Statistics of the polarized submillimetre emission maps from thermal dust in the turbulent, magnetized, diffuse ISM

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ABSTRACT

Context. The interstellar medium (ISM) is now widely acknowledged to display features ascribable to magnetized turbulence. With the public release of Planck data and the current balloon-borne and ground-based experiments, the growing amount of data tracing the polarized thermal emission from Galactic dust in the submillimetre provides choice diagnostics to constrain the properties of this magnetized turbulence.

Aims. We aim to constrain these properties in a statistical way, focussing in particular on the power spectral index $\beta_H$ of the turbulent component of the interstellar magnetic field in a diffuse molecular cloud, the Polaris Flare.

Methods. We present an analysis framework based on simulating polarized thermal dust emission maps using model dust density (proportional to gas density $n_H$) and magnetic field cubes, integrated along the line of sight (LOS), and comparing these statistically to actual data. The model fields are derived from fractional Brownian motion (fBm) processes, which allows a precise control of their one- and two-point statistics. The parameters controlling the model are $\chi^2$-value that is only slightly larger than unity.

Results. We find that the power spectrum of the turbulent component of the magnetic field in the Polaris Flare molecular cloud scales with wavenumber as $k^{-\delta_p}$ with a spectral index $\delta_p \approx 2.8 \pm 0.2$. It complements a uniform field whose norm in the POS is approximately twice the norm of the fluctuations of the turbulent component, and whose position angle with respect to the north-south direction is $\chi_0 \approx -69^\circ$. The density field $n_H$ is well represented by a log-normally distributed field with a mean gas density $\langle n_H \rangle = 40 \text{ cm}^{-3}$, a fluctuation ratio $\sigma_{n_H}/\langle n_H \rangle = 1.6$, and a power spectrum with an index $\beta_n = 1.7^{+0.4}_{-0.3}$. We also constrain the depth of the cloud to be $d = 13 \text{ pc}$, and the polarization fraction $p_0 \approx 0.12$. The agreement between the Planck data and the simulated maps for these best-fitting parameters is quantified by a $\chi^2$-value that is only slightly larger than unity.

Conclusions. We conclude that our fBm-based model is a reasonable description of the diffuse, turbulent, magnetized ISM in the Polaris Flare molecular cloud, and that our analysis framework is able to yield quantitative estimates of the statistical properties of the dust density and magnetic field in this cloud.

Key words. ISM: magnetic fields – ISM: structure – ISM: individual objects: Polaris Flare – polarization – turbulence

1. Introduction

In recent years, a number of experiments have dramatically increased the amount of data pertaining to the polarized thermal emission from Galactic dust in the submillimetre (e.g. Matthews et al. 2009; Ward-Thompson et al. 2009; Dotson et al. 2010; Biermann et al. 2011; Vaillancourt & Matthews 2012; Hull et al. 2014; Koch et al. 2014; Fissel et al. 2016). Chief among these is Planck¹, which provided the first full-sky map of this emission, leading to several breakthrough results. It was thus found that the polarization fraction $p$ in diffuse regions of the sky can reach values above 20% (Planck Collaboration Int. XIX 2015), confirming results previously obtained over one fifth of the sky by the Archeops balloon-borne experiment (Benoît et al. 2004; Ponthieu et al. 2005). Furthermore, the polarization fraction is anti-correlated with the local dispersion $\mathcal{S}$ of polarization angles $\phi$ (Planck Collaboration Int. XIX 2015; Planck Collaboration Int. XX 2015), and the decrease of the maximum observed $p$ with increasing gas column density $N_H$ may be fully accounted for, at the scales probed by Planck ($5'$ at 353 GHz), by the tangling of the magnetic field on the line of collaboration between ESA and a scientific consortium led and funded by Denmark, and additional contributions from NASA (USA).

¹ Planck (http://www.esa.int/Planck) is a project of the European Space Agency (ESA) with instruments provided by two scientific consortia funded by ESA member states and led by principal investigators from France and Italy, telescope reflectors provided through a
sight (LOS; Planck Collaboration Int. XX 2015). Similar anti-
correlations were found with the BLASTPol experiment (Fissel
et al. 2016) at higher angular resolution (a few tens of arc-
seconds) towards a single Galactic molecular cloud (Vela C).
At 10' scales, and over a larger sample of clouds, Planck data
showed that the relative orientation of the interstellar magnetic
field $B$ and filamentary structures of dust emission is consistent
with simulated observations derived from numerical simulations
of sub- or trans-Alfvénic magnetohydrodynamical (MHD) tur-
bulence (Planck Collaboration Int. XXXV 2016), and starlight
polarization data in extinction yield similar diagnostics (Soler
et al. 2016). In diffuse regions, the preferential alignment of
filamentary structures with the magnetic field (Planck Collabora-
tion Int. XXXII 2016; Planck Collaboration Int. XXXVIII 2016)
is linked to the measured asymmetry between the power spec-
tral amplitudes of the so-called $E$- and $B$-modes of polarized
emission. Finally, measurements of the spatial power spec-
trum of polarized dust emission showed that it must be taken
into account in order to obtain reliable estimates of the cos-
omological polarization signal (Planck Collaboration Int. XXX
2016).

With this wealth of data, we may be able to put constraints
on models of magnetized turbulence in the interstellar medium
(ISM), provided we can extract the relevant information from
polarization maps. Of particular interest are the statistical prop-
erties of the Galactic magnetic field (GMF) $B$. Let us write this as a
sum $B = B_\text{turb} + B_\text{f}$, where $B_\text{turb}$ and $B_\text{f}$ have a null spatial average, $(B_\text{turb}) = 0$. The statistical properties in question are then essentially mod-
elled by two quantities, i) the ratio of the turbulent component
to the mean, $y_B = \sigma_B / B_\text{f}$, and ii) the spectral index $\beta_B$, which characterizes the distribution
of power of $B_\text{turb}$ across spatial scales, through the relationship
$P(k) \propto k^{-\beta_B}$, where $k$ is the wavenumber and $P(k)$ is the power
spectrum.

As mentioned above, Planck Collaboration Int. XXXV (2016)
studied the relative orientation between the magnetic field,
probed by polarized thermal dust emission, and fil-
amentary matter in and around nearby molecular clouds. They
find that this relative orientation changes, from mostly paral-
lel to mostly perpendicular, as the total gas column density $N_H$
increases, which is a trend observed in simulations of trans-
Alfvénic or sub-Alfvénic MHD turbulence (Soler et al. 2013).
Using the Davis-Chandrasekhar-Fermi method (Chandrasekhar
& Fermi 1953) improved by Falca&-Gonçalves et al. (2008)
and Hildebrand et al. (2009), and from their results, we estimated
the ratio $y_B$ to be in the range 0.3–0.7. Planck Collaboration Int.
XXXII (2016) studied that same relative orientation in the dif-
fuse ISM at intermediate and high Galactic latitudes, and their
estimate of $y_B$ is in the range 0.6–1.0 with a preferred value
at 0.8. These estimates are confirmed in Planck Collaboration
Int. XLIV (2016), which presents a fit of the distributions of
polarization angles and fractions observed by Planck towards
the southern Galactic cap. They use a model of the GMF involv-
ing a uniform large-scale field $B_\text{f}$ and a small number ($N_f \approx 7$)
of independent “polarization layers” on the LOS, each of which
accounts for a fraction $1/N_f$ of the total unpolarized emission.
Within each layer, the turbulent component $B_\text{turb}$ of the magnetic
field, which is used to compute synthetic Stokes $Q$ and $U$, maps,
is an independent realization of a Gaussian two-dimensional ran-
don field with a prescribed spectral index $\beta_B$. Through these
fits, they confirm the near equipartition of large-scale and turbu-
lent components of $B$, with $y_B \approx 0.9$. They also provide a rough
estimate of the magnetic field’s spectral index $\beta_B$ in the range
two to three. This work was complemented in Vansyngel et al.
(2017), using the same framework, but including observational
constraints on the power spectra of polarized thermal dust emis-
sion. These authors are able to constrain $\beta_B \approx 2.5$, an exponent
which is compatible with the rough estimate of Planck Collabo-
ration Int. XLIV (2016), and close to that measured for the total
intensity of the dust emission. We note that their exploration
of the parameter space does not allow for an estimation of the
uncertainty on $\beta_B$.

In Planck Collaboration Int. XXXII (2016), Planck Collabo-
ration Int. XLIV (2016) and Vansyngel et al. (2017), the
description of structures, in both dust density and magnetic field,
along the LOS is reduced to the bare minimum, while statisti-
cal properties in the plane of the sky (POS) are modelled
through $y_B$ and $\beta_B$. Orthogonal approaches have also been purs-
eued (e.g. Miville-Deschênes et al. 2008; O’Dea et al. 2012), in
which the turbulent component of the magnetic field is modelled
along each LOS independently from the neighbouring ones, as
a realization of a one-dimensional Gaussian random field with a
power-law power spectrum. In this type of approach there is no
correlation from pixel to pixel on the sky, and such studies seek
to exploit the depolarization along the LOS, rather than spatial
correlations in the POS, to constrain statistical properties of the
interstellar magnetic field.

We sought to explore another avenue, taking into account sta-
tistical correlation properties of $B$ in all three dimensions, as well
as properties of the dust density field, building on methods de-
veloped in Planck Collaboration Int. XX (2015) to compare Planck
data with synthetic polarization maps. In that paper, the syn-
thetic maps are computed from data cubes of dust density $n_d$
and magnetic field $B$ produced by numerical simulations of MHD
turbulence. One could imagine generalizing this approach, tak-
ing advantage of the ever-increasing set of such simulations (see,
e.g. Hennebelle et al. 2008; Hennebelle 2013; Hennebelle &
Iffrig 2014; Inutsuka et al. 2015; Seifried & Walch 2015). How-
ever, this would be impractical for two main reasons: i) these
simulations often have a limited inertial range over which the
power spectrum has a power-law behaviour, and ii) a systematic
study exploring a wide range of physical parameters with suffi-
cient sampling is not possible due to the computational cost of
each simulation.

We therefore propose an alternative approach, which is to
build simple, approximate, three-dimensional models for the
dust density $n_d$ and the magnetic field $B$, allowing us to per-
fectly control the statistical properties of these three-dimensional
fields, and to fully explore the space of parameters characteriz-
ing them. With this approach, we are able to perform a statistically
significant number of simulated polarization maps for each set of
parameters. Actual observations may then be compared to these
simulated maps, using least-square analysis methods, to
extract best-fitting parameters, in particular the spectral index
of the magnetic field, $\beta_B$, and the ratio of turbulent to regular
field, $y_B$.

The paper is organized as follows: Sect. 2 presents the
method used to build simulated thermal dust polarized emission
maps using prescribed statistical properties for $n_d$ and $B$. Obs-
erves derived from these maps, serving as statistical diagnostics
of the input parameters, are presented in Sect. 3. In Sect. 4,
we describe the Markov chain Monte Carlo (MCMC) analysis

2 In all generality, several spectral indices may be defined, as one may
consider the power spectrum of any one of the three cartesian compo-
nents of $B_\text{f}$, or of the modulus $|B_\text{f}|$. Assuming that $B_\text{f}$ is isotropic, which
we will, all of these spectral indices are identical.
method devised to explore the space of input parameters for a given set of polarization maps. The validation of the method and its application to actual observations of polarized dust emission from the Polaris Flare molecular cloud observed by Planck are given in Sect. 5. Finally, Sect. 6 discusses our results and offers conclusions. Several appendices complement our work. Appendix A presents further statistical properties of the model dust density fields. Appendix B details the likelihood used in the MCMC analysis. Finally, Appendix C details the χ² parameter used to estimate the goodness of fit.

2. Building synthetic polarized emission maps

In this section, we first describe the synthetic dust density and magnetic field cubes we have used in our analysis. We then explain how simulated polarized emission maps are built from these cubes.

2.1. Fractional Brownian motions

The basic ingredients to synthesize polarized thermal dust emission maps are three-dimensional cubes of dust density n_d and magnetic field B, which we build using fractional Brownian motions (fBm) (Falconer 1990). An N-dimensional fBm X is a random field defined on \( \mathbb{R}^N \) such that \( \langle |X(r_2) - X(r_1)|^2 \rangle \propto |r_2 - r_1|^{2H} \), for any pair of points \( (r_1, r_2) \). H is called the Hurst exponent. These fBm fields are usually built in Fourier space \(^3\),

\[
\tilde{X}(k) = A(k) \exp[i\phi_X(k)],
\]

by specifying amplitudes that scale as a power-law of the wavenumber \( k = ||k|| \),

\[
A(k) = A_0 k^{-\beta_X/2},
\]

with \( \beta_X = 2H + N \) the spectral index, and phases drawn from a uniform random distribution in \( [-\pi, \pi] \), subject to the constraint \( \phi_X(-k) = -\phi_X(k) \) so that \( X \) is real-valued. Their power spectra are therefore power laws,

\[
P_X(k) = \left| \tilde{X}(k) \right|^2 \propto k^{-\beta_X},
\]

where the average is taken over the locus of constant wavenumber \( k \) in Fourier space. Such fields have been used previously as toy models for the fractal structure of molecular clouds, in both density and velocity space (Stutzki et al. 1998; Brunt & Heyer 2002; Miville-Deschênes et al. 2003; Correia et al. 2016).

2.2. Dust density

In our approach, the dust density \( n_d \) is taken to be proportional to the total gas density \( n_H \), so that the dust optical depth within each cell is also proportional to \( n_H \) (see the derivation of polarization maps in Sect. 2.4). Therefore, we intended to model \( n_H \) from numerical realizations of three-dimensional fBm fields built in Fourier space. These have means that are defined by the value chosen for the null-wavevector amplitude \( A(0) \), so if one wished to use such a synthetic random field \( X \) directly as a model for the positive-valued \( n_H \), one would be required to choose \( n_H = X' = X - a \) with \( a \geq \min(X) \) a constant. However, since the distributions of these fields in three dimensions are close to Gaussian, their ratio of standard deviation to mean is typically \( \sigma_X / \langle X' \rangle \lesssim 0.3 \), which is much too small compared to observational values. For instance, the total gas column density fluctuation ratios \( \sigma_{n_H} / \langle n_H \rangle \) in the ten nearby molecular clouds selected for study in Planck Collaboration Int. XX (2015) are in the range 0.3–1, and one should keep in mind that these are only lower bounds for fluctuation ratios in the three-dimensional density field \( n_H \).

We remedied this shortcoming by taking \( X \) to represent the log-density, i.e., \( n_H \) is given by

\[
n_H = n_0 \exp \left( \frac{X}{X'} \right),
\]

where \( X \) is a three-dimensional fBm field with spectral index \( \beta_X \), and \( X' = \langle n_H \rangle / n_0 \) are positive parameters. The \( n_H \) fields built in this fashion have simple and well-controlled statistical properties. First, their probability distribution functions (PDF) are log-normal, which allows, through an adequate choice of \( X' \), to set the fluctuation level of the density field \( y_n = \sigma_{n_H} / \langle n_H \rangle \) to any desired value. Second, their power spectra, as azimuthal averages in Fourier space, retain the power-law behaviour of the original fBm \( X \),

\[
P_{n_H}(k) = \left| \tilde{n_H}(k) \right|^2 \propto k^{-\beta_H},
\]

although the spectral indices \( \beta_H \) may deviate significantly from \( \beta_X \). An example of such a field is shown in Fig. 1, which represents the total gas column density \( n_H \) derived from a gas volume density \( n_{H1} \) built as the exponential of a \( 120 \times 120 \times 120 \) fractional Brownian motion with zero mean, unit variance, and spectral index \( \beta_X = 2.6 \). The parameters of the exponentiation are \( X' = 1.2 \) and \( n_0 = 20 \, \text{cm}^{-3} \), and the grid is chosen so that the extent of the cube is 30 pc on each side, corresponding to a pixel size of 0.25 pc. More detail on the properties of these fields are given in Appendix A.

Fig. 1. Total gas column density \( n_H \), derived from a synthetic density field \( n_{H1} \) built by exponentiation of a fBm field with spectral index \( \beta_X = 2.6 \) and size \( 120 \times 120 \times 120 \) pixels. The volume density fluctuation level is \( y_n = 1 \), and the column density fluctuation level is \( y_{n_H} = 0.25 \).

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\(^3\) Throughout this paper, for any field \( F \) the notation \( \tilde{F} \) represents its Fourier transform.
The density fields built in this fashion are of course only a rough statistical approximation for actual interstellar density fields. For instance, they are unable to reproduce the filamentary structures observed in dust emission maps (André et al. 2010; Miville-Deschênes et al. 2010). These structures cannot be captured by one- and two-point statistics such as those used here, and require a description involving higher-order moments or the Fourier phases (see, e.g., Levrier et al. 2006; Burkhart & Lazarian 2016).

2.3. Magnetic field

To obtain a synthetic turbulent component of the magnetic field \( B_t \), with null divergence and controlled power spectrum, we started from a vector potential \( A \) built as a three-dimensional fractional Brownian motion. To be more precise, each Cartesian component \( A_j \) of \( A \) is a fBm field,

\[
\tilde{A}_j(k) = \mathcal{A}_0 k^{-\beta_A/2} \exp \left[ i \phi_{A_j}(k) \right], \tag{6}
\]

where the spectral index \( \beta_A \) and the overall normalization parameter \( \mathcal{A}_0 \) are independent of the Cartesian component \( j = x, y, z \) considered. Using the definition of the magnetic field from the vector potential \( B_{t,\lambda} = \epsilon_{ijk} \partial_k A_j \), with the Einstein notation, where \( \epsilon_{ijk} \) is the Levi-Civita tensor, and the derivation relation in Fourier space

\[
\partial_\lambda F = ik_\lambda \tilde{F}, \tag{7}
\]

we have the expression of the components of \( B_t \) in Fourier space

\[
\tilde{B}_{t,\lambda}(k) = \mathcal{A}_0 \epsilon_{ijk} k_\lambda k^{-\beta_A/2} \exp \left[ i \phi_{A_j}(k) \right]. \tag{8}
\]

As it should, this expression corresponds to a divergence-free turbulent magnetic field,

\[
\tilde{k}_\lambda \tilde{B}_{t,\lambda} = 0. \tag{9}
\]

Writing \( k_j = k f_j \), with \( f = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \), the power spectrum of each component of \( B_t \) is then

\[
P_{B_{t,\lambda}}(k) = \mathcal{A}_0^2 k^{2-\beta_A} \left( |\epsilon_{ijk} f_\mu k_\nu e^{i \phi_{A_j}(k)/2}|^2 \right)_{|k|=k}. \tag{10}
\]

The last factor is essentially independent of the wavenumber \( k \), so the spectral index of each component of \( B_t \) is \( \beta_{B_t} = \beta_A - 2 \). After Fourier-transforming back to real space, \( B_t \) is shifted and scaled so that it has zero mean and a standard deviation \( \sigma_B \) of 5 \( \mu \)G, a value typical of the interstellar magnetic field (see, e.g., Haverkorn et al. 2008, and references therein).

The model magnetic field \( B \) is obtained by adding a uniform\(^4 \) vector field \( B_0 \) to that turbulent magnetic field \( B_t \). The effect in Fourier space is limited to a modification for \( k = 0 \) only, so the spectral index of each component \( B_{\lambda} \) of the total magnetic field is the same as that of \( B_{t,\lambda} \), i.e.,

\[
P_{B_{\lambda}}(k) \propto k^{2-\beta_A} \tag{11}
\]

This means that the resulting magnetic fields thus only display anisotropy in the \( k = 0 \) mode, and not at the other scales. This is a limitation of our model, which is thus not fully consistent with observations of the magnetic field structure (Planck Collaboration Int. XXXV 2016), but it is sufficient for our purposes.

The uniform field \( B_0 \) which is added to the turbulent field \( B_t \) is defined by its norm \( B_0 \) and a pair of angles, \( \gamma_0 \) and \( \chi_0 \), which are respectively the angle between the magnetic field and the POS, and the position angle of the projection of \( B_0 \) in the POS, counted positively clockwise from the north-south direction (see Fig. 14 of Planck Collaboration Int. XX 2015). The total magnetic field’s direction in three-dimensional space is characterized by angles \( \gamma \) and \( \chi \) defined in the same way. The ratio of the turbulent to mean magnetic field strengths is then defined by

\[
y_B = \frac{\sigma_B \| \tilde{B}_0 \|}{\|B_0\|} = \frac{\sqrt{\langle \tilde{B}_t^2 \rangle} - \langle \tilde{B}_t \rangle^2}{\langle \tilde{B}_0 \rangle^2} \tag{12}
\]

Figure 2 shows an example of a synthetic magnetic field \( B \) generated in this fashion, and defined on the same 120 \( \times \) 120 \( \times \) 120 pixels in the POS, and the position angle of the projection of \( B_0 \) in the POS, counted positively clockwise from the north-south direction (see Fig. 14 of Planck Collaboration Int. XX 2015). The total magnetic field’s direction in three-dimensional space is characterized by angles \( \gamma \) and \( \chi \) defined in the same way. The ratio of the turbulent to mean magnetic field strengths is then defined by

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\]

2.4. Polarization maps

Once cubes of total gas density \( n_\text{H} \) and magnetic field \( B \) are available, maps of Stokes parameters \( I, Q, \) and \( U \) at 353 GHz (the frequency of the Planck channel with the best signal-to-noise ratio in polarized thermal dust emission) are built by integrating along the LOS through the simulation cubes, following the method in Planck Collaboration Int. XX (2015):

\[
I_0 = \int \langle S_\nu \rangle \left( 1 - p_0 \cos^2 \gamma - \frac{2}{3} \right) \sigma_{353} n_\text{H} \, dz; \tag{13}
\]

\[
Q_0 = \int p_0 \langle S_\nu \rangle \cos(2\phi) \cos^2 \gamma \sigma_{353} n_\text{H} \, dz; \tag{14}
\]

\[
U_0 = \int p_0 \langle S_\nu \rangle \sin(2\phi) \cos^2 \gamma \sigma_{353} n_\text{H} \, dz. \tag{15}
\]

In these equations, we take the intrinsic polarization fraction parameter \( p_0 \) to be uniform, and the source function \( S_\nu = B_0(T_\nu) \) to be that of a black body with an assumed uniform dust temperature \( T_\nu \). The dust opacity at this frequency \( \sigma_{353} \) is taken to vary with \( n_\text{H} \), following Fig. 20 in Planck Collaboration XI (2014) for \( X_{\text{CO}} = 2 \times 10^{-4} \) cm\(^{-3} \) K\(^{-1} \) km\(^{-1} \) s, and propagating the associated errors. The order of magnitude of the dust opacity is around \( \sigma_{353} \approx 10^{-20} \) cm\(^2 \). The values of \( n_\text{H} \) considered in our study are typically at most a few 10\(^2\) cm\(^{-3} \), so the optically thin limit applies in the integrals of Eqs. (13)–(15). The angle \( \phi \) is the local polarization angle, which is related to the position angle\(^5 \) \( \chi \) of the magnetic field’s projection on the POS at each position on the LOS by a rotation of 90° (see definitions of angles in Planck Collaboration Int. XX 2015).

The \( n_\text{H} \) and \( B \) cubes are built on a grid which is 132 \( \times \) 132 pixels in the POS and \( n_\text{H} \) pixels in the \( z \) direction (that of the

\(^4\) We do not consider an ordered random or striated random component of the field (Jaffe et al. 2010; Jansson & Farrar 2012), which we justify with the smallness of the field-of-view considered.

\(^5\) Not to be confused with the corresponding position angle \( \chi_0 \) of the uniform component of the magnetic field \( B_0 \).
Fig. 2. Synthetic magnetic field $B$ built using Eq. (8). The spectral index of the vector potential is $\beta_A = 5$ and the size of the cubes is $120 \times 120 \times 120$ pixels, corresponding to 30 pc on each side. Shown are two-dimensional slices through the cubes of the components $B_x$ (top), $B_y$ (middle), and $B_z$ (bottom). The ratio of the fluctuations of the turbulent component $B_t$ to the norm of the uniform component $B_0$ is $y_B = 1$ in this particular case, with angles $\chi_0 = \gamma_0 = 0^\circ$.

Fig. 3. Distribution functions of the components $B_x$, $B_y$, and $B_z$ of a model magnetic field $B = B_0 + B_t$, built on a grid $120 \times 120 \times 120$ pixels using Eq. (8) with $\beta_A = 5$, and a mean, large-scale magnetic field $B_0$ defined by the angles $\chi_0 = 0^\circ$ and $\gamma_0 = 60^\circ$, and a norm $B_0 = 50 \mu G$ such that the fluctuation level is $y_B = 0.1$. The vertical lines represent the projected values of the large scale magnetic field $B_{0x} = B_0 \sin \chi_0 \cos \gamma_0$, $B_{0y} = -B_0 \cos \chi_0 \cos \gamma_0$ and $B_{0z} = B_0 \sin \gamma_0$. See Fig. 14 in Planck Collaboration Int. XX (2015) for the definition of angles.

Fig. 4. Power spectra of the components $B_x$, $B_y$, and $B_z$ of a model magnetic field $B = B_0 + B_t$, built on a grid $120 \times 120 \times 120$ pixels using Eq. (8) with $\beta_A = 3$ (different shades of blue for the three components) and $\beta_A = 5$ (different shades of red for the three components). The power spectra are normalized differently so as to allow comparison between them. The fitted power-laws shown as solid lines yield spectral indices $\beta_B = 1$ and $\beta_B = 3$, in agreement with Eq. (11). These are the power spectra of the same particular realizations shown in Fig. 2.

2.5. Noise and beam convolution

In order to proceed with the analysis of observational data, one cannot use these model Stokes maps directly: it is necessary to
properly take into account noise and beam convolution. Antici-
pating somewhat the description of the Planck data we shall use as
an application of the method, the 353 GHz noise covariance
matrix maps are taken directly from the Planck Legacy Archive\(^6\)
and are part of the 2015 public release of Planck data (Planck
Collaboration I 2016).

\[
\Sigma = \begin{pmatrix}
\sigma_{II} & \sigma_{IQ} & \sigma_{Iu} \\
\sigma_{QI} & \sigma_{QQ} & \sigma_{Qu} \\
\sigma_{UJ} & \sigma_{UQ} & \sigma_{UU}
\end{pmatrix}.
\]

Noise is added to the model Stokes maps pixel by pixel, as
\[
I_n = I_0 + n_I \\
Q_n = Q_0 + n_Q \\
U_n = U_0 + n_U,
\]
where \(n_I, n_Q, \) and \(n_U\) are random values drawn from a three-
dimensional Gaussian distribution with zero mean and character-
ized by the noise covariance matrix \(\Sigma\). To preserve the properties of Planck noise, the random values are directly drawn from the
Healpix (Górski et al. 2005) covariance matrix maps and added to the simulated maps after a gnomonic projection of the region
under study, in our case the Polaris Flare molecular cloud.

The resulting Stokes \(I_m, Q_m, \) and \(U_m\) maps are then placed at a
distance\(^7\) \(D = 140\) pc, so that the angular size of each pixel is about 6\(^\prime\), and then convolved by a circular 15\(^\prime\) full-width
at half maximum (FWHM) Gaussian beam \(B\). To avoid edge
effects, only the central 120 \times 120 pixels of the convolved maps
\(I_m = B \otimes I_m, Q_m = B \otimes Q_m, \) and \(U_m = B \otimes U_m\) are
retained, corresponding to a field of view (FoV) of approximately 12\(^\circ\). With this approach, the model maps \((I_m, Q_m, U_m)\) are fit to be compared to actual Planck data, which we discuss in Sect. 5.2.

3. Observables

From the model maps above, we build an ensemble of derived
maps, starting with the normalized Stokes maps
\[
i = \frac{I_m}{\langle I_m \rangle} \\
q = \frac{Q_m}{\langle Q_m \rangle} \\
u = \frac{U_m}{\langle U_m \rangle},
\]
where \(\langle I_m \rangle\) is the spatial average of the model Stokes \(I_m\) map.
Then we define the polarization fraction, which requires us to
note that since our models include noise, we should not use the
“naïve” estimator (Montier et al. 2015a,b)
\[
p = \sqrt{\frac{\langle Q_m^2 \rangle + \langle U_m^2 \rangle}{\langle I_m^2 \rangle}},
\]
but rather the modified asymptotic (MAS) estimator proposed by
Plaszczynski et al. (2014)
\[
\rho_{\text{MAS}} = p - b^2 \frac{1 - e^{-p^2/2p}}{2p},
\]
where the noise bias parameter \(b^2\) derives from the elements of
the noise covariance matrix \(\Sigma\) (see Montier et al. 2015b). Next, we
define the polarization angle
\[
\psi = \frac{1}{2} \arctan(U_m, Q_m),
\]
with the two-argument atan function lifts the \(\pi\)-degeneracy
of the usual atan function. We note that this expression
means that the polarization angle is defined in the Healpix
convention.

We also build maps of the polarization angle dispersion
function \(S\) (Planck Collaboration Int. XIX 2015; Planck
Collaboration Int. XX 2015; Alina et al. 2016), which quantifies
the local dispersion of polarization angles at a given lag \(\delta\) and is
defined by
\[
S(r, \delta) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} [\psi(r + \delta) - \psi(r)]^2}
\]
where the sum is performed over the \(N\) pixels \(r + \delta\) whose
distance to the central pixel \(r\) lies between \(\delta/2\) and \(3\delta/2\). For
the sake of consistency with the analysis performed on simulated
polarization maps in Planck Collaboration Int. XX (2015), we take \(\delta = 16\)\(^\prime\).

Finally we build the column density and optical depth \(\tau_{353}\)
data from the dust density cube using
\[
N_H = \int n_H \, dz \quad \text{and} \quad \tau_{353} = \sigma_{353} (N_H) \times N_H
\]
with the \(\sigma_{353}(N_H)\) conversion\(^8\) from Planck Collaboration XI
(2014). A temperature map \(T_{\text{obs}}\) is also created using the an-
correlation with the column density \(N_H\) observed in the data.
This “dust temperature map” does not pretend to model reality
but since \(T_d\) is one of the parameters of the model, the
fitting algorithm requires a map whose mean value should
yield \(T_d\).

4. Exploring the parameter space

The goal of this paper is to constrain the physical parameters of
molecular clouds. In particular we aim to constrain the spectral
indices of the dust density and of the turbulent magnetic field,
using Planck polarization maps and a grid of model maps built
as explained in the previous section.

4.1. Parameter space

The nine physical parameters that are explored in this paper
using fBm simulations are summarized in Table 1. They are
sufficient to describe the one-point and two-point statistical
properties of the dust density and magnetic field models. We
note that contrary to the method in Planck Collaboration Int.
XLIV (2016), the FoV of the maps analysed in the following
(approximately 12\(^\circ\)) is too small to contain remarkable features
that could be used to constrain the angle \(\gamma_0\) that the mean
magnetic field makes with the POS. In such small FoVs, there is a
degeneracy between \(\gamma_B\) and \(\gamma_0\) which cannot be lifted. Conse-
quentially, we chose not to try to fit for \(\gamma_0\) and \(\gamma_B\), but for the ratio of
the turbulent magnetic field RMS to the mean magnetic field
in the POS, i.e.,
\[
\frac{y_B}{y_0} = \frac{\sigma_B}{\sigma_0} = \frac{\sqrt{\langle B'^2 \rangle}}{\langle B \rangle \cos \gamma_0} = \frac{y_B}{\cos \gamma_0}.
\]

\(^6\) http://pla.esac.esa.int/pla/
\(^7\) A more recent determination of the distance to the Polaris Flare
places it at 350–400 pc (Schlafly et al. 2014). For the demonstration
of the method presented here, this is not a critical issue, as the power-law
power spectra underline self-similar behaviours, so that a change of the
distance can be compensated by a change in pixel size.

\(^8\) The conversion factor is given for a map of \(N_H\) at a resolution of 30\(^\prime\).
Thus, before applying it pixel by pixel, we smooth the simulated \(N_H\) map to 30\(^\prime\) resolution, apply the conversion, then resample the resulting
\(\tau_{353}\) map at the original resolution.
This analysis was applied to the Polaris Flare (see Sect. 5.2), so the priors were chosen to be flat over a reasonably large range, to cover the expected physical values of the molecular cloud under consideration, but they are necessary for the analysis to converge. The cloud’s average column density is of the order \( N(H) \approx 10^{23} \, \text{cm}^{-2} \) (Planck Collaboration Int. XX 2015). This value was used to set the range for the prior on the depth \( d \) of the cube, with the limits of this range chosen in such a way that the average gas density \( n_{\text{H}} \) lies between 10 and 500 \( \text{cm}^{-3} \), a reasonable assumption for the Polaris Flare molecular cloud. This translates to a total cube depth \( d \) of between 0.5 and 32.5 pc.

The range used for the prior on \( \beta_B \) is justified by a number of observational studies (see, e.g., the review by Hennebelle & Falgarone 2012), and that on \( \beta_B \) is chosen based on the results from Vansyngel et al. (2017), but also on numerical studies of MHD turbulence (see, e.g., Perez et al. 2012; Beresnyak 2014). The fluctuation ratios \( y_n \) and \( y_{n,\text{POS}} \) are explored in a logarithmic scale, as we are mainly interested in order of magnitude estimates for these parameters. The polarization maps being statistically identical when the angle \( \chi_0 \) of the POS projection of the mean magnetic field is shifted by 180°, the prior on this parameter is such that this periodicity is applied when the Metropolis algorithm (see Sect. 4.3) draws values outside the given range. The priors chosen for \( T_d \) and \( p_0 \) are very large and do not play a role in the fitting procedure.

### 4.2. Comparing models with data

To set constraints on the parameters listed in Table 1, we built a likelihood function, which expresses the probability that a given set of synthetic polarization maps reproduce adequately actual observational data. From the model Stokes maps \( I_m, Q_m, \) and \( U_m \), we derived a set of observables that are used in the likelihood function. These observables are given in Table 2. More precisely, we used i) the mean values for the optical depth \( \tau_{353} \) and the dust temperature \( T_{\text{obs}} \), ii) the distribution functions \( F_{\text{DF}} \), iii) the power spectra \( F_{\text{PS}} \), and iv) the pixel-by-pixel anti-correlation between \( S \) and \( p_{\text{MAS}} \) (see Planck Collaboration Int. XIX 2015). Indeed, we find that the shape of this two-dimensional distribution function also depends on the model parameters. Many other observables were tested but we have retained only those which bring constraints on the model parameters.

### Table 1. Parameter space explored in the grid of model polarization maps.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior (^a)</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_B )</td>
<td>[1.4]</td>
<td>Spectral index of the three-dimensional turbulent magnetic field</td>
</tr>
<tr>
<td>( \beta_n )</td>
<td>[1.5]</td>
<td>Spectral index of the three-dimensional dust density field</td>
</tr>
<tr>
<td>( \log_{10} y_n )</td>
<td>[-1, 1]</td>
<td>Log of the RMS-to-mean ratio of dust density</td>
</tr>
<tr>
<td>( \log_{10} y_{n,\text{POS}} )</td>
<td>[-1, 1]</td>
<td>Log of the ratio of the turbulent magnetic field RMS to the mean magnetic field in the POS</td>
</tr>
<tr>
<td>( \chi_0 )</td>
<td>[-90°, 90°]</td>
<td>Position angle of the mean magnetic field in the POS</td>
</tr>
<tr>
<td>( \log_{10} (d/1 \text{ pc}) )</td>
<td>[-0.3, 1.5]</td>
<td>Depth of the simulated cube</td>
</tr>
<tr>
<td>( \log_{10} (m_{n,1}/1\text{ cm}^{-3}) )</td>
<td>[1, 2.7]</td>
<td>Dust temperature</td>
</tr>
<tr>
<td>( T_d )</td>
<td>[5 K, 200 K]</td>
<td>Dusty temperature</td>
</tr>
<tr>
<td>( p_0 )</td>
<td>[0.01, 0.5]</td>
<td>Intrinsic polarisation fraction parameter</td>
</tr>
</tbody>
</table>

\(^a\)Priors are assumed to be flat in the given range for the parametrization given in this table, and zero outside, except \( \chi_0 \), for which a 180° periodicity is applied when the Metropolis algorithm draws values outside the given range. \(^b\)Corresponding to a cube depth in the interval \([0.5 \text{ pc}, 32.5 \text{ pc}]\). \(^c\)Corresponding to a density in the interval \([10 \text{ cm}^{-3}, 500 \text{ cm}^{-3}]\).

### Table 2. Observables from polarization maps used to fit data.

<table>
<thead>
<tr>
<th>Type</th>
<th>From</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean values</td>
<td>( \tau_{353}, T_{\text{obs}} )</td>
</tr>
<tr>
<td>Distribution function</td>
<td>( I_m, Q_m, U_m, p_{\text{MAS}}, \psi, S )</td>
</tr>
<tr>
<td>Power spectrum</td>
<td>( I_m, Q_m, U_m, \tau_{353} )</td>
</tr>
<tr>
<td>Correlation</td>
<td>( {S, p_{\text{MAS}}} )</td>
</tr>
</tbody>
</table>

On the simulation side, \( N_r = 60 \) model realizations per set of parameter values are generated with their observables, to be compared with data. The \( N_r \) models differ by the random phases \( \phi_X \) and \( \phi_A \) used to build the dust density and magnetic field cubes (see Eqs. (1) and (8)), and by the random realization of the noise applied to the model (Eq. (17)). We checked that 60 simulations represent a statistically large enough sample to get robust averages and dispersions for the observables. The statistical properties of the observables derived from the observational polarization data are thus compared with the observables from those 60 models, through the evaluation of a parameter \( D^2 \) which quantifies the distance between data and one random realisation of the model, averaged over the \( N_r \) random realisations, with contributions associated to the various observables listed in Table 2, i.e.,

\[
D^2 = \frac{1}{N_r} \sum_{i=1}^{N_r} \left[ D^2_{\text{DF}} + \sum_o D^2_{\text{DF}(o)} + \sum_o D^2_{\text{PS}(o)} + D^2_{S-p_{\text{MAS}}} \right].
\]

This quantity is slightly different from the usual \( \chi^2 \) (see Appendices B and C). The first term in Eq. (25) covers the observables \( \langle \tau_{353} \rangle \) and \( \langle T_{\text{obs}} \rangle \), and quantifies the difference between these values in the simulated maps and in the data. The second sum extends over the observable maps \( o \) in the set \([I_m, Q_m, U_m, p_{\text{MAS}}, \psi, S, \tau_{353} \) \( / \langle \tau_{353} \rangle \) \] and quantifies the difference between the distribution functions of these observables in synthetic maps and those of the same observables in the data. The third sum extends over the observable maps \( o \) in the set \([I_m, Q_m, U_m] \) and quantifies the difference between the power spectra of the simulated maps and those of the same maps in the data. Finally, the last term quantifies the discrepancy between the two-dimensional joint DFs of \( S \) and \( p_{\text{MAS}} \) in the data and in synthetic maps. We detail the computation of these various terms in Appendix B.
4.3. MCMC analysis

Given the vast parameter space to explore, we built a MCMC analysis (see, e.g. Brooks et al. 2011) that has the advantage to sample specifically the regions of interest in this space. We used a simple Metropolis-Hastings algorithm to build five Markov chains which sample the posterior probability distribution function (PDF) of the parameters listed in Table 1. The likelihood $L$ of a set $s$ of parameters is evaluated thanks to the $D^2$ criterion described in Sect. 4.2 and Appendix B as

$$L(s) \propto e^{-D^2(s)/2} \pi(s),$$

with $\pi(s)$ the prior associated to the parameters.
According to the Metropolis-Hastings algorithm, at each step \( q \) of the chain, parameters are drawn according to a multivariate probability distribution function whose covariance is set to allow for an efficient exploration of the parameter space, with an average given by the parameter values \( s_{q-1} \) at step \( q-1 \). If the likelihood for the new set of parameters, \( s_q \), is larger than for the previous one, then the chain records the new set. Otherwise, the likelihood ratio \( L(s_q)/L(s_{q-1}) < 1 \) is compared to a number \( \alpha \) drawn randomly from a uniform distribution over \([0,1]\). If the likelihood ratio is larger than \( \alpha \), the \( s_q \) set of parameters is kept, otherwise the chain duplicates the \( s_{q-1} \) set, i.e., \( s_q = s_{q-1} \). The posterior probability distribution function is then given by the occurrence frequency of the parameters along the chains, after removal of the initial “burn-in” phase.

The priors used for each parameter are detailed in Table 1. For all parameters, flat priors are set covering a reasonable range of physical interest. If the Metropolis algorithm draws values outside of these priors we set \( \pi(s) = 0 \), except for the position angle of the mean magnetic field, \( \chi_0 \), for which the 180° periodicity is used to bring back the angle inside its definition range when it is by chance drawn outside.

The convergence of the Markov chains is tested using the Gelman-Rubin statistics \( R \) (Gelman & Rubin 1992), which is essentially the ratio of the variance of the chain means to the mean of the chain variances. We estimated that the chains converged when \( R - 1 < 0.03 \) for the least-converged parameter. The convergence is also assessed by visually checking the \( D^2 \) and parameter evolutions along the chains.

The obtained nine-dimensional posterior probability distribution is generally not a multivariate Gaussian distribution. To quote an estimate of the best fit value for any one of the nine parameters and the associated uncertainties, we marginalize over the other eight parameters to obtain the one-dimensional posterior PDF for the remaining parameter. In the following, the quoted best fit value for a parameter is the mean over this posterior PDF (which is less sensitive to binning effects than the maximum likelihood). As the PDFs are usually not Gaussian, we quote asymmetric error bars following the minimum credible interval technique (see, e.g., Hamann et al. 2007).

### 5. Results

#### 5.1. Validation of the method

To validate the fitting method, we simulated four sets of model cubes and computed the corresponding \( I_{\text{tot}} \), \( Q_{\text{tot}} \), and \( U_{\text{tot}} \) maps, including noise, with different values of the input parameters (hereafter simulations A, B, C, and D). The MCMC fitting procedure was run on these mock polarization data sets to check if it was able to recover the statistical properties of the input dust density and magnetic field cubes through the selected observables. The results are presented in Table 3. For the four sets of maps, the fitting method recovered the input values within the quoted uncertainties, after the convergence criteria for all the chains were reached. This shows that this choice of observables is relevant to extract the input values from polarized thermal dust emission data within our model. To assess the goodness of fit of the model to the data, we use an a posteriori \( \chi^2 \) test, as explained in Appendix C. In all four cases, we find that the match is very good, since \( \langle \chi^2_{\text{best}} \rangle \approx 1 \). For illustration, the posterior probability contours for simulation A are presented in Fig. 6. We note that the MCMC procedure reveals correlations between the model parameters, which is not unexpected, for example, between \( y^\text{POS} \) and \( p_0 \), or between \( y_9 \) and \( (n_1) \). These trends are best visualized with the correlation matrix, shown in the upper right corner of Fig. 6.

---

**Table 3.** Best fit values from four fBm simulations using the observables from Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( \beta_B )</th>
<th>( \beta_n )</th>
<th>( \log_{10} y_n )</th>
<th>( \log_{10} y^\text{POS}_B )</th>
<th>( \chi_0 [^\circ] )</th>
<th>( \log_{10} (d/1, \text{pc}) )</th>
<th>( \log_{10} (I\text{cm}^{-2}) )</th>
<th>( T_d [K] )</th>
<th>( p_0 )</th>
<th>( \langle \chi^2_{\text{best}} \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Simulation A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Input parameters</td>
<td>2.6</td>
<td>2.08</td>
<td>-0.10</td>
<td>-0.22</td>
<td>-50</td>
<td>1.00</td>
<td>1.48</td>
<td>18.0</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>Best fit values</td>
<td>( 2.8^{+0.2}_{-0.2} )</td>
<td>( 1.9^{+0.3}_{-0.2} )</td>
<td>( 0.03^{+0.15}_{-0.15} )</td>
<td>( -0.26^{+0.05}_{-0.05} )</td>
<td>( -50^{+2}_{-2} )</td>
<td>( 1.2^{+0.2}_{-0.2} )</td>
<td>( 1.3^{+0.1}_{-0.2} )</td>
<td>( 18.0^{+0.5}_{-0.5} )</td>
<td>( 0.11^{+0.02}_{-0.03} )</td>
<td>( 1.3 )</td>
</tr>
<tr>
<td><strong>Simulation B</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Input parameters</td>
<td>2.6</td>
<td>2.09</td>
<td>-0.10</td>
<td>-0.22</td>
<td>-70</td>
<td>0.70</td>
<td>2.00</td>
<td>20.0</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>Best fit values</td>
<td>( 2.7^{+0.1}_{-0.2} )</td>
<td>( 1.9^{+0.3}_{-0.2} )</td>
<td>( -0.01^{+0.11}_{-0.20} )</td>
<td>( -0.24^{+0.03}_{-0.04} )</td>
<td>( -70^{+2}_{-2} )</td>
<td>( 0.8^{+0.3}_{-0.2} )</td>
<td>( 2.0^{+0.5}_{-0.5} )</td>
<td>( 0.13^{+0.02}_{-0.02} )</td>
<td>( 1.7 )</td>
<td></td>
</tr>
<tr>
<td><strong>Simulation C</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Input parameters</td>
<td>3.0</td>
<td>2.8</td>
<td>0.0</td>
<td>-0.10</td>
<td>-30</td>
<td>1.18</td>
<td>1.30</td>
<td>22.0</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Best fit values</td>
<td>( 2.8^{+0.1}_{-0.2} )</td>
<td>( 2.6^{+0.3}_{-0.2} )</td>
<td>( 0.00^{+0.10}_{-0.11} )</td>
<td>( -0.08^{+0.03}_{-0.03} )</td>
<td>( -30^{+3}_{-4} )</td>
<td>( 1.1^{+0.2}_{-0.2} )</td>
<td>( 1.4^{+0.2}_{-0.2} )</td>
<td>( 22.1^{+0.5}_{-0.5} )</td>
<td>( 0.21^{+0.03}_{-0.03} )</td>
<td>( 1.8 )</td>
</tr>
<tr>
<td><strong>Simulation D</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Input parameters</td>
<td>2.0</td>
<td>1.87</td>
<td>-0.22</td>
<td>-0.10</td>
<td>-10</td>
<td>0.70</td>
<td>2.18</td>
<td>16.0</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Best fit values</td>
<td>( 2.2^{+0.2}_{-0.2} )</td>
<td>( 1.7^{+0.3}_{-0.2} )</td>
<td>( -0.1^{+0.2}_{-0.2} )</td>
<td>( -0.08^{+0.06}_{-0.05} )</td>
<td>( -8^{+2}_{-2} )</td>
<td>( 1.0^{+0.3}_{-0.3} )</td>
<td>( 1.9^{+0.3}_{-0.3} )</td>
<td>( 16.0^{+0.5}_{-0.5} )</td>
<td>( 0.11^{+0.03}_{-0.03} )</td>
<td>( 0.7 )</td>
</tr>
</tbody>
</table>

Notes. The column \( \langle \chi^2_{\text{best}} \rangle \) shows the \( \chi^2 \) values for the best fit parameters averaged over 100 fits (see Appendix C).
5.2. Application to the Polaris Flare

As an application of our method, we wish to constrain statistical properties of the turbulent magnetic field in the Polaris Flare, a diffuse, highly dynamical, non-starforming molecular cloud. There are several reasons for choosing this particular field. First, it has been widely observed: the structures of matter were studied in dust thermal continuum emission by, for example, Miville-Deschênes et al. (2010); the velocity field of the molecular gas was studied down to very small scales through CO rotational lines (Falgarone et al. 1998; Hily-Blant & Falgarone 2009); and the magnetic field was probed by optical stellar polarization data in Panopoulou et al. (2016). Second, as this field does not show signs of star formation, the dynamics of the gas and dust are presumably dominated by magnetized turbulence processes, without contamination by feedback from young stellar objects (YSOs). It therefore seems like an ideal test case for our method.

To this aim, we used the full-mission Planck maps of Stokes parameters \((I_{353}, Q_{353}, U_{353})\) at 353 GHz and the associated covariance matrices from the Planck Legacy Archive. We also used the thermal dust model maps \(T_{353}\) and \(T_{\text{obs}}\) from the 2013 public release (Planck Collaboration XI 2014). All maps are at a native 4.8’ resolution in the Healpix format with \(N_{\text{side}} = 2048\), and the Polaris Flare maps were obtained by projecting these onto a Cartesian grid with 6’ pixels, centred on Galactic coordinates \((l, b) = (120^\circ, 27^\circ)\), with an FoV \(\Delta l = \Delta b = 12^\circ\). The maps of \(I_{353}, Q_{353}, U_{353}\), and \(\tau_{353}\) were then smoothed using a circular Gaussian beam, to obtain maps at a 15’ FWHM resolution. The covariance matrix maps were computed at the same resolution, using a set of Monte-Carlo simulations of pure noise maps, drawn from the original full-resolution covariance maps and smoothed at 15’. The maps of \(I_{353}, Q_{353}, U_{353}, p_{\text{MAS}}, \psi, S\) obtained in this way are shown in Fig. 7.

We note that the features of simulated Stokes maps are not located in the same regions as in the Polaris Flare maps. As the noise covariance matrices are the same for all the simulated maps, this means that the signal-to-noise ratio per pixel in the model \(I_m, Q_m\) and \(U_m\) maps is different for each set of parameters. However, the MCMC procedure is able to choose the parameter sets that give signal-to-noise ratios similar to those in the Planck maps.

The results of the analysis of the Planck polarized thermal dust emission data towards the Polaris Flare are presented in...
The turbulent component of the magnetic field is shown in Fig. 8. We find in particular that the spectral index of the polarization angle dispersion function $S$. The $\tau_{353}$ and $T_{\text{map}}$ maps have the same aspects as the $I_{353}$ map but with their own scales.

Table 4. Best fit values for the Planck Polaris Flare maps, using the observables from Table 2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\beta_b$</th>
<th>$\beta_n$</th>
<th>$\log_{10} y_n$</th>
<th>$\log_{10} y_{\text{POS}}$</th>
<th>$\chi_0$ [$^\circ$]</th>
<th>$\log_{10} \left(\frac{d}{1\text{pc}}\right)$</th>
<th>$\log_{10} \left(\frac{m}{\text{cm}^{-3}}\right)$</th>
<th>$T_{\text{4}}$ [K]</th>
<th>$p_0$</th>
<th>$\langle \chi^2_{\text{best}} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best fit values</td>
<td>2.8$^{+0.2}_{-0.2}$</td>
<td>1.7$^{+0.4}_{-0.3}$</td>
<td>0.2$^{+0.2}_{-0.2}$</td>
<td>$-0.19^{+0.04}_{-0.04}$</td>
<td>$-69^{+2}_{-3}$</td>
<td>1.1$^{+0.3}_{-0.2}$</td>
<td>1.6$^{+0.2}_{-0.3}$</td>
<td>17.5$^{+0.5}_{-0.5}$</td>
<td>0.17$^{+0.02}_{-0.02}$</td>
<td>2.9</td>
</tr>
</tbody>
</table>

Notes. The column $\langle \chi^2_{\text{best}} \rangle$ shows the $\chi^2$ values for the best fit parameters averaged over 100 fits (see Appendix C).

Table 4, and the posterior probability distribution contours are shown in Fig. 8. We find in particular that the spectral index of the turbulent component of the magnetic field is $\beta_b = 2.8 \pm 0.2$, and that the spectral index of the dust density field is around $\beta_n = 1.7$ with a rather large uncertainty. The fluctuation ratio of the density field is about unity, $y_n = 1.6$, and the magnitude of the large-scale magnetic field in the POS dominates slightly the RMS of the turbulent component, $y_{\text{POS}} \approx 0.65$, with a position angle $\chi_0 \approx -69^\circ$. The constraint on the depth of the cloud seems to indicate that $d \approx 13$ pc, with $\langle n_d \rangle \approx 40 \text{ cm}^{-3}$. The temperature $T_d$ is 17.5 K equal to the average of the $T_{\text{obs}}$ Planck map, and the polarization fraction is $p_0 \approx 0.12$. The parameter set for simulation A was chosen a posteriori to give similar best fit parameters and to test our likelihood method in the conditions driven by the Polaris Flare data.

Using the best fitting parameters from Table 4 we performed simulations to visually check the agreement between the model and Planck data. Figure 9 shows the polarization maps from a simulation using these best fitting parameters. The overall similarity with the data maps from Fig. 7 is reasonably good, although spatially coherent structures appear in the data maps which cannot be reproduced by the model maps. The agreement between the best fitting simulation and the data is quantified through plots of the different observables that were used in the fitting procedure (Figs. 10–12). The agreement is excellent for most observables, although substantial deviations are visible in the DFs of the intensity $I_{353}$ and of the normalized optical depth $T_{353}/(\tau_{353})$ and of the polarization angle $\psi$. These deviations are due to the simplifying assumptions of our fBm model. It may be that in the Polaris Flare the large-scale magnetic field has two major components, with global orientations $\chi_0 \approx -70^\circ$ and $\chi_0 \approx 90^\circ$. We note that the DF in Fig. 10 is that of the $\psi$ angle, which differs from $\chi_0$ by 90$^\circ$. Also the exponentiation procedure to model the dust field is a good but incomplete approximation of the reality and it is not able to totally reproduce the shapes of the $I_m$ and $\tau_{353}/(\tau_{353})$ DFs together. These deviations impact the reduced best fit $\langle \chi^2_{\text{best}} \rangle \approx 2.9$, which is somewhat larger than for mock data ($\approx 1$), but still reasonably good.

Concerning the mean values used as observables, the Polaris Flare has a mean optical depth of $\langle \tau_{353} \rangle = (1.25 \pm 0.05) \times 10^{-5}$ and a mean temperature of $\langle T_{\text{obs}} \rangle = 17.5 \pm 0.4$ K. Using the best fitting parameters from Table 4 we obtain optical depth maps with an average of $\langle \tau_{353} \rangle = (1.82 \pm 0.05) \times 10^{-5}$ over 60 realizations which is not fully consistent with the data value, as mentioned above for the $\tau_{353}/(\tau_{353})$ discrepancy. However the
The best fitting parameter for temperature is $T_d = 17.5 \pm 0.5$ K which is exactly the same as in data with a width reflecting the data uncertainties.

The observables we used to extract the statistical properties of the Polaris Flare field are by themselves unable to constrain the $\gamma_0$ angle of the large-scale magnetic field on the LOS. However, the Planck Collaboration Int. XLIV (2016) analysis was able to fit the $\chi_0$ and $\gamma_0$ angle is the southern Galactic cap and found an intrinsic polarization fraction of the gas of $p_{\text{int}} \approx 0.26$. If we believe this latter value is true also in the Polaris Flare, then it is related to our fit as $p_0 \approx p_{\text{int}} \left(\cos^2 \gamma_0\right) \approx p_{\text{int}} \cos^2 \gamma_0$. We can thus constrain the $\gamma_0$ angle to be around $45^\circ$. 

6. Discussion and summary

We have presented an analysis framework for maps of polarized thermal dust emission in the diffuse ISM aimed at constraining the statistical properties of the dust density and magnetic field responsible for this emission. Our framework rests on a set of synthetic models for the dust density and magnetic field, for which we precisely control the one- and two-point statistics, and on a least-squares analysis in which the space of parameters is explored via a MCMC method. The application of the method to Planck maps of the Polaris Flare molecular cloud leads to a spectral index of the turbulent component of the magnetic field $\beta_B = 2.84 \pm 0.10$, in excellent agreement with the measurement by Stutzki et al. (1998) on CO integrated emission at a similar angular resolution.
of turbulent-to-uniform magnetic field are both around unity. Finally, our analysis is able to give a constraint on the polarization fraction, $p_0 \approx 0.12$, and on the depth of the Polaris Flare molecular cloud, $d \approx 13$ pc, which is about half the transverse extent of the FoV, with $\langle n_{\text{H}} \rangle \approx 40$ cm$^{-3}$. The good visual agreement between the Polaris Flare maps and model maps for the best-fitting parameters (Figs. 7 and 9), and the excellent agreement between the two sets of maps for most of the observables used in the analysis (Figs. 10–12), all lead us to conclude that our fBm-based model, although limited, provides a reasonable description of the magnetized, turbulent, diffuse ISM.

In fact, it is quite remarkable to find such a good agreement with the data, considering the limitations of the model. First, it is statistically isotropic, and therefore cannot reproduce the interstellar filamentary structures observed at many scales and over a large range in column densities (see, e.g. Miville-Deschênes et al. 2010; Arzoumanian et al. 2011). Second, our model dust density and magnetic fields are completely uncorrelated, which is clearly not realistic, as it was found that there is a preferential relative orientation between structures of matter and magnetic field, both in molecular clouds (Planck Collaboration Int. XXXV 2016) and in the diffuse, high-latitude sky (Planck Collaboration Int. XXXII 2016; Planck Collaboration Int. XXXVIII 2016). The change in relative orientation, from mostly parallel to mostly perpendicular, as the total gas column density $N_{\text{H}}$ increases, is also not reproducible with our fully-synthetic models. Third, it is now commonly acknowledged that two-point statistics such as power spectra are not sufficient to properly describe the structure of interstellar matter. Improving our synthetic models along these three directions will be the subject of future work.

For completeness, we have also looked into applying our MCMC approach based on fBm models to synthetic polarization maps built from a numerical simulation of MHD turbulence. We used simulation cubes$^{11}$ (Cho & Lazarian 2003; Burkhart et al. 2009, 2014), basing our choice on the simulation parameters, which seemed more or less consistent with the parameters found for the Polaris Flare data. We built simulated Stokes $I$, $Q$, and $U$ maps using the same resolution and noise parameters, and launched the MCMC analysis on these simulated Stokes maps. It turns out that it is more difficult for the Markov chains to converge in this case than when applying the method to the Planck data. It is not yet completely clear why that is so, but we suspect that part of the reason may lie with the limited range of spatial scales over which the fields in the MHD simulation can be accurately described by scale-invariant processes. Indeed, while the fBm models exhibit power-law power spectra over the full range of accessible scales (basically one decade in our case), the MHD simulations are hampered by effects of numerical dissipation at small scales (possibly over nearly ten pixels), and the properties at large scales are dependent on the forcing, which is user-defined. The data, on the other hand, exhibit a much larger “inertial range”. In that respect, our fBm models, despite all their drawbacks, and despite the fact that they lack the physically realistic content of MHD simulations, provide a better framework for assessing the statistical properties of the Planck data than current MHD simulations can. Of course, this conclusion is based on just one simulation, and there would definitely be a point in applying the MCMC approach to assess various instances of MHD simulations with respect to the observational data, based on the same observables, but independently of the grid of fBm models. This project, however, is clearly beyond the scope of this paper.

$^{11}$ http://www.mhdturbulence.com
Fig. 10. Comparison of the DFs extracted from the Planck Polaris Flare maps (black points) with the observables computed from simulations using the best fitting parameters (blue curves). The latter curves are averaged over 60 realizations, as described in Sect. 4.2: the average is given by the central blue curve and the shaded bands give the $1\sigma$ and $2\sigma$ standard deviation in each bin.
Fig. 11. Comparison of the $I_m$, $Q_m$, and $U_m$ power spectra extracted from the Planck Polaris Flare maps (grey points representing the two-dimensional power spectra, and black dots representing the azimuthal averages in Fourier space) with the observables computed from simulations using the best fitting parameters (blue curves). The latter curves are averaged over 60 realizations as described in Sect 4.2: the average is given by the central blue curve and the narrow shaded bands give the 1σ and 2σ standard deviation in each bin.

Fig. 12. Two-dimensional distribution function of $S$ and polarization fraction $p_{\text{MAS}}$ for the Polaris Flare maps (left), for the model maps using the best fitting parameters averaged over 60 realizations (middle), and residuals (right). The polarization angle dispersion function $S$ is computed at a lag $\delta = 16^\prime$. The solid black line shows the mean $S$ for each bin in $p_{\text{MAS}}$ and the dashed black line is a linear fit of that curve, restricted to bins in $p_{\text{MAS}}$ which contain at least 1% of the total number of points ($120 \times 120$).

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Appendix A: Statistical properties of n_H models

A.1. Probability distribution function

The PDF \( f(n_H) \) of the density field \( n_H \) built using Eq. (4) derives from the Gaussian PDF of \( X \), which in all generality has a mean \( \langle X \rangle \) and variance \( \sigma_X^2 \). We thus have a log-normal PDF

\[
f(n_H) = \frac{1}{\sqrt{2\pi\sigma_X^2 n_H}} X_c \exp \left\{ -\frac{1}{2\sigma_X^2} \left[ X, \ln \left( \frac{n_H}{n_0} \right) - \langle X \rangle \right]^2 \right\}, \tag{A.1}
\]

which is defined for \( n_H > 0 \). Figure A.1 presents the distribution function of the \( n_H \) field\(^{12} \) used to build Fig. 1, with the theoretical PDF expected from Eq. (A.1). We note that the distribution function for a single realisation over a finite grid such as the ones used here may deviate from the theoretical PDF, especially for large values of \( \beta_X \), but the mean distribution function over a sufficiently large sample converges to the lognormal form (Eq. (A.1)).

A.2. Moments and fluctuation level

From Eq. (A.1), we may compute moments of any order \( p \) of the PDF of the total gas density \( n_H \)

\[
\langle n_H^p \rangle = \int_0^{n_0} n_H^p f(n_H) dn_H = n_0^p \exp \left\{ p \frac{\langle X \rangle}{X_c} + p^2 \frac{\sigma_X^2}{2X_c^2} \right\}, \tag{A.2}
\]

which allows, in particular, to compute its mean value

\[
\langle n_H \rangle = \int_0^{n_0} n_H f(n_H) dn_H = n_0 \exp \left( \frac{\langle X \rangle}{X_c} + \frac{\sigma_X^2}{2X_c^2} \right) \tag{A.3}
\]

as well as its variance

\[
\sigma_{m_H}^2 = n_0^2 \exp \left( 2 \frac{\langle X \rangle}{X_c} \right) \left[ \exp \left( 2 \frac{\sigma_X^2}{X_c^2} \right) - 1 \right] \tag{A.4}
\]

The density fluctuation level, which is one of the parameters of our model, is therefore

\[
y_n = \frac{\sigma_{m_H}}{\langle n_H \rangle} = \sqrt{2} \exp \left( \frac{\sigma_X^2}{4X_c^2} \right) \left[ \sinh \left( \frac{\sigma_X^2}{2X_c^2} \right) \right]^{1/2}. \tag{A.5}
\]

For instance, the \( n_H \) field whose distribution function is shown in Fig. A.1 has values ranging from 0.3 cm\(^{-3} \) to 880 cm\(^{-3} \), with a mean and standard deviation of \( \langle n_H \rangle = \sigma_{m_H} = 28.3 \text{ cm}^{-3} \), resulting in the desired fluctuation level \( y_n \approx 1 \).

A.3. Power spectra

Figure A.2 shows the azimuthally-averaged power spectra of \( n_H \) fields obtained through Eq. (4), from a 120 × 120 × 120 pixels fractional Brownian motion with spectral index \( \beta_X = 3 \), for various fluctuation levels \( y_n \). The spectra were normalized differently so as to allow comparison between them.

The power-law behaviour is apparent, even at large fluctuation levels, but the spectral index decreases (i.e., the spectra flatten) as the fluctuation level increases. This is quantified in

\(^{12} \) In that case, \( \langle X \rangle = 0, \sigma_X^2 = 1, X_c = 1.2, \) and \( n_0 = 20 \text{ cm}^{-3} \).
Appendix B: Likelihood terms

B.1. $D^2$ terms for mean values

The first term in Eq. (25) is given by

$$D^2_m = \frac{\sigma^2(\tau_{353}^{m}) + \sigma^2(\tau_{353}^{d})}{\sigma^2(T_{obs})} + \frac{(T_{obs} - T_d)^2}{\sigma^2(T_{obs})}. \quad (B.1)$$

The mean value of the optical depth $\tau_{353}$ is evaluated on the simulated $\tau_{353}^{m}$ and Planck data $\tau_{353}^{d}$ maps. These are compared through a standard $\chi^2$ test. The denominator includes the uncertainty $\sigma(\tau_{353})$ on the mean $\tau_{353}^{d}$ value propagated from the uncertainty map provided by the Planck collaboration (Planck Collaboration XI 2014), and the uncertainty $\sigma(\tau_{353}^{m})$ coming from the conversion factor $\sigma_{353}(N_{\Omega})$ used to build the simulated map (Planck Collaboration XI 2014).

In the second term, $T_{obs}$ is the mean value of the temperature map $T_{obs}$ from Planck data, and $\sigma(T_{obs})$ represents the uncertainty on the averaged value propagated from the uncertainty map provided by the Planck collaboration. $T_d$ is directly the model parameter for the dust temperature.

B.2. $D^2$ terms for distribution functions

For a given observable map $o$, we compute its DF over an ensemble of $N_b$ bins. When considering the Planck data, we write this DF as $h_{o,ni}^d$, where $i$ is the bin number, and we estimate the uncertainty on the value of the DF in bin $i$ through the associated Poisson noise $\sigma_{o,ni}^d$. When considering the model, we write $h_{o,ni}^m$ and $\sigma_{o,ni}^m$ to be respectively the bin value and the Poisson noise of the DF in bin $i$, independently for each of the $N_r = 60$ model realizations.

The contribution $D^2_{DF(o)}$ of the observable’s DF to the total $D^2$ in Eq. (25) is then built as an average over the $N_b$ bins.

$$D^2_{DF(o)} = \frac{1}{N_b} \sum_{i=1}^{N_b} \frac{(h_{o,ni}^d - h_{o,ni}^m)^2}{\sigma_{o,ni}^d}.$$  \quad (B.2)

The sum is normalized to the number of bins so that $D^2_{DF(o)}$ is less sensitive to the binning choice and the map noise. If the model fits the data correctly as far as the DF of observable $o$ is concerned, then $D^2_{DF(o)}$ is minimum.

This quantity is different from a standard $\chi^2$ test as it compares data with one random realisation for a given set of model parameters, and this comparison is repeated and averaged $N_r$ times. A standard $\chi^2$ test would compare data with a model prediction that would be the average of the random realizations (see Eq (C.3) in Appendix C). The latter could not be used in our MCMC analysis due to the mathematical relation between the power spectrum of the map and the variance in each bin of the DF: with steep power spectra (high values of the spectral index $\beta$), only the few large-scale modes effectively contribute to the power, leading to a large variance in each DF bin. Thus, the $\chi^2$ test tends to favour these large values of $\beta$, as they yield large denominators and thus allow for a “good” fit. This drives the fit towards a region of parameter space yielding mean DFs that fit the data well but with a huge dispersion: one realization of such a model gives a DF with bin values highly scattered even though the data DF is quite smooth. This means that the data cannot reasonably be interpreted as a random realization using these parameter values, and explains why we had to switch to the $D^2$ function, which directly compares data with one model random realization. In this fashion, we are able to reach the region of parameter space correctly describing the data.

B.3. $D^2$ terms for power spectra

For a given observable map $o$, we first compute its two-dimensional power spectrum

$$P_o(k) = |\tilde{a}(k)|^2 \quad (B.3)$$

then average these within $N_k$ annuli in Fourier space, centred on a set of wavenumbers $|k|$. We write $P_{o,ni}^d$ for this azimuthal average in bin $i$ when considering the Planck data, and $P_{o,ni}^m$ when considering model fields. The uncertainties affecting these quantities strongly depend on the number $N_i$ of wavevectors $k_i$ in each bin. The best estimate for the standard deviation $\sigma_{P_{o,ni}^d}$ of the power spectrum of the data is

$$\sigma_{P_{o,ni}^d} = \sqrt{\frac{1}{N_i(N_i-1)} \sum_{n=1}^{N_i} \left( P_{o,ni}^d - \overline{P_{o,ni}^d} \right)^2} \quad (B.4)$$

where the factor $t_{N_i}$ is the Student coefficient. We thus obtain the best estimate of the true standard deviation in bins with only a few modes (i.e. at large scale). The standard deviation $\sigma_{P_{o,ni}^m}$ for the 60 model realizations is computed in the same way as for data.

The contribution $D^2_{P_{o,ni}^m}$ of the observable’s power spectrum to the total $D^2$ in Eq. (25) is then computed as an average over

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14 We remind the reader that the randomness comes from the fBm itself and the noise addition to the Stokes maps.

15 This is akin to the cosmic variance problem in cosmology.
the \( N_\nu^b \) bins\(^{16} \) in wavenumber space,
\[ D^2_{P(o)} = \frac{1}{N_\nu^b} \sum_{i=1}^{N_\nu^b} \frac{(P_{o,i} - \bar{P}_{o})^2}{(\sigma_{P_{o,i}})^2 + (\sigma_{\bar{P}_{o}})^2}. \] (B.5)

\[ D^2_{S-\text{PMAS}} = \frac{1}{N_{\text{b, tot}}} \sum_{i=1}^{N_{\text{b, tot}}} \sum_{j=1}^{N_{\text{b, tot}}} \frac{(h_{ij}^d - \bar{h}_{ij}^m)^2}{(\sigma_{h_{ij}^d})^2 + (\sigma_{\bar{h}_{ij}^m})^2}. \] (B.6)

In this expression, we note that the total number \( N_{\text{b, tot}} \) of two-dimensional bins considered is less than the product \( N_{\text{b,1}}N_{\text{b,2}} \) of the number of bins in each dimension, which we set to \( N_{\text{b,1}} = N_{\text{b,2}} = 50 \). The reason for this is that we discard the empty bins and those with a signal-to-noise ratio below three\(^{17} \). We thus keep only the significantly populated bins that can drive the fit and contribute to the total \( D^2 \).

**Appendix C: Goodness-of-fit**

To assess the goodness of the fit, we use an a posteriori \( \chi^2 \) test, which we define as
\[ \chi^2 = \frac{1}{N_o} \left[ \chi^2_{\mu} + \sum_o \chi^2_{DF(o)} + \sum_o \chi^2_{P(o)} + \chi^2_{S-\text{PMAS}} \right] \] (C.1)
with \( N_o = 13 \) the total number of observables. Each term is a \( \chi^2 \) test comparing data with the mean of the \( N_f = 60 \) realisations. The first term from Eq \( \chi^2_{\mu} \) is
\[ \chi^2_{\mu} = \frac{(\bar{T}_{\text{obs}} - \bar{T}_\mu)^2}{\sigma(T_{\text{obs}})} \] (C.2)
where \( \bar{T}_{\text{obs}} \) is the ensemble average over the 60 model realisations of the (spatial) mean of the optical depth. In the following, the brackets stand for an average on the pixels while the upper bar represents the average over the \( N_f \) realisations. The \( \Sigma^2_{\tau_{353}} \) quantity is the variance of \( \langle \tau_{353}^m \rangle \) over the \( N_f \) realisations (capital \( \Sigma \) denotes the variance over the random realisations).

The second term is
\[ \chi^2_{DF(o)} = \frac{1}{N_o} \sum_{i=1}^{\nu} \frac{(P_{o,i} - \bar{P}_{o})^2}{(\sigma_{P_{o,i}})^2 + (\sigma_{\bar{P}_{o}})^2} \] (C.3)

where
\[ \bar{h}_{ij}^m = \frac{1}{N_f} \sum_{i=1}^{\nu} h_{ij}^m \] (C.4)

is the ensemble average of the \( i^{th} \) bin of the DF for the observable \( \alpha \), over the \( N_f = 60 \) model realizations, and \( (\sigma_{h_{ij}^m})^2 \) is the associated variance. We note that while \( D^2_{DF(o)} \) in Eq. \( \chi^2 \) is the average of the observable \( \chi^2_{DF(o)} \) of the averaged observable.

The third term is
\[ \chi^2_{P(o)} = \frac{1}{N_o} \sum_{i=1}^{\nu} \frac{(P_{o,i} - \bar{P}_{o})^2}{(\sigma_{P_{o,i}})^2 + (\sigma_{\bar{P}_{o}})^2} \] (C.5)

where \( \bar{P}_{o,i} \) is the averaged power spectrum in bin \( i \) and \( (\sigma_{P_{o,i}})^2 \) its variance.

Finally, the fourth term is
\[ \chi^2_{S-\text{PMAS}} = \frac{1}{N_{\text{b, tot}}} \sum_{i=1}^{N_{\text{b, tot}}} \sum_{j=1}^{N_{\text{b, tot}}} \frac{(h_{ij}^d - \bar{h}_{ij}^m)^2}{(\sigma_{h_{ij}^d})^2 + (\sigma_{\bar{h}_{ij}^m})^2} \] (C.6)

with the same notation conventions as above.

To quantify the goodness of fit, once the MCMC procedure has converged, we perform 100 fits for the set of best fitting parameters, each of these fits comprising 60 model realizations and providing a value of the \( \chi^2 \) quantity defined in Eq. \( \chi^2 \). The average of these 100 \( \chi^2 \) values\(^{18} \) is listed as \( \chi^2_{\text{best}} \) in Tables 3 and 4.

\(^{16} \) To have reliable and smooth power spectra with 120 \times 120 pixel maps, we set initially \( N_\nu^b = 100 \) but later cut off wavenumbers larger than \( k_{\text{max}} = 2\pi/\left(3\times 15^\prime\right) \), which corresponds to scales smaller than three beam sizes. Indeed, for bins \( k_i > k_{\text{max}} \) the power spectrum is completely washed out by the beam convolution (see Fig. 5) and contains no information about the underlying interesting parameters. This uninformative part is thus removed from \( D^2_{P(o)} \), and thus \( N_\nu^b < 100 \).

\(^{17} \) i.e., bins where \( h_{ij}^d = 0, h_{ij}^m = 0, h_{ij}^d/\sigma_{h_{ij}^d} < 3 \), or \( h_{ij}^m/\sigma_{h_{ij}^m} < 3 \).

\(^{18} \) Each simulation is a random realization of the fBm and the noise, then the \( \chi^2 \) values have an intrinsic dispersion even when compiling 60 simulations. To check the agreement with data, we thus average 100 computations of the best \( \chi^2 \) value.