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The internal structure of neutron stars and white dwarfs, and the Jacobi virial equation. II.

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ABSTRACT

Context. The Jacobi virial equation is a very powerful tool for exploring several aspects of the stellar internal structure and evolution. In a previous paper we have shown that the function $[\alpha\beta]_{\rm GR}/\Lambda^{0.9}(R)$ is constant (≈0.4) for pre main-sequence stars (PMS), white dwarfs (WD) and for some neutron star (NS) models, where $\alpha_{\rm GR}$ and $\beta_{\rm GR}$ are the form-factors of the gravitational potential energy and of the moment of inertia.

Aims. To investigate the structural evolution of another type of celestial bodies, we extend these calculations to gaseous planets. We also analyse the cases for which this function is not conserved during some stellar evolutionary phases. Concerning NS, we study the influence of the equation of state (EOS) on this function and refine the exponent of the auxiliary function $\Lambda(R)$. We also present a macroscopic criterion of stability for these stars.

Methods. Non-stop calculations from the PMS to the white dwarf cooling sequences were performed with the MESA code. The covered mass range was 0.1–1.7 $M_\odot$. We used the same code to compute models for gaseous planets with masses between 0.1–50 M_J. Neutron star models were computed using two codes. The first one is a modified version of the NSCool/TOV subroutines. The second code is a plain TOV solver that allows one to use seven previously described EOS. The relativistic moment of inertia and gravitational potential energy were computed through a fourth-order Runge-Kutta method.

Results. By analysing the internal structure of gaseous planets we show that the function $[\alpha\beta]_{\rm GR}/\Lambda^{0.8}(R) \equiv \Gamma(M, {\rm EOS})$ is conserved for all models during the whole planetary evolution and is independent of the planet mass. For the PMS to the white dwarf cooling sequences, we have found a connection between the strong variations of $\Gamma(M, {\rm EOS})$ during the intermediary evolutionary phases and the specific nuclear power. A threshold for the specific nuclear power was found. Below this limit this function is invariant (≈0.4) for these models, i.e., at the initial and final stages (PMS and WD). For NS, we showed that the function $\Gamma(M, {\rm EOS})$ is also invariant (≈0.4) and is independent of the EOS and of the stellar mass. Therefore, we confirm that regardless of the final products of the stellar evolution, NS or WD, they recover the initial value of $\Gamma(M, {\rm EOS}) \approx 0.4$ acquired at the PMS. Finally, we have introduced a macroscopic stability criterion for NS models based on the properties of the relativistic product $[\alpha\beta]_{\rm GR}$.

Key words. stars: evolution – stars: interiors – stars: pre-main sequence – stars: neutron – white dwarfs

1. Introduction

The vectorial equations often used in physics are sometimes too complicated to be solved in a closed form. One of the advantages of the virial theorem is that it reduces the complexity of these equations, but this frequently comes at the expense of losing information. The balance is, however, positive because some complicated problems can be treated and important information can be derived. The moment of inertia and the gravitational potential energy are fundamental elements that are handled by the Jacobi virial equation. This equation is a very useful tool for exploring several aspects of the stellar internal structure and evolution. Given the nature of this equation, a mathematical relation between the moment of inertia and the gravitational potential energy reduces its complexity, which facilitates managing it. Ferronskii et al. (1978) argued that this relationship could be expressed by the constancy of the product $\alpha\beta$ in the Newtonian approximation, where α and β are the form-factors of the gravitational potential energy and of the moment of inertia, respectively. However, the laws of mass distribution used by these authors to evaluate the product $[\alpha\beta]_{\rm Newt}$ covered only a small branch of the Herzsprung-Russel (HR) diagram. To check the constancy of $[\alpha\beta]_{\rm Newt}$ by using realistic stellar models, Claret & Giménez (1989) provided values of α and β for several grids of stellar evolutionary models. The availability of these form-factors allowed deeper investigations into the Jacobi dynamics of stellar evolution (Quiroga & Claret 1992a,b; Quiroga & Mello 1992; Quiroga & Cerqueira 1992). Unfortunately, due to limitations in the opacity tables available at that time, these stellar models only covered the pre main-sequence (PMS), the main sequence and only the first stages of the red giant branch. Therefore, many important aspects of the later stellar evolution could not be investigated in the mentioned papers. Another considerable limitation of these papers concerns mass loss, which is not considered. As we will see in the next sections, this point is very important. For example, a model with an initial mass of 1.7 $M_\odot$ at the PMS reaches the white dwarf stage with only 0.6 $M_\odot$. Modern stellar models available in the literature improved the situation: Claret (2006, 2007) published grids covering a wide range of chemical compositions (Z = 0.001–0.10), masses (0.8–125 $M_\odot$) and also offered detailed tracking of the late phases of stellar evolution.

In a recent paper Claret (2012) showed that the product of the form-factors α and β – neither the Newtonian nor the relativistic one – is not conserved during the core hydrogen-burning phase, helium-burning, thermally pulsating asymptotic giant branch, or "blue loops" for models evolving from the PMS to the white dwarfs stages. In the same paper, it was shown that for very compact objects (neutron stars (NS) – for example) the effects of general relativity must be taken into account and the product $\alpha\beta$ computed using the Newtonian approximation must be superseded by the relativistic product $[\alpha\beta]_{\rm GR}$.

In our formulation the conserved quantity at the beginning and at the end of the stellar evolution is the function $[\alpha\beta]_{\rm GR}/\Lambda^{0.8}(R) \equiv \Gamma(M, {\rm EOS}) \approx 0.4$. The auxiliary function $\Lambda(R)$ is given by $\left[1 - \frac{2GM(R)}{Rc^2}\right]^{-1}$ and is evaluated at the surface of each model, c is the speed of light, R is the radius, $M(R)$ is the gravitational mass, and G is the constant of gravitation. Claret (2012) has also shown that, regardless of the final products of stellar evolution (white dwarfs (WD) or NS), they recovered the fossil value of $[\alpha\beta]_{\rm GR}/\Lambda^{0.9}(R)$ acquired at the PMS phase. The refinement of the exponent in the quoted function using an extended set of more elaborate EOS is given in Sect. 5.

In the present paper we explore some aspects connected to these investigations. First, we extend the calculations of the moment of inertia and the gravitational potential energy to gaseous planets and show that the function $[\alpha\beta]_{\rm GR}/\Lambda^{0.8}(R)$ (hereafter $\Gamma(M, {\rm EOS})$) is conserved during the planetary evolution, regardless of the initial mass. Second, we investigate why this function is not conserved during some evolutionary phases when we consider a complete evolution from the PMS to the WD cooling sequences (hereafter PMS-WD models). Third, we confirm that $\Gamma(M, {\rm EOS})$ is conserved for NS and, in addition, is independent of the equation of state (EOS) and of the stellar mass. We also derive, heuristically, an alternative criterion of stability based on the properties of the gravitational potential energy and of the moment of inertia.

2. Neutron star, pre main-sequence, white dwarf, and gaseous planets models

2.1. Planets

The models of gaseous planets were computed using the MESA code (Paxton et al. 2011, version 4298) for a chemical composition of $X = 0.73$ and $Z = 0.01$. The adopted mixing-length parameter $\alpha_{\rm MLT}$ is 1.5. We computed planet models from 0.1 up to 50 $M_{\rm J}$. All models were followed from the gravitational contraction to an approximate age of 20 Gyr.

2.2. Non-stop calculations: from PMS to white dwarf models

We used the same version of the MESA code to compute a non-stop evolution from the PMS to WD cooling sequences. The adopted chemical composition and the mixing-length parameter $\alpha_{\rm MLT}$ are the same as those adopted for the planet models.

2.3. Neutron star models

The NS models were computed using two different codes. The first is a modified version of the NSCool/TOV subroutines (Page & Reddy 2006; Page et al. 2006). We computed neutron star models adopting the non-relativistic EOS by Akmal et al. (1998, hereafter APR).

Another set of subroutines was used to explore the role of the EOS in the invariance of the function $\Gamma(M, {\rm EOS})$. We considered seven different EOS that were calculated with the model from Hempel & Schaffner-Bielich (2010), using the following relativistic mean-field nucleon interactions: *DD*2 (Typel et al. 2010), *FSUgold* (Todd-Rutel & Piekarewicz 2005), *NL*3 (Lalazissis et al. 1997), *SFHo*, and *SFHx* (Steiner et al. 2012), *TM*1 (Sugahara & Toki 1994), and *TMA* (Toki et al. 1995).

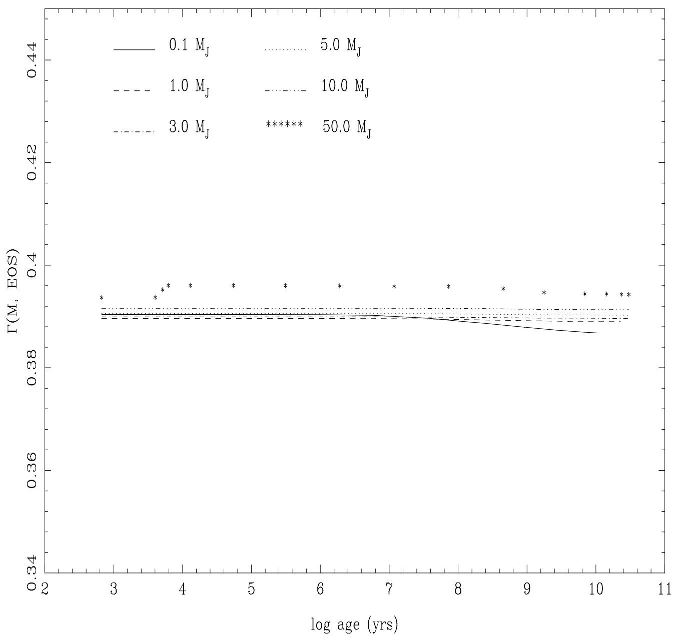

Fig. 1. Time evolution of $\Gamma(M, {\rm EOS})$ for some planets with masses between 0.1 and 50 $M_{\rm J}$.

Some subroutines were added to compute the apsidal-motion constants, the moment of inertia, and the gravitational potential energy.

As we have shown in Claret (2012), the effects of general relativity on the moment of inertia and gravitational potential energy are not important for PMS, main-sequence (MS) stars, WD and, as we will see below, for planets. However, for consistency, we used the relativistic formalism throughout. The moment of inertia $I_{\rm GR}$ and the gravitational potential energy $\Omega_{\rm GR}$ were computed using the following equations (for the approximation of the moment of inertia, see Ravenhall & Pethick 1994, and for the gravitational potential energy, see Misner et al. 1973):

$$J_{\rm GR} = \frac{8\pi}{3}\int_0^R \Lambda(r) r^4 \left[\rho(r) + P(r)/c^2\right] {\rm d}r,$$

$$I_{\rm GR} \simeq \frac{J_{\rm GR}}{\left(1 + \frac{2GJ_{\rm GR}}{R^3c^2}\right)} \equiv (\beta_{\rm GR} R)^2 M \qquad (1)$$

$$\Omega_{\rm GR} = -4\pi \int_0^R r^2 \rho \left[\Lambda^{1/2}(r) - 1\right] {\rm d}r \equiv -\alpha_{\rm GR}\frac{GM^2}{R}, \qquad (2)$$

where $P(r)$ is the pressure, $\rho(r)$ the energy density and the auxiliary function $\Lambda(r)$ is given by $\left[1 - \frac{2Gm(r)}{rc^2}\right]^{-1}$. Equations (1) and (2) were integrated through a fourth-order Runge-Kutta method.

3. Gaseous planets

Figure 1 displays the calculations of $\Gamma(M, {\rm EOS})$ for gaseous planets with masses ranging from 0.1 up to 50 $M_{\rm J}$. Although $\Lambda(R)$ tends to 1.0 for all gaseous planet models, for consistency, we kept the definition of $\Gamma(M, {\rm EOS})$. The refinement of the exponent in the above function is explained in Sect. 5. The evolution of each model was followed from the gravitational contraction to an approximate age of 20 Gyr.

An inspection of Fig. 1 shows that $\Gamma(M, {\rm EOS})$ is constant for all evolutionary phases (≈0.4). This is the same value we

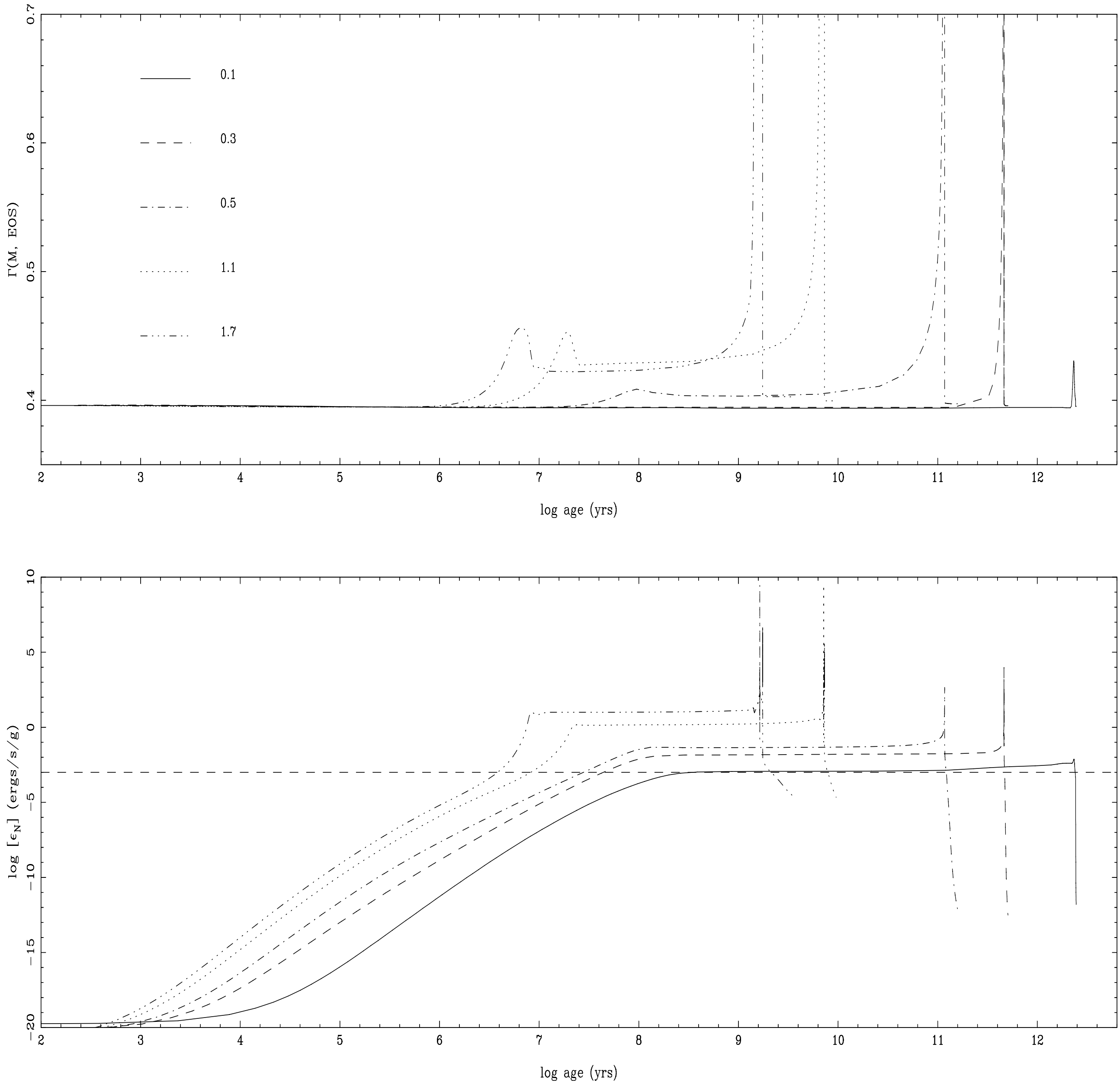

Fig. 2. *Upper panel*: time evolution of $\Gamma(M, \mathrm{EOS})$ for some stellar models evolving from the PMS to the WD stage. The mass range is 0.1–1.7 $M_\odot$ (see symbols in the *left upper corner*). The adopted initial chemical composition is $X = 0.73$, $Z = 0.01$ and the mixing-length parameter $\alpha_{\mathrm{MLT}} = 1.5$. *Lower panel*: specific nuclear power for the same models.

obtained for PMS, WD and NS models (Claret 2012). We performed some tests by varying the initial chemical composition and α_{MLT} and the results are independent of the input physics. With these calculations we extended the invariance of $\Gamma(M, \mathrm{EOS})$ to another category of celestial bodies, the planets.

As we show below, the invariance of the aforementioned function during all planetary evolutionary phases is connected with the specific nuclear power that, for the gaseous planets we analysed here, is below a threshold that is determined in the next section.

4. Connection between the specific nuclear power ϵ_{N} and $\Gamma(\mathrm{M}, \mathrm{EOS})$ for PMS to the WD cooling sequences

In a previous paper (Claret 2012) we argued that the drastic changes in $\alpha\beta$ during the MS, thermally pulsing asymptotic giant branch, and "blue loops" for PMS-WD models could be related to the presence/absence of nuclear reactions, with chemical inhomogeneities, etc. To try to elucidate this point we computed some PMS-WD models with masses varying from 0.1 up to 1.7 $M_\odot$. We found that the specific nuclear power ϵ_{N} is directly correlated with the variation of $\Gamma(M, \mathrm{EOS})$, as shown in Fig. 2. The first peaks in ϵ_N for the models with 1.7 and 1.1 $M_\odot$ just before the ZAMS (zero age main sequence) are due to the reduction in the original ^{12}C content through the nuclear reactions $^{12}\mathrm{C}(\mathrm{p}, \gamma)^{13}\mathrm{N}(\beta^{+}, \nu)^{13}\mathrm{C}(\mathrm{p}, \gamma)^{14}\mathrm{N}$. During the PMS stage the function is conserved for all models and also at the final stages of WD, regardless of the initial mass of the model. Because the PMS duration depends on the initial masses, $\Gamma(M, \mathrm{EOS})$ begins to increase for the most massive models first. For the least massive model the function is conserved almost throughout, except during a very short interval (log age ≈12.4), but the PMS value is recovered a little later.

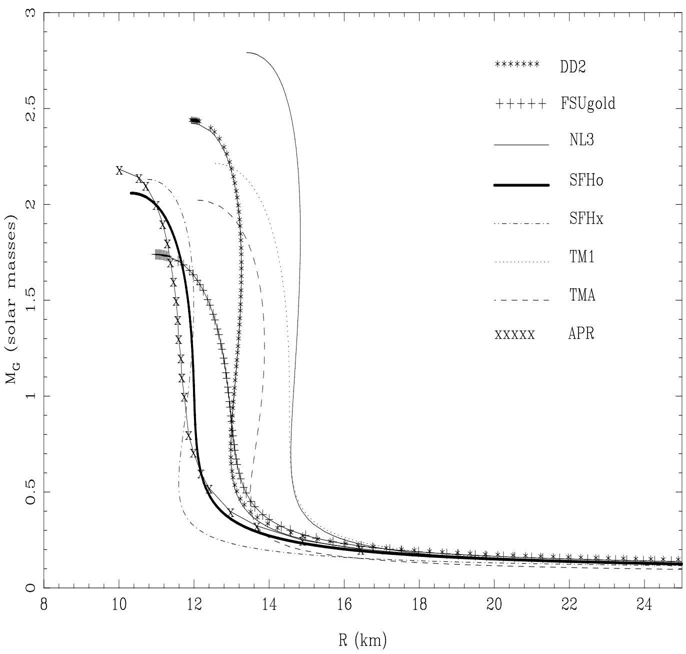

Fig. 3. Mass-radius relation for neutron stars. The adopted EOS are identified by the symbols displayed at the right upper corner of the figure. Only stable models are shown.

By inspecting Fig. 2 we can deduce that there is a threshold below which $\Gamma(M, \text{EOS})$ is invariant. This limit is around 10^{-3} erg s^{-1} g^{-1} (indicated by horizontal dashed line in the second panel of Fig. 2). Above this threshold, the stellar mass distribution changes so drastically due to the presence of nuclear reactions and the subsequent chemical inhomogeneities that $\Gamma(M, \text{EOS})$ increases up to four orders of magnitude (not shown in the figure for the sake of clarity). This occurs mainly during some post-main sequence phases. The behaviour of the configurations during these phases can be understood with the help of the virial theorem. If the accelerative changes in the moment of inertia are not large ($\frac{\partial^2 I}{\partial t^2} \approx 0$), we have 2 E – Ω = 0, where E is the total energy and Ω is the generic gravitational potential energy given by $\alpha GM^2/R$. The form-factor α gives a measure of mass concentration of a model, in a similar way as the apsidal-motion constant does. As the core contracts, α increases and the external layers must expand to keep the gravitational potential energy constant. On the other hand, below the threshold the configurations are changed more smoothly and this function remains constant (≈0.4) as in the cases of PMS, WD, NS, and for the gaseous planets studied in the previous section. Our criterion for quasi-static or stable evolution visible in Fig. 2 differs substantially from that used by Quiroga & Claret (1992a, see their Figs. 1, 3, and 4), who adopted $0.38 \le \alpha\beta \le 0.44$ from the analytical approximations by Ferronskii et al. (1978). In addition, the criterion by Quiroga & Claret (1992a) does not relate the time evolution of $\alpha\beta$ with the nuclear reactions or with the chemical inhomogeneities.

5. Gravitational potential energy and the moment of inertia of neutron stars: the role of the equation of state and the macroscopic stability criterion

The invariance of $[\alpha\beta]_{\rm GR}/\Lambda^{0.9}(R)$ for NS was shown in (Claret 2012) but using only two EOS (Akmal et al. 1998; Shen et al. 2011). Here, we extended these calculations to seven other EOS formulations (see Sect. 2.3 and Fig. 3) to test whether this function is independent of the EOS and also to refine the exponent of this function. First, we analysed each EOS separately and searched for the best exponent to the function $\Lambda(R)$ by means of the least-squares method. Next, we considered all EOS as a whole and repeated the same procedure. The results are listed in Table 1 where the merit function χ^2/N is also tabulated, with N as the number of points. All resulting exponents are close to 0.80, the strongest deviation comes from *FSUgold*, which predicts 0.86. However, this EOS also predicts an $M_{\rm max}$ of 1.74 $M_\odot$, which contradicts with the value of the highest inferred NS mass (1.97 $M_\odot$) obtained by Demorest et al. (2010). Probably this EOS should be ruled out, but we decided to keep it in our calculation of the average exponent (see the last line of Table 1).

Table 1. Neutron stars and EOS: exponents for the function $\Lambda(R)$.

EOS	$M_{\rm max}$ ($M_\odot$)	Exponent	χ^2/N
DD2	2.4223	0.79	6.59E-6
FSUgold	1.7392	0.86	2.13E-5
NL3	2.7911	0.78	2.76E-5
SFHo	2.0587	0.82	1.25E-5
SFHx	2.1301	0.79	2.66E-5
TM1	2.2130	0.82	4.18E-5
TMA	2.0217	0.82	6.52E-5
All EOS	–	0.80	4.31E-5

The individual values of the exponent of $\Lambda(R)$ as well as the average one obtained taking into account all EOS together (Table 1) allow us to consider that the function $\Gamma(M, \text{EOS})$ is invariant for NS and independent of the adopted EOS and of the stellar mass. For WD and NS as the final products of stellar evolution, we have shown that they recover the initial value of $\Gamma(M, \text{EOS})$ characteristic of the PMS (≈0.4), i.e., stars begin and end their lives in different ways but they recover this fossil function at the last stages of stellar evolution. Whether this remains true for black holes has to be examined in future studies.

As we have seen, the function $\Gamma(M, \text{EOS})$ is conserved for all NS models/EOS but $[\alpha\beta]_{\rm GR}$ is no longer constant and is a function of mass (Fig. 4). Some interesting characteristics can be inferred from this figure. First, it can be noticed that for very low NS masses the effects of general relativity are not important and $[\alpha\beta]_{\rm GR}$ tends to the Newtonian value. The second characteristic is related to the stability of NS models. All models displayed in Fig. 4 are in equilibrium, but this does not necessarily mean that they are stable. The main point concerning equilibrium configurations is that they may correspond to a maximum or minimum of energy with respect to compression/expansion. For NS the necessary (but not sufficient) condition of stability is given by (e.g., Harrison et al. 1965)

$$\frac{\partial M}{\partial \rho_c} > 0. \tag{3}$$

An equivalent criterion can be derived considering the classical order-of-magnitude relations for the pressure and density, $P \approx GM^2/R^4$ and $\rho \approx M/R^3$, respectively. From these relations, we obtain (Cox & Giuli 1968)

$$M \approx \left(\frac{P}{G\rho^{4/3}}\right)^{3/2} \Rightarrow \frac{2}{3}\left(\frac{\partial \ln M}{\partial \ln \rho}\right)_S = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S - \frac{4}{3} = \overline{\Gamma_1} - \frac{4}{3}, \tag{4}$$

where the index S refers to the entropy and $\overline{\Gamma_1}$ is an average value over the whole model. From Eq. (4) we derive the condition that a model will be stable for $\overline{\Gamma_1} > 4/3$.

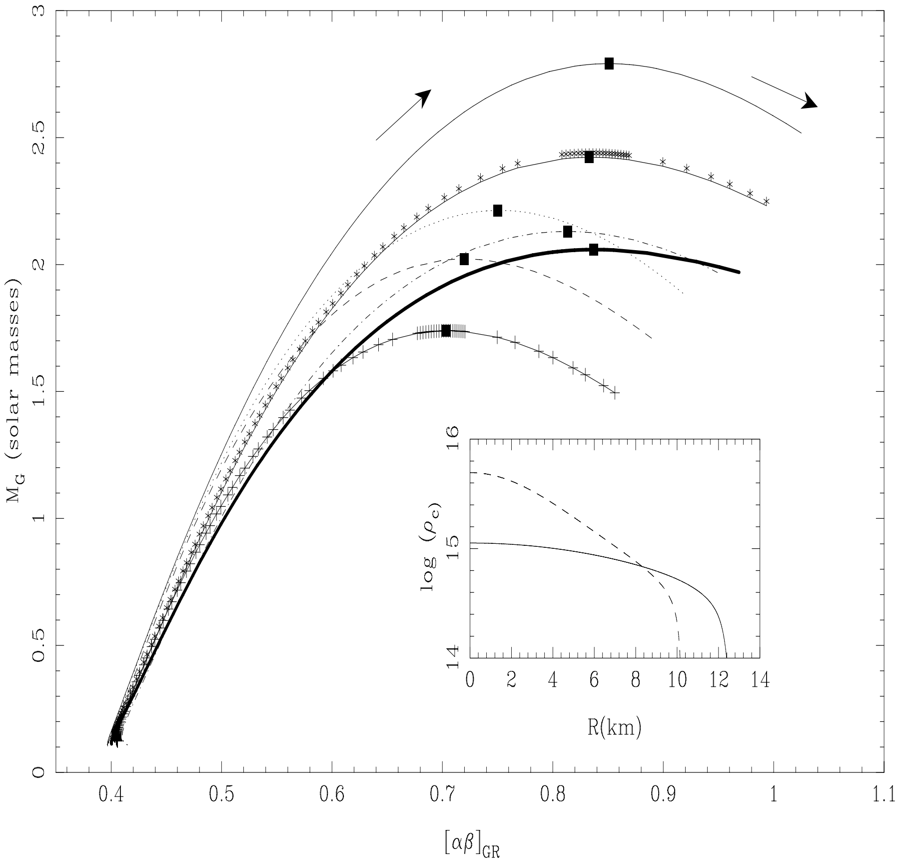

Fig. 4. Mass-$[\alpha\beta]_{\rm GR}$ relation for neutron stars, including unstable models. The two arrows indicate the direction of increasing central density and the full squares denote the $M_{\rm max}$. The same symbols as in Fig. 3. Lower right corner inset: mass-energy distribution for a stable (continuous line) and unstable (dashed line) model with similar masses using *DD*2 EOS.

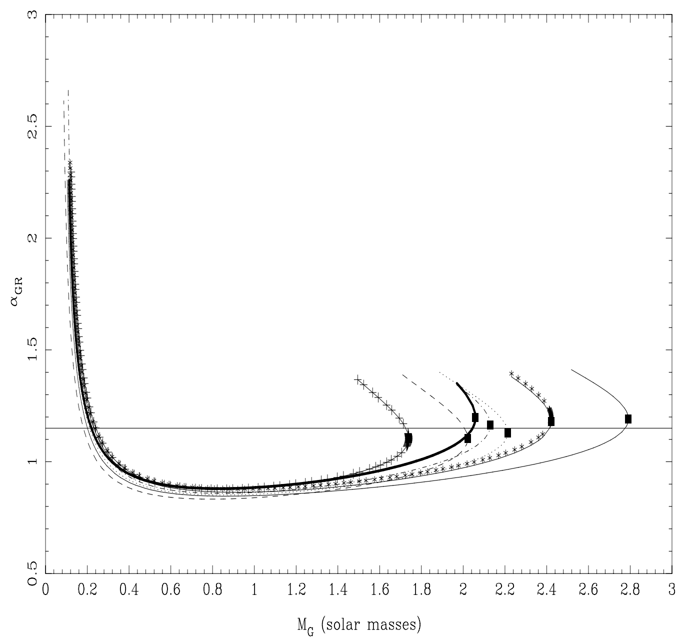

Fig. 5. Mass-$\alpha_{\rm GR}$ relation for neutron stars. The same symbols as in Fig. 3. The horizontal full line indicates the average value of $[\alpha]_{\rm GRmax}$.

Now we use the Jacobi virial theorem to check the stability and derive the dimension of a NS without the need to use numerical calculations like the integration of the TOV equations. We mainly follow the work by Fowler (1966). It can be shown that if we use the Jacobi stability criterion, $\frac{\partial^2 I}{\partial t^2} > 0$, we obtain the inequality

$$\frac{R_S}{R_N}\left(4\zeta_1\ \overline{\Gamma_1 - 1} + 2\zeta_2\ \overline{5 - 3\Gamma_1}\right) < \left(\frac{3\ \overline{3\Gamma_1 - 4}}{2(5-n)}\right), \tag{5}$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius, $R_{\rm N}$ is the radius of a neutron star, the factor $3/(5-n)$ comes from the gravitational potential energy for a polytrope of index n, and $\zeta_n \approx 5.07/(5-n)^2$. This function is connected with the post-Newtonian expansion of the proper internal energy and the gravitational potential energy of polytropes. Using the results of Eq. (4), $\overline{\Gamma_1} = 3/2$, $\zeta_1 = 0.31$, $\zeta_2 = 0.55$, and $n = 2$ we obtain $R_{\rm N} > 4.7\ R_{\rm S}$. Introducing numerical values for 1 $M_\odot$, we obtain that a typical NS must have a radius of the order of 14 km, which agrees well with the average radius inferred from Fig. 3. Indeed, for the more elaborated calculations shown in Fig. 3, the radius of a 1 $M_\odot$ NS model is between 12 and 15 km.

Next, we derive heuristically an alternative macroscopic criterion of stability for NS based on the properties of their gravitational potential energies and moments of inertia through $[\alpha\beta]_{\rm GR}$. We refer to Fig. 4, where several NS models are displayed for seven EOS, including unstable models. In this figure the arrows indicate the direction of increasing of $\rho_{\rm c}$ and the limiting masses are denoted by full squares. We first consider the configurations to the left of the turning points. If a given configuration in this region is compressed (higher $\rho_{\rm c}$), the resulting configuration will present a deficit of gravitational mass if compared with its original position. The gravitational force will act to return it to the original position. Mutatis mutandis, if we decrease $\rho_{\rm c}$, the star also returns to the initial position. For models to the right of the turning points, this force will tend to move them from the original position. Therefore, we introduce a macroscopic criterion of stability: the models will be stable only for positive slopes

$$\frac{\partial M}{\partial[\alpha\beta]_{\rm GR}} > 0. \tag{6}$$

To illustrate this criterion of stability we plotted at the lower right corner of Fig. 4 the mass-energy distribution for two NS models with similar masses (2.23 $M_{\odot}$) but located to the left (stable) and to the right (unstable) of the turning point. The adopted EOS was *DD*2. The steeper profile near the centre corresponding to the model located to the right of the turning point indicates that a perturbation in the star probably induces it to collapse gravitationally to a black hole.

As we have seen (Eq. (4)), the critical value of $\overline{\Gamma_1}$ is 4/3 for Newtonian configurations. However, the effects of the general relativity on $\overline{\Gamma_{\rm cri}}$ can be significant for NS, for example. Shapiro & Teukolsky (1983) have shown that these effects can be expressed as

$$\overline{\Gamma_{\rm cri}} = \frac{4}{3} + a\frac{GM}{Rc^2}, \tag{7}$$

where a is a positive constant of the order of unity. Therefore, the effect of the General Relativity is to increase the value of $\overline{\Gamma_{\rm cri}}$ and making NS more unstable. On the other hand, we have shown in the beginning of this section that $[\alpha\beta]_{\rm GR}/\Lambda^{0.8}(R) \approx 0.4$ for all NS models. Combining Eq. (7) with the definition of the function $\Lambda(R)$, we can show that

$$\overline{\Gamma_{\rm cri}} \approx \frac{4}{3} + a\left(\frac{[\alpha\beta]_{\rm GR} - 0.4}{[\alpha\beta]_{\rm GR}}\right). \tag{8}$$

Equation (8) presents some interesting aspects. The first one is connected with the Newtonian configurations. For PMS and WD sequences we have shown (Claret 2012) that $[\alpha\beta]_{\rm GR} \rightarrow [\alpha\beta]_{\rm Newt} \approx 0.4$. Therefore, for these configurations the relativistic corrections are very small. The other point is that which links Fig. 4, Eq. (6), and Eq. (8). The behaviour of $[\alpha\beta]_{\rm GR}$ for NS is a key for understanding the stability of these stars and also for determining $M_{\rm max}$.

The general features shown in Fig. 4 as well as the alternative criterion of stability are due to the high compactness of NS and are directly related to General Relativity. As mentioned before, new observational data on NS masses gradually act as a discriminator and some EOS should be discarded. At our present level of knowledge, the EOS *FSUgold* must be ruled out. In the framework of Fig. 4 the limiting masses $M_{\rm max}$ are located in a narrow range (0.72–0.85) of $[\alpha\beta]_{\rm GR}$. In Fig. 5 we show the behaviour of $\alpha_{\rm GR}$ as a function of mass. By inspecting this figure we note that $\alpha_{\rm GR}$ is almost independent of the EOS at the turning points, i.e., for the $M_{\rm max}$. The value of $\alpha_{\rm GRmax}$ obtained with a least-square fitting is 1.15: the strongest deviation is 0.05. Surely, $\alpha_{\rm GRmax}$ can be refined with the contribution of improved EOS and new observational data of NS.

In the present investigation, we only included relativistic EOS models and, in part the results for the non-relativistic APR EOS. The seven relativistic mean-field EOS of Hempel et al. cover a broad range of possible neutron star mass-radius relations. Nevertheless, it would be good to include additional non-relativistic EOS and maybe also EOS with exotic degrees of freedom such as hyperons or quarks, to further validate the model-independency of the function $\Gamma(M, \mathrm{EOS})$.

6. Summary

We have investigated some aspects of the stellar evolution concerning the Jacobi virial equation, the function $\Gamma(M, \mathrm{EOS})$, and the stability criterion. We summarise here the main results:

1) We analysed the internal structure of gaseous planets with masses varying from 0.1 up to 50 $M_{\rm J}$. We showed that the function $\Gamma(M, \mathrm{EOS})$ is invariant for all models throughout the planetary evolution. In this way we extended the invariance of this function to gaseous planets.
2) We found a connection between the strong variations of $\Gamma(M, \mathrm{EOS})$ during the intermediary evolutionary phases with the specific nuclear power. We also found a specific nuclear power threshold. Below this limit the function is invariant (≈0.4) for PMS-WD models.
3) It was shown that the function $\Gamma(M, \mathrm{EOS})$ for NS models is independent of the EOS and of the stellar mass.
4) A macroscopic stability criterion for NS models was introduced on the basis of the relativistic product $[\alpha\beta]_{\rm GR}$.
5) Although the observational data for NS does not allow us to distinguish among all EOS yet, we determined that it seems to be a single maximum value of $[\alpha]_{\rm GR}$ for the turning points.
6) We confirmed that regardless of the final products of the stellar evolution, NS or WD, they recover the initial value of $[\alpha\beta]_{\rm GR}/\Lambda^{0.8}(R) \approx 0.4$ acquired at the PMS. The case of black holes will be subject of a future study.

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