work better to determine \( v_{\text{esc}} \) than the top 10,000. Because this halo is more massive than that of Au-06, its escape velocity is higher and there are more stars with extreme velocities. However, because the Au-06 halo is more similar to the Milky Way, we expect that 10,000 is a realistic number of stars for the Milky Way.

Figure 7 shows the results of fitting the tangential velocity tail in the halos of Au-06 (left) and Au-27 (right). In yellow we show the mean \( v_{\text{esc}} \) that is calculated from Eq. (18) by using the pre-computed potential energies of every particle (i.e. the GravPotential parameter). The width of the yellow region indicates that there is a range of escape velocities at a fixed radius. This range exists because the potential is not spherically symmetric. Stars close to the disc experience a stronger potential than those slightly farther away.

The results obtained from a fixed \( v_{\text{cut}} \) are indicated with blue markers, and green markers are for the adaptive \( v_{\text{cut}} \). The top panels of Fig. 7 show that the estimates are systematically too low compared to the expected \( v_{\text{esc}} \) for both halos. However, they match the grey curve well that has been obtained by lowering the yellow region by 10%. This is a reflection of a truncation in the tangential velocity distribution in that it does not extend all the way to \( v_{\text{esc}} \). In both halos, the features in the \( v_{\text{esc}} \) curve are closely matched by the velocity features apparent for the tenth fastest stars shown in Fig. 6. An interesting result is that \( k_t \) varies only weakly over distance, as is shown in the bottom panels of Fig. 7.

The above results mean that the determination of \( v_{\text{esc}} \) with the method described in Sect. 3 is sensitive to the behaviour of the tail of the velocity distribution. This is particularly clear for Au-27, which shows a bump in \( v_{\text{esc}} \) at \( d \gtrsim 15 \text{ kpc} \). An excess of stars (a clump) is visible in the halo of Au-27 that moves at a velocity close to \( v_{\text{esc}} \), as can be seen by comparing the panels of Fig. 8, which plot the velocity distributions for the distance ranges 9–11 kpc and 15–17 kpc.

In summary, the analysis of the Aurigaia experiments analysis shows that \( v_{\text{cut}} \) can be determined from the top 10,000 stars. Secondly, we may assume that \( k_t \) varies only weakly over the distance range probed by the RPM sample. Furthermore, on average, the method underestimates \( v_{\text{esc}} \) by ~10%. This is slightly more than the 7% estimated by Grand et al. (2019), which might be related to differences in the method (e.g., the convolution with an uncertainty distribution and the typically large uncertainties on \( v_t \)). Finally, we note that by determining \( v_{\text{esc}} \) over a range of galactocentric distances, we can determine local ‘biases’.

5. Results: Solar neighbourhood

We determined the escape velocity at the solar position using the two samples of stars described in Sect. 2, one with full 6D information and the other with only tangential velocities (5D). We considered only stars with a heliocentric distance of 2 kpc or less. We evaluated the probability (Eq. (17)) on a grid of 100 × 100 points ranging from 400 km s\(^{-1}\) < \( v_{\text{esc}} \) < 800 km s\(^{-1}\) and 1 < \( k_t \) < 6 (both for the 5D and 6D cases). These ranges bracket the values that are presented in the literature. For the 5D sample, the cut-off velocity is based on the 10,000th fastest star and set to 317 km s\(^{-1}\), and for the 6D sample, it is 250 km s\(^{-1}\). Although the results for the 6D sample are consistent when \( v_{\text{cut}} \) is set to 317 km s\(^{-1}\), in this case, the inference on \( k \) is weaker.

The confidence contours for the two samples are presented in Fig. 9. For the sample with full phase-space information, we plot the results for the Gaia-only data (6D) and also including the additional data from ground-based spectroscopic surveys (6D+). The arrows in the figure indicate the maximum probability values for each sample. The contours correspond to the isolines of the 1, 2, and 3\( \sigma \) levels (see Sect. 4.1). The side panels show the marginalised distributions \( P(v_{\text{esc}}) \) and \( P(k) \) (\( P(k_t) \) for the 5D sample). These distributions are best constrained for the 5D sample (in blue) because of its large number of stars.

The marginal distributions of \( v_{\text{esc}} \) agree very well with each other for all samples. The slight difference in the 5D and 6D curves (the contours are consistent within the 2\( \sigma \) level, however), is driven by the anticipated differences that are the result of using the full velocity modulus or tangential velocity information only.
6. Results: Beyond the solar neighbourhood

6.2. High \( v_{\text{esc}} \) Outside of the solar radius

Several effects could lead to a higher \( v_{\text{esc}} \) outside the solar radius, namely (i) biases in the data (e.g., in the distance estimate), (ii) biases in the method (e.g., sample size), and (iii) variations in the dynamical properties of the stars with distance. We already explored the biases introduced by the first two categories in Sect. 2 (see also KH21) and Sect. 4.1. Nevertheless, we also tested that when the sample is downsized to a random subset of 5000 stars and bins with fewer stars are excluded, the results do not change. We therefore now focus on the third possibility, that the velocity distribution might be different outside of the solar radius.

Careful inspection of Fig. 2 shows that the velocity distribution is not contaminated by single outliers, even though in a relative sense (to the absolute number of objects), there seem to be more extreme velocity values in the outer radial bins. However, as mentioned earlier, the figure does show that the distributions seem to become more exponential with distance.

Curiously, we have seen a similar behaviour for Au-27 of the \( v_{\text{esc}} \) profile as observed for the 5D sample, see Fig. 7. In this case, the increase in \( v_{\text{esc}} \) was tentatively attributed to the presence of tidal debris (or at least lumpiness) that moved with speeds close to the true escape velocity.

(i.e. \( k \neq k_t \) for \( v_{\text{cut}} \) far from \( v_{\text{esc}} \), cf. the contours and red marker in Fig. 4).

Marginalising over \( k_t \), we find a maximum probability value \( v_{\text{esc}} \approx 497^{+8}_{-6} \text{ km s}^{-1} \) for the 5D sample, which we stress is most likely biased low compared to the actual \( v_{\text{esc}} \). The quoted uncertainties correspond to the marginalised 1σ confidence levels (i.e. the \( \sim 1\sigma \) level). Table 1 presents \( v_{\text{esc}} \) and \( k_t \) (or \( k \)) derived for all the samples considered and for the curves shown in Fig. 9.

Table 1. Escape velocity \( (v_{\text{esc}}) \), power-law exponent \( (k) \), for the 5D and \( k \) for the 6D samples), and the number of stars \( (N_{\text{stars}}) \) for different distance estimates in the solar neighbourhood.

<table>
<thead>
<tr>
<th>Sample</th>
<th>( v_{\text{esc}} ) (in km s(^{-1}))</th>
<th>( k_t/k )</th>
<th>( N_{\text{stars}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5D</td>
<td>( 497^{+8}_{-6} )</td>
<td>3.4^{+0.4}_{-0.0}</td>
<td>10 000</td>
</tr>
<tr>
<td>6D+</td>
<td>( 497^{+12}_{-9} )</td>
<td>3.0^{+0.3}_{-0.2}</td>
<td>2067</td>
</tr>
<tr>
<td>6D (Gaia only)</td>
<td>( 505^{+32}_{-16} )</td>
<td>3.0^{+0.7}_{-0.4}</td>
<td>495</td>
</tr>
</tbody>
</table>

Notes. The uncertainties in \( v_{\text{esc}} \) and \( k_t \) (or \( k \)) are given by the marginalised 1σ confidence levels.

Fig. 9. Confidence levels obtained by applying the LT90 method to the 5D and 6D samples in the solar neighbourhood. For each curve the 1, 2, and 3σ levels are shown, and the arrows indicate the maximum probability values. The side panels show the marginalised posterior distributions for \( P(v_{\text{esc}}) \) and \( P(k) \). For the 6D sample we show the results for both the augmented dataset and when using Gaia data alone. For the 5D sample, we recall that the method determines \( k_t \), and this is what is shown on the y-axis of the main panel, while the blue curve in the right panel represents \( P(k) \).

6.1. Determining \( v_{\text{esc}} \)

We now proceed to determine \( v_{\text{esc}} \) as a function of galactocentric distance. As we saw in the Aurigaia halos, the behaviour of \( v_{\text{esc}} \) as a function of distance can help in identifying local ‘biases’ or issues. We assumed as prior for \( k_t \) the marginalised distribution obtained for the solar neighbourhood \( P(k_t)_{\text{SN}} \) that is shown in the right panel of Fig. 9 with the blue curve. Therefore we implicitly assumed that \( k_t \) remains constant over the distance range probed. This assumption is justified by the Aurigaia simulations, as shown in Fig. 7.

We sliced the data into 16 concentric shells of 1 kpc width, with \( 4 < r < 12 \) kpc and centred on the Galactic centre (as in Fig. 2). The number of stars per bin, with velocities higher than \( v_{\text{cut}} \), varies from 58 744 to 1128. In each shell, \( v_{\text{cut}} \) is determined adaptively by selecting the top 10 000 fastest stars. We note that the results do not change significantly when the cut-off is fixed to \( v_{\text{cut}} = 250 \text{ km s}^{-1} \).

Figure 10 shows the trend of our estimate of \( v_{\text{esc}} \) with galactocentric distance. In each bin, the probability map is marginalised over the range 2.6 \( < k_t \) \( < 4.8 \) after applying \( P(k_t)_{\text{SN}} \). This range in \( k_t \) corresponds to the 3σ interval of the posterior distribution of \( P(k_t)_{\text{SN}} \). We note that this is a very similar range to that assumed in S07. The use of the \( P(k_t)_{\text{SN}} \) prior beyond the solar neighbourhood has helped in determining \( v_{\text{esc}} \) for all the distance bins considered, despite the sometimes relatively small number of stars used. With the size of the samples that are currently available, we could not have constrained both \( k_t \) and \( v_{\text{esc}} \) simultaneously for all radial bins.

The behaviour of the escape velocity in the inner halo (\( r < 8 \) kpc) matches the expectation from several Milky Way models well. This can be seen by comparison to the predicted escape velocity plotted in the background of Fig. 10 for the Piff14, McMillan17, BT08 (model I), and MW14 potentials (Piff et al. 2014a; McMillan 2017; Binney & Tremaine 2008; Bovy 2015, all computed using the implementation from AGAMA, Vasiliev 2019). The behaviour for the estimated \( v_{\text{esc}} \) shows small variations: a slight elevation at \( \sim 6 \) kpc and a dip at \( \sim 4.5 \) kpc, although it is fully consistent with a smooth increase towards the inner Galaxy. Furthermore, the amplitude of these variations is of a similar level as we observed in the Aurigaia simulations. Curiously, our estimate of \( v_{\text{esc}} \) is higher outside of the solar radius (i.e. distance \( > 8 \) kpc). This cannot be driven by the mass profile of the Milky Way and can only mean that something biases the determination of \( v_{\text{esc}} \), as we discuss in detail next.
With a two-point velocity correlation function, we tested the statistical clustering of the stars in the tail of the velocity distribution of our 5D sample. An excess of pairs implies that the velocity distribution is not smooth. The two-point velocity correlation function is given by

\[ \xi(\Delta v) = \frac{DD(\Delta v)}{\langle RR(\Delta v) \rangle}, \]

where \( DD(\Delta v) \) is the number of data-data pairs with a velocity separation of \( \Delta v \), and similarly, \( \langle RR(\Delta v) \rangle \) is the mean number of random-random pairs obtained by randomly shuffling the velocities 100 times. To this end, \( v_r \) and \( v_b \) were shuffled and the pseudo-Cartesian velocities were re-calculated from Eq. (6) assuming \( v_{\text{los}} = 0 \). Both the data and re-shuffled samples were cut off at the velocity of the 10,000th star, or 250 km s\(^{-1}\) if there were not enough stars per bin. Because of this re-sampling, most of the bins have an equal number of stars, except for those at large radii.

Figure 11 shows the results of the correlation function \( \xi \) for the 5D sample for the same distance bins as used throughout this paper. The curves in Fig. 11 are coloured by the mean distance of the bin, and the error bars show the uncertainty in \( \xi \) estimated by the Poisson error in the number of counts per bin. A value \( \xi = 1 \) indicates no excess correlation. Inside 8 kpc, \( \xi \) decreases with distance. Meanwhile, the bins just outside of this radius (light red) show the highest level of correlation over the full velocity range probed. The inner and outermost bins (dark colours) show the least correlation, although the uncertainties are large because of the low number of stars in these bins. There might also be an effect associated with the area of the shells increasing with distance squared, which results in the stars in the outer shells being physically more separated than those in the inner shells. This might give rise to gradients in the trajectories of the stars and hence to lower correlation amplitudes.

The analysis of the velocity correlation function confirms that the properties of the velocity distribution change with distance. An indication of velocity clustering at \( r \sim 10 \) kpc in our 5D sample similar (although of lower amplitude) to that seen for Au-27 could thus be responsible for this change.

7. Discussion

7.1. Relating \( v_{\text{esc}} \) to the Milky Way potential

In Eq. (18) we defined the escape velocity as the velocity required to reach \( r = \infty \). A more realistic definition is obtained by taking a different zero-point. No matter how \( \text{’escaping from} \)
the Milky Way' is defined, stars do not have to travel to infinity to be considered as escapees. For example, stars escaping to M31 make a much shorter journey (\(r \approx 800 \text{kpc}\)).

Therefore we used the definition of P14, who took the escape velocity to be the velocity required to reach 3\(r_{200}\),

\[
v_{\text{esc}}(r \to 3r_{200}) = \sqrt{2} \left[ \Phi(r) - \Phi(3r_{200}) \right],
\]

where 3\(r_{200}\) is the radius within which the average halo density is 340 \(\rho_{\text{crit}}\) (which is equal to 3H^2/8\pi G, and where we assume \(H = 73 \text{ km s}^{-1} \text{ Mpc}^{-1}\)). We note that this zero-point is set somewhat arbitrarily, the 'true' value depends on the direction and might be a few km s\(^{-1}\) higher or lower. D19 used a different definition, which is for the star to escape to 2\(r_{200}(\approx 2.5r_{200})\). At the solar position, these two definitions result in a difference of 5 km s\(^{-1}\).

Because the potential is axisymmetric, \(v_{\text{esc}}\) varies as a function of cylindrical \(R\) and \(z\) for a fixed spherical \(r\). In the plane of the disc, the gradient is the steepest, the escape velocity is highest. Using the McMillan (2017) potential, we estimate that \(v_{\text{esc}}\) decreases by \(\approx 20 \text{ km s}^{-1}\) at a distance of 5 kpc away from the plane of the disc, whereas at 10 kpc, the difference is about 50 km s\(^{-1}\).

To develop some intuition of how properties such as the mass of the Milky Way are related to \(v_{\text{esc}}\), we used the following equations: For a spherical potential, the gradient \(dv_{\text{esc}}/dr\) is related to the mass, circular velocity, and potential as

\[
\frac{d\Phi(r)}{dr} = -v_{\text{esc}}(r) \frac{dv_{\text{esc}}(r)}{dr} = \frac{v_{\text{circ}}(r)^2}{r} = \frac{GM(r)}{r^2}.
\]

Another insightful equation, given by Eqs. (2)–(22) of Binney & Tremaine (1987), is

\[
v_{\text{esc}}(r_0)^2 = 2v_{\text{circ}}(r_0)^2 + 8\pi G \int_0^\infty r p(r) \, dr
\]

(see S07). The circular velocity \(v_{\text{circ}}\) at the solar position is a direct measure of the mass inside of the solar radius. On the other hand, the escape velocity \(v_{\text{esc}}\) is a measure of the total gravitational potential. The two are related through a factor of \(\sqrt{2}\) only if there is no mass outside of the radius where both are measured. In other words, the difference \(v_{\text{esc}} - 2v_{\text{circ}}\) in the solar neighbourhood probes the potential, and with it, the mass distribution beyond the solar radius.

### 7.2. Estimating the mass of the Milky Way halo

We used our estimate of \(v_{\text{esc}}\) at the position of the Sun to constrain the mass of the halo of the Milky Way. The escape velocity and the gravitational potential of the Milky Way are related through Eq. (20). A straightforward procedure to derive the mass of the Milky Way is to take an existing model and adjust the parameters of the halo such that it matches the \(v_{\text{esc}}\) measured for the solar neighbourhood. We closely followed the procedure outlined in Sect. 5 of D19, but we used the McMillan (2017) potential and only varied the parameters of its dark halo, which is represented by a Navarro-Frenk-White (NFW) profile (Navarro et al. 1997).

The only issue with this procedure is that \(v_{\text{esc}}(r_0)\) is mostly sensitive to the mass outside of the solar radius. As a result, fitting \(v_{\text{esc}}\) constrains the concentration of mass inside the solar radius only weakly. A solution is to use the circular velocity \(v_{\text{circ}}\), which is sensitive to the mass inside the solar radius, as an additional constraint. That is, when fitting \(v_{\text{esc}}(r_0)\), we forced the model to have a certain \(v_{\text{circ}}(r_0)\).

The best-fitting potential is defined as the one that minimises

\[
\eta = (v_{\text{esc}}(r_0) - v_{\text{esc}})^2 + (v_{\text{circ}}(r_0) - 232.8 \text{ km s}^{-1})^2,
\]

where we take \(v_{\text{esc}}\) to be the maximum probability value found in the solar neighbourhood for the 5D sample (see Table 1). The value for the circular velocity that we assumed, \(v_{\text{circ}}(r_0) = 232.8 \text{ km s}^{-1}\), is the value that was used in the original McMillan (2017) potential. We note that there is no freedom in choosing \(v_{\text{circ}}(r_0)\) because the data are only consistent with the value above because it is used in the correction for the solar motion.

Figure 12 shows the values for Eq. (23) for the ranges of \(M_{200}\) and \(c\) that we explored, namely \(\log_{10}(M_{200}/M_\odot) \in [11.5, 12.5]\) and \(c \in [1, 30]\). The solid line marks all models that have a correct \(v_{\text{esc}}(r_0)\), and the dashed lines mark all models that have the correct \(v_{\text{esc}}\). The best-fitting potential lies at the intersection of the two lines. The curves illustrate the benefit of including the \(v_{\text{circ}}\) in the fit. As we expected, the dashed curve is only weakly sensitive to \(c\). The orange marker highlights the combination of \(M_{200}\) and \(c\) that best fits \(v_{\text{esc}}\) and \(v_{\text{circ}}\). Therefore the best-fitting estimate of the mass is \(M_{200} = 0.67_{-0.08}^{+0.06} \times 10^{12} M_\odot\) and the corresponding concentration parameter is \(c = 15.0_{-2.9}^{+1.2}\).

The uncertainties are derived by calculating the best-fitting \(M_{200}\) and \(c\) for the extreme cases of \(v_{\text{esc}} + 8 \text{ km s}^{-1}\) and \(v_{\text{esc}} - 8 \text{ km s}^{-1}\), which are the limits given by the 1\(\sigma\) level (e.g., Table 1).

As we mentioned above, the LT90 method is likely to underestimate the \(v_{\text{esc}}\). Therefore the mass and concentration parameters quoted above should be seen as a lower limit. It is accordingly consistent with the original potential of McMillan (2017) in the sense that it is smaller and more concentrated. Moreover, this lower limit is also lower than most recent mass estimates (cf. Fig. 7 Callingham et al. 2019, for a recent compilation). If we now use the results from the analysis of the Auriga simulations and adjust for the 10% underestimation of \(v_{\text{esc}}\) (grey dashed curve in Fig. 12), we find that the best-fit mass and concentration parameter are \(M_{200} = 1.11_{-0.07}^{+0.08} \times 10^{12} M_\odot\) and \(c = 11.8_{-0.3}^{+0.3}\).
7.3. Stars that might be unbound

A possibly interesting follow-up project is to measure the radial velocities of the stars in the 5D sample that lie near the truncation of the best-fit power law. Using the maximum probability fit of the velocity distribution, we can calculate which stars have a high probability of being unbound. Given the apparent \( v' \) and its uncertainty, we can calculate the probability of these stars having a true \( v \) higher than the estimated \( v_{\text{esc}} \). We note that strictly speaking, the uncertainties are non-Gaussian, see also Sect. 3. However, we assume that they are small enough such that they may be approximated to be Gaussian.

For the set of stars that have apparent tangential velocities higher than the estimated \( v_{\text{esc}} \), we calculated the probability of the star being bound as

\[
P_{\text{bound}} = \sum_{v_{\text{esc}}} P_{\text{SN}}(v_{\text{esc}}) \int_{v_{\text{esc}}} f_{\text{SN}}(v', v, \sigma_v) \, dv_v,
\]

where \( P_{\text{SN}}(v_{\text{esc}}) \) is the posterior of \( v_{\text{esc}} \) marginalised over \( k \), i.e. the blue curve in the right panel of Fig. 9. The uncertainty distribution \( f_{\text{SN}}(v', v, \sigma_v) \) is defined such that it gives the probability of finding the star with a true velocity \( v \) and uncertainty \( \sigma_v \) with an apparent velocity in the range \( (v', v + \sigma_v) \). The probability of the star being unbound is simply \( P_{\text{bound}} = 1 - P_{\text{bound}} \).

The list of sources that fall outside of the maximum probability value of \( v_{\text{esc}} \) is given in Table 2. We stress that very likely, the actual \( v_{\text{esc}} \) is higher than our best estimate for the values of \( P_{\text{bound}} \) given here should therefore be considered as upper limits. To emphasise this, we also calculated the probability of these stars being unbound after correcting \( v_{\text{esc}} \) for a 10% offset, based on our analysis in Sect. 4.2. Only two sources remain unbound in this case, and one of these just barely. The source with the highest probability of being unbound, with \( \text{Gaia} \) DR2 source_id 2655054950237153664, has been flagged in the faststars \(^2\) database (Guilhon et al. 2017) as a potential hyper-velocity star. The source was first identified by Du et al. (2019) based on its high tangential velocity.

About half of the sources in Table 2 have an inward-pointing velocity vector based on the pseudo-velocities in the galactocentric frame. This makes it likely that the majority of these stars are bound to the Milky Way. A possibility remains, however, that the velocity vectors of the stars point radially outwards when line-of-sight velocities are measured. However, for some stars the vectors will always point inwards, even in the extreme case of \( v_{\text{esc}} = \pm 500 \text{ km s}^{-1} \). One of these stars has the highest probability of being unbound (source_id 2655054950237153664), which has an outward-pointing velocity vector even for adopted line-of-sight velocities of \( \pm 500 \text{ km s}^{-1} \), and therefore it might truly be unbound.

7.4. \( v_{\text{esc}} \) as tracer of the mass distribution

The luminous components of the Milky Way are most definitely not spherically symmetric. Because the escape velocity traces the potential, we should ultimately measure it in axisymmetric coordinates rather than as a function of spherical radius. By estimating \( v_{\text{esc}} \) as a function of \( z \), we can perhaps constrain the flattening of the halo, although with the current sample we are more sensitive to the contribution of the disc to the total potential of the Milky Way. This means that such an analysis would benefit from a large sample of stars that probe the Milky Way halo at a greater depth, such as what may become available with \( \text{Gaia} \) eDR3.

Because of the large number of sources in our 5D sample, it is possible for the first time to explore the escape velocity as a function of cylindrical \( R \) and \( z \). We sliced our 5D sample into overlapping bins of \( 8 \) \( \times \) \( 11 \) volumes of \( |R| < 1 \text{ kpc} \) and \( |z| < 1 \text{ kpc} \), where \( R \) and \( z \) are the centres of the volumes. Assuming that the Milky Way is perfectly axisymmetric, we included sources independent of their azimuthal angles. Bins with fewer than 500 stars were discarded. We used the same method to determine the escape velocity as we used in Sect. 6 and presented in Fig. 10. That is, we again assumed the posterior distribution of \( P_{\text{SN}}(k) \) from the solar neighbourhood as prior on \( k \). For computational reasons, we decreased the size of the grid on which the probability distribution was evaluated to \( 50 \times 22 \) points ranging from \( 400 \text{ km s}^{-1} < v_{\text{esc}} < 800 \text{ km s}^{-1} \) and \( 2.6 < k < 4.8 \) (which corresponds to the \( 3\sigma \) levels in the solar neighbourhood).

Figure 13 shows the escape velocity in these volumes (large coloured markers) with a colour map corresponding to the escape velocity predicted by the McMillan17 model, with the updated lower limit of the halo mass computed in Sect. 7.2. Therefore, this model is based on the estimate for \( v_{\text{esc}} \) that is biased low, and we used it to predict this estimate at other locations for a spherical NFW halo. The large plus markers in Fig. 13 indicate volumes in which the \( 3\sigma \) levels of the \( v_{\text{esc}} \) include the expected value. The large crosses indicate volumes where the \( v_{\text{esc}} \) expected from the updated McMillan17 potential lies outside of the \( 3\sigma \) confidence level of the maximum probability determination of \( v_{\text{esc}} \). Interestingly, the distribution is not fully symmetric in \( z \). The fact that \( v_{\text{esc}} \) does not match the expected value in many locations might indicate a bias in the estimated \( v_{\text{esc}} \) in the solar neighbourhood. Another possibility is that the decrease in the strength of the potential with \( z \) is less steep than expected for a spherical halo (e.g., pointing to a prolate halo or a weaker effect from the disc).

Table 2. High-velocity sources that are close to the escape velocity.

<table>
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<tr>
<th>source_id</th>
<th>( v' ) (( \epsilon_i ))</th>
<th>( P_{\text{bound}} )</th>
<th>( P_{10% \text{ bound}} )</th>
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<tr>
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<td>0.50</td>
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<td>0.13</td>
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<td>3176805236597893248</td>
<td>502 (39)</td>
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<td>0.14</td>
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<td>0.82</td>
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\(^2\) https://faststars.space/
in which the 3
the McMillan17 potential, which assumes a spherical halo, but whose
ground shows expected isocontours for the escape velocity according to
the maximum probability value for $v_{\text{esc}}$. The quoted uncertainties are given by the level where the probability has dropped to 61% of the maximum value (i.e. the $\sim 1\sigma$ level). These values agree well with previous works, but this is the first time that we can determine (a lower limit to) the escape velocity with such high confidence. This value for $v_{\text{esc}}$ agrees remarkably well with the value that is obtained when we use a local sample of halo stars with full phase-space information. Applying the 10% fix would mean that the true escape velocity $v_{\text{esc}}^{10\%} = 552 \text{ km s}^{-1}$.

We also determined $v_{\text{esc}}$ as a function of galactocentric distance. We find that the escape velocity is higher in the inner halo than at the solar radius. This matches the behaviour expected from smooth Milky Way models well. Unexpectedly, for radii beyond 8 kpc, $v_{\text{esc}}$ is also higher than at the solar radius (see Fig. 2). Indications of a similar trend were picked up by M18, but at a much lower significance level because of their limited sample size.

Interestingly, we find that the behaviour of $v_{\text{esc}}$ outside of the solar radius is paired with a change in shape of the velocity distribution. For example, the tail of the velocity distribution becomes more exponential (and less power law-like) with galactocentric distance (see Fig. 2). Moreover, the velocities in the bins outside of 8 kpc show a higher degree of correlation as measured by the velocity correlation function. Therefore we conclude that the bump in $v_{\text{esc}}$ in the outskirts is likely driven by a change in the kinematic properties of the sample as a function of galactocentric distance. Coincidentally, we found a similar effect in one of the Aurigai halos we analysed, where a velocity bump (presumably related to a clump or a non-phase-mixed structure in the halo) dominates the tail near the escape velocity.

The estimated $v_{\text{esc}}$ can be used to provide a very precise estimate of the mass of the halo of the Milky Way. To this end, we adjusted the halo component of the McMillan (2017) Milky Way potential (which is a spherical NFW profile) while keeping the other components fixed. The halo parameters that best fit the estimated $v_{\text{esc}}(r_0)$ are $M_{200} = 0.67^{+0.08}_{-0.06} \cdot 10^{12} M_\odot$ and $c = 15^{+1.2}_{-0.9}$, where we used $v_{\text{esc}}(r_0)$ as an additional constraint. When we applied the tentative 10% fix, we find that the best-fitting halo has $M_{200}^{10\%} = 1.11^{+0.07}_{-0.05} \cdot 10^{12} M_\odot$ and $c^{10\%} = 11.8^{+0.2}_{-0.3}$.

The method for determining $v_{\text{esc}}$ consists of fitting the tail of the velocity distribution with a parametrised model. Using the best-fitting model obtained, we can also establish if there are any unbound stars in the solar neighbourhood. That is, we may calculate which stars have a high probability of having a true velocity that is higher than the determined escape velocity. We list these stars in Table 2. Their pseudo-velocities (without the line-of-sight velocity) suggest they are not all unbound, however; their velocity vectors point both inwards and outwards. If these high-velocity stars were truly escaping, we would expect them to all be on radially outbound trajectories. Nonetheless, it might be interesting to follow these stars up. When the tentative 10% fix is taken into account, only one candidate with a high probability of being unbound remains: Gaia DR2 source_id 2655054950237153664. This star was first flagged as being unbound by Du et al. (2019).

Finally, we discussed a tentative method to probe the mass distribution of the Milky Way by determining $v_{\text{esc}}$ as a function of $(R, z)$. We find that escape velocity values are weakly asymmetric with respect to the Galactic plane, and also a tentative

8. Conclusions

We used a sample of halo stars with high tangential velocities to constrain the escape velocity in the vicinity of the Sun and as a function of galactocentric distance. We applied the well-known LT90 method, which fits the high-velocity tail (i.e. above some velocity $v_{\text{cut}}$) of the velocity distribution with a power law of the form $(v_{\text{esc}} - v)^k$. In the process of applying the method, we identified a number of shortcomings.

The study presented here constitutes the first application of the method to a sample of stars using only tangential velocities. We have found that in practice, the estimated value for the parameter $k$ is not exactly what is predicted by LT90 (namely $k = 1$), except in the very tail of the distribution, in the limit where $v_{\text{cut}}$ differs by 10% from $v_{\text{esc}}$. Unfortunately, the value of $v_{\text{cut}}$ typically chosen is farther away from $v_{\text{esc}}$ because enough stars ($\sim 10^3$) with high velocity need to be present in the sample to precisely estimate $v_{\text{esc}}$. A similar conclusion may be reached when the method is applied to radial velocity samples. This means that care is necessary when the values of $k$ are compared for different studies in the literature. Fortunately, $v_{\text{esc}}$ is not affected.

In addition, and as previously discussed in the literature, the $v_{\text{esc}}$ determined via the LT90 method is most likely a lower limit. To correct for this bias, we tested the method on two mock $Gaia$ catalogues from the Aurigai project (Grand et al. 2018). In these simulated galaxies, the estimated $v_{\text{esc}}$ are $\sim 10\%$ lower than the true values, close to the 7% bias found in a similar study by Grand et al. (2019). Based on this result, we also quote the value obtained by applying a 10% correction when we report our estimates of the escape velocity. However, we note that there is no guarantee that the Milky Way halo is truncated at a similar level as the Aurigai halos. The truncation of the velocity distribution depends on the (recent) assembly history of the Galaxy, and for the simulations, it might depend on the numerical resolution.

In the solar neighbourhood, using a 5D sample, we determine a very precise estimate of the escape velocity, $v_{\text{esc}} = 497^{+8}_{-7} \text{ km s}^{-1}$, and power-law exponent $k_1 = 3.4^{+0.4}_{-0.3}$. The quoted uncertainties are given by the level where the probability has dropped to 61% of the maximum value (i.e. the $\sim 1\sigma$ level).
indication that the halo may be prolate. However, for more robust conclusions, a larger sample with more accurate distances that probes deeper into the Milky Way is necessary. We hope that such a sample will become available with Gaia (e)DR3.

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References
Wolfram Research Inc. 2020.