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Combined calculi for photon orbital and spin angular momenta

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ABSTRACT

Context. Wavelength, photon spin angular momentum (PSAM), and photon orbital angular momentum (POAM), completely describe the state of a photon or an electric field (an ensemble of photons). Wavelength relates directly to energy and linear momentum, the corresponding kinetic quantities. PSAM and POAM, themselves kinetic quantities, are colloquially known as polarization and optical vortices, respectively. Astrophysical sources emit photons that carry this information.

Aims. PSAM characteristics of an electric field (intensity) are compactly described by the Jones (Stokes/Mueller) calculus. Similarly, I created calculi to represent POAM characteristics of electric fields and intensities in an astrophysical context. Adding wavelength dependence to all of these calculi is trivial. The next logical steps are to 1) form photon total angular momentum (PTAM = POAM + PSAM) calculi; 2) prove their validity using operators and expectation values; and 3) show that instrumental PSAM can affect measured POAM values for certain types of electric fields.

Methods. I derive the PTAM calculi of electric fields and intensities by combining the POAM and PSAM calculi. I show how these quantities propagate from celestial sphere to image plane. I also form the PTAM operator (the sum of the POAM and PSAM operators), with and without instrumental PSAM, and calculate the corresponding expectation values.

Results. Apart from the vector, matrix, dot product, and direct product symbols, the PTAM and POAM calculi appear superficially identical. I provide tables with all possible forms of PTAM calculi. I prove that PTAM expectation values are correct for instruments with and without instrumental PSAM. I also show that POAM measurements of "unfactored" PTAM electric fields passing through non-zero instrumental circular PSAM can be biased.

Conclusions. The combined PTAM calculi provide insight into mathematically modeling PTAM sources and calibrating POAM- and PSAM-induced measurement errors.

Key words. instrumentation: miscellaneous – methods: analytical – methods: miscellaneous – methods: observational – techniques: miscellaneous

1. Introduction

Elias (2008) derived propagation calculi to describe astronomical photon orbital angular momentum (POAM; colloquially known as optical vortices). He employed a semi-classical/semi-quantum framework where electric fields are analogous to photon wave functions and intensities are analogous to probabilities. These calculi link POAM quantities on the celestial sphere to POAM quantities at instrument backends. He tacitly assumes that the electric fields on the celestial sphere are spatially uncorrelated (the "Standard Astronomical Assumption", or SAA). Elias (2012) used these calculi to describe POAM and torque metrics for single telescopes and interferometers.

Like most other workers in the POAM field, Elias (2008, 2012) dealt only with optical systems that ignored photon spin angular momentum (PSAM; colloquially known as polarization), in order to simplify calculations. Since POAM and PSAM are complementary properties that will eventually be measured simultaneously, combined calculi are required for modeling source and instrument behavior. Failing to take non-zero PSAM into account will yield incorrect POAM values under certain conditions.

The goals and results of this work are multiple. First, I present the simplest and most general photon total angular momentum (PTAM = POAM + PSAM) electric field forms. Second, I combine the POAM (Elias 2008) and PSAM propagation calculi to create the PTAM propagation calculi. Third, I create the POAM, PSAM, and PTAM operators and calculate the corresponding expectation values for perfect and imperfect instruments. Last, I show that POAM measurements can be biased when unfactored electric fields pass through non-zero instrumental circular PSAM.

2. Electric fields

Elias (2008, 2012) treated the electric field as a scalar quantity when he constructed the POAM state expansions

$$E(\vec{\mathbf{H}};t) = \sum_{m=-\infty}^{\infty} E_m(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad E_m(H;t) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\chi\,\mathrm{e}^{-\mathrm{j}m\chi}\,E(\vec{\mathbf{H}};t), \tag{1}$$

where $\vec{\mathbf{H}} = (H\cos\chi, H\sin\chi)$ is the vector in a plane (e.g., celestial sphere, image plane, etc.), t is time, the m are the POAM quantum numbers ($-\infty \leq m \leq \infty$), and the $E_m(H;t)$ are the POAM states (azimuthal Fourier components of the electric field) for each radius (perpendicular to the propagation direction). An azimuthal Fourier series is performed for each H and t.

The standard way of describing the PSAM behavior of an electric field is the Jones vector

$$\vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \begin{bmatrix} E_\mathrm{R}(\vec{\mathbf{H}};t) \\ E_\mathrm{L}(\vec{\mathbf{H}};t) \end{bmatrix}, \tag{2}$$

where $E_\mathrm{R}(\vec{\mathbf{H}};t)$ and $E_\mathrm{L}(\vec{\mathbf{H}};t)$ are the right-circular and left-circular components. The circular PSAM basis is ideal for the subsequent analyses of this paper.

A "factored" electric field, where the POAM and PSAM parts are separate factors, is the simplest and likely the most well known PTAM form

$$\vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \vec{\epsilon}_\mathrm{POAM}(\vec{\mathbf{H}};t)\,\epsilon_\mathrm{PSAM}(\vec{\mathbf{H}};t) = \begin{bmatrix} \epsilon_\mathrm{R}(\vec{\mathbf{H}};t) \\ \epsilon_\mathrm{L}(\vec{\mathbf{H}};t) \end{bmatrix} \sum_{m=-\infty}^{\infty} \epsilon_M(H;t)\,\mathrm{e}^{\mathrm{j}m\chi}. \tag{3a}$$

The most general "unfactored" PTAM form, on the other hand, comes from independently expanding each PSAM component into POAM components, or

$$\vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \sum_{m=-\infty}^{\infty} \vec{\mathbf{E}}_m(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} = \sum_{m=-\infty}^{\infty} \begin{bmatrix} E_{\mathrm{R},m}(H;t) \\ E_{\mathrm{L},m}(H;t) \end{bmatrix} \mathrm{e}^{\mathrm{j}m\chi} = \begin{bmatrix} \sum_{m=-\infty}^{\infty} E_{\mathrm{R},m}(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} \\ \sum_{m=-\infty}^{\infty} E_{\mathrm{L},m}(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} \end{bmatrix}. \tag{3b}$$

Although POAM and PSAM appear completely intertwined, these expressions work with the PTAM calculi (Sect. 3) and lead to the correct operators and expectation values (Sects. 4 and 5). This type of electric field can be prepared in the laboratory. At present there are no known astrophysical mechanisms that generate unfactored PTAM, but I use this form anyway for the sake of mathematical completeness and in the event that such mechanisms will eventually be found.

3. PTAM calculi

Elias (2008) created POAM propagation calculi for electric fields and intensities using SAA. He also treated the electric fields as scalars, ignoring PSAM. Their time-averaged square magnitudes are intensities, which are analogous to Stokes I.

In this section, I combine the Elias (2008) POAM calculi with the electric-field PSAM calculi of Jones (1941) and the intensity PSAM calculi of Stokes (1852) and Mueller (1948) to form the PTAM calculi. I also employ the mathematics of Schmeider (1969) and Barakat (1981), hereafter collectively SB, to more easily link Jones vectors, Jones matrices, Stokes vectors, and Mueller matrices via direct products and coherence matrices (assuming no system depolarization).

3.1. POAM correlations

According to SB, the coherence vector is the direct product of the electric field from Eq. (2)

$$\vec{\mathbf{C}}(\vec{\mathbf{H}}) = \left\langle \frac{1}{2}\vec{\mathbf{E}}(\vec{\mathbf{H}};t) \otimes \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle = \begin{bmatrix} \left\langle \frac{1}{2}E_\mathrm{R}(\vec{\mathbf{H}};t)E_\mathrm{R}^*(\vec{\mathbf{H}};t) \right\rangle \\ \left\langle \frac{1}{2}E_\mathrm{R}(\vec{\mathbf{H}};t)E_\mathrm{L}^*(\vec{\mathbf{H}};t) \right\rangle \\ \left\langle \frac{1}{2}E_\mathrm{L}(\vec{\mathbf{H}};t)E_\mathrm{R}^*(\vec{\mathbf{H}};t) \right\rangle \\ \left\langle \frac{1}{2}E_\mathrm{L}(\vec{\mathbf{H}};t)E_\mathrm{L}^*(\vec{\mathbf{H}};t) \right\rangle \end{bmatrix} = \begin{bmatrix} C_\mathrm{R,R}(\vec{\mathbf{H}}) \\ C_\mathrm{R,L}(\vec{\mathbf{H}}) \\ C_\mathrm{L,R}(\vec{\mathbf{H}}) \\ C_\mathrm{L,L}(\vec{\mathbf{H}}) \end{bmatrix} = \begin{bmatrix} C_\mathrm{R,R}(\vec{\mathbf{H}}) \\ C_\mathrm{R,L}(\vec{\mathbf{H}}) \\ C_\mathrm{R,L}^*(\vec{\mathbf{H}}) \\ C_\mathrm{L,L}(\vec{\mathbf{H}}) \end{bmatrix}, \tag{4}$$

where $\otimes$ is the direct (outer) product, and $\langle\,\rangle$ is the time average. If I substitute Eq. (3b) (instead of Eq. (2)) into Eq. (4), I obtain

$$\vec{\mathbf{C}}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \left\langle \frac{1}{2}\vec{\mathbf{E}}_m(H;t) \otimes \vec{\mathbf{E}}_n^*(H;t) \right\rangle \mathrm{e}^{\mathrm{j}(m-n)\chi} = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \vec{\mathbf{C}}_{m,n}(H)\,\mathrm{e}^{\mathrm{j}(m-n)\chi},$$

where

$$\vec{\mathbf{C}}_{m,n}(H) = \begin{bmatrix} \left\langle \frac{1}{2}E_{\mathrm{R},m}(H;t)E_{\mathrm{R},n}^*(H;t) \right\rangle \\ \left\langle \frac{1}{2}E_{\mathrm{R},m}(H;t)E_{\mathrm{L},n}^*(H;t) \right\rangle \\ \left\langle \frac{1}{2}E_{\mathrm{L},m}(H;t)E_{\mathrm{R},n}^*(H;t) \right\rangle \\ \left\langle \frac{1}{2}E_{\mathrm{L},m}(H;t)E_{\mathrm{L},n}^*(H;t) \right\rangle \end{bmatrix} = \begin{bmatrix} C_{(\mathrm{R},m),(\mathrm{R},n)}(H) \\ C_{(\mathrm{R},m),(\mathrm{L},n)}(H) \\ C_{(\mathrm{L},m),(\mathrm{R},n)}(H) \\ C_{(\mathrm{L},m),(\mathrm{L},n)}(H) \end{bmatrix} \tag{5a}$$

is the (m, n)th POAM correlation of the coherence vector. By comparing Eqs. (4) and (5a-b), I find that the individual correlations can be expanded into double sums

$$C_{\rm R,R}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{({\rm R},m),({\rm R},n)}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{6a}$$

$$C_{\rm R,L}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{({\rm R},m),({\rm L},n)}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{6b}$$

$$C_{\rm L,R}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{({\rm L},m),({\rm R},n)}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{6c}$$

and

$$C_{\rm L,L}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{({\rm L},m),({\rm L},n)}(H)\, {\rm e}^{{\rm j}(m-n)\chi}. \tag{6d}$$

The $\vec{\mathbf{C}}_{m,n}(H)$ have a similar form to and the identical units as the $I_{m,n}(H)$ POAM correlations defined by Elias (2008).

Coherence vectors are used mostly by engineers because some instruments, such as radio interferometers, employ right- and left-circular feeds. Astronomers prefer Stokes vectors because their components represent the total intensity and the polarization parameters required for scientific analysis. The Stokes vector is related to the coherence vector via a simple matrix transformation (SB)

$$\vec{\mathbf{S}}(\vec{\mathbf{H}}) = \overleftrightarrow{\mathbf{T}} \cdot \vec{\mathbf{C}}(\vec{\mathbf{H}}) = \begin{bmatrix} C_{\rm R,R}(\vec{\mathbf{H}}) + C_{\rm L,L}(\vec{\mathbf{H}}) \\ C_{\rm R,L}(\vec{\mathbf{H}}) + C^*_{\rm R,L}(\vec{\mathbf{H}}) \\ -{\rm j}\left[C_{\rm R,L}(\vec{\mathbf{H}}) - C^*_{\rm R,L}(\vec{\mathbf{H}})\right] \\ C_{\rm R,R}(\vec{\mathbf{H}}) - C_{\rm L,L}(\vec{\mathbf{H}}) \end{bmatrix} = \begin{bmatrix} C_{\rm R,R}(\vec{\mathbf{H}}) + C_{\rm L,L}(\vec{\mathbf{H}}) \\ 2\,{\rm Re}\,C_{\rm R,L}(\vec{\mathbf{H}}) \\ 2\,{\rm Im}\,C_{\rm R,L}(\vec{\mathbf{H}}) \\ C_{\rm R,R}(\vec{\mathbf{H}}) - C_{\rm L,L}(\vec{\mathbf{H}}) \end{bmatrix} = \begin{bmatrix} I(\vec{\mathbf{H}}) \\ Q(\vec{\mathbf{H}}) \\ U(\vec{\mathbf{H}}) \\ V(\vec{\mathbf{H}}) \end{bmatrix}, \tag{7a}$$

where $\cdot$ is the dot (inner) product, and

$$\overleftrightarrow{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -{\rm j} & {\rm j} & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \tag{7b}$$

is the coherence-to-Stokes transformation matrix in the circular basis. If I substitute Eqs. (5a-b) into Eq. (7a), I obtain

$$\vec{\mathbf{S}}(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\overleftrightarrow{\mathbf{T}} \cdot \vec{\mathbf{C}}_{m,n}(H)\right] {\rm e}^{{\rm j}(m-n)\chi} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \vec{\mathbf{S}}_{m,n}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{8a}$$

where

$$\vec{\mathbf{S}}_{m,n}(H) = \begin{bmatrix} C_{({\rm R},m),({\rm R},n)}(H) + C_{({\rm L},m),({\rm L},n)}(H) \\ C_{({\rm R},m),({\rm L},n)}(H) + C_{({\rm L},m),({\rm R},n)}(H) \\ -{\rm j}\left[C_{({\rm R},m),({\rm L},n)}(H) - C_{({\rm L},m),({\rm R},n)}(H)\right] \\ C_{({\rm R},m),({\rm R},n)}(H) - C_{({\rm L},m),({\rm L},n)}(H) \end{bmatrix} = \begin{bmatrix} I_{m,n}(H) \\ Q_{m,n}(H) \\ U_{m,n}(H) \\ V_{m,n}(H) \end{bmatrix} \tag{8b}$$

is the (m, n)th POAM correlation of the Stokes vector. By comparing Eqs. (7a) and (8a-b), I find that the individual Stokes parameters can be expanded into double sums

$$I(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{m,n}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{9a}$$

$$Q(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{m,n}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{9b}$$

$$U(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} U_{m,n}(H)\, {\rm e}^{{\rm j}(m-n)\chi}, \tag{9c}$$

and

$$V(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{m,n}(H)\, {\rm e}^{{\rm j}(m-n)\chi}. \tag{9d}$$

The $\vec{\mathbf{S}}_{m,n}(H)$ have a similar form to and the same units as the $I_{m,n}(H)$ POAM correlations defined by Elias (2008). As a matter of fact, Eq. (9a) is identical to the expansion derived by Elias (2008) using scalar electric fields.

The Stokes Q, U, and V expansions are unnecessary, so I rewrite the Stokes vector as

$$\vec{\mathbf{S}}(\vec{\mathbf{H}}) = \begin{bmatrix} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{m,n}(H)\, \mathrm{e}^{\mathrm{j}(m-n)\chi} \\ Q(\vec{\mathbf{H}}) \\ U(\vec{\mathbf{H}}) \\ V(\vec{\mathbf{H}}) \end{bmatrix}. \tag{10}$$

This PTAM form maintains both the POAM and PSAM information while minimizing complications.

3.2. POAM rancors

Elias (2008) defined a quantity called rancor, which is the azimuthal Fourier series versus radius of the intensity

$$I(\vec{\mathbf{H}}) = \sum_{m=-\infty}^{\infty} \mathcal{I}_m(H)\, \mathrm{e}^{\mathrm{j}m\chi} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \mathcal{I}_m(H) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\chi\, \mathrm{e}^{-\mathrm{j}m\chi}\, I(\vec{\mathbf{H}}), \tag{11a}$$

where $\mathcal{I}_m(H)$ is the mth POAM rancor. This quantity is interesting because it identical to the infinite sum over a subset of POAM correlations

$$\mathcal{I}_m(H) = \sum_{k=-\infty}^{\infty} I_{k,k-m}(H). \tag{11b}$$

Rancors, which may be easier to determine in some cases, contain a limited amount of POAM information. As an analogy, I point out that squared visbilities and closure phases in optical interferometry can provide important physical data about astronomical sources, in spite of the fact that they contain less information than complex visibilities.

In Sect. 3.1, I combined POAM correlations with PSAM Stokes vectors. Since rancors can be expressed in terms of correlations, it follows that all intensity formulae in Sect. 3.1 can be written in terms of rancors. I will not list all possible expressions here, since those expansions are identical to those in Elias (2008, Sects. 4, 5, Appendix C) apart from the fact that scalar quantities are replaced by vectors and matrices and scalar products are replaced by dot and direct products.

3.3. Propagating POAM quantities

Elias (2008) derived scalar electric-field and intensity calculi for propagating POAM from celestial sphere to image plane and listed them in several tables. He employed system forms and SAA. In this section, I extend these expressions to combine POAM and PSAM propagation calculi, thus creating PTAM propagation calculi.

Consider the system form for propagation of the scalar electric field from celestial sphere to image plane

$$E(\vec{\mathbf{\Omega}}'; t) = \int \mathrm{d}^2\Omega\, D(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}})\, E(\vec{\mathbf{\Omega}}; t), \tag{12a}$$

where $\vec{\mathbf{\Omega}}' = (\rho' \cos\phi', \rho' \sin\phi')$ is the coordinate in the image plane, $\vec{\mathbf{\Omega}} = (\rho\cos\phi, \rho\sin\phi)$ is the coordinate on the celestial sphere,

$$D(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) = \int \mathrm{d}^2 r\, \mathrm{e}^{j2\pi\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{\Omega}}' - \vec{\mathbf{\Omega}}\right)}\, D(\vec{\mathbf{r}}) \tag{12b}$$

is the diffraction function, $\mathrm{e}^{j2\pi\vec{\mathbf{r}}\cdot\left(\vec{\mathbf{\Omega}}' - \vec{\mathbf{\Omega}}\right)}$ is the Fraunhofer propagator (it can be replaced with the Fresnel propagator), $\vec{\mathbf{r}} = (r\cos\psi, r\sin\psi)$ is the coordinate in the pupil plane normalized by wavelength, and $D(\vec{\mathbf{r}})$ is the pupil function which describes the telescope aberrations, atmospheric turbulence, etc. If these scalar electric fields are changed to 2×1 Jones vectors, the diffraction function must become a 2×2 Jones matrix

$$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) = \int \mathrm{d}^2\Omega\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \cdot \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t). \tag{12c}$$

In principle, the scalar and matrix diffraction functions can also be functions of time, although their variability time scales are much slower than those of the electric fields.

If the Jones vector components are expanded into independent POAM states (unfactored form, Eq. (3b)), Eq. (12c) becomes the PTAM state expansion

$$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) = \sum_{p=-\infty}^{\infty} \vec{\mathbf{E}}_p(\rho'; t)\, \mathrm{e}^{\mathrm{j}p\phi'} = \sum_{p=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} 2\pi \int_0^{\infty} \mathrm{d}\rho\, \rho\, \overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho) \cdot \vec{\mathbf{E}}_m(\rho; t) \right] \mathrm{e}^{\mathrm{j}p\phi'}, \tag{13a}$$

Table 1. POAM expansions of $\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t)$ in terms of POAM expansions of $\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})$ (Table 2) and $\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t)$.

POAM expansion type	Expression
Input	$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t) = \sum_{m=-\infty}^{\infty} \vec{\mathfrak{E}}_m(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t)$ where $\vec{\mathfrak{E}}_m(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t) = 2\pi \int_0^{\infty} \mathrm{d}\rho\, \rho\, \overleftrightarrow{\mathbf{D}}^{-m}(\vec{\mathbf{\Omega}}', \rho; \vec{\mathbf{a}}) \cdot \vec{\mathbf{E}}_m(\rho; t)$
Output	$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t) = \sum_{p=-\infty}^{\infty} \vec{\mathbf{E}}_p(\rho'; \vec{\mathbf{a}}, t)\, \mathrm{e}^{\mathrm{j}p\phi'}$ where $\vec{\mathbf{E}}_p(\rho'; \vec{\mathbf{a}}, t) = \int \mathrm{d}^2\Omega\, \overleftrightarrow{\mathbf{D}}_p(\rho', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}}) \cdot \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t)$
Input/Output	$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t) = \sum_{p=-\infty}^{\infty} \vec{\mathbf{E}}_p(\rho'; \vec{\mathbf{a}}, t)\, \mathrm{e}^{\mathrm{j}p\phi'}$ where $\vec{\mathbf{E}}_p(\rho'; \vec{\mathbf{a}}, t) = \sum_{m=-\infty}^{\infty} 2\pi \int_0^{\infty} \mathrm{d}\rho\, \rho\, \overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}}) \cdot \vec{\mathbf{E}}_m(\rho; t)$

Notes. Vector $\vec{\mathbf{a}}$ is a generic representation of optional configuration parameters.

Table 2. POAM expansions of $\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})$.

POAM expansion	Expression
Input sensitivity:	
Integral form (forward)	$\overleftrightarrow{\mathbf{D}}^{-m}(\vec{\mathbf{\Omega}}', \rho; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi}\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty} \overleftrightarrow{\mathbf{D}}^{-m}(\vec{\mathbf{\Omega}}', \rho; \vec{\mathbf{a}})\, \mathrm{e}^{-\mathrm{j}m\phi}$
Output sensitivity:	
Integral form (forward)	$\overleftrightarrow{\mathbf{D}}_p(\rho', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \overleftrightarrow{\mathbf{D}}_p(\rho', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})\, \mathrm{e}^{\mathrm{j}p\phi'}$
Input/Output gain:	
Integral form (forward)	$\overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi}\, \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}})\, \mathrm{e}^{-\mathrm{j}m\phi}\, \mathrm{e}^{\mathrm{j}p\phi'}$

where

$$\vec{\mathbf{E}}_p(\rho'; t) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) \tag{13b}$$

is the output POAM state p,

$$\vec{\mathbf{E}}_m(\rho; t) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{-\mathrm{j}m\phi}\, \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t) \tag{13c}$$

is the input POAM state m, and

$$\overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi}\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \tag{13d}$$

is the diffraction function gain between output POAM state p and the input POAM state m. I summarize all PTAM electric field expansions in Tables 1 and 2. $\vec{\mathfrak{E}}_m(\vec{\mathbf{\Omega}}'; \vec{\mathbf{a}}, t)$ is not a true PTAM state, which means that the input expansion is of limited use but included for the sake of completeness.

The intensity is the squared magnitude of the electric field. Using SAA and Eq. (12a), the scalar intensity becomes

$$I(\vec{\mathbf{\Omega}}') = \left\langle \frac{1}{2} \left| E(\vec{\mathbf{\Omega}}'; t) \right|^2 \right\rangle = \int \mathrm{d}^2\Omega\, P(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}})\, I(\vec{\mathbf{\Omega}}), \tag{14a}$$

where $P(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) = \left| D(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \right|^2$ is the point-spread function (PSF), and $I(\vec{\mathbf{\Omega}}) = \left\langle \frac{1}{2} \left| E(\vec{\mathbf{\Omega}}; t) \right|^2 \right\rangle$. SAA collapses one of the integrals over the celestial sphere. If I employ Eqs. (4), (7a), and (12c) as well as SAA, the scalar Eq. (14a) becomes the vector equation

$$\begin{aligned} \vec{\mathbf{S}}(\vec{\mathbf{\Omega}}') &= \overleftrightarrow{\mathbf{T}} \cdot \left\langle \frac{1}{2} \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) \otimes \vec{\mathbf{E}}^*(\vec{\mathbf{\Omega}}'; t) \right\rangle = \int \mathrm{d}^2\Omega \left\{ \overleftrightarrow{\mathbf{T}} \cdot \left[\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \otimes \overleftrightarrow{\mathbf{D}}^*(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \right] \cdot \overleftrightarrow{\mathbf{T}}^{-1} \right\} \cdot \left\{ \overleftrightarrow{\mathbf{T}} \cdot \left\langle \frac{1}{2} \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t) \otimes \vec{\mathbf{E}}^*(\vec{\mathbf{\Omega}}; t) \right\rangle \right\} \\ &= \int \mathrm{d}^2\Omega\, \overleftrightarrow{\mathbf{P}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \cdot \vec{\mathbf{S}}(\vec{\mathbf{\Omega}}). \end{aligned} \tag{14b}$$

Table 3. POAM expansions of $\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}})$, for a spatially incoherent source, in terms of POAM expansions of $\overleftrightarrow{\mathbf{P}}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}};\vec{\mathbf{a}})$ (Table 4) and $\vec{\mathbf{S}}(\vec{\mathbf{\Omega}})$.

POAM expansion type	Expression
Input (correlated)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\vec{\mathfrak{S}}_{m,n}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}})$ where $\vec{\mathfrak{S}}_{m,n}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = 2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}^{-m+n}(\vec{\mathbf{\Omega}}',\rho;\vec{\mathbf{a}})\cdot\vec{\mathbf{S}}_{m,n}(\rho)$
Input (rancored)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}\vec{\mathfrak{S}}_{m}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}})$ where $\vec{\mathfrak{S}}_{m}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = 2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}^{-m}(\vec{\mathbf{\Omega}}',\rho;\vec{\mathbf{a}})\cdot\vec{\mathcal{S}}_{m}(\rho)$
Output (correlated/unexpanded)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}(p-q)\phi'}$ where $\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}}) = \int\mathrm{d}^2\Omega\,\overleftrightarrow{\mathbf{P}}_{p,q}(\rho',\vec{\mathbf{\Omega}};\vec{\mathbf{a}})\cdot\vec{\mathbf{S}}(\vec{\mathbf{\Omega}})$
Output (rancored/unexpanded)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}p\phi'}$ where $\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}}) = \int\mathrm{d}^2\Omega\,\overleftrightarrow{\mathcal{P}}_{p}(\rho',\vec{\mathbf{\Omega}};\vec{\mathbf{a}})\cdot\vec{\mathbf{S}}(\vec{\mathbf{\Omega}})$
Input/Output (correlated/correlated)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}(p-q)\phi'}$ where $\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}_{p,q}^{-m+n}(\rho',\rho;\vec{\mathbf{a}})\cdot\vec{\mathbf{S}}_{m,n}(\rho)$
Input/Output (correlated/rancored)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}(p-q)\phi'}$ where $\vec{\mathbf{S}}_{p,q}(\rho';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}_{p,q}^{-m}(\rho',\rho;\vec{\mathbf{a}})\cdot\vec{\mathcal{S}}_{m}(\rho)$
Input/Output (rancored/correlated)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}p\phi'}$ where $\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}_{p}^{-m+n}(\rho',\rho;\vec{\mathbf{a}})\cdot\vec{\mathbf{S}}_{m,n}(\rho)$
Input/Output (rancored/rancored)	$\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty}\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}})\,\mathrm{e}^{\mathrm{j}p\phi'}$ where $\vec{\mathcal{S}}_{p}(\rho';\vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty}2\pi\int_0^{\infty}\mathrm{d}\rho\,\rho\,\overleftrightarrow{\mathcal{P}}_{p}^{-m}(\rho',\rho;\vec{\mathbf{a}})\cdot\vec{\mathcal{S}}_{m}(\rho)$

Notes. Vector $\vec{\mathbf{a}}$ is a generic representation of optional configuration parameters.

The point spread function is now a 4×4 Mueller matrix. I summarize all PTAM intensity expansions in Tables 3 and 4. $\vec{\mathfrak{S}}_{m,n}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}})$ and $\vec{\mathfrak{S}}_{m}(\vec{\mathbf{\Omega}}';\vec{\mathbf{a}})$ are not true PTAM quantities, which means that the input expansions are of limited use but included for the sake of completeness. Also, note that the intensity equations are cannot be derived from the electric field equations when a system has depolarization (Mueller matrices cannot be uniquely determined from Jones matrices).

Now consider the Stokes-I parameter the image plane

$$\begin{aligned} I(\vec{\mathbf{\Omega}}') &= \hat{\boldsymbol{d}}^{\mathrm{T}}\cdot\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}') = \int\mathrm{d}^2\Omega\,\hat{\boldsymbol{d}}^{\mathrm{T}}\cdot\overleftrightarrow{\mathbf{P}}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})\cdot\vec{\mathbf{S}}(\vec{\mathbf{\Omega}}) \\ &= \int\mathrm{d}^2\Omega\,\mathbf{P}^{I,I}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})\,I(\vec{\mathbf{\Omega}}) + \int\mathrm{d}^2\Omega\,\mathbf{P}^{I,Q}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})\,Q(\vec{\mathbf{\Omega}}) + \int\mathrm{d}^2\Omega\,\mathbf{P}^{I,U}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})\,U(\vec{\mathbf{\Omega}}) + \int\mathrm{d}^2\Omega\,\mathbf{P}^{I,V}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})\,V(\vec{\mathbf{\Omega}}), \end{aligned} \tag{15}$$

where $\hat{\boldsymbol{d}}^{\mathrm{T}} = [1,0,0,0]$ is the detector operator, and the $\mathbf{P}^{I,x}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})$ are the elements of the top row of the Mueller matrix PSF. In Sects. 3.1 and 3.2, I point out that only the Stokes-I parameter must be expanded in terms of POAM correlations and rancors, even though the complete derivations involve POAM-like expansions of the other Stokes parameters. Similarly, only the upper-left element of the Mueller matrix $\mathbf{P}^{I,I}(\vec{\mathbf{\Omega}}',\vec{\mathbf{\Omega}})$ must be expanded in terms of POAM correlations or rancors. Equation (15) indicates that non-zero Stokes- Q, U, and V terms could introduce measurement biases which must be calibrated when measuring the POAM of the Stokes-I parameter. I present a simple example in Sect. 5 using operators and expectation values.

4. Operators and expectation values

Expectation values are specific quantities that can be measured by instruments. In this section, I: 1) define the POAM, PSAM, and PTAM operators; 2) derive the corresponding expectation values; 3) show how the operators and expectation values are modified by imperfect instruments.

Table 4. POAM expansions of $\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}})$.

POAM expansion	Expression
Input sensitivity (separate):	
Integral form (forward)	$\overleftrightarrow{\mathbf{P}}^{-m,-n}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) = \overleftrightarrow{\mathbf{T}} \cdot \left[\overleftrightarrow{\mathbf{D}}^{-m}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) \otimes \overleftrightarrow{\mathbf{D}}^{-n,*}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}})\right] \cdot \overleftrightarrow{\mathbf{T}}^{-1}$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}^{-m,-n}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}(m-n)\phi}$
Output sensitivity (separate):	
Integral form (forward)	$\overleftrightarrow{\mathbf{P}}_{p,q}(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \overleftrightarrow{\mathbf{T}} \cdot \left[\overleftrightarrow{\mathbf{D}}_p(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) \otimes \overleftrightarrow{\mathbf{D}}^{*}_q(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}})\right] \cdot \overleftrightarrow{\mathbf{T}}^{-1}$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{p,q}(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) \mathrm{e}^{\mathrm{j}(p-q)\phi'}$
Input/Output gain (separate):	
Integral form (forward)	$\overleftrightarrow{\mathbf{P}}_{p,q}^{-m,-n}(\rho', \rho; \vec{\mathbf{a}}) = \overleftrightarrow{\mathbf{T}} \cdot \left[\overleftrightarrow{\mathbf{D}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}}) \otimes \overleftrightarrow{\mathbf{D}}_q^{-n,*}(\rho', \rho; \vec{\mathbf{a}})\right] \cdot \overleftrightarrow{\mathbf{T}}^{-1}$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{p,q}^{-m,-n}(\rho', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}(m-n)\phi} \mathrm{e}^{\mathrm{j}(p-q)\phi'}$
Input sensitivity (combined):	
Integral form (forward)	$\overleftrightarrow{\mathcal{P}}^{-m}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi}\, \overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{k=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}^{-k,-k+m}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{m=-\infty}^{\infty} \overleftrightarrow{\mathcal{P}}^{-m}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}m\phi}$
Output sensitivity (combined):	
Integral form (forward)	$\overleftrightarrow{\mathcal{P}}_p(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{\mathrm{L}=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{\mathrm{L,l-p}}(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \overleftrightarrow{\mathcal{P}}_p(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) \mathrm{e}^{\mathrm{j}p\phi'}$
Input/Output gain (combined #1):	
Integral form (forward)	$\overleftrightarrow{\mathcal{P}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{k=-\infty}^{\infty} \sum_{\mathrm{L}=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{\mathrm{L,l-p}}^{-k,-k+m}(\rho', \rho; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overleftrightarrow{\mathcal{P}}_p^{-m}(\rho', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}m\phi} \mathrm{e}^{\mathrm{j}p\phi'}$
Input/Output gain (combined #2):	
Integral form (forward)	$\overleftrightarrow{\mathcal{P}}_{p,q}^{-m}(\rho', \rho; \vec{\mathbf{a}}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{j}m\phi}\, \overleftrightarrow{\mathbf{P}}_{p,q}(\rho', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{k=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{p,q}^{-k,-k+m}(\rho', \rho; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \overleftrightarrow{\mathcal{P}}_{p,q}^{-m}(\rho', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}m\phi} \mathrm{e}^{\mathrm{j}(p-q)\phi'}$
Input/Output gain (combined #3):	
Integral form (forward)	$\overleftrightarrow{\mathcal{P}}_p^{-m,-n}(\rho', \rho) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi'\, \mathrm{e}^{-\mathrm{j}p\phi'}\, \overleftrightarrow{\mathbf{P}}^{-m,-n}(\vec{\boldsymbol{\Omega}}', \rho; \vec{\mathbf{a}}) = \sum_{\mathrm{L}=-\infty}^{\infty} \overleftrightarrow{\mathbf{P}}_{\mathrm{L,l-p}}^{-m,-n}(\rho', \rho; \vec{\mathbf{a}})$
Sum form (reverse)	$\overleftrightarrow{\mathbf{P}}(\vec{\boldsymbol{\Omega}}', \vec{\boldsymbol{\Omega}}; \vec{\mathbf{a}}) = \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \overleftrightarrow{\mathcal{P}}_p^{-m,-n}(\rho', \rho; \vec{\mathbf{a}}) \mathrm{e}^{-\mathrm{j}(m-n)\phi} \mathrm{e}^{\mathrm{j}p\phi'}$

4.1. Perfect instrument

In the paraxial case, the scalar quantum mechanical POAM operator along the $+z$ propagation axis is

$$L_Z(\vec{\mathbf{H}}) \;\rightarrow\; L_Z(\chi) = \mathrm{j}\hbar \frac{\partial}{\partial \chi}, \tag{16}$$

where $j = \sqrt{-1}$, and $\hbar$ is Planck's constant h divided by 2π. The POAM expectation value is measured when this operator is applied to the scalar electric field

$$\hat{L}_Z = \frac{1}{I_\mathrm{s}} \int \mathrm{d}^2 H \left\langle \frac{1}{2} E(\vec{\mathbf{H}}; t)\, L_Z(\vec{\mathbf{H}})\, E^*(\vec{\mathbf{H}}; t) \right\rangle, \tag{17a}$$

where

$$I_\mathrm{s} = \int \mathrm{d}^2 H\, I_\mathrm{s}(\vec{\mathbf{H}}) = \int \mathrm{d}^2 H \left\langle \frac{1}{2} E(\vec{\mathbf{H}}; t)\, E^*(\vec{\mathbf{H}}; t) \right\rangle = \int \mathrm{d}^2 H \left\langle \frac{1}{2} \left| E(\vec{\mathbf{H}}; t) \right|^2 \right\rangle \tag{17b}$$

is the integrated intensity of the scalar electric field. The numerator is a quantum-mechanics-like product of states and matrix elements, and the denominator is the normalization. Substituting Eqs. (1) and (16) into Eq. (17a), I obtain

$$\hat{L}_Z = M_\mathrm{eff} \hbar = \left\{ \sum_{m=-\infty}^{\infty} m p_{m,m} \right\} \hbar, \tag{18a}$$

where $M_{\rm eff}$ is the effective quantum number,

$$p_{m,m} = \frac{I_{m,m}}{I_{\rm s}} = \frac{1}{I_{\rm s}} 2\pi \int_0^{r_{\rm max}} {\rm d}H H\, I_{m,m}(H) = \frac{1}{I_{\rm s}} 2\pi \int_0^{r_{\rm max}} {\rm d}H H \left\langle \frac{1}{2} |E_m(H;t)|^2 \right\rangle \tag{18b}$$

is the probability of a photon (or an enesmble of photons) being in state m, $r_{\rm max}$ is the maximum radius which contains all of the flux, and $I_{m,m}$ is the radially integrated autocorrelation of POAM state m. The expection value is simply the effective quantum number times $\hbar$.

Similarly, the quantum mechanical PSAM operator along the $+z$ propagation axis is

$$\overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}}) \rightarrow \overleftrightarrow{\mathbf{S}}_Z = \hbar \overleftrightarrow{\sigma}_3 = \hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{19}$$

where $\overleftrightarrow{\sigma}_3$ is the third Pauli spin matrix. The PSAM expectation value is measured when this operator is applied to the vector electric field

$$\hat{S}_Z = \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}};t) \cdot \overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle, \tag{20a}$$

where

$$I = \int {\rm d}^2 H\, I(\vec{\mathbf{H}}) = \int {\rm d}^2 H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}};t) \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle = \int {\rm d}^2 H\, \hat{\boldsymbol{a}}^{\rm T} \cdot \overleftrightarrow{\mathbf{T}} \cdot \left\langle \frac{1}{2} \vec{\mathbf{E}}(\vec{\mathbf{H}};t) \otimes \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle \tag{20b}$$

is the integrated intensity of the vector electric field, and the T superscript indicates the transpose. Substituting Eqs. (2) and (19) into Eq. (20a), I obtain

$$\hat{S}_Z = v\hbar = \{p_{\rm R,R} - p_{\rm L,L}\}\hbar, \tag{21a}$$

where v is the normalized Stokes-V parameter,

$$p_{\rm R,R} = \frac{I_{\rm R,R}}{I} = \frac{1}{I} \int {\rm d}^2 H\, I_{\rm R,R}(\vec{\mathbf{H}}) = \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} |E_{\rm R}(\vec{\mathbf{H}};t)|^2 \right\rangle \tag{21b}$$

and

$$p_{\rm L,L} = \frac{I_{\rm L,L}}{I} = \frac{1}{I} \int {\rm d}^2 H\, I_{\rm L,L}(\vec{\mathbf{H}}) = \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} |E_{\rm L}(\vec{\mathbf{H}};t)|^2 \right\rangle \tag{21c}$$

are the probabilities of a photon (or an ensemble of photons) being in the RCP and LCP states, and $I_{\rm R,R}$ and $I_{\rm L,L}$ are the integrated autocorrelations of the RCP and LCP states. For an unpolarized and/or linearly polarized source $v = 0$, which means that $p_{\rm R,R} = p_{\rm L,L} = \frac{1}{2}$. For a fully circularly polarized source, $v = +1$ ($v = -1$), $p_{\rm R,R} = 1$ and $p_{\rm L,L} = 0$ ($p_{\rm R,R} = 0$ and $p_{\rm L,L} = 1$).

The PTAM expectation value is the sum of the POAM and PSAM expectation values, or $\hat{J}_Z = \hat{L}_Z + \hat{S}_Z$. The PTAM expectation value can be measured directly with the PTAM operator $\overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}})$ instead, but the POAM operator must first be converted to a matrix

$$L_Z(\vec{\mathbf{H}}) \rightarrow L_Z(\chi) \Rightarrow \overleftrightarrow{\mathbf{L}}_Z(\vec{\mathbf{H}}) \rightarrow \overleftrightarrow{\mathbf{L}}_Z(\chi) = \overleftrightarrow{\sigma}_0 {\rm j}\hbar \frac{\partial}{\partial \chi}, \tag{22}$$

where $\overleftrightarrow{\sigma}_0 = 1$ is the zeroth Pauli spin matrix (2×2 identity matrix). With this redefined POAM operator, the PTAM operator becomes

$$\begin{aligned} \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) &= \overleftrightarrow{\mathbf{L}}_Z(\vec{\mathbf{H}}) + \overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}}) \rightarrow \overleftrightarrow{\mathbf{J}}_Z(\chi) = \overleftrightarrow{\mathbf{L}}_Z(\chi) + \overleftrightarrow{\mathbf{S}}_Z \\ &= \hbar \left[{\rm j} \overleftrightarrow{\sigma}_0 \frac{\partial}{\partial \chi} + \overleftrightarrow{\sigma}_3 \right] = \hbar \begin{bmatrix} {\rm j}\frac{\partial}{\partial \chi} + 1 & 0 \\ 0 & {\rm j}\frac{\partial}{\partial \chi} - 1 \end{bmatrix}. \end{aligned} \tag{23a}$$

Thus,

$$\begin{aligned} \hat{J}_Z &= \hat{L}_Z + \hat{S}_Z \\ &= \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}};t) \cdot \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle \\ &= \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}};t) \cdot \overleftrightarrow{\mathbf{L}}_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle + \frac{1}{I} \int {\rm d}^2 H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}};t) \cdot \overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle \\ &= [M_{\rm eff} + v]\,\hbar = \left[\left\{ \sum_{m=-\infty}^{\infty} m p_{m,m} \right\} + \{p_{\rm R,R} - p_{\rm L,L}\} \right] \hbar. \end{aligned} \tag{23b}$$

The choice of measuring $\hat{J}_Z$ using separate $\overleftrightarrow{\mathbf{L}}_Z(\chi)$ and $\overleftrightarrow{\mathbf{S}}_Z$ operators or the combined $\overleftrightarrow{\mathbf{J}}_Z(\chi)$ operator depends on the application.

4.2. Imperfect instrument

An instrument with non-zero instrumental PSAM, subject to the equations of Sect. 3.3, modifies the expectation values derived in Sect. 4.1. For the sake of simplicity, I assume that the circular telescope aperture is uniformly unaberrated with a non-zero instrumental PSAM, which means that

$$\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{r}}) \rightarrow \overleftrightarrow{\mathbf{D}} = \begin{bmatrix} D^{A,A} & D^{A,B} \\ D^{B,A} & D^{B,B} \end{bmatrix}, \tag{24a}$$

$$\overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) = \overleftrightarrow{\mathbf{D}}\, \pi R_{\rm tel}^2 \,\mathrm{jinc}\left(2\pi\, R_{\rm tel} \left|\vec{\mathbf{\Omega}}' - \vec{\mathbf{\Omega}}\right|\right), \tag{24b}$$

and

$$\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) = \int \mathrm{d}^2\Omega\, \overleftrightarrow{\mathbf{D}}(\vec{\mathbf{\Omega}}', \vec{\mathbf{\Omega}}) \cdot \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t) = \overleftrightarrow{\mathbf{D}} \cdot \int \mathrm{d}^2\Omega \left\{\pi R_{\rm tel}^2 \,\mathrm{jinc}\left(2\pi R_{\rm tel} \left|\vec{\mathbf{\Omega}}' - \vec{\mathbf{\Omega}}\right|\right)\right\} \vec{\mathbf{E}}(\vec{\mathbf{\Omega}}; t) = \overleftrightarrow{\mathbf{D}} \cdot \vec{\mathcal{E}}(\vec{\mathbf{\Omega}}'; t), \tag{24c}$$

where $R_{\rm tel}$ is the telescope radius in units of wavelength, $\mathrm{jinc}\,(x) = 2J_1\,(x)\,/x$, and $J_1\,(x)$ is the Bessel function of the first kind of order one. The quantity in the curly braces approaches the Dirac delta function $\delta(\vec{\mathbf{\Omega}}' - \vec{\mathbf{\Omega}})$ when $R_{\rm tel} \rightarrow \infty$. A perfect instrument implies that $\overleftrightarrow{\mathbf{D}} = \eta \overleftrightarrow{\mathbf{1}}$, where η is a complex constant ($0 < |\eta| \leq 1$). Conversely, when $\overleftrightarrow{\mathbf{D}} \neq \eta \overleftrightarrow{\mathbf{1}}$ the instrument mixes the PSAM components.

To keep the notation consistent with Sect. 4.1, I let $\vec{\mathbf{\Omega}}' \rightarrow \vec{\mathbf{H}}$, $\vec{\mathbf{E}}(\vec{\mathbf{\Omega}}'; t) \rightarrow \vec{\mathbf{E}}'(\vec{\mathbf{H}}; t)$, and $\vec{\mathcal{E}}(\vec{\mathbf{\Omega}}'; t) \rightarrow \vec{\mathbf{E}}(\vec{\mathbf{H}}; t)$. The PTAM expectation value for this imperfect instrument is

$$\begin{aligned} \hat{J}'_Z &= \frac{1}{I'} \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}'^{\rm T}(\vec{\mathbf{H}}; t) \cdot \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}'^{*}(\vec{\mathbf{H}}; t) \right\rangle = \frac{1}{I'} \int \mathrm{d}^2H \left\langle \frac{1}{2} \left[\overleftrightarrow{\mathbf{D}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{H}}; t)\right]^{\rm T} \cdot \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) \cdot \left[\overleftrightarrow{\mathbf{D}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{H}}; t)\right]^{*} \right\rangle \\ &= \frac{1}{I'} \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}}; t) \cdot \left[\overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) \cdot \overleftrightarrow{\mathbf{D}}^{*}\right] \cdot \vec{\mathbf{E}}^{*}(\vec{\mathbf{H}}; t) \right\rangle = \frac{1}{I'} \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}}; t) \cdot \overleftrightarrow{\mathbf{J}}'_Z(\vec{\mathbf{H}}) \cdot \vec{\mathbf{E}}^{*}(\vec{\mathbf{H}}; t) \right\rangle, \end{aligned} \tag{25a}$$

where $\overleftrightarrow{\mathbf{J}}'_Z$ is the operator that includes PSAM mixing effects from the imperfect instrument,

$$\begin{aligned} I' &= \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}'^{\rm T}(\vec{\mathbf{H}}; t) \cdot \vec{\mathbf{E}}'^{*}(\vec{\mathbf{H}}; t) \right\rangle = \int \mathrm{d}^2H \left\langle \frac{1}{2} \left[\overleftrightarrow{\mathbf{D}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{H}}; t)\right]^{\rm T} \cdot \left[\overleftrightarrow{\mathbf{D}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{H}}; t)\right]^{*} \right\rangle \\ &= \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}}; t) \cdot \left[\overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{D}}^{*}\right] \cdot \vec{\mathbf{E}}^{*}(\vec{\mathbf{H}}; t) \right\rangle = \int \mathrm{d}^2H \left\langle \frac{1}{2} \vec{\mathbf{E}}^{\rm T}(\vec{\mathbf{H}}; t) \cdot \overleftrightarrow{\mathcal{D}}_0 \cdot \vec{\mathbf{E}}^{*}(\vec{\mathbf{H}}; t) \right\rangle, \end{aligned} \tag{25b}$$

is the integrated intensity through the imperfect instrument, and

$$\overleftrightarrow{\mathcal{D}}_0 = \overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{D}}^{*} = \begin{bmatrix} \left|D^{A,A}\right|^2 + \left|D^{B,A}\right|^2 & D^{A,A} D^{A,B*} + D^{B,A} D^{B,B*} \\ D^{A,A*} D^{A,B} + D^{B,A*} D^{B,B} & \left|D^{A,B}\right|^2 + \left|D^{B,B}\right|^2 \end{bmatrix} \tag{25c}$$

is a 2×2 matrix. Because of the linearity of the POAM and PSAM operators, the mixed PTAM operator becomes

$$\overleftrightarrow{\mathbf{J}}'_Z(\vec{\mathbf{H}}) = \overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}}) \cdot \overleftrightarrow{\mathbf{D}}^{*} = \overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{L}}_Z(\vec{\mathbf{H}}) \cdot \overleftrightarrow{\mathbf{D}}^{*} + \overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}}) \cdot \overleftrightarrow{\mathbf{D}}^{*} = \overleftrightarrow{\mathbf{L}}'_Z(\vec{\mathbf{H}}) + \overleftrightarrow{\mathbf{S}}'_Z(\vec{\mathbf{H}}), \tag{26a}$$

where

$$\overleftrightarrow{\mathbf{L}}'_Z(\vec{\mathbf{H}}) = \left[\overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\sigma}_0 \cdot \overleftrightarrow{\mathbf{D}}^{*}\right] \mathrm{j}\hbar \frac{\partial}{\partial \chi} = \left[\overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\mathbf{D}}^{*}\right] \mathrm{j}\hbar \frac{\partial}{\partial \chi} = \overleftrightarrow{\mathcal{D}}_0\, \mathrm{j}\hbar \frac{\partial}{\partial \chi} \tag{26b}$$

is the POAM operator including the imperfect instrument,

$$\overleftrightarrow{\mathbf{S}}'_Z(\vec{\mathbf{H}}) = \left[\overleftrightarrow{\mathbf{D}}^{\rm T} \cdot \overleftrightarrow{\sigma}_3 \cdot \overleftrightarrow{\mathbf{D}}^{*}\right] \hbar = \overleftrightarrow{\mathcal{D}}_3\, \hbar \tag{26c}$$

is the PSAM operator including the imperfect instrument, and

$$\overleftrightarrow{\mathcal{D}}_3 = \begin{bmatrix} \left|D^{A,A}\right|^2 - \left|D^{B,A}\right|^2 & D^{A,A} D^{A,B*} - D^{B,A} D^{B,B*} \\ D^{A,A*} D^{A,B} - D^{B,A*} D^{B,B} & \left|D^{A,B}\right|^2 - \left|D^{B,B}\right|^2 \end{bmatrix} \tag{26d}$$

is another 2×2 matrix. When $\overleftrightarrow{\mathbf{D}} = \eta\, 1$: $\overleftrightarrow{\mathcal{D}}_0 = |\eta|^2\, \overleftrightarrow{\sigma}_0 = |\eta|^2\, 1$, $\overleftrightarrow{\mathcal{D}}_3 = |\eta|^2\, \overleftrightarrow{\sigma}_3$, $\overleftrightarrow{\mathbf{J}}'_Z(\vec{\mathbf{H}}) = |\eta|^2\, \overleftrightarrow{\mathbf{J}}_Z(\vec{\mathbf{H}})$, $\overleftrightarrow{\mathbf{L}}'_Z(\vec{\mathbf{H}}) = |\eta|^2\, \overleftrightarrow{\mathbf{L}}_Z(\vec{\mathbf{H}})$, $\overleftrightarrow{\mathbf{S}}'_Z(\vec{\mathbf{H}}) = |\eta|^2\, \overleftrightarrow{\mathbf{S}}_Z(\vec{\mathbf{H}})$, $\hat{J}'_Z = \hat{J}_Z$, $\hat{L}'_Z = \hat{L}_Z$, and $\hat{S}'_Z = \hat{S}_Z$. The η factor does not modify the expectation values because they are normalized quantities.

5. Simple example

In Sect. 1, I show that the most general unfactored PTAM electric field has PSAM states with different POAM expansions. In this section, I demonstrate how the measured POAM expectation value can be affected by source and instrumental PSAM using the simplest unfactored PTAM electric field

$$\vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \begin{bmatrix} E_A(\vec{\mathbf{H}};t) \\ E_B(\vec{\mathbf{H}};t) \end{bmatrix} = \begin{bmatrix} E_{A,m}(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} \\ E_{B,n}(H;t)\,\mathrm{e}^{\mathrm{j}n\chi} \end{bmatrix}. \tag{27}$$

The PTAM expectation value ultimately depends on the behavior of three intensity-based quantities,

$$\begin{aligned} I_{A,A} &= \int \mathrm{d}^2H\, I_{A,A}(\vec{\mathbf{H}}) = \int \mathrm{d}^2H \left\langle \frac{1}{2}\left|E_A(\vec{\mathbf{H}};t)\right|^2 \right\rangle \\ &= 2\pi \int \mathrm{d}H\, H\, I_{A,A,m,m}(H) = 2\pi \int \mathrm{d}H\, H \left\langle \frac{1}{2}\left|E_{A,m}(H;t)\right|^2 \right\rangle, \end{aligned} \tag{28a}$$

$$\begin{aligned} I_{B,B} &= \int \mathrm{d}^2H\, I_{B,B}(\vec{\mathbf{H}}) = \int \mathrm{d}^2H \left\langle \frac{1}{2}\left|E_B(\vec{\mathbf{H}};t)\right|^2 \right\rangle \\ &= 2\pi \int \mathrm{d}H\, H\, I_{B,B,n,n}(H) = 2\pi \int \mathrm{d}H\, H \left\langle \frac{1}{2}\left|E_{B,n}(H;t)\right|^2 \right\rangle, \end{aligned} \tag{28b}$$

and

$$\begin{aligned} I_{A,B}\,\delta_{m,n} &= \int \mathrm{d}^2H\, I_{A,B}(\vec{\mathbf{H}}) = \int \mathrm{d}^2H \left\langle \frac{1}{2} E_A(\vec{\mathbf{H}};t)\, E_B^*(\vec{\mathbf{H}};t) \right\rangle \\ &= 2\pi \int \mathrm{d}H\, H\, I_{A,B,m,n}(H)\,\delta_{m,n} = 2\pi \int \mathrm{d}H\, H \left\langle \frac{1}{2} E_{A,m}(H;t)\, E_{B,n}^*(H;t) \right\rangle \delta_{m,n}, \end{aligned} \tag{28c}$$

where $\delta_{m,n}$ is the Kronecker delta function. They can be rearranged to become more familiar quantities, namely the Stokes parameters $I = I_{A,A} + I_{B,B}$, $Q = 2\mathrm{Re}\{I_{A,B}\}\,\delta_{m,n}$, $U = 2\mathrm{Im}\{I_{A,B}\}\,\delta_{m,n}$, and $V = I_{A,A} - I_{B,B}$. This electric field contains linear polarization only when the PSAM states are temporally correlated have the same POAM state, or $m = n$.

Using the definitions in Sects. 2 and 4, the PTAM expectation value becomes

$$\hat{J}_Z = \hat{L}_Z + \hat{S}_Z = \left[\left\{m\frac{I_{A,A}}{I} + n\frac{I_{B,B}}{I}\right\} + \left\{\frac{I_{A,A}}{I} - \frac{I_{B,B}}{I}\right\}\right]\hbar = [\{m p_{A,A} + n p_{B,B}\} + \{p_{A,A} - p_{B,B}\}]\,\hbar = [M_{\mathrm{eff}} + v]\,\hbar. \tag{29}$$

Note that $p_{A,A}$ and $p_{B,B}$ are part of both the POAM and PSAM expectation values. For a purely unpolarized and/or linearly polarized source $p_{A,A} = p_{B,B} = \frac{1}{2}$ and $\hat{J}_Z = \frac{1}{2}(m+n)\,\hbar$. Also, $\hat{J}_Z = (m+1)\,\hbar$ for a purely right-handed circularly polarized source ($p_{A,A} = 1$ and $p_{B,B} = 0$) and $\hat{J}_Z = (n-1)\,\hbar$ for a purely left-handed circularly polarized source ($p_{A,A} = 0$ and $p_{B,B} = 1$). If $m = n$ (factored electric field), those PTAM expectation values become $\hat{J}_Z = m\hbar$ (PTAM expectation value is independent of PSAM expectation value), $(m+1)\,\hbar$ (PTAM expectation value is POAM expectation value plus RCP PSAM expectation value), and $(m-1)\,\hbar$ (PTAM expectation value is POAM expectation value minus LCP PSAM expectation value), respectively.

An instrument with non-zero instrumental PSAM modifies the result of Eq. (29). Using a Jones matrix in the form of Eq. (24a), the electric field becomes

$$\vec{\mathbf{E}}'(\vec{\mathbf{H}};t) = \begin{bmatrix} E'_A(\vec{\mathbf{H}};t) \\ E'_B(\vec{\mathbf{H}};t) \end{bmatrix} = \overleftrightarrow{\mathbf{D}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \begin{bmatrix} D^{A,A}\, E_A(\vec{\mathbf{H}};t) + D^{A,B}\, E_B(\vec{\mathbf{H}};t) \\ D^{B,A}\, E_A(\vec{\mathbf{H}};t) + D^{B,B}\, E_B(\vec{\mathbf{H}};t) \end{bmatrix}. \tag{30}$$

When $\overleftrightarrow{\mathbf{D}} \neq \eta\overleftrightarrow{\mathbf{1}}$ ($D^{A,B} = D^{B,A} \neq 0$), the instrument mixes the PSAM components. The integrated intensity of this electric field (cf. Eq. (25b)) can be rewritten in terms of the source Stokes parameters

$$\begin{aligned} I' &= \int \mathrm{d}^2H\, I'(\vec{\mathbf{H}}) = \int \mathrm{d}^2H \left\langle \frac{1}{2}\vec{\mathbf{E}}^{\mathrm{T}}(\vec{\mathbf{H}};t) \cdot \overleftrightarrow{\mathcal{D}}_0 \cdot \vec{\mathbf{E}}^*(\vec{\mathbf{H}};t) \right\rangle \\ &= \left[\left|D^{A,A}\right|^2 + \left|D^{B,A}\right|^2\right] I_{A,A} + \left[\left|D^{A,B}\right|^2 + \left|D^{B,B}\right|^2\right] I_{B,B} + 2\mathrm{Re}\left\{\left[D^{A,A}D^{A,B*} + D^{B,A}D^{B,B*}\right] I_{A,B}\right\}\delta_{m,n} \\ &= M^{I,I}\, I + M^{I,Q}\,\delta_{m,n}\, Q + M^{I,U}\,\delta_{m,n}\, U + M^{I,V}\, V = \left[M^{I,I} + M^{I,Q}\,\delta_{m,n}\, q + M^{I,U}\,\delta_{m,n}\, u + M^{I,V}\, v\right] I = \mathcal{M} I, \end{aligned} \tag{31a}$$

where

$$M^{I,I} = \frac{1}{2}\left[\left|D^{A,A}\right|^2 + \left|D^{B,A}\right|^2 + \left|D^{A,B}\right|^2 + \left|D^{B,B}\right|^2\right], \tag{31b}$$

$$M^{I,Q} = \mathrm{Re}\left[D^{A,A}D^{A,B*} + D^{B,A}D^{B,B*}\right], \tag{31c}$$

$$M^{I,U} = -\mathrm{Im}\left[D^{A,A}D^{A,B*} + D^{B,A}D^{B,B*}\right] \tag{31d}$$

and

$$M^{I,V} = \frac{1}{2}\left[\left|D^{A,A}\right|^2 + \left|D^{B,A}\right|^2 - \left|D^{A,B}\right|^2 - \left|D^{B,B}\right|^2\right] \quad (31e)$$

are instrument-dependent coefficients, and $q = Q/I$, $u = U/I$, and $v = V/I$ are the normalized source Stokes parameters. These equations are interesting, because 1) source circular polarization contributes to I'; and 2) if $m = n$ (unfactored electric field) the linear PSAM contributes to I'. Note that the system gain $\mathcal{M} \rightarrow \mathcal{M}(q, u, v)$, i.e., it depends on both the instrumental and source PSAM.

Using Eqs. (30) and (31a) and the mathematics of Sect. 4.2, the POAM and PSAM expectation values including instrumental PSAM are

$$\begin{aligned}\hat{L}'_Z &= \frac{M^{I,I} + M^{I,V}}{\mathcal{M}} p_{A,A}\, m\hbar + \frac{M^{I,I} - M^{I,V}}{\mathcal{M}} p_{B,B}\, n\hbar + \frac{M^{I,Q} - \mathrm{j}M^{I,U}}{\mathcal{M}} p_{A,B}\, n\hbar\, \delta_{m,n} + \frac{M^{I,Q} + \mathrm{j}M^{I,U}}{\mathcal{M}} p^*_{A,B}\, m\hbar\, \delta_{m,n} \\ &= \hat{L}_Z + \left[\frac{M^{I,I} + M^{I,V} - \mathcal{M}}{\mathcal{M}} p_{A,A}\, m\hbar + \frac{M^{I,I} - M^{I,V} - \mathcal{M}}{\mathcal{M}} p_{B,B}\, n\hbar + \frac{M^{I,Q} - \mathrm{j}M^{I,U}}{\mathcal{M}} p_{A,B}\, n\hbar\, \delta_{m,n} + \frac{M^{I,Q} + \mathrm{j}M^{I,U}}{\mathcal{M}} p^*_{A,B}\, m\hbar\, \delta_{m,n}\right] \\ &= \hat{L}_Z + \Delta\hat{L}_Z \end{aligned} \quad (32a)$$

and

$$\begin{aligned}\hat{S}'_Z &= \frac{M^{V,I}}{\mathcal{M}}\hbar + \frac{M^{V,Q}}{\mathcal{M}} q\hbar\, \delta_{m,n} + \frac{M^{V,U}}{\mathcal{M}} u\hbar\, \delta_{m,n} + \frac{M^{V,V}}{\mathcal{M}} v\hbar \\ &= \hat{S}_Z + \left[\frac{M^{V,I}}{\mathcal{M}}\hbar + \frac{M^{V,Q}}{\mathcal{M}} q\hbar\, \delta_{m,n} + \frac{M^{V,U}}{\mathcal{M}} u\hbar\, \delta_{m,n} + \frac{M^{V,V} - \mathcal{M}}{\mathcal{M}} v\hbar\right] \\ &= \hat{S}_Z + \Delta\hat{S}_Z, \end{aligned} \quad (32b)$$

where

$$M^{V,I} = \frac{1}{2}\left[\left|D^{A,A}\right|^2 - \left|D^{B,A}\right|^2 + \left|D^{A,B}\right|^2 - \left|D^{B,B}\right|^2\right], \quad (32c)$$

$$M^{V,Q} = \mathrm{Re}\left[D^{A,A}D^{A,B*} - D^{B,A}D^{B,B*}\right], \quad (32d)$$

$$M^{V,U} = -\mathrm{Im}\left[D^{A,A}D^{A,B*} - D^{B,A}D^{B,B*}\right], \quad (32e)$$

and

$$M^{V,V} = \frac{1}{2}\left[\left|D^{A,A}\right|^2 - \left|D^{B,A}\right|^2 - \left|D^{A,B}\right|^2 + \left|D^{B,B}\right|^2\right] \quad (32f)$$

are other instrument-dependent coefficients, and $p_{A,B} = I_{A,B}/I = \frac{1}{2}(q + \mathrm{j}u)$ is the "transitional probability". Equation (32b) shows that instrumental PSAM changes the PSAM expectation value. Elias (2008) showed that instrumental POAM changes the POAM expectation value. These results are not unexpected and not particularly exciting. On the other hand, changes in the POAM expectation value due to instrumental and source PSAM deserve further analysis. I call this effect "P*S*AM-*M*odified P*O*AM *M*easurement", or SMOM.

After analysing the complete set of use cases (unpolarized, linearly polarized, circularly polarized, elliptically polarized source and instrument polarization; $m \neq n$ or $m = n$), I found that

$$\Delta\hat{L}_Z = \frac{1}{2}\frac{m^{I,V}}{1 + m^{I,V}\, v}\left(1 - v^2\right)(m - n)\,\hbar, \quad (33)$$

where $m^{I,V} = M^{I,V}/M^{I,I}$ is the normalized circular PSAM gain. This equation completely describes the conditions required for SMOM in this simplified example.

SMOM is possible only when $m \neq n$, i.e., for unfactored PTAM electric fields. If the PTAM electric field is factored ($m = n$), on the other hand,

$$\vec{\mathbf{E}}(\vec{\mathbf{H}}; t) = \begin{bmatrix} E_A(H; t) \\ E_B(H; t) \end{bmatrix} \mathrm{e}^{\mathrm{j}m\chi} \quad (34)$$

does not lead to SMOM. SMOM only occurs in the presence of instrumental circular PSAM, or $m^{I,V} \neq 0$, because only it can mix the different PSAM states leading to modified POAM expectation values. No PSAM or partial/full linear PSAM corresponds to $v = 0$, which leads to the maximum $\Delta\hat{L}_Z$ for a given instrumental circular PSAM. The $0 < v < 1$ cases correspond to circular source PSAM plus a combination of unpolarized and/or linear PSAM. Increasing $|v|$ decreases the $\Delta\hat{L}_Z$. The $v = +1$ and $v = -1$ cases correspond to

$$\vec{\mathbf{E}}(\vec{\mathbf{H}}; t) = \begin{bmatrix} E_A(H; t)\, \mathrm{e}^{\mathrm{j}m\chi} \\ 0 \end{bmatrix} = \begin{bmatrix} E_A(H; t) \\ 0 \end{bmatrix} \mathrm{e}^{\mathrm{j}m\chi} \quad (35a)$$

and

$$\vec{\mathbf{E}}(\vec{\mathbf{H}};t) = \begin{bmatrix} 0 \\ E_B(H;t)\,\mathrm{e}^{\mathrm{j}m\chi} \end{bmatrix} = \begin{bmatrix} 0 \\ E_B(H;t) \end{bmatrix} \mathrm{e}^{\mathrm{j}m\chi}, \tag{35b}$$

respectively. Because only one PSAM component is non-zero, no SMOM is possible. These electric fields are similar to Eq. (34) because the POAM exponential can also be factored outside the PSAM vector.

6. Conclusions

I present the most general "unfactored" PTAM electric field form, where each PSAM component has its own POAM expansion. It is slightly more general than the more commonly invoked "factored" PTAM electric field form where the PSAM and POAM components are separable. I then combine the POAM and PSAM calculi to obtain the PTAM calculi. Apart from the vectors, matrices, dot products, and direct products, the PTAM and POAM calculi appear superficially identical. I derive the PTAM operator and expectation value in terms of POAM/PSAM operators and expectation values for systems with and without instrumental PSAM. Last, I prove using a simple example that POAM measurements of sources with unfactored PTAM electric fields passing through instrumental circular PSAM yield systematic POAM measurement errors.

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