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Which fundamental constants for cosmic microwave background and baryon-acoustic oscillation?

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ABSTRACT

We use the three-scale framework of Hu et al. to show how the cosmic microwave background (CMB) anisotropy spectrum depends on the fundamental constants. As expected, the spectrum depends only on dimensionless combinations of the constants, and we emphasize the points that make this generally true for cosmological observables. Our analysis suggests that the CMB spectrum shape is mostly determined by $\alpha^2 m_e/m_p$ and by m_p/m_χ, the proton-CDM-particle mass ratio. The distance to the last-scattering surface depends on $Gm_p m_\chi/\hbar c$, so published CMB observational limits on time variations of the constants implicitly assume the time independence of this quantity, as well as a flat-ΛCDM cosmological model. On the other hand, low-redshift baryon-acoustic oscillation, H_0, and baryon-mass-fraction measurements can be combined with the shape of the CMB spectrum to give information that is largely independent of these assumptions. In particular, we show that the pre-recombination values of $Gm_\chi^2/\hbar c$, m_p/m_χ, and $\alpha^2 m_e/m_p$ are equal to their present values at a precision of $\sim$15%.

Key words. cosmic background radiation

1. Introduction

The cosmic microwave background (CMB) anisotropy spectra are primarily used to determine cosmological parameters (Planck Collaboration XVI 2014; Planck Collaboration XIII 2015a), but the spectra can also give information on the values of the fundamental constants in the early universe. Limits on the difference between the pre-recombination and present values of the fine structure constant, α, were first obtained in studies using CMB data from BOOMeranG and MAXIMA (Kaplinghat et al. 1999; Avelino et al. 2000) and WMAP (Rocha et al. 2004). The limits were generalized to combined limits on (α, m_e) using WMAP data (Ichikawa et al. 2006; Scóccola et al. 2008, 2009; Nakashima et al. 2010; Landau & Scóccola 2010; Scóccola et al. 2013) and *Planck* data (Planck Collaboration Int. XXIV 2015b). These limits are based on the effects of (α, m_e) on the recombination process (Kaplinghat et al. 1999; Hannestad 1999; Seager et al. 2000). While the procedure used to obtain these limits is not obviously incorrect, the publication of a limit on the variation in m_e is perplexing since it is generally admitted that only dimensionless fundamental constants are physically meaningful (Dicke 1962). This is manifestly true for laboratory measurements, which consist of comparing quantities of a given dimension with standards of the same dimension (Rich 2003). It is less obviously true for cosmological measurements where two times are typically involved. For example, CMB measurements concern the time of photon-matter decoupling, t_{dec}, and the measurement time, t_0, and one can form dimensionless quantities like $m_e(t_{dec})/m_e(t_0)$. In fact, CMB-based limits like those of Planck Collaboration Int. XXIV (2015b) are generally expressed as limits on the deviation from unity of this dimensionless quantity. Similarly, limits from other studies on time variations of Newton's constant G (for a review, see Uzan 2011) are typically expressed as measurements of $G(t)/G(t_0)$. In this paper we show how a proper analysis gives only measurements of equal-time dimensionless quantities like $m_e(t)/m_p(t)$.

Part of the problem with using CMB data is that the phenomenology is rather complicated so it is difficult to include the effects of all relevant fundamental constants in compact formulas. This is one reason that results are expressed in terms of dimensioned constants like m_e, since the standard numerical procedures like CAMB[1] and RECFAST (Seager et al. 2000) use such quantities. Here, this problem is avoided by using the qualitative model of Hu et al. (Hu et al. 1997, 2001; Hu & White 1997) to give the dominant dependencies of the spectrum on the relevant physical and cosmological parameters. This allows us to give a general analysis of the problem, while the published studies leading to limits in (m_e, α) space assume the time independence of all non-electronic masses and of G. Because of these assumptions, Planck Collaboration Int. XXIV (2015b) interpreted their limits on m_e as limits on $Gm_e^2/\hbar c$, to which one must add the caveat that all non-electronic masses are held constant. Quoting limits on $Gm_e^2/\hbar c$ is troubling because gravitational interactions of electrons should have negligible effects on the spectrum. In fact, the analysis presented here suggests that the natural dimensionless variables for studying the shape of the spectrum are $\alpha^2 m_e/m_p$, m_p/m_χ and $Gm_\chi m_p/\hbar c$, where m_χ is the mass of the cold dark matter (CDM) particles. The introduction of m_χ into the problem reminds us that not even the present values of all relevant fundamental constants are known. However, this does not prevent us from studying their time variation.

In the following analysis, Sect. 2 defines the fundamental and cosmological parameters, and Sect. 3 applies the model of Hu et al. to determine the dependencies of the CMB spectrum on those parameters. Section 4 describes the information that can be derived from an analysis of the spectrum. Section 5 combines

[1] http://camb.info

the CMB-derived quantities with low-redshift measurements to derive limits on the time variations of fundamental constants. Finally, Sect. 6 concludes with some thoughts on why cosmological observations always conspire to give information only on dimensionless constants.

2. The fundamental constants and cosmological parameters

We first define the physical and cosmological model that we use. For the CMB, the five most important coupling constants and masses are

$$\alpha \qquad G \qquad m_\chi \qquad m_{\rm p} \qquad m_{\rm e}. \tag{1}$$

Since we allow for time variations, the current values are given with a zero subscript, e.g. $m_{\rm p0}$. Of the five, only α is dimensionless and our goal is to show that observable quantities depend only on dimensionless combinations of the last four like $m_{\rm e}/m_{\rm p}$ and $Gm_\chi^2/\hbar c$. (In this paper, the factors of $\hbar$ and c are generally omitted, so Gm_χ^2 is dimensionless.)

As emphasized, for example, in Uzan (2011), simply knowing the dependence of observable quantities on fundamental constants in the absence of time-variations does not mean that one can reliably calculate the cosmological consequences of time variations. This is because the physical introduction of time-variations of constants generally requires the introduction of extra degrees of freedom, like time-varying scalar fields. This adds additional terms to the Friedman equation, modifying the expansion rate. In the absence of a specific model, one has to avoid these complications by making simplifying assumptions. As was done in the WMAP and *Planck* studies (Planck Collaboration Int. XXIV 2015b) we assume that time variations of fundamental constants are such that they are effectively time-independent at high redshift, where they determine the recombination process. They then quickly "relax" to their post-recombination values where they determine the distance to the last-scattering surface and provide standards for local measurements of the CMB temperature, T_0, and the expansion rate, H_0. We ignore the modifications of the expansion dynamics that necessarily occur during the relaxation. This does not significantly affect our results since we are concerned mostly with distance-independent constraints.

We assume that the universe at recombination contains baryons, CDM particles, photons and neutrinos. Such a universe is described by $\eta_{\rm b}$, the baryon-photon number density ratio, η_χ, the same quantity for dark-matter particles, and N_ν, the number of neutrino species that were in thermal equilibrium with the photons for $T >\sim$ MeV. We suppose throughout this paper that $\eta_{\rm b}$ and η_χ are time-independent. At least two parameters are necessary to describe the primordial fluctuations but these have only a small effect on our discussion. The important cosmological parameters are therefore

$$\eta_\chi \qquad \eta_{\rm b} \qquad N_\nu \qquad H_0 \qquad T_0 \tag{2}$$

where H_0 and T_0 are the current expansion rate and temperature. The proton and CDM masses only enter through the gravitational effects of their densities, $\propto m_{\rm p}\eta_{\rm b}$ and $\propto m_\chi\eta_\chi$. The most important combinations of physical and cosmological parameters are therefore H_0, T_0 and

$$Gm_\chi\eta_\chi \qquad Gm_{\rm p}\eta_{\rm b} \qquad N_\nu \qquad \alpha^2 m_{\rm e} \tag{3}$$

where we have anticipated that the combination of $(\alpha, m_{\rm e})$ that is most relevant is $\alpha^2 m_{\rm e}$. We note also that standard studies replace $m_\chi\eta_\chi$ with $\Omega_\chi H_0^2$ by assuming that $G = G_0$:

$$(\Omega_\chi H_0^2)_{\rm no-var} = 2.04 G_0 m_\chi \eta_\chi T_0^3, \tag{4}$$

where here and throughout the subscript no−var denotes results assuming no time variations of fundamental constants.

Because we are mostly concerned with the shape of the CMB spectrum, the density of dark energy is not be an important parameter, since it only enters into the distance to the last-scattering surface, determining the angular scale of the spectrum. However, we sometimes give results that depend on this scale, assuming a flat-ΛCDM universe. In this case, the vacuum energy density is $\Omega_\Lambda H_0^2 = H_0^2 - \Omega_{\rm M} H_0^2$ where $\Omega_{\rm M} = \Omega_\chi + \Omega_{\rm B}$.

3. The CMB anisotropy spectrum

To understand the CMB anisotropy spectrum, we use the qualitative model of Hu et al. (Hu et al. 1997, 2001; Hu & White 1997) based on three length scales that are imprinted on the spectrum. The scales are the Hubble length at matter-radiation equality, $r_{\rm eq}$; the acoustic scale, $r_{\rm A}$, equal to the distance a sound wave can travel before photon-matter decoupling; and the damping scale, $r_{\rm damp}$, due to photon random walks near decoupling. In the anisotropy power spectrum, C_ℓ, the three length scales are reflected in three inverse-angular scales, $\ell_i \sim \pi D(z_{\rm dec})/r_i$, ($i$ = eq, A, damp) where $D(z_{\rm dec})$ is the co-moving angular-diameter distance to the last-scattering surface.

Besides the three scales, the spectrum depends on four other parameters: the primordial amplitude of scalar perturbations and its spectral index $(A_{\rm s}, n_{\rm s})$; the effective number of neutrino species, N_ν; and the baryon-photon ratio at photon-matter decoupling

$$R_{\rm dec} = \frac{3\rho_{\rm B}(T_{\rm dec})}{4\rho_\gamma(T_{\rm dec})} = 0.278\frac{m_{\rm b}\eta_{\rm b}}{T_{\rm dec}}\cdot \tag{5}$$

The shape of the spectrum depends on distance-independent quantities: $r_{\rm eq}/r_{\rm A}$, $r_{\rm damp}/r_{\rm A}$, $R_{\rm dec}$, N_ν and $n_{\rm s}$.

Hu et al. propose an approximate form for C_ℓ which depends on these parameters. The characteristic peak-trough structure is described by A_ℓ^2 where

$$A_\ell \propto [1 + R_{\rm dec}T(\ell/\ell_{\rm eq})]\,\cos\pi(\ell/\ell_{\rm A} + \delta) - R_{\rm dec}T(\ell/\ell_{\rm eq}). \tag{6}$$

The peaks in the spectrum are at integer values of $\ell/\ell_{\rm A} + a = n$ where $a \sim 0.267$ has only a weak dependence on fundamental and cosmological parameters. The cross-term in A_ℓ^2 favors odd-n (compression) peaks compared to even-n (rarefaction) peaks with the amplitude difference governed by $R_{\rm dec}T(\ell_{\rm A}/\ell_{\rm eq})$. Here, T is the matter transfer function expressed in angular variables, i.e. $T(k/k_{\rm eq})$ with $k = \ell/D(z_{\rm dec})$.

Averaged over peaks and troughs, the amplitude of the spectrum is determined by the other scales, with $r_{\rm eq}$ governing the rise with ℓ above the low-ℓ Sachs-Wolfe plateau and $r_{\rm damp}$ governing the decline at high ℓ:

$$C_\ell \propto \ell^{n_{\rm s}-1} D_\ell^2 P_\ell \left[\frac{A_\ell^2 - 1}{1 + (\ell_{\rm A}/2\ell)^6} + 2\right] \tag{7}$$

where $n_{\rm s} \sim 0.97$ is the spectral index and the "radiation driving" and damping envelopes are

$$P_\ell = \ell^{n_{\rm s}-1}\left[1 + B\exp\left(1.4\ell_{\rm eq}/\ell\right)\right] \quad D_\ell = \exp\left[-\left(\ell/\ell_{\rm damp}\right)^{1.2}\right] \tag{8}$$

Table 1. Scales relevant for the CMB temperature anisotropy spectrum.

T	$H(T) \times (T_0/T)$
$T_{\rm eq} \sim m_\chi \eta_\chi$	$r_{\rm eq}^{-1} \sim \sqrt{G m_\chi^2 \eta_\chi}\; T_0$
$T_{\rm p\gamma} \sim m_{\rm p} \eta_{\rm b}$	$r_{\rm p\gamma}^{-1} \sim \sqrt{G m_\chi m_{\rm p}}\, \sqrt{\eta_\chi \eta_{\rm b}}\; T_0$
$T_{\rm dec} = \alpha^2 m_{\rm e} f_{\rm dec}$	$r_{\rm dec}^{-1} \sim \sqrt{G m_\chi m_{\rm e} \alpha^2 f_{\rm dec}}\, \sqrt{\eta_\chi}\; T_0$ $\sim r_{\rm p\gamma}^{-1}/\sqrt{R_{\rm dec}}$
	$D^{-1} \sim \sqrt{G_0 m_{\chi 0} m_{\rm p0}}\, \sqrt{\eta_\chi T_0/m_{\rm p0}}\; T_0$

Notes. Column 1: the temperature scale. Column 2: the associated distance scale, $1/H(T)$, redshifted to the present epoch. The table shows the simplified dependencies on cosmological and fundamental parameters. (Numerical factors and factors of $\hbar$ and c are omitted.) The redshifting in Col. 2 leaves only dimensionless combinations of fundamental constants. The subscript zero refers to present values and its absence refers to pre-recombination values. CDM domination is assumed ($m_\chi \eta_\chi \gg m_{\rm p} \eta_{\rm b}$). The factor $f_{\rm dec} \sim 0.01$ is a logarithmic function of cosmological and fundamental parameters, Eq. (16). The fourth line shows the co-moving distance to the last-scattering surface in the flat-ΛCDM model.

where $B \sim 12$ depends on N_ν and $R_{\rm dec}$ (Hu & White 1997). Roughly speaking, for $n_{\rm s} \sim 1$, a measurement of the amplitude of the first peak relative to the Sachs-Wolfe plateau determines $\ell_{\rm eq}/\ell_{\rm A}$ and a measurement of the ratio the higher peaks to the first determines $\ell_{\rm damp}/\ell_{\rm A}$. For models approximating with the observed CMB spectrum, the values are $(\ell_{\rm eq}, \ell_{\rm A}, \ell_{\rm dec}) \sim (150, 300, 1300)$ (Hu et al. 2001).

We now discuss how the parameters in the expression for C_ℓ depend on the fundamental and cosmological parameters. The three length scales ($r_{\rm eq}$, r_A, $r_{\rm damp}$) are closely related to the Hubble lengths at matter-radiation equality, $1/H_{\rm eq}$, at baryon-photon equality, $1/H_{\rm p\gamma}$, and at photon-matter decoupling, $1/H_{\rm dec}$. They have the simple dependencies on fundamental and cosmological parameters shown in Table 1. The first column gives the temperatures at the redshift where the scales are defined. The second column gives the inverse scales redshifted to present epoch where, along with the distance $D(z_{\rm dec})$, they determine the observed spectrum. We note the important fact that after this redshift only dimensionless combinations of fundamental constants appear in the second column.

The matter-radiation equality scale, $r_{\rm eq}$, determines the minimum ℓ that benefited from radiation driving (early-time Sachs-Wolfe effect), resulting an enhancement of the temperature anisotropies over the primordial value $\Delta T/T \sim 10^{-5}$. The temperature at equality is

$$T_{\rm eq} = \frac{m_\chi \eta_\chi + m_{\rm p} \eta_{\rm b}}{2.7(1 + 0.68 N_\nu/3)} \tag{9}$$

where $N_\nu \sim 3$ is the number of neutrino species. The equality scale is then

$$r_{\rm eq} \equiv \frac{c}{H_{\rm eq}} \frac{T_{\rm eq}}{T_0} = \left[0.95\, \frac{\sqrt{G}(m_\chi \eta_\chi + m_{\rm p} \eta_{\rm b})}{1 + 0.13 \Delta N_\nu} T_0 \right]^{-1} \tag{10}$$

where $\Delta N_\nu = N_\nu - 3$.

The acoustic scale, $r_{\rm A}$, is the distance a sound wave can travel before recombination and determines the positions of the peaks in the spectrum. It is determined by two scales: the Hubble scale at the epoch of baryon-photon equality (when the sound speed starts to fall below its high-temperature value of $c_{\rm s} = c/\sqrt{3}$) and recombination (drag epoch) when the waves stops. The first factor is

$$r_{\rm p\gamma} = \left[\sqrt{G(m_\chi \eta_\chi + m_{\rm p} \eta_{\rm b}) m_{\rm p} \eta_{\rm b} T_0} \right]^{-1}. \tag{11}$$

Including the propagation at reduced speed until decoupling gives (Eisenstein & Hu 1998)[2]

$$r_{\rm A} = 1.53 r_{\rm p\gamma} F_{\rm A}(R_{\rm dec}, R_{\rm eq}) \tag{12}$$

where

$$F_{\rm A} = \ln \left[\frac{\sqrt{1 + R_{\rm dec}} + \sqrt{R_{\rm dec} + R_{\rm eq}}}{1 + \sqrt{R_{\rm eq}}} \right]. \tag{13}$$

Here, $3\rho_{\rm B}/4\rho_\gamma$ at matter-radiation equality is

$$R_{\rm eq} = (3/4)(1 + .68 N_\nu/3) \frac{m_{\rm p} \eta_{\rm b}}{m_\chi \eta_\chi + m_{\rm p} \eta_{\rm b}}. \tag{14}$$

The value of R at decoupling

$$R_{\rm dec} = 0.278 \frac{m_{\rm p}}{T_{\rm dec}} \eta_{\rm b} = 0.278 \frac{m_{\rm p} \eta_{\rm b}}{\alpha^2 m_{\rm e} f_{\rm dec}} \tag{15}$$

where the decoupling temperature has the form $T_{\rm dec} = \alpha^2 m_{\rm e} f_{\rm dec}$ with $f_{\rm dec}$ being a factor that depends weakly on the fundamental and cosmological parameters and which we now estimate.

There is no simple approximate formula for $T_{\rm dec}$ because decoupling happens simultaneously with recombination. It therefore depends in a complicated way on the relative rates of recombination, ionization, and photon scattering. Simple approximate formulas can be found if one modifies the numerical factors in the relevant cross sections so that one of two extreme conditions is satisfied. In the first, the recombination rates are sufficiently high to maintain equilibrium abundances of electron and atoms when decoupling occurs. In the second, the Compton scattering cross-section is sufficiently high to decouple the photons after recombination has "frozen". In both cases, one finds that $T_{\rm dec} = \alpha^2 m_{\rm e} f_{\rm dec}$ with $f_{\rm dec}$ a logarithmic function of physical and cosmological parameters.

We first consider the case of equilibrium abundances of electrons and atoms, so the free-electron density is determined by the Saha equation. The decoupling temperature is defined by equating the photon-electron (Thompson) scattering rate, $n_{\rm e} \sigma_{\rm T} c$, and the expansion rate. Using $\sigma_{\rm T} = (8\pi/3)\alpha^2/m_{\rm e}^2$ we get

$$f_{\rm dec}^{-1} - 3 \ln f_{\rm dec} = 2 \ln \left[\frac{8\pi}{3(2\pi)^{3/2}} \frac{y_{\rm e} \eta_{\rm b}}{\eta_\chi} \frac{\alpha^7}{G m_\chi m_{\rm e}} \right], \tag{16}$$

where $y_{\rm e}$ is the electron-to-baryon ratio. For our universe with $m_{\rm p} \eta_{\rm b} \sim m_\chi \eta_\chi / 5$, this gives $f_{\rm dec}^{-1} \sim 2 \ln(\alpha^7 / G m_{\rm p}^2) \sim 107$.

In the other extreme, decoupling occurs after recombination reactions stop. In this case, one fixes the electron-photon ratio at its value at "freeze out", defined by $H(T_{\rm freeze}) = \Gamma({\rm e^- p} \rightarrow H)$. As before with the $T_{\rm dec}$, one finds $T_{\rm freeze} = \alpha^2 m_{\rm e} f_{\rm freeze}$ where $f_{\rm freeze}$ is a logarithmic function of physical and cosmological parameters. The decoupling temperature is then set by diluting the electron density until $H(T_{\rm dec}) = \sigma_{\rm T} n_{\rm e}$ with the result that $(T_{\rm dec}/T_{\rm freeze})^3 = (\langle \sigma v \rangle / \sigma_{\rm T})^2$ where $\langle \sigma v \rangle$ is the capture cross-section time velocity at $T_{\rm freeze}$. As it turns out, the ratio

[2] In this paper, we are not concerned with the small differences between the acoustic scale $r_{\rm A}$ and the sound horizon at the drag epoch, $r_{\rm d}$, relevant for BAO studies.

for capture to any bound state is $(\langle\sigma v\rangle/\sigma_{\rm T})^2 = \alpha^2 m_{\rm e}/T_{\rm freeze}$ and this results in $T_{\rm dec} = \alpha^2 m_{\rm e} f_{\rm dec}$ with $f_{\rm dec} = f_{\rm freeze}^{2/3}$ still being a logarithmic function of physical and cosmological parameters.

In the intermediate, realistic case, numerical calculations (see e.g. Kaplinghat et al. 1999) integrate the Boltzmann equation to find the decoupling temperature. Studies using *Planck* and WMAP data use the RECFAST code (Seager et al. 2000) which can be modified to include all expected dependencies on the recombination process on fundamental constants. Presumably, such calculations would give a slowly varying dependence of $f_{\rm dec}$ on fundamental constants as in Eq. (16). The combination would necessarily be dimensionless and (16) suggests that it would be $Gm_\chi m_{\rm e}$ times a power of α.

The estimate of $T_{\rm dec}$ determines the value of $R_{\rm dec}$ (Eq. (15)) and the damping, $r_{\rm damp}$. The damping scale is the geometric mean of the photon mean free path and Hubble scale at decoupling, but at this time the two are forced to be of the same order of magnitude. The result is

$$r_{\rm damp} \sim r_{\rm p\gamma}\sqrt{R_{\rm dec}}. \tag{17}$$

The shape of the CMB spectrum is determined by the distance-independent ratios of the scales in the second column of Table 1, along with $R_{\rm dec}$:

$$R_{\rm dec} = \frac{m_{\rm p}\eta_{\rm b}}{\alpha^2 m_{\rm e} f_{\rm dec}} \sim \left(\frac{r_{\rm damp}}{r_{\rm p\gamma}}\right)^2 \tag{18}$$

$$\frac{r_{\rm A}}{r_{\rm eq}} = \left(\frac{m_\chi\eta_\chi + m_{\rm p}\eta_{\rm b}}{m_{\rm p}\eta_{\rm b}}\right)^{1/2} F_{\rm A}(R_{\rm dec}, R_{\rm eq}). \tag{19}$$

Apart from the weak dependence on ΔN_ν and $n_{\rm s}$, we see that the spectrum shape is determined by two parameters, $m_{\rm p}\eta_{\rm b}/m_\chi\eta_\chi$ and $\alpha^2 m_{\rm e}/m_{\rm p}\eta_{\rm b}$. Note that N_ν enters both in the radiation-matter ratio (through $r_{\rm eq}$) and in the neutrino-photon ratio (through B in Eq. (8)) so it cannot be absorbed into the other two parameters.

While we are primarily concerned with distance-independent features in the CMB spectrum, for completeness, we note that the use of the angular positions of the features induced by these three scales requires the introduction of the fourth length scale, the distance to the last-scattering surface. For flat-ΛCDM models, this is give by

$$D(z_{\rm dec}) = \frac{1}{\sqrt{\Omega_{\rm M} H_0^2}} \int_0^{z_{\rm dec}} \frac{{\rm d}z}{[(1-\Omega_{\rm M})/\Omega_{\rm M} + (1+z)^3]^{1/2}}. \tag{20}$$

Most of the integral is in the matter dominated redshift range and the integral is not far from its value, 1.94, for $\Omega_{\rm M} = 1$. We therefore write

$$D(z_{\rm dec}) = \frac{1.94}{\sqrt{\Omega_{\rm M} H_0^2}}[1 - f_0(\Omega_{\rm M})] \tag{21}$$

where the small correction ranges from $f_0(1) = 0$ to $f_0(0.2) = 0.13$.

In terms of our adopted cosmological parameters, the distance is given by

$$D(z_{\rm dec})^{-1} = \frac{0.82T_0}{1-f_0}\left(G_0(m_{\chi0}\eta_\chi + m_{\rm p0}\eta_{\rm b})\, m_{\rm p0}\frac{T_0}{m_{\rm p0}}\right)^{1/2}. \tag{22}$$

The distance depends on the dimensionless combinations of parameters $G_0 m_{\chi0} m_{\rm p0}$ and $G_0 m_{\rm p0}^2$ and on the measured ratio of the temperature and the proton mass.

The angular scales associated with the three distance scales are the ratios between the length scales and $D(z_{\rm dec})$. Usually, one refers to the peaks in ℓ-space which are near harmonics of $D(z_{\rm dec})/r_{\rm A}$. Using (22) and (12) we get

$$\frac{D(z_{\rm dec})}{r_{\rm A}} \sim \left(\frac{Gm_\chi m_{\rm p}}{G_0 m_{\chi0} m_{\rm p0}}\right)^{1/2}\left(\frac{\eta_{\rm b}}{T_0/m_{\rm p0}}\right)^{1/2}\frac{1-f_0}{F_{\rm A}} \times \left(\frac{1+m_{\rm p}\eta_{\rm b}/m_\chi\eta_\chi}{1+m_{\rm p0}\eta_{\rm b}/m_{\chi0}\eta_\chi}\right)^{1/2}. \tag{23}$$

The angular scale thus depends on the ratio of $Gm_\chi m_{\rm p}$ in the early universe to the same quantity today.

4. Analysis of CMB spectra

We now reverse the discussion in the previous section and discuss the information that can be obtained from the study of the observed CMB spectrum. What one deduces depends on the assumptions made about the time-dependence of the fundamental constants and about the characteristics of the dark energy. We consider the three cases: (1) flat-ΛCDM and no variations of the constants, (2) flat-ΛCDM with variations of α and $m_{\rm e}/m_{\rm p}$ but none of m_χ or G, and (3) all variations allowed and no assumptions on the dark energy or curvature.

The first case corresponds to the standard CMB studies that assume no variations and $N_\nu = 3$, (e.g. Planck Collaboration XIII 2015a). The CMB spectrum shape can be fit to determine $m_{\rm p}\eta_{\rm b}/m_\chi\eta_\chi$ and $\alpha^2 m_{\rm e}/m_{\rm p}\eta_{\rm b}$. Imposing the low-redshift value of α, $m_{\rm e}$ and $m_{\rm p}$ then determines $\eta_{\rm b}$ and $m_\chi\eta_\chi$. Then assuming no evolution of $m_\chi\eta_\chi$ and using $G = G_0$ one determines $\Omega_{\rm B}H_0^2 \propto m_{\rm p}\eta_{\rm b}$ and $\Omega_\chi H_0^2 \propto m_\chi\eta_\chi$. This is consistent the well-known fact that the CMB shape determines precisely these two cosmological parameters, if one assumes that the fundamental constants have not varied. That they are determined only by the shape is attested by the fact that fits allowing curvature do not change significantly the central values or errors on $\Omega_{\rm B}h^2$, $\Omega_{\rm M}h^2$ or $r_{\rm A}$ (Planck Collaboration XVI 2014) Allowing curvature would permit compensating changes in $D(z_{\rm dec})$ and $r_{\rm A}$ so as to maintain the angular scale, but this is not seen because it is the shape that determines $(\Omega_{\rm B}h^2, \Omega_\chi h^2)$ and, hence, $r_{\rm A}$. We note, however, that not requiring $N_\nu = 3$ increases $\Omega_\chi h^2$ by $\sim 5\%$ and doubles its error. These changes, and the corresponding changes in $r_{\rm A}$ are sufficiently small to ignore for the limits we find in Sect. 5.

The second case corresponds to the traditional studies of time variations, e.g. Planck Collaboration Int. XXIV (2015b), where one does not impose the local values of α or $m_{\rm e}/m_{\rm p}$. In this case, the shape-determined values of $m_{\rm p}\eta_{\rm b}/m_\chi\eta_\chi$ and $\alpha^2 m_{\rm e}/m_{\rm p}\eta_{\rm b}$ are not sufficient to separately measure the cosmological and fundamental parameters. These studies therefore also use the angular scale, assuming that it is given by the flat-ΛCDM result (23) and assume that $Gm_\chi m_{\rm p}$ has not varied in time. In this case, Eq. (23) provides a third constraint, determining $\eta_{\rm b}$. The shape-determined value of $\alpha^2 m_{\rm e}/m_{\rm p}\eta_{\rm b}$ then determines $\alpha^2 m_{\rm e}/m_{\rm p}$. This pre-recombination value can then be compared with the $(\alpha^2 m_{\rm e}/m_{\rm p})_0$. This is a simplified version of what is done in traditional CMB studies of time variations. Studies using WMAP data (Ichikawa et al. 2006; Scóccola et al. 2008, 2009; Nakashima et al. 2010; Landau & Scóccola 2010; Scóccola et al. 2013) confirm that in the $(\alpha, m_{\rm e})$ space, the best determined combination is indeed $\sim\alpha^2 m_{\rm e}$. (Those studies assume a fixed $m_{\rm p}$.)

The *Planck* data extends to sufficiently high ℓ to give tight constraints on other combinations of (α, m_e) (Planck Collaboration Int. XXIV 2015b).

We now turn to the last case, what can be learned if one makes no assumptions about the time variations of the fundamental constants or the dark energy. Lacking a consistent analysis of the CMB spectrum leaving all constants free, we must look for scaling relations that say how the announced results would be modified if variations are allowed. Equation (18) suggests that the CMB measurement[3] of $m_p\eta_b$ ($\propto \Omega_B h^2 = 0.02222 \pm 0.00023$) comes from the baryon-photon ratio R_{dec} and should therefore be understood as a measurement of $m_p\eta_b/\alpha^2 m_e$, if we ignore the weak parameter dependence of f_{dec}. We can interpret the CMB measurement as

$$\left(m_p\eta_b\right)_{no-var} = m_p\eta_b \frac{(\alpha^2 m_e)_0}{\alpha^2 m_e} \tag{24}$$

where the subscript no−var refers to values reported assuming no time variations. This formula should be regarded as a first-order approximation, since we neglect the logarithmic dependence of f_{dec} on the parameters. CMB studies convert $m_p\eta_b$ to $\Omega_B H_0^2$ using the laboratory value of Newton's constant:

$$\left(\Omega_B h^2\right)_{no-var} = \frac{2.04 T_0^3 G_0 m_p \eta_b}{(100\ \mathrm{km\,s^{-1}\,Mpc^{-1}})^2} \frac{(\alpha^2 m_e)_0}{\alpha^2 m_e} = 0.02222 \pm 0.00023 \tag{25}$$

where $h = H_0/100\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. The baryon mass fraction measured with the CMB spectrum does not use the value of the proton mass measured at low redshift so

$$\left(\frac{\Omega_B h^2}{\Omega_\chi h^2}\right)_{no-var} = \frac{m_p\eta_b}{m_\chi\eta_\chi} = 0.1856 \pm 0.004. \tag{26}$$

This implies with (25)

$$\left(\Omega_\chi h^2\right)_{no-var} = \frac{2.04 T_0^3 G_0 m_\chi \eta_\chi}{(100\ \mathrm{km\,s^{-1}\,Mpc^{-1}})^2} \frac{(\alpha^2 m_e)_0}{\alpha^2 m_e} = 0.1197 \pm 0.0022. \tag{27}$$

Finally, expressing r_A in (12) in terms of the directly measured quantities $\alpha^2 m_e/m_p\eta_b$ and $m_p\eta_b/m_\chi\eta_\chi$, one finds

$$(r_A)_{no-var} = r_A \left(\frac{(G m_e^2 \alpha^4)_0}{G m_e^2 \alpha^4}\right)^{1/2} = (147.33 \pm 0.49)\ \mathrm{Mpc}. \tag{28}$$

Relations (26)–(28) are used in the next section to set limits on time variations of the fundamental constants.

5. Limits on time variations

The CMB derived values in the expressions (26)–(28) can be compared with measurements of the analogous quantities at low redshift to set limits on time variations of the fundamental constants that appear in the expressions. The fact that measurements of cosmological parameters generally agree with the "concordance ΛCDM model" at the 10% level tells us to expect constraints at this level. All of these limits use the locally measured value of the Hubble constant: $H_0 = (72 \pm 3)\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ (Humphreys et al. 2013).

[3] We use throughout the "TT+lowP" values from Planck Collaboration XIII (2015a).

The most direct limit comes from comparing (26) with the same quantity derived from the baryon mass-fraction in galaxy clusters. Mantz et al. (2014) found $h^{3/2}\Omega_B/\Omega_M = 0.089 \pm 0.012$, implying $\Omega_B/\Omega_M = 0.145 \pm 0.02$ and

$$\frac{\Omega_B}{\Omega_\chi} \equiv \frac{m_{p0}\eta_b}{m_{\chi 0}\eta_\chi} = 0.170 \pm 0.023. \tag{29}$$

This measurement assumes that galaxy clusters are sufficiently large to contain a representative sample of all massive species, an assumption justified by simulations of structure formation. Dividing (26) by (29) and assuming that η_b and η_χ are time independent gives

$$\frac{m_p/m_\chi}{m_{p0}/m_{\chi 0}} = 1.09 \pm 0.15. \tag{30}$$

While we do not know the value of m_χ, this shows that it is stable in time, relative to the proton mass. We note however, that there is a controversy concerning cluster masses (Simet et al. 2015) so this result should be considered as provisional.

The use of Eq. (27) is delicate because there are no direct low-redshift measurements of the matter density as there are of the photon density. The simplest constraints come from Hubble diagrams using type Ia supernovae or the baryon-acoustic-oscillation (BAO) standard ruler. These measurements of the matter density are, of course, complicated by the fact that dark-energy dominates at low redshift so the deceleration expected from matter turns out to be an acceleration. It is necessary to make some simplifying assumptions about the dark energy and we make the usual assumption that it is sufficiently well described by a cosmological constant, though we make no assumptions about the curvature, i.e. we do not require $\Omega_M + \Omega_\Lambda = 1$.

The most useful measurements for our purpose is the BAO Hubble diagram unconstrained by the CMB calibration of r_A. The physics that leads to the peaks in the CMB spectrum also generates the BAO peak seen in the correlation function of tracers of the matter density. While the non-linear processes leading to structure formation make the correlation function more complicated to interpret than the CMB spectrum, the position of the BAO peak is believed to be placed reliably at r_A to a precision of better than 1%. Unlike the CMB spectrum which is only observed in the transverse (angular) direction, the BAO feature can be observed in both the transverse and radial (redshift) directions. The observable peaks in (redshift,angle) space in the correlation function at redshift z are at

$$\Delta\theta_{BAO} = \frac{r_A}{D(z)} \qquad \Delta z_{BAO} = \frac{r_A}{c/H(z)} \tag{31}$$

where we ignore the small difference between r_A and r_d, the sound horizon at the drag epoch (slightly after photon decoupling). If averaged over all directions (longitudinal and transverse), the BAO peak measures $r_A/D_V(z)$ where $D_V(z)^3 \equiv (z/H(z))D(z)^2$.

Using the available measurements of $D(z)/r_A$ and $c/H(z)/r_A$ one can fit for the two density parameters $(\Omega_M, \Omega_\Lambda)$ and the sound horizon relative to the present Hubble scale $(c/H_0/r_A)$. The results (Fig. 3 of Aubourg et al. 2014) is

$$\Omega_M = 0.29 \pm 0.05 \qquad \frac{c/H_0}{r_A} = 29 \pm 1. \tag{32}$$

We note that the sensitivity for Ω_M is enhanced by the measurement of $c/H(z = 2.34)/r_A = 9.18 \pm 0.28$ by Delubac et al. (2015) at a redshift where the universe is expected to be matter dominated. The precise measurement of $c/H_0 r_A$ is driven by the $r_A/D_V(z = 0.106) = 0.336 \pm 0.015$ from (Beutler et al. 2011) at a redshift where all distances are to good approximation proportional to c/H_0.

Using $H_0 = (72 \pm 3)$ km s^{-1} Mpc^{-1} (Humphreys et al. 2013) gives

$$\Omega_M h^2 = 0.150 \pm 0.026 \qquad r_A = 143.5 \pm 5.9. \tag{33}$$

Removing the baryonic component from $\Omega_M h^2$ gives $\Omega_\chi h^2 = 0.128 \pm 0.021 \propto G_0 \eta_\chi m_{\chi 0}$. Comparing this value with the *Planck* result (27) gives

$$\frac{\left(\Omega_\chi H_0^2\right)_{\text{no-var}}}{\left(\Omega_\chi H_0^2\right)_{low-z}} = \frac{\left(\alpha^2 m_e/m_\chi\right)_0}{\alpha^2 m_e/m_\chi} = 0.93 \pm 0.16. \tag{34}$$

Finally, comparing the CMB calculated sound horizon (28) with the low-redshift value (33), we get

$$\frac{r_A}{(r_A)_{\text{no-var}}} = \left(\frac{G m_e^2 \alpha^4}{(G m_e^2 \alpha^4)_0}\right)^{1/2} = 0.97 \pm 0.04. \tag{35}$$

The three limits (30), (34), and (35) exhaust the information that we can obtain from the three-scale model. For example, we could derive a limit analogous to (35) with r_{eq} instead of r_A using a the position of r_{eq} in the matter power spectrum at low redshift (Padmanabhan et al. 2007; Blake et al. 2007). However, this would not give an independent limit since we have already used the ratio r_{eq}/r_A in the other limits.

The three limits can be combined to limit time variations on other interesting combinations, like $G m_\chi^2$ and $G m_p^2$. In fact, the limits can be summarized as excluding large variations of all ratios of the four mass scales that enter the problem:

$$\frac{m_i/m_j}{(m_i/m_j)_0} \sim 1.0 \pm \sim 0.15 \qquad m_i, m_j = m_{pl},\ m_\chi,\ m_p,\ \alpha^2 m_e \tag{36}$$

where the *Planck* mass is $m_{pl} = \sqrt{\hbar c/G}$. The 15% precision on these limits is dominated by the precision of the low-redshift measurements and relatively insensitive to small modifications of the pre-recombination physics. For example, not requiring $N_\nu = 3$ increases the uncertainty in the CMB-derived CDM density to ~5%, still small compared to the low-redshift uncertainties.

Our limits assume that there are no large changes in the fundamental constants during late times that would invalidate the interpretation of the low-redshift measurements. They could therefore be evaded if the late-time variations somehow canceled the pre-recombination variations. All three limits use distance-ladder measurements of H_0 and the use of this ladder assumes no variations of the electromagnetic or gravitational interactions of ordinary matter, which would affect the luminosities of Cepheid variable stars and supernovae. There are strict limits on variations of such interactions at the level of 10^{-12} yr^{-1} for gravitational interactions (Williams et al. 2004) and 10^{-16} yr^{-1} for electromagnetic interactions (Uzan 2011). These are stronger that those presented here which are of order 10^{-11} yr^{-1}. This suggests that the limit (35), which uses only the distance ladder, is insensitive to our assumption of no low-redshift variations. On the other hand, the two other limits use the gravitational interaction of dark-matter particles in galaxy clusters and in cosmological deceleration. As such, one cannot appeal to strong limits on current variations to argue against compensating variations. Most conservatively, the limits (30), (34), and (35), should then be interpreted as constraints on theories that predict both early- and late-time variations.

6. Conclusion

The prime motivation of this study was to clear up the question of what fundamental constants determine the CMB anisotropy spectrum and to show that they consist of dimensionless combinations. In this context, the striking result of this study is seen in the second column of Table 1: all three length scales of the CMB spectrum, after redshifting to the present epoch, depend on dimensionless combinations of constants in the pre-recombination universe. Before the redshifting, the dimensionality was contained in the fundamental constants. The redshifting transferred the inverse-length dimension to T_0. This means that even if the distance to the last-scattering surface were somehow known, the angular features would depend only on dimensionless combinations in the pre-recombination universe.

In fact, the distance to the last scattering surface must be calculated. For the flat-ΛCDM model, it is shown in the fourth line of Table 1. It also depends on a dimensionless combinations, this time at the present epoch. This came about by the "trick" of writing $G m_{\chi 0} T_0$ as $G m_{\chi 0} m_{p0} \times T_0/m_{p0}$. This just corresponds to our freedom to express measured quantities like T_0 as multiples of fundamental quantities. In fact, this "freedom" is an obligation since it takes into account the dependence of our SI standards on fundamental constants. Expressing results in such manifestly dimensionless forms avoids all discussion about what units are being used.

The transfer of the inverse-length dimension to T_0 works for any standard ruler, so our conclusion that only dimensionless combinations are relevant for length scales is quite general. A similar reasoning works for standard candles (Rich 2013). For example, if one can express the total energy output of a supernova, Q_{SN}, in terms of fundamental constants (e.g. $Q_{SN} \sim (m_{Pl}/m_p)^3 Q_{56}$, where Q_{56} is the energy liberated in the β-decay of ^{56}Co), then one can also work with the dimensionless energy output, $Q_{SN}/\alpha^2 m_e$. This quantity gives the number of photons that would be produced if all energy were converted to Lyα photons. It can be related to the true number of photons by scaling by the observed ratio of the mean supernova photon energy to the energy of Lyα photons from the same redshift. Therefore, the supernova photon output depends only on the dimensionless combination $Q_{SN}/\alpha^2 m_e$ and a directly measurable energy ratio.

The CMB observables studied here are the distance independent quantities (18) and (19) which provide a tidy way of summarizing the first-order cosmological and physical information contained in the CMB spectrum. The combinations of parameters seen in these expressions reflect the degeneracies between fundamental and cosmological parameters that can be broken by explicitly assuming a flat-ΛCDM, constant-G model (Planck Collaboration Int. XXIV 2015b). Here, we have shown how combinations of CMB data with low-redshift measurements of cosmological parameters lead to the more model-independent limits summarized by (36). It will be a challenge to incorporate these qualitative results into a rigorous analysis of the CMB spectrum. Such an analysis would certainly modify two of the scaling relations we have used, (27) and (28), because of the complications in the α dependence of recombination that we have not taken into account. This would modify the effective dimensionless combination of constants that are probed so the limits (34) and (35) should be viewed as first order results. The limit (30) is more robust cosmologically because baryons and CDM enter the system only through their densities. In this case, the limit is accurate only to the extent that the interpretation of the low-redshift data is reliable.

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