Cross-sectional area and intensity variations of sausage modes[★]

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ABSTRACT

Context. The observations obtained using the Rapid Oscillations in the Solar Atmosphere instrument (ROSA) show variations in both cross-sectional area and intensity for magnetic pores in the photosphere.

Aims. We study the phase behaviour between cross-sectional area and intensity variations for sausage modes in a photospheric context. We aim to determine the wave mode by looking at the phase difference between the cross-sectional area and intensity variations.

Methods. We used a straight cylinder as a model for the flux tube. The plasma is uniform both inside and outside the flux tube with a possible jump in the equilibrium values at the boundary, the magnetic field is directed along the flux tube. We derived analytic expressions for the cross-sectional area variation and the total intensity variation. Using these analytic expressions, we calculated the phase differences between the cross-sectional area and the intensity variations. These phase differences were then used to identify the wave mode.

Results. We found that for slow sausage modes the cross-sectional area and intensity variations are always in phase, while for fast sausage modes the variations are in antiphase.

Key words. magnetohydrodynamics (MHD) – methods: analytical – Sun: photosphere – Sun: oscillations

1. Introduction

Solar oscillations were discovered in 1960 (Leighton 1960), and since then there have been numerous observations of wave motions in the solar atmosphere (Stein & Leibacher 1974). Recently, modern ground- and space-based observational instruments show ubiquitous magnetohydrodynamic (MHD) waves in different regions of the solar atmosphere. In the corona MHD waves were detected using the Extreme ultraviolet Imaging Telescope (EIT) onboard the Solar and Heliospheric Observatory (SOHO; Thompson et al. 1999), and also with the imaging telescope onboard the Transition Region and Coronal Explorer (TRACE) spacecraft (Nakariakov et al. 1999). More recent observations of MHD waves in the corona have been made with the Coronal Multi-Channel Polarimeter (CoMP; Tomczyk et al. 2007), with the Extreme-ultraviolet Imaging Spectrometer (EIS) onboard Hinode (Van Doorselaere et al. 2008; Banerjee et al. 2009), and with the Solar Dynamics Observatory Atmospheric Imaging Assembly (SDO/AIA; Liu et al. 2010; Morton et al. 2012). Magnetohydrodynamic waves have recently been detected in the photosphere using the Solar Optical Telescope (SOT; Fujimura & Tsuneta 2009) and using the Rapid Oscillations in the Solar Atmosphere imaging system (ROSA; Morton et al. 2011; Jess et al. 2012a,b).

The nature of MHD waves in sunspots and pores has been thoroughly investigated. Sunspots have been modelled as thin, gravitationally stratified flux tubes (Roberts & Webb 1978), while the theory of Edwin & Roberts (1983) described MHD waves in straight, uniform flux tubes and has

been extensively used in the solar corona. Dispersion diagrams for the photosphere (Edwin & Roberts 1983; Moreels & Van Doorselaere 2013) show the nature of waves in a photospheric waveguide. In this article we are interested in axisymmetric $m = 0$ modes, which should exhibit periodic variations in the cross-sectional area of the flux tube. These periodic variations were first reported by Dorotović et al. (2008) using observations obtained with the Swedish Solar Telescope (SST). However, there was no information on the variations in intensity. The first simultaneous observations of variations in the cross-sectional area and in the intensity were reported by Morton et al. (2011) using data obtained with ROSA. In that article the authors did not identify the wave mode, i.e. whether it was fast or slow. Here we propose a theoretical framework for identifying the wave mode by looking at the phase difference between the cross-sectional area and the intensity variation.

We study the phase relations between the intensity and the cross-sectional area variation for sausage modes in the photosphere. To keep the mathematical formulation tractable we used a simple model for a flux tube, i.e. uniform plasma both inside and outside the flux tube with a jump in the equilibrium values at the flux-tube boundary (Edwin & Roberts 1983; Moreels & Van Doorselaere 2013). In this simple setup we analytically calculated the cross-sectional area and the integrated intensity variations for different sausage modes, from which we then deduced the phase differences. We also numerically visualised different sausage modes to verify the phase behaviour.

2. Mathematical formulation

We model a flux tube as a straight cylinder with radius R . The plasma is uniform both inside and outside the cylinder with a possible jump in the equilibrium values at the

[★] Appendix A is available in electronic form at <http://www.aanda.org>

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boundary (cf. [Edwin & Roberts 1983](#)). The magnetic field is directed along the axis of the flux tube and is given by $B_{0,i}$ inside the flux tube and $B_{0,e}$ outside the flux tube. The plasma pressure and density are $p_{0,i}$ and $\rho_{0,i}$ inside the flux tube and $p_{0,e}$ and $\rho_{0,e}$ outside the flux tube. We assume that the plasma has no background flow, i.e. the equilibrium velocity is $v_0 = 0$ both inside and outside the flux tube. The ideal MHD equations for this equilibrium configuration have been solved in the past ([Edwin & Roberts 1983](#); [Sakurai et al. 1991](#)). The wave modes in such a flux tube can be solved in terms of the radial component of the Lagrangian displacement ξ_r and the total Eulerian pressure perturbation P' . Thus all we need is a link between the cross-sectional area, the integrated intensity variations on the one hand and ξ_r , P' on the other. We point out that this model omits important physics, e.g. density stratification and loop expansion ([Osterbrock 1961](#); [Rosenthal et al. 2002](#); [Erdélyi et al. 2007](#)). For this work we are mainly interested in the radial eigenfunctions that need to be integrated over a surface. Previous work (e.g. [Andries & Cally 2011](#)) has shown that the perpendicular eigenfunctions remain similar for slowly expanding flux tubes. Therefore we believe that in a first approximation the use of a uniform flux-tube model is justified.

2.1. General case

We begin the derivation in a more general case than the uniform flux tube, namely a flux tube where the equilibrium parameters also vary in the radial direction. To calculate the cross-sectional area and the integrated intensity variations we need to calculate a surface integral over a moving surface. For the cross-sectional area we need to evaluate $\int_S dS$ and for the integrated intensity we need to evaluate $\int_S I dS$. We are aware that for the intensity it makes more sense to integrate over the surface where the optical depth $\tau = 1$. In Appendix A we show that in a linear theory integrating over a constant τ surface is the same as integrating over a surface that follows the motion of the plasma. More generally, we show that the difference is second order (in the perturbations) between integrating over a circular cross-section, a co-moving surface, or the $\tau = 1$ layer. This also confirms that it is justified to neglect gravity since this would not affect the radial displacement of the wave mode. Figure 1 represents a sketch of the surface of constant optical depth for a slow surface sausage mode. The surface that follows the motion of the plasma has the same shape as this surface, as is shown in Appendix A. In a linear approximation we can model the equation of the surface as $f(r, \varphi, z, t) \equiv z - \xi_z(r, \varphi, z = 0, t)$. Because we are studying sausage modes, there is no φ dependence, and we drop this variable. An elemental surface element dS is easily related to an elemental surface element in the horizontal plane dS_h , i.e.

$$dS = dS_h \sqrt{1 + \left(\frac{\partial \xi_z}{\partial r}\right)^2}.$$

In a linear theory this simplifies to $dS = dS_h = r dr d\varphi$. To calculate the cross-sectional area $S(R)$, we need to evaluate

$$S(R) = \int_S dS,$$

where S is the surface that follows the motion of the plasma. In the equilibrium state it corresponds to the surface S_0 that is bounded by the circle C_0 with radius R . In other words, our surface integral becomes a one-dimensional integral where the line elements are moving, i.e.

$$S(R) = 2\pi \int_0^{R+\xi_r(R)} r dr.$$

Fig. 1. Sketch of the surface of constant optical depth for a slow surface mode.

If we substitute $r = r_0 + \xi_r(r_0)$, we find

$$S(R) = 2\pi \int_0^R (r_0 + \xi_r(r_0)) \left(1 + \frac{\partial \xi_r(r_0)}{\partial r_0}\right) dr_0.$$

Neglecting higher-order terms shows

$$S(R) = 2\pi \int_0^R \left\{ r_0 + \frac{\partial(r_0 \xi_r)}{\partial r_0} \right\} dr_0,$$

resulting in $S(R) = \pi R^2 + 2\pi R \xi_r(R)$. The Lagrangian variation of the cross-sectional area is $\delta S = S(R) - S_0 = 2\pi R \xi_r(R)$. Finally, we normalise the cross-sectional area variation with the equilibrium value to find

$$\frac{\delta S}{S_0} = 2 \frac{\xi_r(R)}{R}. \quad (1)$$

This equation shows that the cross-sectional area variation depends only on the radial component of the Lagrangian displacement, as we would expect for axisymmetric modes.

Next we calculate the integrated intensity, which is given by

$$I(R) = \int_S I dS,$$

where S is again the surface that follows the motion of the plasma. As before, the integrated intensity simplifies to a one-dimensional integral

$$I(R) = 2\pi \int_0^{R+\xi_r(R)} I r dr.$$

If we denote $F(\mathbf{r}, t) = 2\pi r I(\mathbf{r}, t)$, then

$$I(R) = \int_0^{R+\xi_r(R)} F(\mathbf{r}, t) dr.$$

We know that $\mathbf{r} = \mathbf{r}_0 + \xi(\mathbf{r}_0, t)$, which leads us to the same substitution as before, namely $r = r_0 + \xi_r(r_0)$ to find

$$I(R) = \int_0^R F(\mathbf{r}_0 + \xi(\mathbf{r}_0, t), t) \left(1 + \frac{\partial \xi_r(r_0)}{\partial r_0}\right) dr_0.$$

We now expand $F(\mathbf{r}_0 + \xi(\mathbf{r}_0, t), t)$ around $\mathbf{r}_0$, i.e. $F(\mathbf{r}_0 + \xi(\mathbf{r}_0, t), t) = F(\mathbf{r}_0, t) + \xi(\mathbf{r}_0, t) \cdot \nabla F(\mathbf{r}_0, t)$. We can also expand

$F(\mathbf{r}_0, t)$ around the equilibrium to find $F(\mathbf{r}_0, t) = F_0(r_0) + F'(\mathbf{r}_0, t)$, where 0 indicates the equilibrium and the prime indicates the Eulerian perturbation. Combining this and neglecting higher-order terms shows

$$I(R) = \int_0^R \left\{ F_0(r_0) + F'(r_0) + \xi_r(r_0) \frac{\partial F_0(r_0)}{\partial r_0} + F_0(r_0) \frac{\partial \xi_r(r_0)}{\partial r_0} \right\} dr_0.$$

This can be integrated to find

$$I(R) = \int_0^R F_0(r_0) dr_0 + \int_0^R F'(r_0) dr_0 + [F_0(r_0) \xi_r(r_0)]_0^R.$$

The Lagrangian integrated intensity variation is $\delta I = I - I_0$, i.e.

$$\delta I = \int_0^R F'(r_0) dr_0 + [F_0(r_0) \xi_r(r_0)]_0^R.$$

When we substitute $F(r_0)$, we find

$$\delta I = 2\pi \int_0^R I'(r_0) r_0 dr_0 + 2\pi R I_0(R) \xi_r(R).$$

Normalising with I_0 shows

$$\frac{\delta I}{I_0} = \frac{\int_0^R I'(r_0) r_0 dr_0}{\int_0^R I_0(r_0) r_0 dr_0} + \frac{R I_0(R) \xi_r(R)}{\int_0^R I_0(r_0) r_0 dr_0}. \quad (2)$$

This equation shows that the integrated intensity variation depends on I_0 , I' , and ξ_r . In the uniform case the integrals including I_0 will be easy to calculate, therefore we only need a link between the Eulerian perturbation of the intensity I' and the eigenfunctions ξ_r , P' .

We use the same method to calculate the magnetic flux ϕ_B , which is equal to

$$\phi_B(R) = \int_S \mathbf{B} \cdot \mathbf{1}_n dS,$$

where S is again the surface that follows the motion of the plasma. We easily find that

$$\mathbf{1}_n = \frac{1}{\sqrt{1 + \left(\frac{\partial \xi_z}{\partial r}\right)^2}} \left(-\frac{\partial \xi_z}{\partial r} \mathbf{1}_r + \mathbf{1}_z \right).$$

Because the equilibrium magnetic field $\mathbf{B}_0$ has no radial component $\mathbf{B} \cdot \mathbf{1}_n = B_z$. As before, $dS = r dr d\varphi$, thus

$$\phi_B(R) = 2\pi \int_0^{R+\xi_r(R)} B_z r dr.$$

When we denote $F(\mathbf{r}, t) = 2\pi r B_z(\mathbf{r}, t)$ we can use the same reasoning as for the intensity to find

$$\delta \phi_B = 2\pi \int_0^R B'_z(r_0) r_0 dr_0 + 2\pi R B_{z,0}(R) \xi_r(R),$$

which after normalising becomes

$$\frac{\delta \phi_B}{\phi_{B,0}} = \frac{\int_0^R B'_z(r_0) r_0 dr_0}{\int_0^R B_{z,0}(r_0) r_0 dr_0} + \frac{R B_{z,0}(R) \xi_r(R)}{\int_0^R B_{z,0}(r_0) r_0 dr_0}. \quad (3)$$

This equation shows that the magnetic flux variation depends on $B_{z,0}$, B'_z , and ξ_r . In the uniform case the integrals including $B_{z,0}$ will be easy to calculate, therefore we only need a link between

the Eulerian perturbation of the z -component of the magnetic field B'_z and the eigenfunctions ξ_r , P' .

We still need a link between the intensity and the eigenfunctions ξ_r and P' . Using the fact that the photosphere is optically thick and by assuming local thermodynamic equilibrium, we find that the continuum intensity is given by (e.g. Rutten 2003)

$$I = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/k_B T) - 1},$$

where h is the Planck constant, k_B is the Boltzmann constant, T is the temperature of the plasma, ν is the frequency, and c is the speed of light. We now linearise this expression. We know that $T(r) = T_0(r) + T'(\mathbf{r}, t)$ with $T'/T_0 \ll 1$, where 0 indicates the equilibrium value while the prime denotes the Eulerian perturbed value. Using this, we find

$$I = I_0(r) + \frac{h\nu}{k_B T_0(r)} I_0 \frac{T'}{T_0}, \quad (4)$$

where I_0 is given by

$$I_0(r) = 2 \frac{h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_B T_0(r)}\right).$$

We linearise the ideal gas law to find

$$\frac{T'}{T_0} = \frac{p'}{p_0} - \frac{\rho'}{\rho_0}.$$

Conservation of mass and energy shows

$$\begin{aligned} \frac{\rho'}{\rho_0} &= -\nabla \cdot \xi - \frac{1}{\rho_0} \xi \cdot \nabla \rho_0 \\ \frac{p'}{p_0} &= \gamma \frac{\rho'}{\rho_0} + \gamma \frac{1}{\rho_0} \xi \cdot \nabla p_0 - \frac{1}{p_0} \xi \cdot \nabla p_0, \end{aligned}$$

where γ is the ratio of specific heats. Combining the above equations results in

$$\frac{T'}{T_0} = -(\gamma - 1) \nabla \cdot \xi + \left(\frac{1}{\rho_0} \xi \cdot \nabla p_0 - \frac{1}{p_0} \xi \cdot \nabla p_0 \right).$$

Thus Eq. (4) becomes

$$\frac{I'}{I_0} = \frac{h\nu}{k_B T_0} (-(\gamma - 1) \nabla \cdot \xi + X(r) \xi_r), \quad (5)$$

where $X(r)$ is given by

$$X(r) = \frac{1}{\rho_0} \frac{d\rho_0}{dr} - \frac{1}{p_0} \frac{dp_0}{dr}.$$

We would like to stress that Eq. (5) shows that in a uniform plasma I' is proportional to $\nabla \cdot \xi$, which coincides with the equation in Moreels & Van Doorsselaere (2013). However, in a non-uniform plasma I' also has a contribution depending on ξ_r . Using Sakurai et al. (1991), we find an expression for $\nabla \cdot \xi$ in terms of the total pressure perturbation P' , meaning that we have a link between the intensity and the eigenfunctions ξ_r , P' .

The z -component of the Eulerian perturbation of the magnetic field B'_z can be expressed with the use of ξ_r and P' , i.e.

$$B'_z = ik B_{z,0} \xi_z - B_{z,0} \nabla \cdot \xi - \frac{dB_{z,0}}{dr} \xi_r, \quad (6)$$

where we only need an expression for the z -component of the Lagrangian displacement ξ_z . We have

$$\rho_0 (\omega^2 - k^2 c_A^2) \xi_z = ik P' + ik \frac{B_{z,0}^2}{\mu} \nabla \cdot \xi.$$

We can express $\nabla \cdot \xi$ in terms of the eigenfunctions ξ_r , P' (Sakurai et al. 1991) and combining with the above equations we thus have the z -component of the Eulerian perturbation of the magnetic field B'_z in terms of the eigenfunctions ξ_r , P' .

2.2. Uniform tube

Up to now the derivation has been general in the sense that the equilibrium quantities were allowed to vary in the radial direction. Now we assume a uniform flux tube, thus $X(r) = 0$ and in Eq. (2) the integral $\int_0^R I_0(r_0) r_0 dr_0$ can be evaluated to find $I_0 R^2/2$. In the uniform case Eq. (2) becomes

$$\frac{\delta I}{I_0} = \frac{-2(\gamma-1)}{R^2} \frac{h\nu}{k_B T_0} \int_0^R \nabla \cdot \xi r_0 dr_0 + 2 \frac{\xi_r(R)}{R}.$$

Substituting $\nabla \cdot \xi$ in terms of P' shows

$$\frac{\delta I}{I_0} = 2 \left\{ \frac{h\nu}{k_B T_0} \frac{(\gamma-1)\omega^2}{\rho_0 (c_s^2 + c_A^2) (\omega^2 - k^2 c_T^2)} \frac{1}{R^2} \int_0^R P' r_0 dr_0 + \frac{\xi_r(R)}{R} \right\}. \quad (7)$$

In the above equation c_T is the tube speed, defined as

$$c_T = \frac{c_s c_A}{(c_s^2 + c_A^2)^{1/2}},$$

where $c_s = (\gamma p_0 / \rho_0)^{1/2}$ and $c_A = B_0 / (\mu \rho_0)^{1/2}$ are the sound and Alfvén speeds. Equation (3) becomes

$$\frac{\delta \phi_B}{\phi_{B,0}} = 2 \left\{ \frac{(\omega^2 - k^2 c_s^2)}{\rho_0 (c_s^2 + c_A^2) (\omega^2 - k^2 c_T^2)} \frac{1}{R^2} \int_0^R P' r_0 dr_0 + \frac{\xi_r(R)}{R} \right\}. \quad (8)$$

We are interested in the phase behaviour of the integrated intensity and the cross-sectional area variations and therefore focus on Eqs. (1) and (7). We start with a surface mode with $P' = I_0(\kappa r)$ where κ is given by

$$\kappa^2 = \frac{(k^2 c_s^2 - \omega^2)(k^2 c_A^2 - \omega^2)}{(c_s^2 + c_A^2)(k^2 c_T^2 - \omega^2)}.$$

Note that κ^2 is positive for surface modes. The radial component of the Lagrangian displacement

$$\xi_r = \frac{1}{\rho_0 (\omega^2 - k^2 c_A^2)} \frac{dP'}{dr} = \frac{\kappa}{\rho_0 (\omega^2 - k^2 c_A^2)} I_1(\kappa r).$$

We can easily calculate $\int_0^R P' r_0 dr_0$ using identity 11.3.25 from Abramowitz & Stegun (1972) and find

$$\int_0^R P' r_0 dr_0 = \frac{R}{\kappa} I_1(\kappa R).$$

Thus Eqs. (1), (7), and (8) become

$$\frac{\delta I}{I_0} = 2 \left\{ \frac{h\nu}{k_B T_0} \frac{(\gamma-1)\omega^2}{\rho_0 (c_s^2 + c_A^2) (\omega^2 - k^2 c_T^2)} + \frac{\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \right\} \frac{I_1(\kappa R)}{\kappa R} \quad (9)$$

$$\frac{\delta S}{S_0} = \frac{2\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \frac{I_1(\kappa R)}{\kappa R} \quad (10)$$

$$\frac{\delta \phi_B}{\phi_{B,0}} = 2 \left\{ \frac{(\omega^2 - k^2 c_s^2)}{\rho_0 (c_s^2 + c_A^2) (\omega^2 - k^2 c_T^2)} + \frac{\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \right\} \frac{I_1(\kappa R)}{\kappa R}. \quad (11)$$

This last equation can be simplified by substituting κ^2 and shows $\frac{\delta \phi_B}{\phi_{B,0}} = 0$. Why the Lagrangian variation of the magnetic flux is zero is obvious. We have calculated the Lagrangian variation of the magnetic flux over a surface S that moves with the plasma. In ideal MHD the magnetic field lines are frozen in the plasma, thus the variation of the magnetic flux is zero. This result clearly shows that the observations in Fujimura & Tsuneta (2009), which have a non-zero magnetic flux perturbation, were taken in an Eulerian way and not in a Lagrangian way. The authors claimed that they tracked the region of interest (ROI), e.g. a magnetic pore, in a Lagrangian way for an extended period of time. However, they also explained that for a magnetic pore the ROI was typically only a portion of the pore and the size of the ROI did not change in time. This last statement clearly shows that when the magnetic pore expanded, the ROI did not. This confirms that it was justified to study Eulerian perturbations in Moreels & Van Doorselaere (2013), since the goal was to apply the model to the observations made in Fujimura & Tsuneta (2009).

We now turn to body modes with $P' = J_0(nr)$ and

$$\xi_r = \frac{-n}{\rho_0 (\omega^2 - k^2 c_A^2)} J_1(nr),$$

where $n^2 = -\kappa^2$ and is again positive. Using identity 11.3.20 from Abramowitz & Stegun (1972), we find

$$\int_0^R P' r_0 dr_0 = \frac{R}{n} J_1(nR).$$

Thus Eqs. (1), (7), and (8) become

$$\frac{\delta I}{I_0} = 2 \left\{ \frac{h\nu}{k_B T_0} \frac{(\gamma-1)\omega^2}{\rho_0 (c_s^2 + c_A^2) (\omega^2 - k^2 c_T^2)} + \frac{\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \right\} \frac{J_1(nR)}{nR} \quad (12)$$

$$\frac{\delta S}{S_0} = \frac{2\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \frac{J_1(nR)}{nR} \quad (13)$$

$$\frac{\delta \phi_B}{\phi_{B,0}} = 2 \left\{ \frac{(\omega^2 - k^2 c_s^2)}{\rho_0 (c_s^2 + c_A^2) (\omega^2 + k^2 c_T^2)} + \frac{\kappa^2}{\rho_0 (\omega^2 - k^2 c_A^2)} \right\} \frac{J_1(nR)}{nR}. \quad (14)$$

This last equation can be simplified by substituting κ^2 and as expected $\frac{\delta \phi_B}{\phi_{B,0}} = 0$. Note that the terms in Eqs. (12)–(14) before $J_1(nR)/nR$ are the same as the terms in Eqs. (9)–(11) before $I_1(\kappa R)/\kappa R$. We shall use I_1 to indicate the first term in Eq. (9), I_2 to indicate the second term in Eq. (9), and S_1 to indicate the first term in Eq. (10). Looking at Eqs. (9) and (10), it is clear that for surface modes the intensity and the cross-sectional area variations are either in phase or in antiphase, depending only on the sign in front of $I_1(\kappa R)/\kappa R$ in Eqs. 9 and 10. A similar argument holds for body modes.

The signs of I_1 , I_2 , and S_1 can be determined by looking at a phase diagram for wave modes under photospheric conditions (Edwin & Roberts 1983). The result is shown in Fig. 2. To obtain this phase diagram we used an equilibrium model with $c_{A,i} = 2c_{s,i}$, $c_{A,e} = 0.5c_{s,i}$, and $c_{s,e} = 1.5c_{s,i}$. We can determine different wave modes. Starting from the top, we first have the fast surface mode, second we have slow body modes, and third we have the slow surface mode. For the slow body modes we plotted two lines corresponding to different radial overtones. For more information on these modes and their eigenfunctions see Moreels & Van Doorselaere (2013).

Fig. 2. Phase speed diagram of wave modes under photospheric conditions. We have taken $c_{A,i} = 2c_{s,i}$, $c_{A,e} = 0.5c_{s,i}$, and $c_{s,e} = 1.5c_{s,i}$. The Alfvén speeds are not indicated in the graph because no modes with real frequencies appear in that vicinity. The modes with phase speeds between $C_{T,i}$ and $c_{s,i}$ are body modes and the other modes are surface modes. Note that we only plotted two body modes, while there are infinitely many radial overtones.

Looking at the phase diagram (Fig. 2), we find that for the slow surface mode I_1 , I_2 , and S_1 are all negative. This shows that the intensity and the cross-sectional area variations are in phase for a slow surface mode. For a slow body mode (i.e. $\kappa^2 < 0$) we find that I_1 , I_2 , and S_2 are all positive, showing that the intensity and the cross-sectional area variations are in phase for a slow body mode. For fast surface modes we find that I_1 is positive while I_2 and S_2 are negative. Thus we do not yet know if the intensity and the cross-sectional area variations are in phase or in antiphase. Therefore we look at the ratio of the I_1 to I_2 and find

$$\left| \frac{I_1}{I_2} \right| = \frac{h\nu}{k_B T_0} \frac{(\gamma - 1)\omega^2/k^2}{|c_s^2 - \omega^2/k^2|}.$$

Because we have integrated the intensity over the interior of the flux tube, we substitute T_0 with T_i , which is the equilibrium temperature inside the flux tube. Because the phase speed ω/k decreases when kR increases (see Fig. 2), we know that $|I_1/I_2|$ increases when kR increases. We therefore look at the thin tube limit (i.e. $kR \ll 1$), for which we know $\omega/k \rightarrow c_{s,e}$, and find

$$\left| \frac{I_1}{I_2} \right| = \frac{h\nu}{k_B T_i} \frac{(\gamma - 1)}{|T_i/T_e - 1|}, \quad (15)$$

where T_i and T_e are the internal and external equilibrium temperatures. In this derivation we used that the sound speeds are proportional to the temperature. Typical values for the photosphere can be found in the literature (Fujimura & Tsuneta 2009; Morton et al. 2011), and using these, we find

$$\left| \frac{I_1}{I_2} \right| > 1,$$

showing that I_1 is higher in absolute value than I_2 . The sign of I_1 is positive while the sign of S_1 is negative and thus we have an antiphase behaviour.

We now investigate the equilibrium temperatures we need to have $|I_1/I_2| < 1$ in Eq. (15). Because the internal temperature is

Table 1. Phase differences between the cross-sectional area variation and the intensity perturbation for different sausage wave modes.

Wave mode	sign of I_1	Sign of I_2	sign of S_1	Phase behaviour
Slow surface	–	–	–	in phase
Slow body	+	+	+	in phase
Fast surface	+	–	–	in antiphase

lower than the external temperature for photospheric flux tubes, we can rewrite $|I_1/I_2| < 1$ to

$$0 < -(\gamma - 1) \frac{hc}{k_B \lambda} + T_i - \frac{1}{T_e} T_i^2 = f(T_i),$$

where λ is the wavelength of the observations. For this inequality to be valid, $f(T_i)$ must have a real zero, i.e. the discriminant must be positive, resulting in

$$\lambda T_e > 4(\gamma - 1) \frac{hc}{k_B} = 0.038. \quad (16)$$

Typical values for the wavelength λ in the photosphere are 400 to 650 nm (Fujimura & Tsuneta 2009; Morton et al. 2011), indicating external temperatures of the order 7×10^4 K, which are unrealistic for the photosphere. This argument shows that for the photosphere $|I_1/I_2| > 1$, and thus we have an antiphase behaviour between the cross-sectional area and the intensity variation for fast sausage modes. These results are summarised in Table 1.

3. Visualisations

We show visualisations for different sausage modes to illustrate the phase differences. We computed sausage modes using the same equilibrium values as before, namely $c_{A,i} = 2c_{s,i}$, $c_{A,e} = 0.5c_{s,i}$, and $c_{s,e} = 1.5c_{s,i}$. We fixed the value of $kR = 2$ and fixed the external density $\rho_{0,e} = 1$. The first mode is the fast surface mode; the result can be seen in Fig. 3. There are two panels in this figure, representing different times in the computation. The left panel is at $t = 0$ and the right panel is at $t = P/2$, where P is the period of the wave. The background colour indicates the density, which is a proxy for the intensity, the flux-tube boundary is indicated by a black line, and the vectors indicate the Lagrangian displacement. We notice that when the surface area decreases, i.e. the radius of the flux tube decreases, the intensity increases. This is no confirmation of the antiphase behaviour, because the intensity increases but the area over which the intensity is integrated decreases. Therefore we have plotted the radial displacement of the flux-tube boundary and the total intensity over the flux tube (see Fig. 4). This figure clearly demonstrates the antiphase behaviour.

The second mode is the slow body mode; the result can be seen in Fig. 5. This figure contains two panels corresponding to different times in the computation. The left panel is at $t = 0$ and the right panel is at $t = P/2$. As before, the background colour indicates the density, which is a proxy for the intensity, the flux-tube boundary is indicated by a black line, and the vectors indicate the displacement. We notice that when the surface area decreases, i.e. the radius of the flux decreases, the intensity also decreases. Thus, we can conclude that the total intensity decreases when the surface area decreases, confirming the in-phase behaviour of slow body modes.

The third mode is the slow surface mode; the result can be seen in Fig. 6. In this figure there are two panels corresponding

Fig. 3. Longitudinal cut of a fast sausage surface wave at two different times. The background colour indicates the density, which is used as a proxy for the intensity. The flux-tube boundary is indicated by a black line. The arrows indicate the plasma displacement.

Fig. 4. The radial displacement of the flux-tube boundary and the total intensity over the flux tube as a function of time at a fixed height for the fast surface mode. The displacement and the intensity vary in antiphase with one another.

to different times in the computation. The left panel is at $t = 0$ and the right panel is at $t = P/2$. The background colour indicates the density, which is a proxy for the density, the flux-tube boundary is indicated by a black line, and the vectors indicate the displacement. We notice that when the surface area decreases, i.e. the radius of the flux decreases, the intensity also decreases. Thus, we can conclude that the total intensity decreases when the surface area decreases, confirming the in-phase behaviour of slow surface modes.

We emphasise that Figs. 5 and 6 show that for slow modes the longitudinal displacement is much more pronounced than the radial displacement. This indicates that it is unlikely that the surface area variation is observable for slow modes. This result is also confirmed in Moreels & Van Doorselaere (2013), where we showed that the polarisation for slow modes in highly longitudinal. Figure 5 (Moreels & Van Doorselaere 2013) clearly shows that the longitudinal displacement $\xi_{\parallel}$ is much higher than the transverse displacement $\xi_{\perp}$ for slow modes. Another confirmation are the data in Fujimura & Tsuneta (2009). From the

observations in Fujimura & Tsuneta (2009) we find that a typical value for the radius of the flux tube R is 2000 km. We also find that the line-of-sight velocity amplitude is of the order 80 m/s and the period is of the order 350 s. Since $\mathbf{v} = \partial \xi / \partial t = i\omega \xi$, the amplitude of ξ_r is equal to the amplitude of the velocity perturbation divided by the angular frequency. Using the values in Fujimura & Tsuneta (2009), we find that the order of magnitude of the amplitude of ξ_r is 4.5 km, which is below the current resolution of the observations. Based on these three sources we conclude that if we observe a surface area variation, we are most likely observing a fast sausage mode. The fact that the longitudinal displacement is much higher than the radial displacement for slow modes also justifies the use of the simple one-dimensional approach for modelling slow magnetoacoustic oscillations in the chromosphere, as in Botha et al. (2011).

Finally, we briefly discuss the observations taken in Morton et al. (2011). In that article observations for magnetic pores were reported and the authors compared the variations in pore size with the variations in intensity. They concluded that the observations were sausage modes but were not able to determine the wave mode, i.e. slow or fast. The authors are convinced that the observations had a strong out-of-phase behaviour. Applying our model to these observations shows that the observations are fast sausage modes. This is also confirmed by the previous paragraph where we showed that if we observe a cross-sectional area variation, we are most likely dealing with a fast mode.

4. Conclusions

We have shown that cross-sectional area variations and intensity variations are in phase for slow sausage modes while they are in antiphase for fast sausage modes. We did this by modelling the flux tube as a straight cylinder with constant radius and constant plasma parameters both inside and outside the flux tube. We then calculated the cross-sectional area and the integrated intensity variations. From these variations we were able to deduce

Fig. 5. Longitudinal cut of a slow sausage body wave at two different times. The background colour indicates the density, which is used as a proxy for the intensity. The flux-tube boundary is indicated by a black line. The arrows indicate the plasma displacement.

Fig. 6. Longitudinal cut of a slow sausage surface wave at two different times. The background colour indicates the density, which is used as a proxy for the intensity. The flux-tube boundary is indicated by a black line. The arrows indicate the plasma displacement.

the phase difference between the variations. We verified these results using some visualisations.

In our derivation we found that the Lagrangian perturbation of the magnetic flux is zero. This was to be expected in ideal MHD. The surface over which we integrated the magnetic flux was following the motion of the plasma, and in ideal MHD the magnetic field lines are frozen in the plasma. This result showed that the ROI used in Fujimura & Tsuneta (2009) to study the photospheric pore oscillations did not follow the plasma motions

perfectly. Their observations are probably better described with Eulerian variations (e.g. Moreels & Van Doorselaere 2013).

The visualisations showed that for slow modes the longitudinal displacement is much higher than the transverse displacement. This result was also found in Moreels & Van Doorselaere (2013) by studying the polarisation of different wave modes. Another confirmation of this result was obtained from the observations in Fujimura & Tsuneta (2009), from which we deduced that the order of magnitude of the amplitude of ξ_r is

4.5 km, which is below the current resolution of the observations. Therefore we conclude that if one observes a cross-sectional area variation, the most likely interpretation is a fast sausage mode.

Finally, we also applied our model to the observations in Morton et al. (2011), where the authors claimed to find an out-of-phase behaviour between variations in the cross-sectional area and the intensity. The fact that the cross-sectional area variation is observable indicates that the observations are most likely fast sausage modes. The out-of-phase behaviour confirms that the observation is a fast sausage mode.

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Appendix A: Surface with constant optical depth

The optical depth τ_ν can be calculated using (Rybicki & Lightman 1986; Rutten 2003)

$$\tau_\nu = \int_0^D \alpha_\nu ds. \quad (\text{A.1})$$

This shows that the optical depth depends on the frequency ν , the LOS, and the extinction coefficient α , which also depends on the photon frequency. To simplify the calculations we assume that the LOS is directed along the z -axis, i.e. $ds = dz$. The goal is to find the height D for which the optical depth at a given frequency τ_ν is constant. Because we are using observations of the photosphere, we assume that the intensity observations are taken in the continuum and not in the line core. In Rutten (2003) we can find different sources for opacity in the continuum each with their own extinction coefficient. We have bound-free transitions, free-free transitions, Thomson scattering, and Rayleigh scattering. The most important source for the solar photosphere comes from bound-free transitions where the hydride H^- absorb radiation to form hydrogen and a free electron (Marshall 2003; Pradhan & Nahar 2011). From Rutten (2003) we find that for bound-free transitions $\alpha_\nu = \sigma n_i \left\{ 1 - \exp\left(-\frac{h\nu}{k_B T}\right) \right\}$, where σ is a constant that depends on the atom/ion involved in the bound-free transition, n_i is the number density in the ionising level, h is the Planck constant, k_B is the Boltzmann constant, T is the temperature of the plasma, and ν is the frequency. We assume that n_i is proportional to the density ρ , thus the monochromatic optical depth is

$$\tau_\nu = a\sigma \int_0^D \rho \left\{ 1 - \exp\left(-\frac{h\nu}{k_B T}\right) \right\} dz,$$

where a indicates the fraction of ionisation. We can now linearise both the density and the temperature to find

$$\rho \left\{ 1 - \exp\left(-\frac{h\nu}{k_B T}\right) \right\} \approx \rho_0(1 - I_0) + \rho_0 A(r) P' + \rho_0 B(r) \xi_r,$$

where I_0 , $A(r)$, and $B(r)$ are given by

$$\begin{aligned} I_0(r) &= 2 \frac{h\nu^3}{c^2} \exp\left(-\frac{h\nu}{k_B T_0(r)}\right) \\ A(r) &= \frac{\omega^2}{\rho_0(c_s^2 + c_A^2)(\omega^2 - k^2 c_T^2)} \left((1 - I_0) - (\gamma - 1) \frac{h\nu}{k_B T_0(r)} I_0 \right) \\ B(r) &= -\frac{h\nu}{k_B T_0(r)} I_0 \left(\frac{1}{\rho_0} \frac{d\rho_0}{dr} - \frac{1}{p_0} \frac{dp_0}{dr} \right) - (1 - I_0) \frac{1}{\rho_0} \frac{d\rho_0}{dr}. \end{aligned}$$

The optical depth (Eq. (A.1)) is thus given by

$$\tau_\nu = a\sigma\rho_0 \left[(1 - I_0) \int_0^D dz + A(r) \int_0^D P' dz + B(r) \int_0^D \xi_r dz \right]. \quad (\text{A.2})$$

In general, D is a function of r, φ , and t , but since we are dealing with axisymmetric modes, we have $D(r, t)$. We linearise $D(r, t) = D_0 + D'(r, t)$ and substitute in Eq. (A.2) to find

$$\tau_\nu = a\sigma\rho_0 \left[(1 - I_0)(D_0 + D') + A(r) \int_0^{D_0} P' dz + B(r) \int_0^{D_0} \xi_r dz \right], \quad (\text{A.3})$$

where we neglected higher-order terms. Because the optical depth τ_ν is constant, the zeroth-order terms in Eq. (A.3) show

$$D_0(r) = \frac{\tau_\nu}{a\sigma\rho_0(1 - I_0)}. \quad (\text{A.4})$$

To find D' we look at the first-order terms in Eq. (A.3) and find

$$D'(r, t) = \frac{-1}{(1 - I_0)} \left(A(r) \int_0^{D_0} P' dz + B(r) \int_0^{D_0} \xi_r dz \right).$$

Since both P' and ξ_r are proportional to $\exp\{i(kz - \omega t)\}$,

$$D'(r, t) = \frac{-1}{(1 - I_0)} \left(A(r) P'(r, t) + B(r) \xi_r(r, t) \right) \frac{\exp(ikD_0) - 1}{ik}. \quad (\text{A.5})$$

The equation for a surface with constant optical depth is given by $g(r, z, t) \equiv z - D(r, t)$. In the uniform case we have D_0 constant, $A(r)$ constant, and $B(r) = 0$, thus $g(r, z, t)$ has a similar form as the surface that follows the motions of the plasma (i.e. $f(r, \varphi, z, t) \equiv z - \xi_z(r, \varphi, z = 0, t)$). We calculate the normal to the surface g

$$\mathbf{1}_n \sim \left(\frac{A}{(1 - I_0)} \frac{(\exp(ikD_0) - 1)}{ik} \frac{\partial P'}{\partial r} \mathbf{1}_r + \mathbf{1}_z \right).$$

We relate a surface element dS to an elemental surface element in the horizontal plane dS_h , i.e.

$$dS = dS_h \sqrt{1 + \left(\frac{A}{(1 - I_0)} \frac{(\exp(ikD_0) - 1)}{ik} \frac{\partial P'}{\partial r} \right)^2},$$

which in a linear approximation is $dS = dS_h = r dr d\varphi$. This shows that integrating over the surface where τ_ν is constant or integrating over the surface that follows the motions of the plasma is identical, since for a uniform equilibrium and in a linear expansion the elemental surface elements are the same.