Saturation mechanism and generated viscosity in gravito-turbulent accretion disks

L. Löhner{1}, S. Krätschmer{2}, and A. G. Peeters{1}

1 Physics Department of Bayreuth, Universitätstrasse 30, Bayreuth, Germany
e-mail: lucas.loehnert@uni-bayreuth.de
2 Alfred Wegener Institute, Helmholtz Centre for Polar and Marine Research, Bremerhaven, Germany

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ABSTRACT

Here, we address the turbulent dynamics of the gravitational instability in accretion disks, retaining both radiative cooling and irradiation. Due to radiative cooling, the disk is unstable for all values of the Toomre parameter, and an accurate estimate of the maximum growth rate is derived analytically. A detailed study of the turbulent spectra shows a rapid decay with an azimuthal wave number stronger than \( k^{-1} \), whereas the spectrum is more broad in the radial direction and shows a scaling in the range \( k^{-1} \) to \( k^{-2} \). The radial component of the radial velocity profile consists of a superposition of shocks of different heights, and is similar to that found in Burgers’ turbulence. Assuming saturation occurs through nonlinear wave steepening leading to shock formation, we developed a mixing-length model in which the typical length scale is related to the average radial distance between shocks. Furthermore, since the numerical simulations show that linear drive is necessary in order to sustain turbulence, we used the growth rate of the most unstable mode to estimate the typical timescale. The mixing-length model that was obtained agrees well with numerical simulations.

The model gives an analytic expression for the turbulent viscosity as a function of the Toomre parameter and cooling time. It predicts that relevant values of \( \alpha \approx 10^{-4} \) can be obtained in disks that have a Toomre parameter as high as \( Q \approx 10 \).

Key words. accretion, accretion disks – protoplanetary disks – hydrodynamics – instabilities – turbulence

1. Introduction

Self-gravity in accretion disks can lead to gravitational instability (GI), which was originally studied in the context of galaxies (Toomre 1964; Lin & Shu 1964; Lynden-Bell & Kalnajs 1972), but is relevant also for accretion disks around young stellar objects and protoplanetary disks (YSOs; PPDs) (Gammie 2001; Kratter & Lodato 2016). The gravitational instability sets in when the Toomre parameter,

\[
Q = \frac{c_s k}{\pi G \Sigma} \tag{1}
\]

is smaller than one \( (Q < 1) \) (Toomre 1964). Here, \( c_s \) is the sound speed, \( \Sigma \) is the mass surface density, and \( k \) is the epicyclic frequency, or the angular frequency \( (\kappa = \Omega_{\ast}) \) in the case of a Keplerian disk. The Toomre parameter expresses that a higher temperature (or equivalently a higher sound speed) will stabilize the disk, whereas a higher surface density \( (\Sigma) \) has a destabilizing effect (Kratter & Lodato 2016; Lin & Kratter 2016). Additional physics has been considered since the original derivation, and it has been shown that radiative cooling and viscosity can destabilize the disk for \( Q > 1 \) (Lin & Kratter 2016).

Numerical simulations have clarified some aspects of the nonlinear evolution of the gravitational instability. In the case of sufficiently fast cooling, fragmentation occurs (Johnson & Gammie 2003; Rice et al. 2003, 2005; Kratter & Murray-Clay 2011; Booth & Clarke 2019), which may be relevant for the formation of massive exoplanets. If the cooling is less efficient, a gravito-turbulent state is obtained, in which the radiation losses are compensated by the heating of the disk through dissipation in shocks (Gammie 2001; Kratter & Lodato 2016). Hence, cooling strongly effects the nonlinear saturation of the gravitational instability (Cossins et al. 2009), and different implementations of the cooling prescription have been tested. These include the so-called \( \beta \) cooling prescription (Gammie 2001), an irradiated version of this latter model (Rice et al. 2011; Baehr & Klahr 2015), and also the solution of the full radiative transfer problem (Hirose & Shi 2019).

One major quality of gravito-turbulence is its ability to transport angular momentum, leading to accretion. A common measure for the angular momentum transport through turbulence is the \( \alpha \)-parameter (Shakura & Sunyaev 1973). In a stationary state, the turbulent dissipation connected with the effective viscosity described by the \( \alpha \) parameter is then balanced by the energy loss through radiation \( (\langle U \rangle / \tau_c) \), where \( \tau_c \) is the cooling timescale on which the thermal energy is lost), yielding (Gammie 2001; Rice et al. 2011)

\[
\alpha \approx \frac{4}{9\gamma(\gamma - 1)\tau_c \Omega_{\ast}} \left( 1 - \frac{U_0}{\langle U \rangle} \right), \tag{2}
\]

where \( \gamma \) is the adiabatic index, \( U_0 \) the stationary energy density obtained in the absence of turbulence through the combination of irradiation and radiation loss, and \( \langle U \rangle \) is the averaged energy density in the presence of turbulence. Although the relation above is a powerful restriction on \( \alpha \), it does not allow a direct prediction because the averaged energy density \( \langle U \rangle \) is not a priori known. As \( U_0 / \langle U \rangle \approx \frac{Q_0^2}{Q^2} \), a prediction of \( \alpha \) is possible when the Toomre parameter of the saturated turbulent state \( \langle Q \rangle \) is known. It is generally accepted that saturation occurs close to marginal stability \( (Q \approx 1) \) (Kratter & Lodato 2016). However, simulations suggest saturated values higher than one \( (1 \leq \langle Q \rangle \leq 2) \) (Rice et al. 2011; Vanon 2018), and those are
2. The model

The evolution of the disk is described with the local two-dimensional shearing sheet approximation (Goldreich & Lynden-Bell 1965; Balbus & Hawley 1998; Gammie 2001). The model equations consist of the continuity equation, Euler’s equation, and the Poisson equation for the gravitational potential ($\Phi$) due to the mass density in the disk:

$$\partial_t \Sigma + \nabla \cdot (\Sigma \mathbf{e}) = 0 \quad (3a)$$
$$\partial_t \rho + (\rho \cdot \nabla) \mathbf{v} = -\frac{1}{\Sigma} \nabla P - 2\Omega_0 \times \mathbf{v} + 3\Omega_0^2 \Sigma \mathbf{e}_t - \nabla \Phi \quad (3b)$$
$$\partial_t U + (\rho \mathbf{v} \cdot \nabla) U = -\gamma U (\nabla \cdot \mathbf{v}) - \frac{U - U_0}{\tau_c} \quad (3c)$$
$$\nabla^2 \Phi = 4\pi G \Sigma \cdot \partial^2 (z). \quad (3d)$$

In the equations above, $\mathbf{v}$ is the fluid velocity, $G$ is the gravitational constant, and $P$ is the pressure, with the latter linked to the internal energy density ($U$) through the equation of state

$$P = (\gamma - 1) U. \quad (4)$$

The shearing box coordinates ($x, y$) used in the equations above represent the radial $x$ and azimuthal $y$ direction. The forces per unit mass on the right hand side of the Euler equation, Eq. (3b), include the pressure gradient and the Coriolis, tidal, and self-gravitational forces in that order. The equilibrium is given by a uniform surface mass density ($\Sigma_0$) and uniform internal energy density ($U_0$). This equilibrium incorporates the shear in the Keplerian velocity profile $b_0 = -3\Omega_0^2 \Sigma_0 \mathbf{e}_y/2$, which develops through a balance of the Coriolis ($-2\Omega_0 \times \mathbf{v}$) and tidal ($3\Omega_0^2 \Sigma_0 \mathbf{e}_t$) force.

For consistency with the literature (e.g., Gammie 2001) the adiabatic index is chosen as $\gamma = 2$, which agrees with the two-dimensional nature of the system. A model equation for the internal energy rather than an adiabatic closing relation for the pressure ($P = \text{const.} \cdot \Sigma^\gamma$) is necessary, because the $\beta$-cooling prescription to mimic the loss of thermal energy due to radiation (Gammie 2001) is applied. Then, $\beta = \tau_2 \Omega_0$ is the dimensionless cooling timescale. The cooling term incorporates a fiducial thermal energy density, $U_0$, that the system would obtain in the absence of turbulence. Here, $U_0$ is motivated by the irradiation of the disk by the young star (see e.g., Rice et al. 2011), which, in combination with the radiation losses from the disk, leads to a floor in the thermal energy density.

The equilibrium values $\Sigma_0$ and $U_0$ also define initial values for the Toomre parameter ($Q_0$) and sound speed $c_s(0)$, defined as

$$c_s^2 = \frac{P}{\Sigma} = \gamma(\gamma - 1) \frac{U}{\Sigma}. \quad (5)$$

3. Linear stability in the presence of cooling

The Toomre stability criterion $Q > 1$ was derived assuming the gas behaves adiabatically or barotropically, and expresses the fact that the gas pressure has a stabilizing effect. It is therefore natural to assume that radiative cooling leads to further destabilization and indeed this has been found (see e.g., Lin & Kratter 2016). As the linear instability plays a role for the nonlinear saturated state, it is investigated in some detail in this section.

The model equations of the previous section are linearized around the equilibrium ($\Sigma_0, U_0, v_0 = -3/2(\Omega_0 \Sigma_0 x \mathbf{e}_y), \Phi = 0$,

$$U = U_0 + \tilde{U}, \quad \Sigma = \Sigma_0 + \tilde{\Sigma}, \quad \mathbf{v} = \frac{3}{2} \Omega_0 \Sigma_0 x \mathbf{e}_y + \tilde{\mathbf{v}} \mathbf{e}_x + \tilde{v}_y \mathbf{e}_y, \quad \Phi = \tilde{\Phi}, \quad (6)$$

whereby the tilde denotes a perturbed quantity. Neglecting all quadratic terms in the perturbations then gives a set of equations for the evolution of the perturbations:

$$\partial_t \tilde{\Sigma} + \partial_t \Sigma_0 \partial^2 \tilde{u}_x = 0 \quad (7a)$$
$$\partial_t \tilde{v}_x + \frac{\gamma - 1}{\Sigma_0} \partial_t \tilde{U} - 2\Omega_0 \tilde{v}_y + \partial_x \tilde{\Phi} = 0 \quad (7b)$$
$$\partial_t \tilde{v}_y + \frac{\Omega_0}{2} \tilde{v}_x = 0 \quad (7c)$$
$$\partial_t \tilde{U} + \gamma U_0 \partial_x \tilde{v}_x + \frac{\tilde{U}}{\tau_c} = 0 \quad (7d)$$

where the perturbations have been assumed to be axisymmetric. Substituting $\tilde{f} = \tilde{f} \cdot \exp(\mathbf{g} \cdot \mathbf{x})$ for all perturbed quantities $(f)$ yields a set of algebraic equations from which the dispersion relation is obtained:

$$g^2 = \frac{-\Omega_0^2 + 2\tau_c \Omega_0 k}{Q_0} - \frac{c_s^2(0)^2 \tau_c g^2}{1 + \tau_c g^2}. \quad (8)$$

The dispersion relation is similar to that found by Lin & Kratter (2016). Solutions for the growth rate as function of the wave vector for different values of $Q_0$ and a normalized cooling time, $\beta \equiv \tau_c \Omega_0 = 10$, are shown in Fig. 1.

The finite $\beta$ follows from the dispersion relation that unstable modes occur for all values of $Q_0$. The $k$ domain is divided into two regions, with instability only occurring for $k > Q_0/2$ as larger structures (smaller wave numbers) are stabilized by the Coriolis force. One might expect large wave numbers (small structures) to be stabilized as well because the critical Jeans mass cannot be met. However, the cooling extracts the excess thermal energy that is generated by the compression of the fluid, and exponential growth is obtained on timescales that are sufficiently long for this cooling to be effective. This is expressed in Eq. (8) through the occurrence of $\tau_c g$ in the stabilizing term connected with the pressure response. Therefore, the observation that the value of the Toomre parameter of the saturated turbulent state is larger than one ($Q > 1$) does not imply linear stability. Although, it should be mentioned that the instability for $Q_0$ might sensitively depend on the exact choice of the cooling model (see Lin & Kratter 2016, and the discussion in Sect. 7).

The fastest growing mode can be obtained by taking the partial derivative of the dispersion relation towards $k$ and setting
2.5 simulations (τg mial in k cases are unstable for all Q tions with β ◦. The analytic Q the interest is mostly for cases with β ◦. It is surprising as one usually finds kcs0/Ω0 = 1/Q0 according to the classical Toomre criterion in the absence of cooling. The latter is equivalent to a Jeans criterion, stating that for smaller mass densities (larger Q0) one needs a larger volume (size ∝ 1/k) to reach the critical mass. In fact, the result in Eq. (11) is assumed to hold for small g. It turns out that for gβ >> 1 one actually finds kcs0/Ω0 = 1/Q0 as for the case without cooling. This backs up the above given interpretation that, for a given Q0, the growth rate is sufficiently small in comparison to 1/β to allow further collapse.

4. Code and benchmarking

The simulations presented here use the code DiskFlow which is a grid-based finite-difference code. The equations solved are the two-dimensional compressible fluid equations outlined above. The fluid equations are integrated using a fourth-order Runge-Kutta method. DiskFlow assumes shearing periodic boundary conditions, with the box being periodic in azimuthal y-direction and sheared periodic in the radial x-direction (see e.g., Balbus & Hawley 1998). The Poisson equation for self-gravity is solved using a Fourier transform (Gammie 2001). At each time-step the surface density Σ is transformed to Fourier space ˆΣk, and the Fourier amplitude of the gravitational potential (Φ) is updated through

\[ \Phi_k = -\frac{2\pi G}{k} \hat{\Sigma}_k, \]  

where k := |k| and k = (kx, ky) is the two-dimensional wave vector. The gravitational potential in real space is then obtained through a backward transformation. To avoid difficulties with the Fourier transform in combination with the shearing periodic boundary conditions, a back mapping of the surface mass density, undoing the effect of the shearing and ensuring the solution is periodic in both directions, is performed before the Fourier transform is calculated. Furthermore, a cut-off wave vector \( k_{\text{max}} = \sqrt{\frac{k_x^2 + k_y^2}{\Omega_0^2 - \frac{\Omega^2}{c_s^2}}} \) is introduced, following (Gammie 2001). The cut-off acts as a smoothing factor for small-scale gravity. Although this is possibly problematic for cases with clumping (see e.g., Young & Clarke 2015) it has been shown to work well for the gravito-turbulent state.

An artificial viscous pressure is included in DiskFlow to guarantee numerical stability

\[ P_{\text{vis}} = \zeta (\nabla \cdot \mathbf{v})^2. \]  

This artificial viscous pressure is especially useful in the case of shocks (see e.g., Gammie 2001), as it acts as a viscosity that responds to volumetric changes in the fluid. Moreover, it does not extract energy from the system, as the dissipated kinetic energy is consistently transferred to thermal energy. For the simulations shown here, ζ = 0.006 is chosen.

The code also provides the possibility of using artificial viscosity, which is implemented as a second-order diffusion scheme. More precisely, all of the Eqs. (3a)–(3c) contain an additional dissipative term (second spatial derivatives):

\[ \partial_t f + \cdots = \cdots + D(2) \left( \Delta \lambda \right) \nabla^2 f, \]  

where the maximum wave vector is obtained by inserting the estimate for the growth rate in Eq. (9). The result of Eq. (11) is relatively simple and therefore an easy-to-use estimate of the growth rate that works well for \( Q \geq 1.2 \). The wave vector is

\[ k_{\text{max}} = \sqrt{\frac{k_x^2 + k_y^2}{\Omega_0^2 - \frac{\Omega^2}{c_s^2}}} \]  

Substituting this back into Eq. (8) leads to a cubic polynomial \( g \) only

\[ \left( \frac{g}{\Omega_0} \right)^3 - \left( \frac{1}{Q_0^2} - \frac{1}{\beta Q_0^2} \right) \frac{g}{\Omega_0} - \frac{1}{\beta Q_0^2} = 0. \]  

Solutions of this equation are shown as solid lines in Fig. 2 for cooling times \( \beta \in [6, 10, 100, \infty] \). The last case (\( \beta \rightarrow \infty \)) is the classical Toomre criterion without cooling. The analytic expression of the solution \( g(k) \) is somewhat extensive, but here the interest is mostly for cases with \( Q_0 > 1 \) for which the growth rate is relatively small. In this case, the first term in Eq. (10) can be neglected against the second, and one readily obtains

\[ \frac{g}{\Omega_0} = \frac{1}{\beta Q_0^2 - 1} \frac{kM c_s0}{\Omega_0} = Q_0. \]  

where \( Q_0 \) is the classical Toomre criterion without cooling. The latter is equivalent to a Jeans criterion, stating that for smaller mass densities (larger \( Q_0 \)) one needs a larger volume (size \( \propto 1/k \)) to reach the critical mass. In fact, the result in Eq. (11) is assumed to hold for small \( g \). It turns out that for \( g\beta > 1 \) one actually finds \( k c_s0/\Omega_0 = 1/\Omega_0 \) as for the case without cooling. This backs up the above given interpretation that, for a given \( Q_0 \), the growth rate is sufficiently small in comparison to \( 1/\beta \) to allow further collapse.

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\[ \partial_t f + \cdots = \cdots + D(2) \left( \Delta \lambda \right) \nabla^2 f, \]  

1 For more details as well as the source code see https://bitbucket.org/astro_bayreuth/accretion-disk-flow/src/master/
with a damping coefficient $D_{(2)}$ and resolution $\Delta x = L/N$. Typical values used for the simulations are $D_{(2)} \sim (0.07-0.8)$. Empirically, it is found that $D_{(2)} \leq 0.1$ is problematic as small-scale perturbations cannot be damped sufficiently and the simulation might eventually become numerically unstable. The coefficient $D_{(2)}$ is made variable in time in order to capture the violent transition from the linear growth phase to the nonlinearly saturated state. In this latter state, the damping coefficient is kept as low as possible. By construction, the damping predominantly acts on small scales (i.e., roughly grid-scale).

All quantities have been made dimensionless using the angular frequency ($\Omega_0$), the sound speed at initialisation ($c_0$), and the gravitational constant $G$. With this choice the surface mass density is normalized with $\Sigma_{\text{ch}} = c_0^2\Omega_0/G$, the internal energy density with $U_{\text{ch}} = c_0^2\Omega_0/G$, and the gravitational potential with $\Phi_{\text{ch}} = c_0^2\Omega_0$. Furthermore, the characteristic length scale $L_{\text{ch}} = c_0\Omega_0$ is equal to the disk scale height, $L_{\text{ch}} = H$, that would be obtained if the vertical force balance were considered. Timescales are normalized with the characteristic time $t_{\text{ch}} = \Omega_0^{-1}$, which is consistent with the dimensionless cooling parameter, $\beta \equiv \tau_c\Omega_0$, used before. Unless stated otherwise, all quantities below are dimensionless.

As one of the code benchmarks, the growth rate of the most unstable mode is calculated as a function of the Toomre parameter ($Q_0$) for different normalized cooling times ($\beta$). The numerical results, shown in Fig. 2, are in excellent agreement with the analytic formula obtained in Sect. 3. The simulations use a box size comparable to the analytically derived wavelength of the fastest growing mode and are initialised with random density perturbations. The growth rates are then determined for an interval of exponential growth $t_1 < t < t_2$, using

$$g = \frac{\ln \left( \frac{\Sigma_2}{\Sigma_1} \right)}{2(t_2 - t_1)}. \quad \text{(15)}$$

Figure 3 shows that a stationary turbulent state is obtained in the simulations and gives the time evolution of the box-averaged perturbed kinetic, gravitational, and thermal energy densities:

![Figure 3](image_url)

**Fig. 3.** Temporal evolution of the averaged perturbed energy densities for a simulation with $Q_0 = 1$ and $\beta = 10$. Shown are the mean thermal energy density $E_{\text{th}}$, the mean kinetic energy density $E_{\text{kin}}$, and the mean gravitational potential energy density $E_{\text{grav}}$, as defined in Eq. (16).

### Table 1. Specifications of the example simulation.

<table>
<thead>
<tr>
<th>Quantity-name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ (grid points)</td>
<td>768</td>
</tr>
<tr>
<td>$L_xH^{-1} = L_yH^{-1}$ (box length)</td>
<td>30</td>
</tr>
<tr>
<td>$Q_0$ (Toomre at initialisation)</td>
<td>1.0</td>
</tr>
<tr>
<td>$\beta = \tau_c\Omega_0$ (cooling time)</td>
<td>10</td>
</tr>
</tbody>
</table>

The corresponding simulation parameters are specified in Table 1. The corresponding surface density $\Sigma(x, y)$ at time $t = 200$ for a simulation with $Q_0 = 1$ and $\beta = 10$.

![Figure 4](image_url)

**Fig. 4.** Snapshot of the surface density $\Sigma(x, y)$ at time $t = 200$ for a simulation with $Q_0 = 1$ and $\beta = 10$.

The corresponding simulation parameters are specified in Table 1. The corresponding surface density $\Sigma(x, y)$ and radial velocity $v_r(x, y)$ at $t = 200$ for this simulation are shown in Figs. 4 and 5, respectively. As can be inferred from the images, the system is subject to shock formation. The shocks provide the mechanism for dissipating kinetic energy and to increase the thermal energy which is then lost through radiative cooling.

The $\alpha$ parameter is directly linked to the $xy$-component of the total stress ($S_{xy}$). Here, the definition of Ref. (Gammie 2001) is used

$$\alpha = \frac{2(S_{xy})}{3(\Sigma c_s^2)} = \frac{3}{2}\left(\frac{S_{xy}}{\Sigma c_s^2}\right)$$

where the average $\langle \cdot \rangle$ is over both the simulation domain and time. The total stress consists of the Reynolds and Gravitational stress $S_{xy} = S_{xy}^{(R)} + S_{xy}^{(G)}$, with

$$S_{xy}^{(R)} = \Sigma \partial_x \tilde{v}_y \quad \text{(Reynolds)},$$

$$S_{xy}^{(G)} = \frac{1}{4\pi} \int_{-\infty}^{\infty} (\partial_t \Phi \cdot \partial_y \Phi) \, dz \quad \text{(gravitation)}$$

(For a derivation of the gravitational stress, see e.g., Lynden-Bell & Kalnajs 1972). A stringent benchmark for the nonlinear
saturated state is then provided by Eq. (2), which links \( \alpha \) to the averaged internal energy \( \langle U \rangle \), which can be directly measured in the simulations. The two methods are compared in Fig. 6, which shows that they agree very well. As Eq. (2) is derived from energy conservation, the agreement between the two methods of determining \( \alpha \) shows that energy conservation is adequately satisfied numerically.

5. Nonlinear state

To better understand the nature of the gravito-turbulent state, in this section, the nonlinear saturated state is investigated in some detail. Important insights can be obtained through the study of the power spectra in Fourier space. The power spectrum of the perturbed mass surface density \( \hat{\Sigma}(k_x, k_y) \), where \( \hat{\Sigma}(k_x, k_y) \) is the Fourier amplitude, is shown in Fig. 7 as a function of the wave vector \( k = (k_x, k_y) \). This spectrum is obtained by averaging over a time interval of \( \Delta t = 400 \) of the fully developed turbulent state.

![Fig. 5. Snapshot of the radial velocity \( v_r(x, y) \) at time \( t = 200 \) for a simulation with \( Q_0 = 1 \) and \( \beta = 10 \).](image)

![Fig. 6. \( \alpha \)-values obtained via two different approaches. Values of \( \alpha \) directly obtained from the simulations via averaging of the stresses over both the box and time, according to Eq. (17) (\( \boldsymbol{\bigcirc} \)). The values of \( \alpha \) were obtained via Eq. (2) (\( \times \)). All simulations use \( \beta = 10 \) and the sizes of time averaging intervals are chosen in the range 100–400.](image)

![Fig. 7. Two-dimensional power spectrum of the surface density \( \mathcal{E}_k = |\hat{\Sigma}(k)|^2 \) for the case \( Q_0 = 2.2, \beta = 10, L_x = L_y = 20, N = 512 \). The image was obtained by averaging all time-snapshots over an interval of orbits \( \Delta t = 400 \). The parameters of the simulation are: \( Q_0 = 2.2 \) and \( \beta = 10 \), and a 512 \( \times \) 512 grid is used with box sizes \( L_x = L_y = 20 \). Although the surface mass density spectrum is shown here, the spectrum of the kinetic energy is qualitatively similar. As can be seen in Fig. 7, the spectrum is not isotropic in the \((k_x, k_y)\) plane. Contours of constant intensity are tilted ellipses, with the spectrum being more extended in the x- than in the y-direction. A similar result has been found Lesur & Longaretti (2011), Mamatsashvili et al. (2014), and Gogichaishvili et al. (2017) for the case of magnetohydrodynamic turbulence. The anisotropy can be explained through the equilibrium flow which sets a unique direction to the system. More specifically, due to the equilibrium shear flow, each structure with a finite \( k_y \) develops over smaller wavelengths in the x-direction or, in other words, the x-component of the wave vector \( (k_x) \) is time dependent,

\[
k_x = k_{x,0} + \frac{3}{2} k_y t, \tag{19}
\]

(see e.g., Lesur & Longaretti 2011; Goldreich & Lynden-Bell 1965). It has been shown for the case of incompressible magnetohydrodynamic turbulence that the shearing can lead to an anomalous energy transfer in Fourier space in the direction of larger radial wave vectors (Lesur & Longaretti 2011).

In order to study the dependency on \( k_x \) (\( k_y \)), the spectra are projected onto the \( k_x \) (\( k_y \)) axis by integrating over \( k_y \) (\( k_x \))

\[
\mathcal{P}_x \mathcal{E}_x(k_x, k_y) = \frac{1}{k_y} \int_{0}^{k_y} \mathcal{E}_x(k_x, k_y) dk_y.
\]

Depicted in Fig. 8 are the \( \mathcal{P}_x \)-projections of the power spectra for the kinetic energy density per mass \( \hat{u}(k) \) and the surface density \( \hat{\Sigma}(k) \), for both \( Q_0 = 1.2 \) and \( Q_0 = 2.4 \). As expected, the turbulent intensity drops with increasing \( Q_0 \). However, the qualitative shape of the spectrum remains unaltered. The spectra appear to obey a scaling law for the range of radial wave numbers \( 3 \leq k_x \leq 30 \). Depending on whether one studies the surface density or the kinetic energy density per mass, the scaling is between \( k^{-2} \) and \( k^{-3} \), as can be seen in Fig. 8. The \( \mathcal{P}_y \) projection is shown for comparison in Fig. 9 for an initial Toomre
The gravito-turbulent response, neither enstrophy nor potential vorticity are conserved. Furthermore, the turbulence is strongly anisotropic. The strong cascade in the decay of the energy in the $k_x$ modes is not entirely consistent with the steep decrease in $\mathcal{P}_x E_k$ with $k_y$, when only shearing over a fixed time interval is considered. Additional physics is therefore expected to further broaden the spectrum in the radial direction.

Further insights can be drawn by investigating the radial velocity profiles. Depicted in Fig. 10 is a slice $y = 0$ of the radial velocity $v_r(x, y = 0)$ from a simulation with $Q_0 = 2.8, \beta = 10$. One characteristic of the radial velocity snapshots is the appearance of shocks or velocity discontinuities.

The radial profile of the radial velocity is in agreement with the nonlinear wave steepening obtained in Burgers’ turbulence (see e.g., Bec & Khanin 2007):

$\partial_t v_r + v_r \partial_x v_r = 0. \tag{21}$

An initially unstable sine profile of the radial velocity, $v_r = v_0 \sin(k_r x)$, develops through the nonlinearity with the two adjacent maxima (positive and negative) approaching each other and eventually coalescing. In this process, the original sine profile will change its morphology to a structure similar to those seen in the small image in Fig. 11.
The shock spectrum has been scaled vertically for better compar-
ison with the spectra obtained from the simulations. The shock
spectrum of the shock with this assumption is shown in Fig. 11.

\begin{equation}
\langle \Sigma \rangle = -\langle \Sigma \rangle \cdot \frac{\partial (v_y)}{\partial x},
\end{equation}

where a Keplerian velocity profile in the local approximation
\( \partial (v_y)/\partial x = -3/2 \) is used. The kinematic viscosity of the non-
linear state can be estimated using a mixing length approach Shukura (2018). Consider the two dimensional \((x, y)\) shear flow
in the local shearing box approximation. A turbulent velocity \(\tilde{v}_s\)
will move a fluid parcel in the radial direction over a distance \(\delta_x\),
with the parcel keeping its original \(y\)-component of the velocity
\(v_y\). Hence, the \(y\)-velocity perturbation \(\delta v_y\) is (see Shukra 2018)

\begin{equation}
\delta v_y = \langle \delta v_x \rangle - \langle \delta v_y \rangle + \delta v_x \approx -\delta_x \frac{\partial (v_y)}{\partial x}.
\end{equation}

The averaging brackets \(\langle \rangle\) are assumed to be spatial and
temporal averages (more technically, one can also assume them
to be ensemble averages, see e.g., Shukra 2018). Using the
equation above in the stress yields

\begin{equation}
\langle \Sigma \rangle \cdot \delta v_x \approx -\langle \Sigma \rangle \cdot \frac{\partial (v_y)}{\partial x},
\end{equation}

and comparison with equation (25) then gives the mixing length estimate of the kinematic viscosity
\(v_t = \langle \delta_x \rangle = \langle \delta^2_x / \delta_t \rangle\),

where in the second step the radial velocity is expressed through a
typical timescale of \(v_t = \delta_x / \delta_t\),

In order to predict \(\alpha\), Eq. (2) is used to eliminate \(\langle U \rangle\) in
Eq. (17), yielding

\begin{equation}
\alpha = \frac{2}{3y(y-1)(S_{xy})} \left( 1 - \frac{2}{y(y-1)} \beta \right),
\end{equation}

Subsequently, expressing the stress in the kinematic viscosity
using Eq. (25), gives

\begin{equation}
\alpha = \frac{v_t}{1 + \frac{3y}{2} \gamma(y-1) \beta v_t} = \frac{v_t}{1 + \frac{3y}{2}},
\end{equation}

whereby \(\alpha_0 \equiv \beta \) is used for the last step. We would like to point out that \(v_t \neq \alpha\), as \(v\) is normalized with the background sound speed \(c_{s0}\) rather than saturated speed of sound \(c_{s}\).

Using the mixing length model for \(v_t\) in the equation above gives an expression for \(\alpha\).

As discussed in Sect. 5, saturation occurs through the nonlinear
steepening of radial waves. Since the maximum (minimum)
of the wave has to move over roughly a quarter wavelength to
generate the observed shock structures, the mixing length can be
taken to be a quarter wavelength of the dominant radial mode:

\begin{equation}
\delta_x = \frac{\lambda}{4} = \frac{\pi}{2} k^{-1}.
\end{equation}

The spectra of the nonlinear state decrease in amplitude with increasing \(Q_0\), but the functional dependence on \(k_x\) remains nearly unaltered. Therefore, the typical wave vector appearing in the mixing length estimate is relatively insensitive to the \(Q_0\) value, and a typical value can be obtained by averaging over the spectrum:

\begin{equation}
\langle k_x \rangle = \frac{\int \left| \mathcal{X}(k) \right|^2 k_x \, d^2 k}{\int \left| \mathcal{X}(k) \right|^2 d^2 k}.
\end{equation}
This procedure yields
\[ \tilde{k} \approx \frac{3}{\bar{k}} \]

as a typical value for \( \tilde{k} \). It then follows that \( \lambda = 2\pi/\tilde{k} \approx 4 \), which fits with the average distance between two shock fronts in Fig. 5, and agrees with the results of previous work (e.g., Kratter & Lodato 2016; Cossins et al. 2009) showing that dissipation occurs at wave numbers \( \tilde{k} \approx 1 \) through sonic shocks.

The typical timescale can be argued to be linked to the growth rate. Although, for many turbulent systems, linear theory is not particularly relevant, we show in the previous section that the nonlinear state of the gravito-turbulence is not strongly turbulent. The relevance of linear theory can be further justified by the following observations:

- There must be a mechanism providing an ongoing exchange of energy from the background (shear flow) to the actual turbulent motion of the fluid. A linear instability is a candidate for this mechanism. Indeed, the strength of turbulence drops with increasing values of \( Q_0 \) as can be seen in Fig. 8 in agreement with the dependence of the linear growth rate \( g \) on \( Q_0 \).
- The relevance of linear theory for the nonlinear state can be further assessed by investigating cases close to the threshold of the linear instability. With radiative cooling, the disk is not particularly relevant, we show in the previous section that the nonlinear state of the gravito-turbulence is not strongly turbulent. The relevance of linear theory for the nonlinear state can be further justified by the following observations:
  - The prediction is completed using Eq. (2), which provides a relation between \( \alpha \) and \( \langle Q \rangle \) depending on irradiation equilibrium \( Q_0 \).

The relevant timescale in the mixing length model is set by the linear growth rate. However, the latter must describe the growth in the saturated turbulent state. Consequently, the growth rate must be considered a function of \( \langle Q \rangle \) and \( \delta^{-1} = g\langle Q \rangle \beta \).

As \( \langle Q \rangle > 1 \), the approximate growth rate from Eq. (11) is used and substituted into Eq. (34):
\[ \alpha = \left( \frac{\pi}{3} \right)^2 \frac{g}{\bar{k}^2} \frac{1}{\beta \langle Q \rangle^2} - 1 \]
\[ \alpha = \frac{\nu_i}{1 + \nu_i/\alpha_0} \]
where \( \bar{k} = 3/2 \) is used. This latter equation relates \( \alpha \) with \( \langle Q \rangle \).

The prediction of \( \langle Q \rangle \) as derived above is shown as the black solid line in Fig. 12. One conclusion that can be drawn directly from Eq. (37) is that \( \langle Q \rangle > Q_0 \) always holds. Furthermore, as \( g \to 0 \) for \( \langle Q \rangle \to \infty \), we conclude that \( \langle Q \rangle \to Q_0 \) for \( Q_0 \to \infty \) by considering Eq. (36). The prediction for \( \alpha \) depending on \( Q_0 \) is shown in Fig. 13.

The predictions seem to fit the data well except for values of \( Q_0 > 3.2 \). This discrepancy is due to the above-mentioned numerical damping, which prevents a turbulent state from being sustained as the linear driving is compromised. More precisely, the simulations for \( Q_0 = 3.4 \) and 4.0 could not maintain a turbulent state. The corresponding values for \( \alpha \) and \( \langle Q \rangle \) are obtained by averaging the decaying turbulence, in the absence of a persistent nonlinear state.
We note that the analytic model predicts comparatively large \( \alpha \) values for high background Toomre parameters \( Q_0 \). To see that, one can approximate Eq. (30) assuming small growth rates,

\[
\alpha = \left( \frac{\pi}{3} \right)^{\frac{1}{2}} \frac{1}{\beta} \frac{Q_0^2}{1}.
\]

(38)

whereby \( \langle Q \rangle \approx Q_0 \) for large \( Q_0 \). Assuming \( \beta = 10 \), a relevant value \( \alpha = 10^{-3} \) is obtained for \( \langle Q \rangle \approx 10 \), that is, an order of magnitude above the Toomre stability criterion. Of course, shorter cooling times lead to larger values of \( \alpha \) at fixed \( \langle Q \rangle \). However, shorter cooling may also lead to clumping (Johnson & Gammie 2003; Rice et al. 2003, 2005, 2011; Kratter & Murray-Clay 2011). The effects of clumping are beyond the scope of this work but a study of this effect at the higher values of \( \langle Q \rangle \) obtained in this paper when compared with the literature is worthy of further study.

It is noted that the prediction of \( \alpha > 10^{-3} \) for \( \langle Q \rangle \approx 10 \) is connected with the stability analysis in Sect. 3. The latter might depend on the cooling model that is applied. Here, the fiducial cooling level is a constant background thermal energy density \( U_0 \). Alternatively, one can choose the reference cooling level to be density dependent, leading to a cooling term (Rice et al. 2011)

\[
-\frac{\Sigma c_s^2 - c_s^2_{hot}}{\gamma(y-1)\tau_c} = -\frac{U}{\tau_c} + \frac{\Sigma c_s^2}{\gamma(y-1)\tau_c}.
\]

(39)

Linearization leads to an additional term \( -\bar{\Sigma} \) (see e.g., Lin & Kratter 2016), when compared with the analysis of Sect. 3. This additional term has stabilizing effects as locally compressed areas are cooled less efficiently than locally expanding areas. Indeed, linear stability analysis reveals that the cooling model presented in Eq. (39) yields positive growth rates only for \( Q_0 < \sqrt{7} \) or in the case discussed here: \( Q_0 \lesssim 1.4 \). Nevertheless, the \( \beta \)-cooling description is a strong simplification and one could argue for both cooling models. To fully elucidate the problem, one would have to solve the full radiative transfer equation.

8. Conclusion

In this paper, the gravito-turbulent state of a razor-thin irradiated disk is studied in detail. We show that, depending on the cooling prescription, a linear instability occurs for all values of the equilibrium Toomre parameter \( Q_0 \), and we derive an accurate analytic estimate of the maximum growth rate for \( Q_0 > 1 \). A detailed study of the spectra reveals that the gravito-turbulent state is not strongly turbulent. The spectra are anisotropic with the anisotropy increasing with the wave vector. The spectra drop off rapidly with \( k_p \) (stronger than \( k_p^{-3} \)) and show no clear power-law scaling with \( k_p \). This suggests that no cascade in the \( y \)-direction takes place and saturation is connected with dynamics in the radial direction. In contrast, the spectrum as a function of the radial wave vector does show a power law with a scaling in the range \( k^{-2} \sim k^{-3} \), which can be explained through the existence of shocks. The radial velocity profile as a function of the radial coordinate is consistent with Burgers’ turbulence, consisting of several shocks of different height. The observations suggest that linearly unstable modes grow in amplitude until the nonlinearity is strong enough, leading to wave steepening and shock formation. The small radial scales connected with the shock then allow for efficient dissipation and, consequently, saturation of the mode.

Using these observations, a mixing length model is developed using a quarter wave length as the radial step length and the growth rate of the most unstable mode as the typical time. This model gives an analytic prediction of the viscosity parameter \( \alpha \) as a function of the Toomre parameter and cooling time, and it compares very well with the numerical simulations. The model predicts relevant values of \( \alpha = 10^{-3} \) for Toomre parameters an order of magnitude larger than the original Toomre limit (i.e., \( \langle Q \rangle \approx 10 \)). It is noted that this result can change when using a different cooling description and more accurate models with radiative transfer would be useful here.

References

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