

Toward detailed prominence seismology

II. Charting the continuous magnetohydrodynamic spectrum

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ABSTRACT

Context. Starting from accurate magnetohydrodynamic flux rope equilibria containing prominence condensations, we initiate a systematic survey of their linear eigenoscillations. This paves the way for more detailed prominence seismology, which thus far has made dramatic simplifications about the prevailing magnetic field topologies.

Aims. To quantify the full spectrum of linear MHD eigenmodes, we require knowledge of all flux-surface localized modes, charting out the continuous parts of the MHD spectrum. We combine analytical and numerical findings for the continuous spectrum for realistic prominence configurations, where a cool prominence is embedded in a hotter cavity, or where the flux rope contains multiple condensations supported against gravity.

Methods. The equations governing all eigenmodes for translationally symmetric, gravitating equilibria containing an axial shear flow, are analyzed, along with their flux-surface localized limit. The analysis is valid for general 2.5D equilibria, where either density, entropy, or temperature vary from one flux surface to another. We analyze the intricate mode couplings caused by the poloidal variation in the flux rope equilibria, by performing a small gravity parameter expansion. We contrast the analytical results with continuous spectra obtained numerically.

Results. For equilibria where the density is a flux function, we show that continuum modes can be overstable, and we present the stability criterion for these convective continuum instabilities. Furthermore, for all equilibria, a four-mode coupling scheme between an Alfvénic mode of poloidal mode number m and three neighboring $(m - 1, m, m + 1)$ slow modes is identified, occurring in the vicinity of rational flux surfaces. For realistically structured prominence equilibria, this coupling is shown to play an important role, from weak to stronger gravity parameter g values. The analytic predictions for small g are compared with numerical spectra, and progressive deviations for larger g are identified.

Conclusions. The unstable continuum modes could be relevant for short-lived prominence configurations. The gaps created by poloidal mode coupling in the continuous spectrum need further analysis, as they form preferred frequency ranges for global eigenoscillations.

Key words. Sun: filaments, prominences – instabilities – magnetohydrodynamics (MHD) – plasmas

1. Prominence seismology from MHD modeling

Motivated by the recent interest and progress made in diagnosing oscillations in solar prominences, we have started an effort (Blokl and Keppens 2011, hereafter Paper I) to close the gap between observed prominence properties, and the models used to quantify their oscillations. From observations, increasingly detailed knowledge of thermodynamical properties in filaments is available. This relies on inverting spectroscopic data, where departures from local thermodynamic equilibrium in the radiative transfer through one- and two-dimensional prominence models are nowadays routinely incorporated (Labrosse et al. 2010). At the same time, high resolution observations (Lin et al. 2005), computations of three-dimensional (3D) magnetohydrostatic equilibrium configurations without gravity (Aulanier et al. 2002), or modern magnetic field extrapolations from photospheric magnetograms (DeRosa et al. 2009), together with the application of spectro-polarimetric techniques (Lin et al. 1998; Trujillo Bueno et al. 2002), have brought insight into the complex three-dimensional topology of the surrounding magnetic field. Simultaneously, increased spatio-temporal resolution has quantified a host of oscillation modes observed in prominences, with velocity amplitudes of a few km s^{-1} . Both short- and long-period signals, from minutes to hours, can be detected, and regions that oscillate coherently with similar periods allow us to quantify damping rates for seemingly global eigenoscillations (Terradas et al. 2002).

Theoretical studies in prominence seismology have meanwhile identified the most likely candidates explaining these motions, in terms of fundamental sausage, kink, and other mode types present in the MHD spectrum of magnetized slabs or cylinders. To explain the observed damping of the eigenmodes, one can on the one hand study wave damping as a result of non-adiabatic effects, such as anisotropic, thermal conduction, and radiative losses. For cylindrically structured coronal loops and prominences, with prevailing helical fields, early work by Keppens et al. (1993) demonstrated how these non-adiabatic effects can give rise to damped and/or overstable global, discrete non-adiabatic Alfvén waves. Soler et al. (2008) revisited these effects for eigenmodes of more simplified, uniform cylinder models in homogeneous magnetic fields. By using uniform internal/external conditions, the

possibility of wave resonant behavior was excluded artificially. This resonant behavior, where global waves have eigenfrequencies in the range of the MHD continuous spectrum, provides an alternative means of explaining the observed damping rates. The theory for resonant MHD waves is reviewed in many papers, e.g. [Goossens et al. \(2011\)](#), and references therein. In other reviews of prominence oscillations (e.g. [Mackay et al. 2010](#); [Arregui & Ballester 2011](#)), many of the accumulated theoretical findings are discussed, and a striking observation is that all these efforts dramatically simplify the underlying magnetic topology, in order to allow analytic progress. Non-uniformity, and the associated possibility of resonant absorption, is at most introduced by varying the density profiles (e.g. [Soler et al. 2010](#)), sticking to uniform field topologies. At the same time, efforts to account for the multi-stranded nature of actual solar filaments have also been initiated, in similarly simple magnetic fields ([Luna et al. 2010](#)).

To pave the way for more detailed prominence seismology, true MHD spectroscopy ([Goedbloed et al. 1993](#)), of the kind pioneered in fusion-related plasma physics, is called for. This relies on accurate computations of the underlying equilibria, with full account of the complex magnetic topology, and subsequent computation of its eigenspectrum. In our accompanying Paper I, we started this effort by demonstrating our ability to compute accurate translationally symmetric flux rope equilibria containing prominence condensations, governed by the magnetohydrostationary equations balancing Lorentz forces, pressure gradients, and gravity. We here continue this effort, and use these equilibria, where we incorporate helical fields, axial shear flows, and varying plasma beta conditions consistent with observed filament properties. We here concentrate on quantifying all eigenmodes defining the continuous parts of the MHD eigenspectrum. This is a worthwhile effort in its own right, partly because the global, discrete modes are known to be organized about this continuous spectrum (see e.g. modern textbooks such as [Goedbloed & Poedts 2004](#); [Goedbloed et al. 2010](#)), and because the possible damping of these global modes due to resonant absorption relies on knowledge of the frequency ranges that they occupy.

Charting out the continuous MHD spectrum for realistic flux rope structures with prominences has never been attempted before, in sharp contrast to the vast body of knowledge on MHD eigenmodes in axisymmetric, tokamak configurations. Our effort considers flux ropes idealized to be invariant along the filament axis, with an outer cylindrical cross-section, for simplicity. Since the gravitational stratification will act to deform a pure cylindrical flux model to one with still nested, but downwards displaced flux surfaces, the resulting poloidal variation in the equilibrium quantities will induce coupling between the different branches of the continuous MHD spectrum. These branches are labeled by their axial, k , and azimuthal, m , mode numbers. When charting the mode frequencies from the filament axis to the flux rope edge, the spectrum can help us to determine the avoided crossings, which are akin to the poloidal mode couplings discussed for equilibria without gravity, such as the nested elliptic cylinders studied by [Chance et al. \(1977\)](#), or configurations with non-circular cross-sections and/or density variations along field lines analyzed by [Poedts & Goossens \(1991\)](#). We extend these latter studies to translationally invariant, gravitating equilibria with possibly shear flow $v_z(\psi)$, restricted to be a flux function. The actual equilibria are presented and discussed in Paper I. The analysis of their continuous spectra presented here, benefits from the detailed knowledge of the spectrum of the more complex, axisymmetric topologies ([Goedbloed 1975](#)). The case for self-gravitating, static axisymmetric magnetized tori was treated analytically in [Poedts et al. \(1985\)](#). [Goedbloed et al. \(2004b\)](#) analyzed of the continuous MHD spectrum for magnetized accretion tori, where external gravity, nested toroidal flux surfaces, and both poloidal and toroidal (Keplerian-like) rotation is included, and identified a potential for unstable localized modes called trans-slow Alfvén continuum modes. The approach of [Goedbloed et al. \(2004b\)](#), which combined analytic findings for the equilibria, the poloidal mode couplings that were confronted with numerically computed equilibria, and eigenmodes is analogous to our efforts in this series.

Our paper is then organized as follows. In Sect. 2, we present the equations governing all eigenmodes for these gravitating, 2.5D MHD equilibria. Since we allow for axial flow, the Frieman-Rosenbluth formalism ([Frieman & Rosenbluth 1960](#)) is revisited, and using the straight field line coordinates (see Paper I) describing our equilibria, we present them in a form where a field line projection can be made. This form can directly be compared with all other cases studied previously, such as the cylindrical limit, or the analogy with tokamak configurations that contain toroidal flow. In Sect. 3, we then derive the equations for the continuous spectrum, and find that prominence equilibria where density is a flux function can also be affected by convective continuum instabilities ([Blokland et al. 2007a](#); [van der Holst et al. 2000b,a](#)). The coupled system of differential equations can be converted to algebraic relations between poloidal Fourier harmonics, which clearly identifies how Alfvén and slow modes can couple. This coupling is then studied analytically in Sect. 4, where the small gravity parameter expansion previously used (Paper I) to analyze equilibrium effects due to gravity, re-enters to analyze the leading order coupling effects. For low frequencies, we then demonstrate that a gap can arise in the Alfvén continuum branches, from a four-mode coupling scheme. In Sect. 5, we briefly summarize the numerical approach used in the remaining sections. There, the equilibria presented in Paper I are all diagnosed spectroscopically, by charting out their continuous spectrum. The analytic findings are used to check and analyze in detail the derived eigenfrequencies. This is done for cool prominences embedded in hot cavities in Sect. 6, and for complex, double layered prominences without and with axial flows in Sect. 7. Conclusions and an outlook to future work is given in Sect. 8.

2. Spectral formulation

In what follows, we analyze the equations governing linear, ideal MHD perturbations about a stationary equilibrium. The equations for the equilibrium have been presented and solved numerically in Paper I. The (time-independent) equilibrium quantities are denoted by the magnetic field vector \mathbf{B} , the pressure p , the density ρ , and the external gravitational potential Φ , which for the prominence application at hand are all functions of the poloidal coordinates (x, y) . The background equilibrium may contain a purely longitudinal shear flow $\mathbf{v} = v_z(\psi)\mathbf{e}_z$, where the axial flow is constant on each flux surface. Each flux label ψ identifies a single flux surface, which form nested surfaces defined by the magnetic field topology within the prescribed outer flux rope shape. We first recapitulate the governing equations, and then present a representation exploiting a projection involving straight field line coordinates.

2.1. Frieman-Rosenbluth formalism

The formalism introduced by [Frieman & Rosenbluth \(1960\)](#) is used here to investigate the stability properties of stationary MHD equilibria. From the full set of linearized MHD equations, these authors derived one second-order differential equation for the Lagrangian displacement field vector ξ

$$\mathbf{G}(\xi) - 2\rho\mathbf{v} \cdot \nabla \frac{\partial \xi}{\partial t} - \rho \frac{\partial^2 \xi}{\partial t^2} = 0, \quad (1)$$

where the generalized force operator $\mathbf{G}(\xi)$ is

$$\mathbf{G}(\xi) = \mathbf{F}(\xi) + \nabla\Phi\nabla \cdot (\rho\xi) + \nabla \cdot [\rho\xi\mathbf{v} \cdot \nabla\mathbf{v} - \rho\mathbf{v}\mathbf{v} \cdot \nabla\xi]. \quad (2)$$

Here,

$$\mathbf{F}(\xi) = -\nabla\Pi + \mathbf{B} \cdot \nabla\mathbf{Q} + \mathbf{Q} \cdot \nabla\mathbf{B} \quad (3)$$

is the force operator for static equilibria without gravity, which was derived by [Bernstein et al. \(1958\)](#). These expressions contain the Eulerian perturbation of the total pressure

$$\Pi = -\gamma p\nabla \cdot \xi - \xi \cdot \nabla p + \mathbf{B} \cdot \mathbf{Q}, \quad (4)$$

where γ is the ratio of specific heats, and the Eulerian perturbation of the magnetic field

$$\mathbf{Q} = \nabla \times (\xi \times \mathbf{B}). \quad (5)$$

The time-dependence of the displacement field is assumed to be exponential with normal mode frequencies ω , $\xi = \hat{\xi} \exp(-i\omega t)$. Using this assumption, the Frieman-Rosenbluth equation can be written as

$$\mathbf{G}(\hat{\xi}) + 2i\rho\omega\mathbf{v} \cdot \nabla\hat{\xi} + \rho\omega^2\hat{\xi} = 0. \quad (6)$$

We base our derivations below on the axial symmetry of the solar prominence equilibrium by writing the axial dependence of the displacement field as $\hat{\xi} \sim \exp(ikz)$, where k is the axial mode number. We then neglect the hat on $\hat{\xi}$ and suppress the implied $\exp(-i\omega t + ikz)$ dependence for all linear quantities.

2.2. Field-line projection and representation

As in all stability studies in fusion research, we need to exploit a projection based on the nested topology of the magnetic surfaces, on which the helical field lines reside. This projection ensures that on each flux surface, one can make a distinction between two wave directions: one parallel and the other perpendicular to the magnetic field. In the short wavelength limit, these directions correspond to the polarizations of the slow and Alfvén waves, respectively. To perform this projection, we adopt straight field line, non-orthogonal coordinates (ψ, ϑ, z) that are introduced in our accompanying paper ([Bloklund & Keppens 2011](#)), which are known once the equilibrium is fully determined. The triad of unit vectors based on the projection is

$$\mathbf{n} \equiv \frac{\nabla\psi}{|\nabla\psi|}, \quad \boldsymbol{\pi} \equiv \mathbf{b} \times \mathbf{n}, \quad \mathbf{b} \equiv \frac{\mathbf{B}}{B}, \quad (7)$$

where the gradient of the magnetic flux ψ is perpendicular to the flux surfaces. Using these unit vectors, the components of the displacement field ξ can be replaced by the three related variables

$$X \equiv B_\vartheta \xi \cdot \mathbf{n}, \quad Y \equiv i \frac{B}{B_\vartheta} \xi \cdot \boldsymbol{\pi}, \quad Z \equiv i \frac{1}{B} \xi \cdot \mathbf{b}, \quad (8)$$

and the projected gradient operators become

$$\mathcal{D} \equiv \frac{1}{B_\vartheta} \mathbf{n} \cdot \nabla = \partial_\psi - \frac{g_{12}}{g_{22}} \partial_\vartheta, \quad (9)$$

$$\mathcal{G} \equiv -iB_\vartheta B \boldsymbol{\pi} \cdot \nabla = \frac{-iI}{J} \partial_\vartheta - kB_\vartheta^2, \quad (10)$$

$$\mathcal{F} \equiv -i\mathbf{B} \cdot \nabla = \frac{-i}{J} \partial_\vartheta + \frac{kq}{J}. \quad (11)$$

The second equality in the expressions for \mathcal{F} and \mathcal{G} only holds when these operators act on a component of the displacement field ξ . These expressions contain several quantities already introduced in Paper I, such as the metric elements g_{ij} and the Jacobian J for the non-orthogonal straight field line coordinates, and quantities shown to be flux functions to realize equilibrium, such as the poloidal stream function $I = B_z$ and the safety factor $q(\psi) = JI$.

The Frieman-Rosenbluth equation in Eq. (1) can be written as

$$(\mathbf{A} + \mathbf{G})\mathbf{x} - \rho\bar{\omega}^2\mathbf{B}\mathbf{x} = 0, \quad \text{with} \quad \mathbf{x} \equiv \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad (12)$$

by exploiting the straight field line coordinates and applying the projection defined by Eq. (8), where $\bar{\omega}(\psi) \equiv \omega - kv_z(\psi)$ is the flux-surface localized Doppler-shifted eigenfrequency. Furthermore, the matrices \mathbf{A} and \mathbf{B} are 3×3 matrix operators that are also present in the case of a static equilibrium without gravity. The matrix operator \mathbf{G} contains the elements associated with the external gravitational potential. The non-vanishing matrix elements of \mathbf{A} and \mathbf{B} are

$$\begin{aligned} A_{11} &\equiv -\mathcal{D}\frac{\gamma p + B^2}{J}\mathcal{D}^\dagger J + \mathcal{F}\frac{1}{B_\theta^2}\mathcal{F} + 2\left[\mathcal{D}(B_\theta\kappa_p)\right], \\ A_{12} &\equiv -\mathcal{D}\gamma p\mathcal{G}\frac{1}{B^2} - \mathcal{D}\mathcal{G} + 2kB_\theta\kappa_p, \\ A_{13} &\equiv -\mathcal{D}\gamma p\mathcal{F}, \\ A_{21} &\equiv \frac{1}{B^2}\mathcal{G}\frac{\gamma p}{J}\mathcal{D}^\dagger J + \mathcal{G}\frac{1}{J}\mathcal{D}^\dagger J + 2kB_\theta\kappa_p, \\ A_{22} &\equiv \frac{1}{B^2}\mathcal{G}\gamma p\mathcal{G}\frac{1}{B^2} + \mathcal{G}\frac{1}{B^2}\mathcal{G} + \mathcal{F}\frac{B_\theta^2}{B^2}\mathcal{F}, \\ A_{23} &\equiv \frac{1}{B^2}\mathcal{G}\gamma p\mathcal{F}, \\ A_{31} &\equiv \mathcal{F}\frac{\gamma p}{J}\mathcal{D}^\dagger J, \\ A_{32} &\equiv \mathcal{F}\gamma p\mathcal{G}\frac{1}{B^2}, \\ A_{33} &\equiv \mathcal{F}\gamma p\mathcal{F}, \end{aligned} \quad (13)$$

and

$$B_{11} \equiv \frac{1}{B_\theta^2}, \quad B_{22} \equiv \frac{B_\theta^2}{B^2}, \quad B_{33} \equiv B^2. \quad (14)$$

We used the poloidal curvature of the magnetic surfaces κ_p from Paper I, along with the operators

$$\mathcal{D}^\dagger \equiv \mathcal{D} - \left[\partial_\theta \left(\frac{g_{12}}{g_{22}} \right) \right], \quad {}^\dagger\mathcal{D} \equiv \mathcal{D} + \left[\partial_\theta \left(\frac{g_{12}}{g_{22}} \right) \right]. \quad (15)$$

These are related to the normal gradient operator \mathcal{D} , originally introduced by Goedbloed (1997) for the spectral analysis of static tokamak equilibria. The square brackets in the expressions in Eq. (13) for the matrix elements of \mathbf{A} and the gradient operators defined by (15) indicate that the differential operator only acts on the term inside the bracket. This notation is also used in the expressions below. As expected, the matrices \mathbf{A} and \mathbf{B} can also be derived from the expressions for wave modes about static axisymmetric MHD equilibria, found in the papers by Goedbloed (1975, 1997), by formally setting the toroidal distance R to unity. The matrix \mathbf{G} enters if there is an external gravitational potential present. The expression for this matrix is

$$\mathbf{G} = \begin{pmatrix} -\rho \left\{ J^\dagger \mathcal{D} \left(\frac{1}{J} \lambda \right) \right\} & (i\mathcal{D}I\mu + \lambda\mathcal{G})\frac{\rho}{B^2} & (i\mathcal{D}\mu + \lambda\mathcal{F})\rho \\ \frac{\rho}{B^2} \left(iI\mu \frac{1}{J} \mathcal{D}^\dagger J + \mathcal{G}\lambda \right) & -\frac{\rho I^2}{B^4} \left[\frac{1}{J} \partial_\theta \mu \right] & -\rho \left(\frac{I}{B^2} \left[\frac{1}{J} \partial_\theta \mu \right] - ik\mu \right) \\ \rho \left(i\mu \frac{1}{J} \mathcal{D}^\dagger J + \mathcal{F}\lambda \right) & -\rho \left(\frac{I}{B^2} \left[\frac{1}{J} \partial_\theta \mu \right] + ik\mu \right) & -\rho \left[\frac{1}{J} \partial_\theta \mu \right] \end{pmatrix}, \quad (16)$$

where

$$\mu \equiv -\frac{1}{J}\partial_\theta\Phi = \frac{1}{\rho J}\partial_\theta p = \frac{1}{\rho}\mathbf{B}_\theta \cdot \nabla p, \quad (17)$$

$$\lambda \equiv -\mathcal{D}\Phi. \quad (18)$$

We note that the term μ represents the pressure variation on a flux surface. This matrix \mathbf{G} can also be derived from the expression for the matrix operator \mathbf{R} found in the paper by Blokland et al. (2007a), where axisymmetric, gravitating, magnetized accretion tori with purely toroidal flow were spectroscopically analyzed. The equivalence involves (1) setting their radial distance $R = 1$; (2) producing the toroidal curvature $\kappa_t = 0$ for a translationally symmetric solar prominence; and (3) replacing the toroidal mode number n by the axial wave number k . In that analysis, as well as in the related one for toroidally rotating tokamak plasmas by

van der Holst et al. (2000a), another matrix \mathbf{C} related to the Coriolis effect appeared, which naturally vanishes for our translationally invariant configurations.

Another interesting, related limit is the cylindrical limit known from standard textbook treatments (Goedbloed & Poedts 2004). If we ignore gravity altogether, then matrix \mathbf{G} will be zero and the operators \mathcal{F} and \mathcal{G} will just be algebraic expressions. In this case, the spectral equation given in Eq. (12) reduces exactly to the well-known equation presented by Hain & Lüst (1958). They rewrote this matrix form to one second-order differential equation for the radial displacement field, assuming a ratio of specific heats $\gamma = 1$. This was generalized by Goedbloed (1971) for arbitrary γ , and rewritten, for cases involving equilibrium flow in Hameiri (1981) and Bondeson et al. (1987). A cylindrical limit involving flow and external gravity, suitable for analyzing accretion disk stability, was presented by Keppens et al. (2002), and can also be easily shown to be directly related to the above expressions.

3. Continuous MHD spectrum

In the previous section, we derived the spectral equation in Eq. (12) governing all MHD waves and instabilities in translational symmetric solar prominences. This equation is the starting point to all following MHD spectral computations. In particular, we can derive the equations for the continuous MHD spectrum by first considering modes localized on a particular flux surface $\psi = \psi_0$. We can then obtain a non-singular eigenvalue problem in the local Alfvén and slow-like eigenfunctions $\eta(\vartheta)$, $\zeta(\vartheta)$. This form allows us to derive a stability criterion for continuum modes, where it is found that for an axially symmetric prominence, gravity can cause the continuous spectrum to become unstable. By expanding the periodic ϑ -dependence of the eigenfunctions, we get an equivalent algebraic set of coupled equations.

3.1. General formalism

To derive the equation for the continuous spectrum, we assume that the normal derivative ($\partial/\partial\psi$) of the eigenfunctions is large compared to the eigenfunctions themselves and the local Doppler-shifted eigenfrequency $\tilde{\omega}(\psi)$ is finite. Using these assumptions, the first row of the spectral equation in Eq. (12) can be solved approximately

$$\frac{1}{J} \mathcal{D}^\dagger JX \approx \frac{-\gamma p}{\gamma p + B^2} \left(\mathcal{G} \frac{1}{B^2} Y + \mathcal{F} Z \right) - \frac{1}{\gamma p + B^2} \mathcal{G} Y + \frac{i\rho\mu}{\gamma p + B^2} \left(\frac{1}{B^2} Y + Z \right). \quad (19)$$

We note that X is small compared to either Y or Z . This implies that the continuum modes are mainly tangential to a particular flux surface. Furthermore, we note that $\partial X/\partial\psi$, Y , and Z are of the same order.

Inserting the approximated solution given by Eq. (19) into the second and third row of the spectral equation in Eq. (12) results in an eigenvalue problem that is independent of the normal derivative. Hence, the reduced problem becomes non-singular. Exploiting this property, we can write the projected displacement field components Y and Z as

$$\begin{aligned} Y &\approx \delta(\psi - \psi_0) \eta(\vartheta), \\ Z &\approx \delta(\psi - \psi_0) \zeta(\vartheta), \end{aligned} \quad (20)$$

where $\delta(\psi - \psi_0)$ is the Dirac delta function. Here, Y and Z have been divided into an improper (ψ -dependence) and proper part (ϑ -dependence). This kind of splitting was introduced by Goedbloed (1975) for static axisymmetric equilibria without gravity. The reduced, non-singular eigenvalue problem becomes

$$(\mathbf{a} + \mathbf{g} - \rho\tilde{\omega}^2 \mathbf{b}) \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = 0, \quad (21)$$

where

$$\mathbf{a} = \begin{pmatrix} \mathcal{F} \frac{B_\vartheta^2}{B^2} \mathcal{F} + 4 \frac{\gamma p}{\gamma p + B^2} (B_\vartheta \kappa_g)^2 & -2i \frac{\gamma p B}{\gamma p + B^2} (B_\vartheta \kappa_g) \mathcal{F} \\ 2i \mathcal{F} \frac{\gamma p B}{\gamma p + B^2} (B_\vartheta \kappa_g) & \mathcal{F} \frac{\gamma p B^2}{\gamma p + B^2} \mathcal{F} \end{pmatrix}, \quad (22)$$

$$\mathbf{g} = \begin{pmatrix} -4 \frac{I}{B^2} \frac{\rho\mu B}{\gamma p + B^2} (B_\vartheta \kappa_g) + \frac{\rho I^2}{B^2} N_{m,\text{pol}}^2 & iI \frac{\rho\mu B}{\gamma p + B^2} B^2 \mathcal{F} \frac{1}{B^2} + \rho I N_{m,\text{pol}}^2 \\ -i \frac{I}{B^2} \mathcal{F} \frac{\rho\mu}{\gamma p + B^2} B^2 + \rho I N_{m,\text{pol}}^2 & -B^2 \left[\frac{1}{JB^2} \partial_\vartheta \frac{\rho\mu}{\gamma p + B^2} B^2 \right] + \rho B^2 N_{m,\text{pol}}^2 \end{pmatrix}, \quad (23)$$

$$\mathbf{b} = \begin{pmatrix} \frac{B_\vartheta^2}{B^2} & 0 \\ 0 & B^2 \end{pmatrix}. \quad (24)$$

In the expressions above, κ_g is the geodesic curvature,

$$\kappa_g = \boldsymbol{\kappa} \cdot \boldsymbol{\pi} = \frac{I}{JB_\theta B^2} \partial_\theta B, \quad (25)$$

where $\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b}$ is the field line curvature. Furthermore, $N_{\text{m,pol}}^2$ is the magnetically modified Brunt-Väisälä frequency projected onto a flux surface,

$$\begin{aligned} N_{\text{m,pol}}^2 &= \frac{\mu}{B^2} \left[\frac{1}{J\rho} \partial_\theta \rho - \frac{\rho\mu}{\gamma p + B^2} \right] \\ &= \left[\frac{\mathbf{B}_\theta \cdot \nabla p}{\rho B} \right] \left[\frac{\mathbf{B}_\theta}{\rho B} \cdot \left(\nabla \rho - \frac{\rho}{\gamma p + B^2} \nabla p \right) \right]. \end{aligned} \quad (26)$$

This magnetically modified Brunt-Väisälä frequency has also been found in various related cases. For example, [Poedts et al. \(1985\)](#) and [Beliën et al. \(1997\)](#) found this frequency for static, axisymmetric gravitationally stratified plasmas. [van der Holst et al. \(2000b,a\)](#) discussed its role for tokamak plasmas with purely toroidal flow. [Blokland et al. \(2007a\)](#) generalized the expression by including an arbitrary external gravitational potential in its application to rotating, magnetized accretion disks. When we examine the matrix \mathbf{g} closely, we see that the poloidal variation in the pressure is present in all its matrix elements. This has the consequence that the diagonal dominance of matrix \mathbf{a} will be destroyed, which causes possible mode coupling between Alfvén and slow modes.

The non-singular eigenvalue problem presented in Eq. (21) is solved for each flux surface, using poloidally periodic boundary conditions for the eigenfunctions η and ζ . Typically, one solves the problem for a given axial mode number k , and we then get a set of discrete eigenvalues for each flux surface. When we quantify these eigenfrequencies for all flux surfaces, they together map out the continuous spectra. For completeness, we point out that if one uses a formulation based on the primitive variables ρ , \mathbf{v} , p , and \mathbf{B} , an additional entropy continuum kv_z is found. [Goedbloed et al. \(2004a\)](#) showed that this Eulerian entropy continuum, in ideal MHD, does not couple to any of the other continua, nor to the discrete stable and/or unstable modes.

3.2. Spectral properties and stability criterion

We now derive a general stability criterion for the continua. For this derivation, a Hilbert space with appropriate inner product needs to be constructed. The parallel gradient operator \mathcal{F} is a Hermitian operator under the inner product

$$\oint Jv^* (\mathcal{F}w) d\vartheta = \oint J (\mathcal{F}v)^* w d\vartheta \quad (27)$$

for poloidally periodic functions v and w . Inspired by this property, a Hilbert space can be defined with the inner product

$$\langle v, w \rangle \equiv \oint v^* \cdot w J d\vartheta. \quad (28)$$

In this Hilbert space, the matrix operators \mathbf{a} , \mathbf{b} , and \mathbf{g} are Hermitian operators. Setting \mathbf{v} and \mathbf{w} equal to the eigenfunctions $(\eta, \zeta)^T$, one can derive a simple quadratic polynomial for the eigenfrequency $\tilde{\omega}$ from the spectral equation in Eq. (21) for the continua, namely

$$a\tilde{\omega}^2 - c = 0, \quad (29)$$

where the coefficients are

$$a \equiv \oint \rho \left(\frac{B_\theta^2}{B^2} |\eta|^2 + B^2 |\zeta|^2 \right) J d\vartheta, \quad (30)$$

$$c \equiv \oint \left\{ \frac{B_\theta^2}{B^2} |\mathcal{F}\eta|^2 + \frac{\gamma p B^2}{\gamma p + B^2} \left| \mathcal{F}\zeta + 2i \frac{B_\theta \kappa_g}{B} \eta - i \frac{\rho\mu}{\gamma p} \left(\frac{I}{B^2} \eta + \zeta \right) \right|^2 + \rho B^2 N_{\text{BV,pol}}^2 \left| \frac{I}{B^2} \eta + \zeta \right|^2 \right\} J d\vartheta. \quad (31)$$

Here, $N_{\text{BV,pol}}^2$ is the Brunt-Väisälä frequency projected onto a flux surface

$$\begin{aligned} N_{\text{BV,pol}}^2 &= \frac{\mu}{B^2} \left[\frac{1}{J\rho} \partial_\theta \rho - \frac{\rho\mu}{\gamma p} \right] \\ &= \left[\frac{\mathbf{B}_\theta \cdot \nabla p}{\rho B} \right] \left[\frac{\mathbf{B}_\theta}{\rho B} \cdot \left(\nabla \rho - \frac{\rho}{\gamma p} \nabla p \right) \right] \\ &= - \left[\frac{\mathbf{B}_\theta \cdot \nabla p}{\rho B} \right] \left[\frac{\mathbf{B}_\theta \cdot \nabla S}{\gamma B S} \right]. \end{aligned} \quad (32)$$

The coefficient a is real and always greater than zero. As for a , the coefficient c is real but can be zero or even negative. The solutions of this second order polynomial in Eq. (29) are simply $\tilde{\omega} = \pm \sqrt{c/a}$. If c is negative, a damped stable wave and an overstable mode

are obtained. In this case, its absolute value is found from $|\bar{\omega}| = \sqrt{-c/a}$, while $\text{Re}(\omega) = kv_z$, so the oscillation frequency is set by the local Doppler shift. By looking at the expression for c , we realize that this coefficient is equal to or greater than zero if

$$N_{\text{BV,pol}}^2 \geq 0 \quad (33)$$

is satisfied everywhere in the plasma. Therefore, equilibria for which the entropy or the temperature (assuming $\gamma > 1$) is a flux function, will always be stable according to this criterion. However, when density is a flux function, the equilibrium variation may violate this criterion. This implies that such equilibria have unstable MHD continua, a situation identified in earlier, related work, as being caused by the convective continuum instability (CCI) (van der Holst et al. 2000b,a; Blokland et al. 2007a). This CCI instability criterion in Eq. (33) is analogous to the Schwarzschild criterion for convective instability, with one important difference. The Schwarzschild criterion deals with normal derivatives, while this criterion deals with tangential ones. This difference has also been found by various authors (Hellsten & Spies 1979; Hameiri 1983; Poedts et al. 1985; van der Holst et al. 2000a; Blokland et al. 2007a).

The relevance of this CCI instability to solar prominences may well be significant. We note that it is common practice to study the (slow) evolution of prominences carrying flux ropes by taking an equilibrium configuration, and subjecting it to e.g. photospheric-boundary-driven changes, and relax it thereby repeatedly to successive equilibrium states (see e.g., Amari et al. 1999). As a result, the internal flux rope variation proceeds through a series of equilibria. For short-lived prominences, equilibria with density as a flux function may occur, and in this evolutionary process, the equilibrium can transit from the stable to the unstable regime as governed by the CCI criterion. As we later demonstrate, the CCI growth rate also varies with increasing gravity parameter g . As explained in Paper I, this g -increase could relate to an increase in size, internal density, or the weaker overall magnetic field strength of the prominence carrying flux rope.

3.3. Expansion in poloidal harmonics

We now continue to transform the two coupled differential equations for the MHD continua in Eq. (21) to an algebraic set, by expanding the eigenfunctions η and ζ in poloidal Fourier harmonics. This expansion has the advantage that we can perform a detailed investigation of how different poloidal harmonics may couple to each other. This coupling could either be between modes of the same kind, for example Alfvén-Alfvén coupling, or between modes of different kind such as Alfvén-slow coupling. In the next section, we show that in an actual solar prominence, more complex mode coupling is present.

Expanding the eigenfunctions η and ζ for the MHD continua in a finite Fourier series,

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \sum_{m'} \begin{pmatrix} \eta_{m'} \\ \zeta_{m'} \end{pmatrix} e^{im'\vartheta}, \quad (34)$$

and applying the operator

$$\oint J e^{-im'\vartheta} d\vartheta \quad (35)$$

to the spectral equation in Eq. (21), one obtains the algebraic coupled equations

$$\begin{pmatrix} K_{mm'} & M_{mm'} \\ N_{mm'} & L_{mm'} \end{pmatrix} \begin{pmatrix} \eta_{m'} \\ \zeta_{m'} \end{pmatrix} = 0. \quad (36)$$

The operator in Eq. (35) is the result of exploiting the inner product in Eq. (28) of the previously defined Hilbert space. The poloidal mode numbers m and m' have the same range and the 2×2 submatrices are given by

$$\sum_{m'} \begin{pmatrix} K_{mm'} & M_{mm'} \\ N_{mm'} & L_{mm'} \end{pmatrix} = \frac{1}{2\pi} \oint \begin{pmatrix} K(\vartheta) & M(\vartheta) \\ N(\vartheta) & L(\vartheta) \end{pmatrix} e^{-i(m'-m)\vartheta} d\vartheta. \quad (37)$$

The explicit expressions for the integrand functions $K(\vartheta)$, $L(\vartheta)$, $M(\vartheta)$, and $N(\vartheta)$ are

$$K(\vartheta) = J \frac{B_\vartheta^2}{B^2} \left[\frac{(m+kq)(m'+kq)}{J^2} + 4 \frac{\gamma p B^2}{\gamma p + B^2} \kappa_g^2 - \rho \tilde{\omega}^2 - 4 \frac{I}{B_\vartheta} \frac{\rho \mu B}{\gamma p + B^2} \kappa_g + \frac{I^2}{B_\vartheta^2} \rho N_{\text{m,pol}}^2 \right], \quad (38)$$

$$M(\vartheta) = JB_\vartheta \left[-2i \frac{\gamma p B}{\gamma p + B^2} \kappa_g \frac{m'+kq}{J} + \frac{\rho \mu}{\gamma p + B^2} \left(i \frac{I}{B_\vartheta} \frac{m'+kq}{J} - 2B\kappa_g \right) + \frac{I}{B_\vartheta} \rho N_{\text{m,pol}}^2 \right], \quad (39)$$

$$N(\vartheta) = JB_\vartheta \left[2i \frac{\gamma p B}{\gamma p + B^2} \kappa_g \frac{m'+kq}{J} + \frac{\rho \mu}{\gamma p + B^2} \left(-i \frac{I}{B_\vartheta} \frac{m'+kq}{J} - 2B\kappa_g \right) + \frac{I}{B_\vartheta} \rho N_{\text{m,pol}}^2 \right], \quad (40)$$

$$L(\vartheta) = JB^2 \left[\frac{\gamma p}{\gamma p + B^2} \frac{(m+kq)(m'+kq)}{J^2} - \rho \tilde{\omega}^2 - i \frac{\rho \mu}{\gamma p + B^2} \frac{m-m'}{J} + \rho N_{\text{m,pol}}^2 \right]. \quad (41)$$

These expressions are derived by exploiting the Hermitian property in Eq. (27) of the parallel gradient operator \mathcal{F} .

Once again, we consider the cylindrical limit in combination with zero gravity, which is appropriate for many earlier seismological studies of prominences. In this limit, all equilibrium quantities are independent of the poloidal angle ϑ , therefore the geodesic

curvature will be zero. This means that the Alfvén and slow continuum decouple fully from each other. Furthermore, only one single harmonic is required for the expansion of the Fourier series of the eigenfunctions η and ζ . Solving the spectral equation for this case, one finds the well-known local Doppler-shifted Alfvén continua

$$\Omega_{A,m}^{\pm} = kv_z \pm \omega_{A,m}, \quad \omega_{A,m} = \left(\frac{m}{r} B_{\vartheta} + kB_z \right) / \sqrt{\rho}, \quad (42)$$

and the local Doppler-shifted slow continua

$$\Omega_{s,m}^{\pm} = kv_z \pm \omega_{s,m}, \quad \omega_{s,m} = \sqrt{\frac{\gamma p}{\gamma p + B^2}} \omega_{A,m}. \quad (43)$$

Looking at the expressions in Eqs. (38)–(41) for the submatrix elements, we can make some general statements about the MHD continua of solar prominences. In the absence of gravity ($\mu = N_{m,\text{pol}}^2 = 0$), the Alfvén and slow continuum can be coupled by the geodesic curvature present in the off-diagonal terms $M(\vartheta)$ and $N(\vartheta)$. With gravity, these off-diagonal terms can become more pronounced. The second statement is that an Alfvén continuum at a resonant surface, associated with the element $K(\vartheta)$, may shift away from the local Doppler-shifted frequency kv_z because of the (1) geodesic curvature, involving its square κ_g^2 ; (2) compressibility and finite pressure (γp); and (3) the pressure variation on a flux surface. All these effects suggest that an Alfvén gap can appear around the local Doppler-shifted frequency and that Alfvén-slow coupling should be taken into account for these gaps. This kind of coupling is investigated in the next section.

4. Small gravity expansion

In previous sections, we have derived equations for the continuous MHD spectrum of a translationally symmetric equilibrium. These derivations were completely general, and allowed us to identify an instability criterion on the one hand, as well as to identify the terms responsible for Alfvén-slow type coupling. In this section, we quantify this coupling effect, using the small gravity parameter expansion for the equilibrium, similar to those discussed in Paper I. In the remainder of this paper, we assume a gravitational potential in which the prominence is embedded given by

$$\Phi(x, y) = (x - x_0)g, \quad (44)$$

where x_0 is the location of the center of the last closed flux surface of the prominence and the role of gravity is represented by the constant g . This gravity parameter g denotes a dimensionless quantification of the relative importance of gravity on the equilibrium structure, and we explained in Paper I that this parameter can meaningfully vary by orders of magnitude for solar prominence configurations. For small g and assuming a circular cross-section of the prominence plasma, we quantified in Paper I the change induced in the equilibrium (the downward shift of the flux surfaces), and the corresponding straight field line coordinates. This kind of expansion is similar to the small inverse aspect ratio expansion for tokamak plasmas. Using this analytical result for the equilibrium, we can now demonstrate that gaps (or even instabilities) may appear in the continuous MHD spectrum because of mode coupling, as a direct consequence of the force of gravity.

4.1. Continuous MHD spectrum for equilibria weakly affected by gravity

We apply the small gravity expansion to the expressions found in Eqs. (38)–(41) for the matrix elements $K(\vartheta)$, $M(\vartheta)$, $N(\vartheta)$, and $L(\vartheta)$, respectively. For this expansion, we rely on the metric elements that are valid for straight field line coordinates exploiting the Shafranov shift $\Delta(r)$, given by Eq. (26) in our accompanying paper (Blokland & Keppens 2011). In addition, one can derive the quantities

$$\rho \approx \rho_0 \left[1 - t_e \frac{r\rho_0 g}{p_0} \cos(\vartheta) \right] = O(1), \quad (45)$$

$$p \approx p_0 \left[1 - \frac{r\rho_0 g}{p_0} \cos(\vartheta) \right] = O(1), \quad (46)$$

$$B_{\vartheta} \approx \psi' [1 + \Delta' \cos(\vartheta)] = O(1), \quad (47)$$

$$\kappa_g \approx -t_g \frac{\psi'^2}{B_0^2} \frac{I}{B_0} \frac{\Delta'}{r} \sin(\vartheta) = O(g), \quad (48)$$

$$\frac{\rho\mu}{\gamma p + B^2} \approx \frac{\rho_0 g}{\gamma p_0 + B_0^2} \psi' \sin(\vartheta) = O(g), \quad (49)$$

$$\rho N_{m,\text{pol}}^2 \approx \frac{\psi'^2}{B_0^2} (\rho_0 g)^2 \left(\frac{t_e}{p_0} - \frac{1}{\gamma p_0 + B_0^2} \right) \sin^2(\vartheta) = O(g^2), \quad (50)$$

where the total magnetic field $B_0^2 = \psi'^2 + I^2$. The parameter $t_g = 1$ has been introduced to track the linear and quadratic dependence on the geodesic curvature. Furthermore, the parameter t_e allows us to distinguish the three options for the flux function dependence

in the equilibrium, namely

$$t_e = \begin{cases} 1 & \text{for } T(\psi), \\ 0 & \text{for } \rho(\psi), \\ 1/\gamma & \text{for } S(\psi). \end{cases} \quad (51)$$

These flux functions lead to different variants of the extended Grad-Shafranov equation in Eq. (10) for the equilibrium, as presented in our accompanying paper (Blokland & Keppens 2011). We again note that the magnetically modified Brunt-Väisälä frequency $N_{m,\text{pol}}^2$ from Eq. (50) can only be negative if the density is a flux function. This once more confirms that for this particular choice of the flux function, the MHD continua may become unstable as already found to be true in general above.

Applying the small gravity expansion, the eigenvalue equation in Eq. (36) can be written as

$$\sum_{m'} \begin{pmatrix} \overline{K}_{mm'} & \overline{M}_{mm'} \\ \overline{N}_{mm'} & \overline{L}_{mm'} \end{pmatrix} \begin{pmatrix} \overline{\eta}_{m'} \\ \overline{\zeta}_{m'} \end{pmatrix} = 0, \quad (52)$$

where

$$\overline{K}_{mm'} \equiv \frac{qB_0^2}{r^2 I} K_{mm'}, \quad \overline{M}_{mm'} \equiv \frac{1}{r} M_{mm'}, \quad \overline{\eta}_{m'} \equiv \frac{r}{q} \eta_{m'}, \quad (53)$$

$$\overline{N}_{mm'} \equiv \frac{1}{r} N_{mm'}, \quad \overline{L}_{mm'} \equiv \frac{I}{qB_0^2} L_{mm'}, \quad \overline{\zeta}_{m'} \equiv \frac{B_0^2}{I} \zeta_{m'}. \quad (54)$$

We note that the first and second rows of the eigenvalue equation in Eq. (52) have been multiplied by B_0^2/rI and q^{-1} , respectively, to get rid of the dimensional factors in front of the square brackets in the definitions given in Eqs. (38)–(41). The approximated expressions of the integrands of the matrix elements used in the spectral equation in Eq. (52) are now

$$\overline{K}(\vartheta) = \frac{I^2}{q^2} (m+kq)(m'+kq) - \rho_0 \widetilde{\omega}^2 \quad (55)$$

$$+ 2 \frac{I^2}{B_0^2} \left[\frac{I^2}{q^2} (m+kq)(m'+kq) \Delta' - \rho_0 \left(\Delta' - t_e \frac{B_0^2}{I^2} \frac{r \rho_0 g}{p_0} \right) \widetilde{\omega}^2 \right] \cos(\vartheta) \\ + 4 \frac{I^2}{\gamma p_0 (\gamma p_0 + B_0^2)} t_g \alpha (t_g \alpha + \rho_0 g) \sin^2(\vartheta) + \frac{q^2}{r^2} \overline{\rho N_{m,\text{pol}}^2} \sin^2(\vartheta),$$

$$\overline{M}(\vartheta) = i \frac{I^2}{q} \frac{m'+kq}{\gamma p_0 + B_0^2} (2t_g \alpha + \rho_0 g) \sin(\vartheta) + 2t_g \frac{r^2 I^4}{q^3 B_0^2} \frac{\rho_0 g \Delta'}{\gamma p_0 + B_0^2} \sin^2(\vartheta) + \frac{q}{r} \overline{\rho N_{m,\text{pol}}^2} \sin^2(\vartheta), \quad (56)$$

$$\overline{N}(\vartheta) = -i \frac{I^2}{q} \frac{m+kq}{\gamma p_0 + B_0^2} (2t_g \alpha + \rho_0 g) \sin(\vartheta) + 2t_g \frac{r^2 I^4}{q^3 B_0^2} \frac{\rho_0 g \Delta'}{\gamma p_0 + B_0^2} \sin^2(\vartheta) + \frac{q}{r} \overline{\rho N_{m,\text{pol}}^2} \sin^2(\vartheta), \quad (57)$$

$$\overline{L}(\vartheta) = \frac{\gamma p_0}{\gamma p_0 + B_0^2} \frac{I^2}{q^2} (m+kq)(m'+kq) - \rho_0 \widetilde{\omega}^2 \quad (58)$$

$$+ \left[\frac{\gamma p_0}{\gamma p_0 + B_0^2} \frac{r I^2}{q^2 (\gamma p_0 + B_0^2)} \left(2\alpha - \frac{B_0^2}{p_0} \rho_0 g \right) (m+kq)(m'+kq) - \frac{r \rho_0}{B_0^2} \left(2 \frac{I^2}{q^2} r \Delta' - t_e \frac{B_0^2}{p_0} \rho_0 g \right) \right] \cos(\vartheta) \\ - i \frac{r \rho_0 g}{\gamma p_0 + B_0^2} \frac{I^2}{q^2} (m-m') \sin(\vartheta) + \overline{\rho N_{m,\text{pol}}^2} \sin^2(\vartheta),$$

where

$$\alpha \equiv \frac{\gamma p_0}{B_0^2} \frac{I^2}{q^2} r \Delta', \quad (59)$$

$$\overline{\rho N_{m,\text{pol}}^2} \equiv \frac{r^2 I^2}{q^2 B_0^2} (\rho_0 g)^2 \left(\frac{t_e}{p_0} - \frac{1}{\gamma p_0 + B_0^2} \right). \quad (60)$$

Once more, the magnetically modified Brunt-Väisälä frequency $\overline{\rho N_{m,\text{pol}}^2}$ (60) is only negative if the density is a flux function ($t_e = 0$).

4.2. Low frequency $\Delta m = 0$ gap

The approximate relations just given can now be analyzed in terms of mode coupling possibilities. We focus on low frequencies, $\widetilde{\omega} = O(g)$, and show that a $\Delta m = 0$ gap can be created in the neighborhood of a rational surface $m+kq=0$, where q quantifies the flux-surface specific local twist in the field lines on this surface. For such a surface, we determine the leading order coupling

on an Alfvén continuum mode $\bar{\eta}_m$ with the same poloidal mode number m . Close to the rational surface, we can safely neglect the third term of the expression in Eq. (55) for \bar{K} and Eq. (58) for \bar{L} . We are left with the eigenvalue equation for the MHD continua in Eq. (52), which reduces in essence to a four-mode coupling between the central Alfvén mode, and the three neighboring slow modes given by

$$\begin{pmatrix} \rho_0 \hat{\omega}_{s,m-1}^2 - \rho_0 \bar{\omega}^2 & b_{m-1} & c_{m-1,m} & d \\ b_{m-1} & \rho_0 \hat{\omega}_{A,m}^2 - \rho_0 \bar{\omega}^2 & e & -b_{m+1} \\ c_{m-1,m} & e & \rho_0 \hat{\omega}_{s,m}^2 - \rho_0 \bar{\omega}^2 & c_{m,m+1} \\ d & -b_{m+1} & c_{m,m+1} & \rho_0 \hat{\omega}_{s,m+1}^2 - \rho_0 \bar{\omega}^2 \end{pmatrix} \begin{pmatrix} \bar{\zeta}_{m-1} \\ \bar{\eta}_m \\ \bar{\zeta}_m \\ \bar{\zeta}_{m+1} \end{pmatrix} = 0, \quad (61)$$

where

$$\rho_0 \hat{\omega}_{s,m}^2 \equiv \rho_0 \omega_{s,m}^2 + \frac{1}{2} \rho N_{m,\text{pol}}^2, \quad (62)$$

$$\rho_0 \hat{\omega}_{A,m}^2 \equiv \rho_0 \omega_{A,m}^2 + 2 \frac{I}{\gamma p (\gamma p + B^2)} t_g \alpha (t_g \alpha + \rho_0 g) + \frac{q^2}{2r^2} \rho N_{m,\text{pol}}^2, \quad (63)$$

$$b_m \equiv \frac{I}{2(\gamma p + B^2)} (2t_g \alpha + \rho_0 g) \sqrt{\rho_0} \omega_{A,m}, \quad (64)$$

$$c_{m,m'} \equiv \frac{r}{2(\gamma p + B^2)} \left(2\alpha - \frac{B_0^2}{\rho_0} \rho_0 g \right) \rho_0 \omega_{s,m} \omega_{s,m'} - \frac{r}{2(\gamma p + B^2)} \frac{I^2}{q^2} \rho_0 g, \quad (65)$$

$$d \equiv -\frac{1}{4} \rho N_{m,\text{pol}}^2, \quad (66)$$

$$e \equiv \frac{r}{q} \left(t_g \alpha \frac{I^2}{\gamma p (\gamma p + B^2)} \rho_0 g + \frac{1}{2} \frac{q^2}{r^2} \rho N_{m,\text{pol}}^2 \right). \quad (67)$$

In this derivation, we made use of the finding that the integral of $\sin(\vartheta)$ and $\sin^2(\vartheta)$ over ϑ gives $\frac{1}{2}i(\delta_{m,m'+1} + \delta_{m,m'-1})$ and $-\frac{1}{4}(\delta_{m,m'-2} - 2\delta_{m,m'} + \delta_{m,m'+2})$, respectively. We note that for this four-mode coupling scheme, we neglected any coupling to the $m \pm 1$ Alfvén modes, an assumption that is valid as long as the associated frequencies are far away in the spectrum. The same argument also holds for the coupling to the $m \pm 2$ Alfvén and slow modes. By careful examination of this 4×4 matrix, one notices that the coupling between $\bar{\zeta}_{m\pm 1}$ with either $\bar{\eta}_m$ or $\bar{\zeta}_m$ is on the order of $\mathcal{O}(g)$, while all the other types of coupling are on the order of $\mathcal{O}(g^2)$. Because the main modes involved have the same poloidal mode number m , this coupling is known as a $\Delta m = 0$ gap possibility.

The matrix of the reduced eigenvalue problem presented in Eq. (61) is symmetric and therefore its eigenvalues $\bar{\omega}^2$ will be real. This is consistent with the properties of the quadratic polynomial (29) for the eigenfrequency $\bar{\omega}$. If gravity drives a certain MHD continuum unstable, it means that the corresponding eigenfrequency $\bar{\omega}^2$ will be smaller than zero, resulting in a complex eigenvalue $\bar{\omega}$. Once the eigenvalue $\bar{\omega}$ is known, it is straightforward to determine the dominant character of the MHD mode by determining the dominant component of the eigenvector of the eigenvalue equation in Eq. (61). This analytic result will be used in what follows to verify the numerically obtained continua.

5. The spectral code PHOENIX

While the equilibria presented in Paper I were computed with the FINESSE code (Beliën et al. 2002), the stability analysis of the prominence models is subsequently done using the spectral code PHOENIX (Blokland et al. 2007b). Although this code was originally developed for analyzing axisymmetric plasma tori such as tokamaks and accretion disks, the design is flexible enough to analyze axially symmetric, gravitating plasma tubes. This possibility is exploited in the present PHOENIX version 1.2. For a given two-dimensional equilibrium computed by FINESSE and converted to straight field line coordinates, we then need to select an axial wave number k and a range of poloidal wave numbers m for which we determine the complete MHD spectrum numerically. A range of poloidal wave numbers is required to allow for the anticipated mode coupling caused by the combinations of non-circular cross-sections of the flux surfaces and/or the presence of a gravitational potential. Instead of solving the spectral equation in Eq. (12), PHOENIX solves the linearized version of the full set of non-linear MHD equations. This allows us to extend the eigenspectrum computations for future incorporation of non-adiabatic effects, while it in addition will contain the trivial solution for the entropy modes known as the Eulerian entropy continuum. The discretization of these linearized MHD equations exploits a finite element method in the radial direction, while in the poloidal direction a spectral method is applied. The finite elements used are a combination of quadratic and cubic Hermite elements. This combination prevents the creation of spurious eigenvalues (Rappaz 1977). The resulting generalized non-Hermitian eigenvalue problem is then solved using the Jacobi-Davidson method (Sleijpen & van der Vorst 1996). For such an eigenspectrum computation, one needs to supply boundary conditions at both the magnetic axis (set by symmetry) and the outer flux surface. There, we assume a perfectly conducting wall for simplicity. Strictly speaking, this condition is not applicable to true flux ropes carrying embedded solar prominences. However, since we focus on the continuous spectrum alone, the boundary conditions in the radial direction have hardly any influence, since those waves or instabilities are by definition localized on flux surfaces.

Exploiting this realization, Poedts & Schwarz (1993) introduced a more efficient method than the Jacobi-Davidson method to compute the continuous MHD spectrum. This method replaces the radial dependence of the cubic Hermite and the quadratic elements with $\log(\epsilon)$ and $1/\epsilon$, respectively, on each individual flux surface. In this way, the singular behavior of the continuous

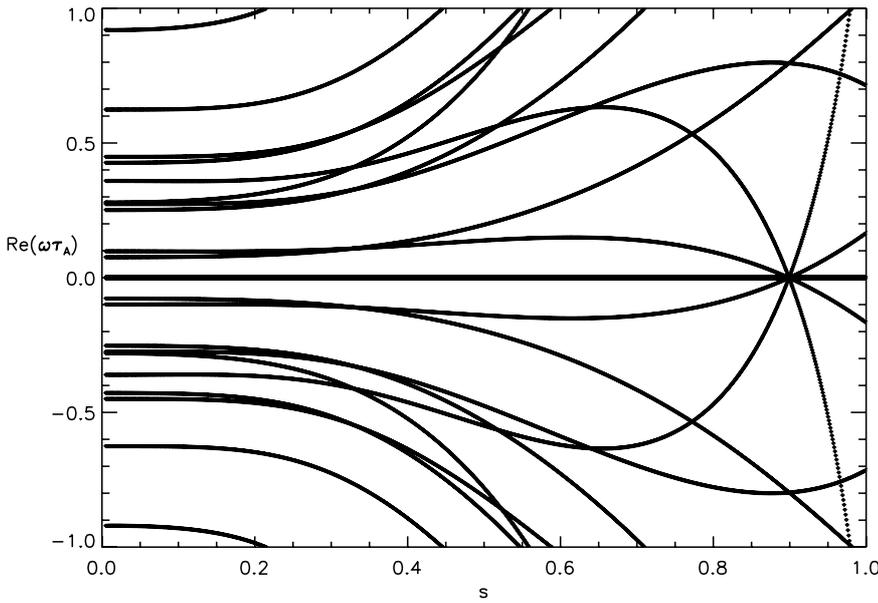


Fig. 1. Part of the continuous MHD spectrum for a cool solar prominence with gravity parameter $g = 0.001$ computed by PHOENIX. Shown are eigenfrequencies normalized to the Alfvén time τ_A as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$, all corresponding to flux surface localized eigenoscillations.

modes can be approximated. Goedbloed (1975) and Pao (1975) describe in detail the reasons for this type of singular behavior in the continua for static plasma tori without gravity. Goedbloed et al. (2004b) pointed out that similarly singular behavior for the MHD continua prevails in plasma tori including toroidal plus poloidal flows and gravity. The same behavior is thus also expected for our translationally symmetric case. After replacing the elements, the eigenvalue problem becomes a small algebraic problem on each flux surface, with a size determined by the number of poloidal mode numbers taken along. For each flux surface, this can be solved using a direct QR method. All spectra below are computed in this way, for an axial wavenumber $k = -1$ and using seven poloidal mode numbers centered around $m = 1$. A larger number of poloidal Fourier harmonics can be used, which results in a more accurate solution but also in a longer computation time. The typical radial resolution used is either 101 or 1001 points, depending on how much detail is required in identifying each branch of the MHD continua.

6. Cool prominence surrounded by hot medium

The theory presented will be applied to the two equilibrium classes presented in Paper I, cool prominences embedded in hot medium and double layered prominences. We now discuss the first class. The equilibria and their associated profiles are discussed in our accompanying paper (Blokl and Keppens 2011). As described in our accompanying paper, we analyze the continuous MHD spectrum for increasing values of the gravity parameter g .

6.1. Gap creation when increasing g

Figure 1 shows part of the continuous MHD spectrum as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ computed by PHOENIX for a prominence equilibrium with gravity $g = 0.001$ and assuming that the temperature is a flux function. The frequencies are normalized to the Alfvén time $\tau_A \equiv L \sqrt{\mu_0 \rho_{\text{mag}}}/B_{\text{mag}}$, where μ_0 , L , ρ_{mag} , and B_{mag} are the permeability, the radius of the prominence, the density on the magnetic axis, and the magnetic field on the magnetic axis, respectively. Near the magnetic axis where $s = 0$, each branch can be labeled immediately using the analytical expressions in Eqs. (42) and (43) for the MHD Alfvén and slow continua. At the location $s \approx 0.9$, the safety factor q of the equilibrium becomes the value of one (see Paper I). At this location, the $m = 1$ Alfvén and the $m = 1$ slow continuum all become zero. At the low gravity value for Fig. 1, no true gap can be detected, and for progressively stronger gravity, we expect mode coupling to occur, ultimately inducing a possible gap or even instabilities in the continua. At the same location, the $m = 0$ slow and $m = 2$ slow continuum have the same frequency, namely approximately 0.797.

We next take a closer look at the continuous MHD spectrum about the $q = 1$ surface, to compare the PHOENIX results with the ones predicted by the four-mode coupling scheme in Eq. (61). This comparison is presented in Fig. 2. Results from PHOENIX are represented by crosses, while the colored lines result from solving the eigenvalue equation in Eq. (61). The colors indicate the dominant characters, which are easily determined from the associated eigenvector. The colors red, green, blue, and yellow correspond to dominant $\bar{\eta}_1$ (Alfvén), $\bar{\zeta}_1$ (slow), $\bar{\zeta}_0$ (slow), and $\bar{\zeta}_2$ (slow) characters, respectively. The plot shows that there is excellent agreement between both methods, as expected for small gravity. Furthermore, the $m = 1$ Alfvén continuum (red) intersects the $m = 0$ (blue) and $m = 2$ slow continuum (yellow). This intersection may lead to mode coupling, resulting in a $\Delta m = 1$ gap. The analysis of this kind of gap is left to future work. The crosses at zero frequency correspond to the Eulerian entropy continuum, as also found numerically by PHOENIX.

We now repeat the same analysis, for a case with intermediate gravity $g = 0.500$ and still assuming that the temperature is a flux function. The results of PHOENIX results are in excellent agreement with the solutions of the four-mode coupling scheme

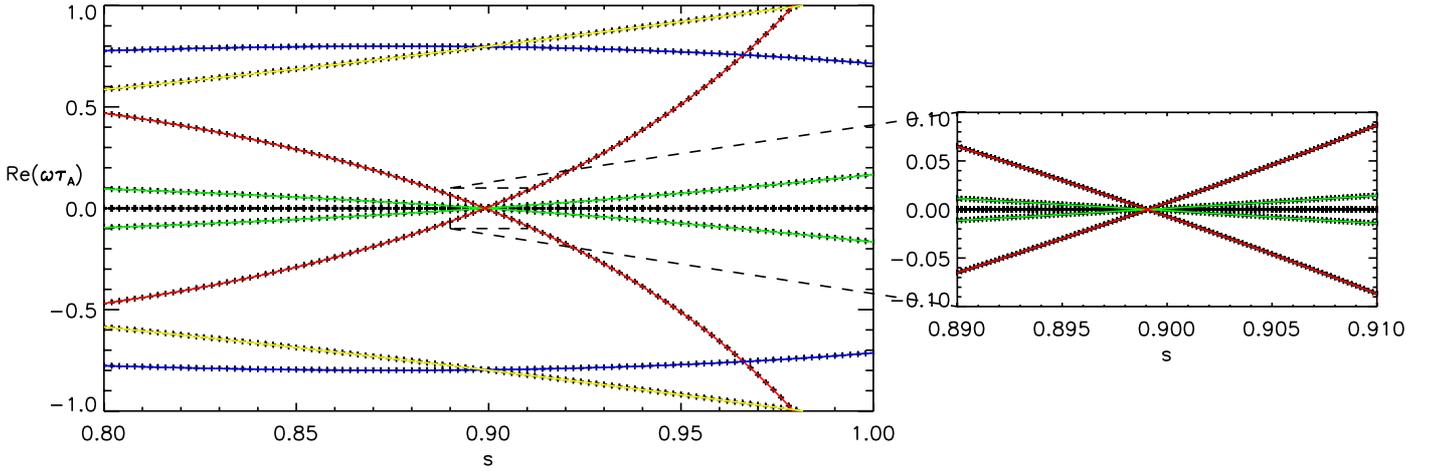


Fig. 2. The real part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with small gravity ($g = 0.001$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. The temperature is assumed to be a flux function. The crosses are PHOENIX results, while the colored lines compare them with the four-mode coupling scheme. The colors red, green, blue, and yellow correspond to dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ components of the eigenvector, respectively.

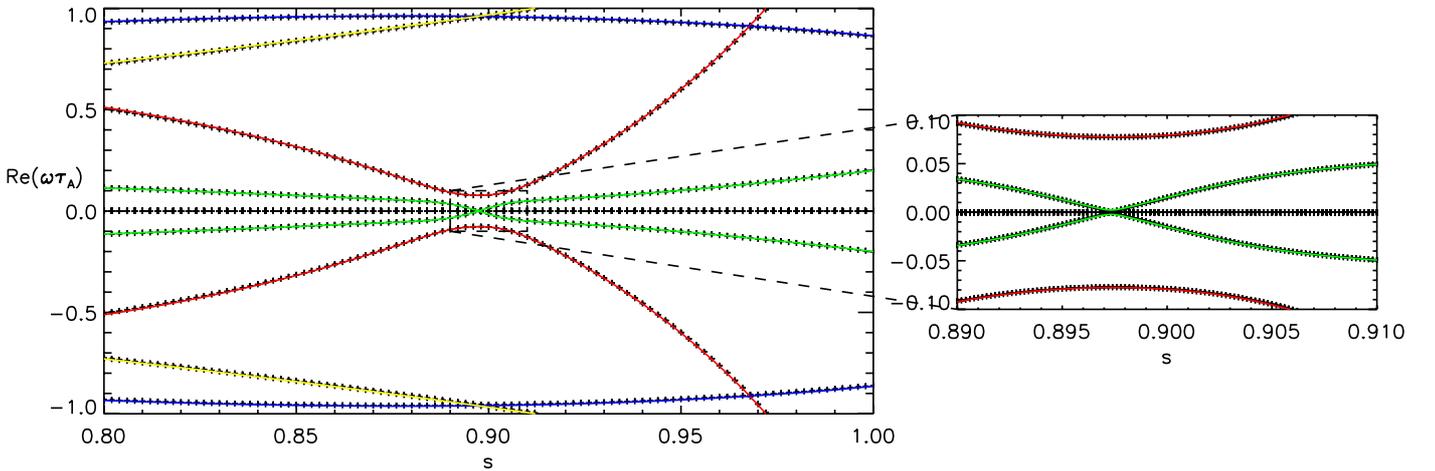


Fig. 3. The real part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with intermediate gravity $g = 0.500$ as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Temperature is assumed to be a flux function. Crosses are the PHOENIX results, while the colored lines give the four-mode coupling scheme. The colors red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ components of the eigenvector.

in Eq. (61), as shown in Fig. 3. For this larger g value, the results now clearly illustrate that a gap appears in the $m = 1$ Alfvén continuum, while no gap appears in the $m = 1$ slow continuum. We note that the radial flux coordinate s of the $q = 1$ surface is shifted inwards relative to the previously discussed equilibrium. However, the radial location r of this surface remains almost the same (see Paper I).

If we continue to increase the gravity parameter g , for a cool prominence equilibrium where temperature is a flux function, we find progressive deviations between numerical and (approximate) analytical predictions. The plot presented in Fig. 4 shows the results from PHOENIX (crosses) and the four-mode coupling scheme (colored lines). Near $q = 1$, the results of the four-mode coupling scheme corresponding to the Alfvén continuum (red) show good agreement with PHOENIX, while for a slow continuum (green) there is a discrepancy. The four-mode coupling scheme indicates that a gap in the slow continuum may result, while PHOENIX does not find such a gap. The gap in the Alfvén continuum clearly becomes larger if gravity is increased. The radial location r of the $q = 1$ surface remains almost the same as in the two previously discussed cases, while its radial flux coordinate s is shifted outward.

In the previous three equilibria, which only differ in terms of the increased role of gravity, we observed that the gap appearing in the Alfvén continuum becomes larger if the gravity is increased. This dependence was investigated by determining the Alfvén frequency at the $q = 1$ surface, for various values of the gravity parameter. The result of this parametric investigation (involving the construction of numerical equilibria, subsequently diagnosed for their MHD continua by PHOENIX) has been plotted in Fig. 5. The plot shows excellent agreement between PHOENIX and the growth of the “gap” predicted by the four-mode coupling scheme. It is seen that the Alfvén gap frequency depends non-linearly on the gravity parameter g . Despite this excellent agreement for the Alfvén continuum gap up to $g = 1$, successive differences occur, e.g. as noted for the slow continua shown in Fig. 4. In Fig. 5, the horizontal

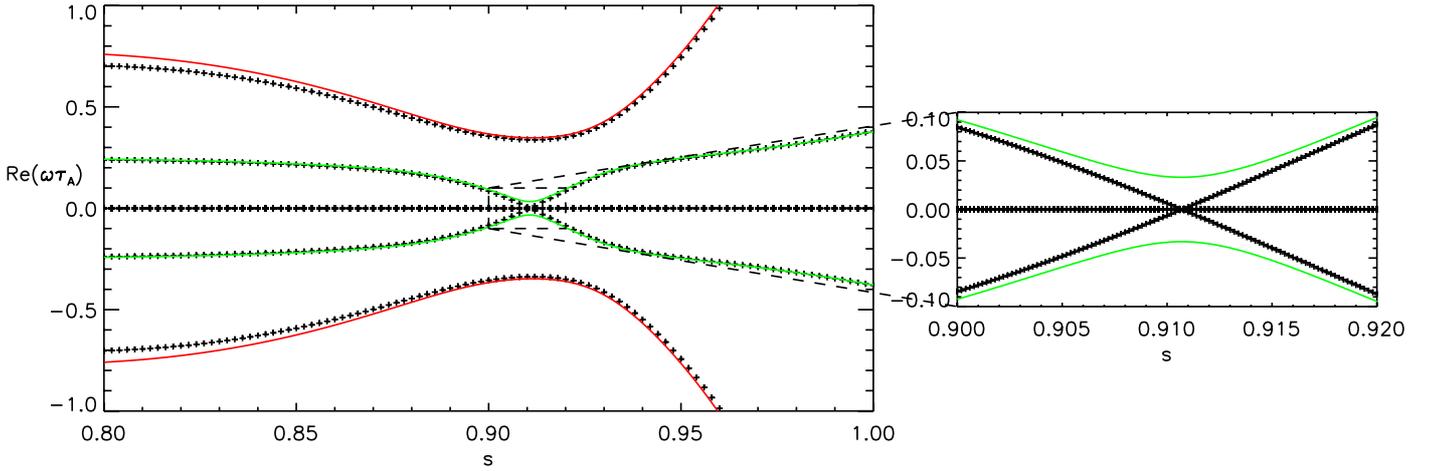


Fig. 4. The real part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with strong gravity ($g = 1.000$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Here, the temperature is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ components of the eigenvector.

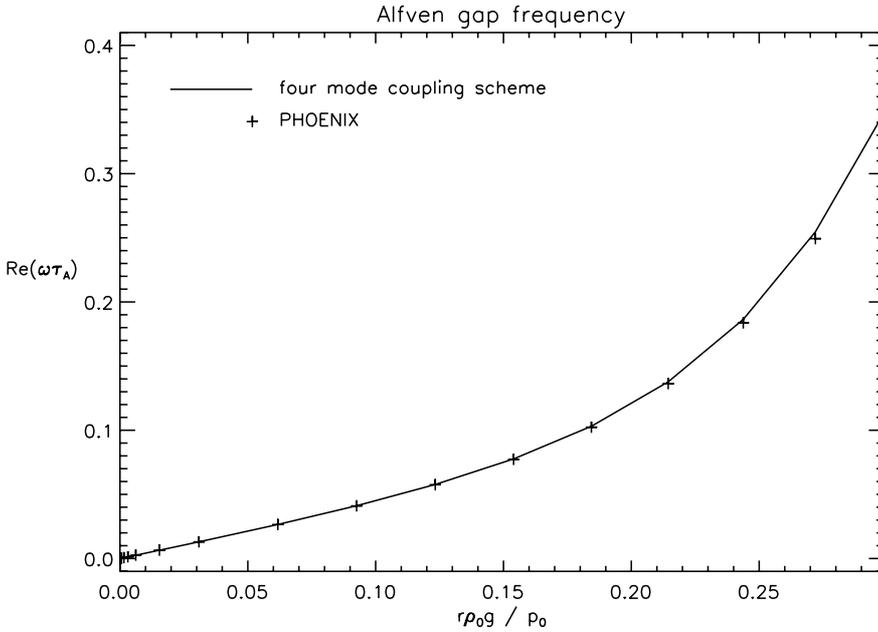


Fig. 5. The frequency of the Alfvén continuum mode $\bar{\eta}_1$ at the resonant $q = 1$ surface as a function of the gravity. The temperature is assumed to be a flux function.

axis does not use the gravity parameter g itself, but rather a scaled value. The equilibrium with gravity $g = 0.001$, $g = 0.500$, and $g = 1.000$ correspond to $r\rho_0g/p_0 = 0.0003$, $r\rho_0g/p_0 = 0.154$, and $r\rho_0g/p_0 = 0.300$, respectively.

6.2. Unstable continua

Finally, to illustrate the effect of the chosen flux function in the equilibrium on the MHD continua, we now discuss in the intermediate gravity case $g = 0.500$, the continuous spectra when the density or the entropy is a flux function. We recall that when the density is a flux function the continuous MHD spectrum may become unstable. The continuous MHD spectrum of the density case is presented in Fig. 6. The figure shows there is again an excellent agreement between the four-mode coupling scheme and PHOENIX. The plot shows two distinct features, that are not present when the temperature is a flux function. Firstly, no gap appears in the $m = 1$ Alfvén continuum. Near the $q = 1$ surface ($s \approx 0.90$), an interesting observation can be made, where Alfvén and slow mode characters mingle. A mode at for example $s = 0.89$ with dominant $m = 1$ Alfvén character slowly changes to a dominant $m = 1$ slow character when it is traced closer to the $q = 1$ surface. The transition points are indicated by the arrows in the plot. The second distinct feature is the merger of the two $m = 1$ slow branches to one at $s \approx 0.852$ and $s \approx 0.94$. Exactly at these locations, the $m = 1$ slow continuum turns unstable as shown in Fig. 7. This figure quantifies the imaginary parts of the continuum eigenfrequencies, corresponding to (normalized) growth rates. This instability is the convective continuum instability (CCI) as discussed by Blokland et al. (2007a). There is again excellent agreement between PHOENIX and the four-mode coupling scheme. Furthermore, the transition points of

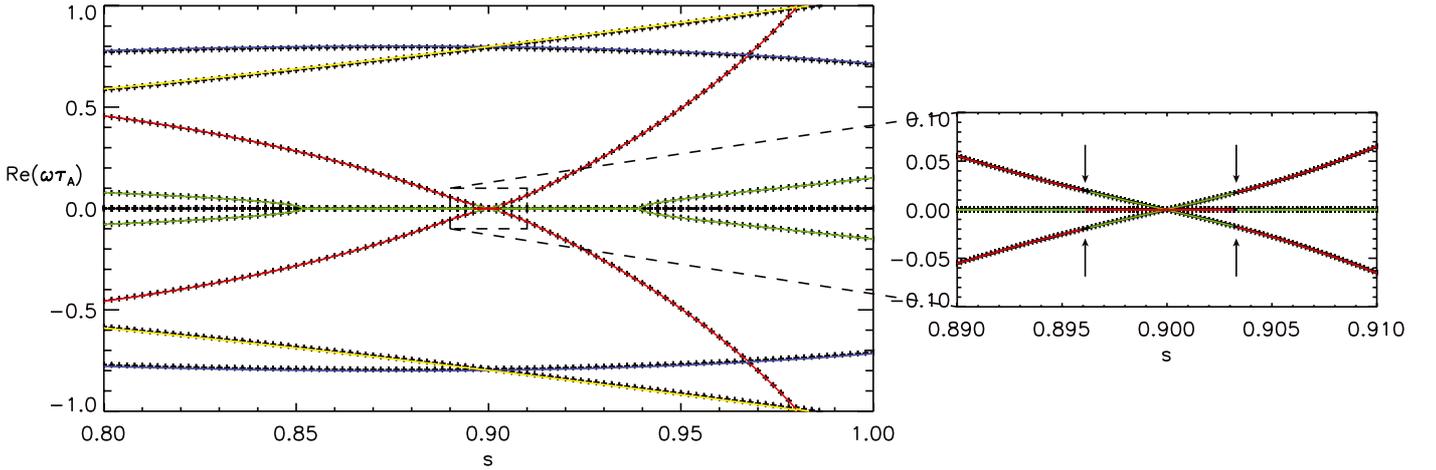


Fig. 6. The real part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with intermediate gravity ($g = 0.500$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Density is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

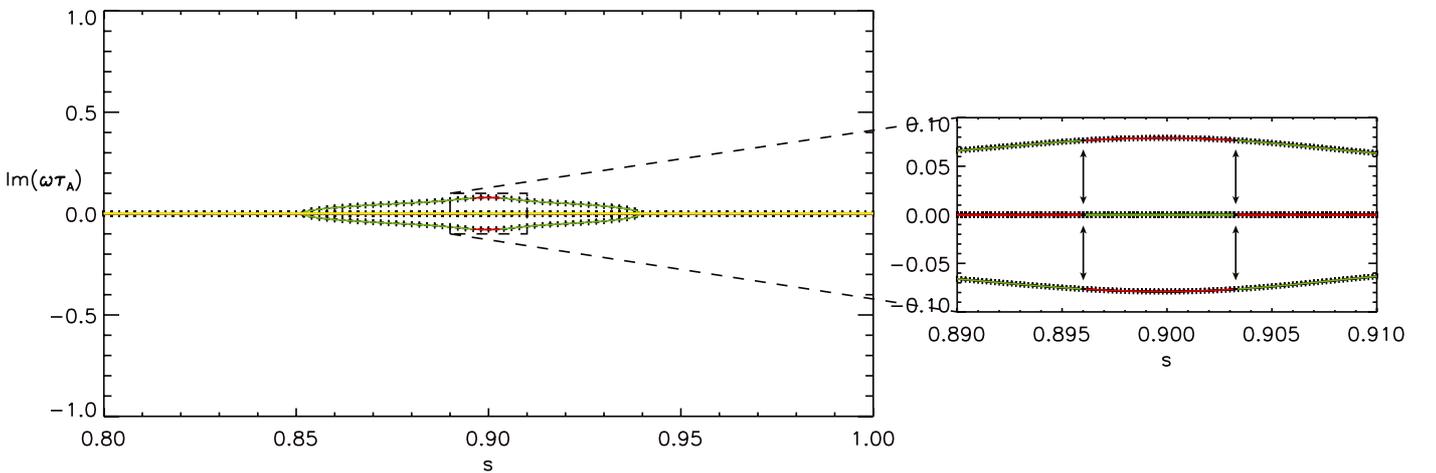


Fig. 7. The imaginary part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with intermediate gravity ($g = 0.500$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. These quantify the growth rates of the CCI instabilities. Density is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

the dominant character of the mode indicated by arrows in Fig. 7 are exactly at the same locations as those indicated in Fig. 6. From this analysis, we conclude that the CCI at the $q = 1$ surface has a dominant Alfvén character.

As in the case where temperature is a flux function, we can parametrically investigate the dependence of the growth rate of the CCI on the gravity parameter g . This analysis again involves repeated FINESSE computations for increasing values of g , all diagnosed with PHOENIX, and the extraction of the growth rate for the most unstable mode in each case. These results are presented in Fig. 8. Once more, there is excellent agreement between the two methods. Furthermore, our analysis clearly identifies a linear dependence between the growth rate and the gravity parameter g . Owing to our non-dimensionalization, this implies that there is a larger growth of the CCI mode for flux ropes with embedded prominences that (1) gradually grow in cross-sectional extent; and/or (2) become more heavy (density increase); and/or (3) decrease in magnetic field strength.

In the context of prominence dynamics, the existence of equilibria viable to unstable continua may have relevance to the complex dynamics observed in high resolution prominence movies (Berger et al. 2008), as well as to the sudden disappearance of prominence plasma, the so-called *disparition-brusque* phenomenon (see e.g., Tandberg-Hanssen 2011, and references therein). It should be emphasized that the onset of the CCI instability will be localized on those flux surfaces where the CCI criterion is violated. As shown in Fig. 7, growth rates can vary from one flux surface to another in a given unstable case, while in addition, the maximal growth rate also varies because of gradual equilibrium changes, as demonstrated in Fig. 8. Whether this linear prediction for the onset of internal, continuum-driven dynamics leads to the rather turbulent prominence dynamics revealed by contemporary observations, or even to full disruption and the disappearance of prominence structures, must be answered by follow-up high resolution, nonlinear numerical simulations. This requires initializing multi-dimensional nonlinear simulations with the spectrally diagnosed equilibria.

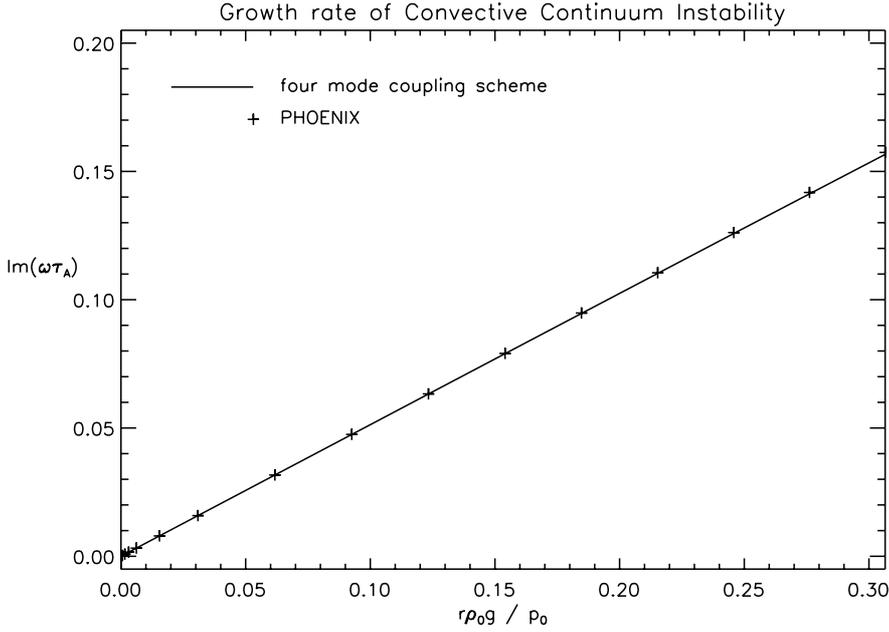


Fig. 8. The growth rate of the CCI mode at the resonant $q = 1$ surface as a function of the gravity parameter. The density is assumed to be a flux function.

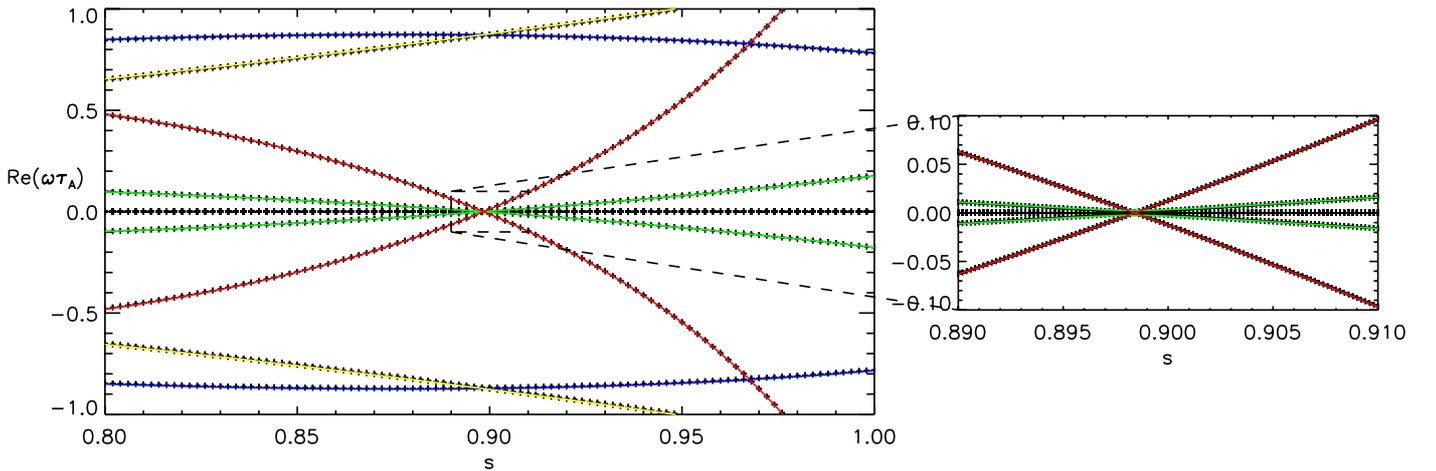


Fig. 9. The real part of the sub-spectrum of the MHD continua of a cool solar prominence equilibrium with intermediate gravity ($g = 0.500$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Here, entropy is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

6.3. Spectra for equilibria in which entropy is a flux function

We now consider the case where entropy is a flux function, for an equilibrium with intermediate gravity ($g = 0.500$). The resulting continuous MHD spectrum is presented in Fig. 9. There is again excellent agreement between both methods. It can be clearly seen that no gap in the Alfvén continuum appears. The continua for the strong gravity $g = 1.000$ case have also been computed, which also show no appearance of a gap. This can be understood by examining Eq. (29) for the eigenfrequency. When the entropy is a flux function, the Brunt-Väisälä frequency projected on a flux surface $N_{\text{BV,pol}}$ is zero, thereby transforming the eigenfrequency given by Eq. (29), to one similar to a case without gravity, for which no gap can appear.

7. Double layered prominences

We discuss the continuous MHD spectrum of the second equilibrium class, the double layered prominences. As for cool prominences, the equilibria and the associated profiles are discussed in our accompanying paper (Blokland & Keppens 2011). For this equilibrium class, we also investigate the influence of the gravity on the continuous MHD spectrum by slowly increasing the gravity.

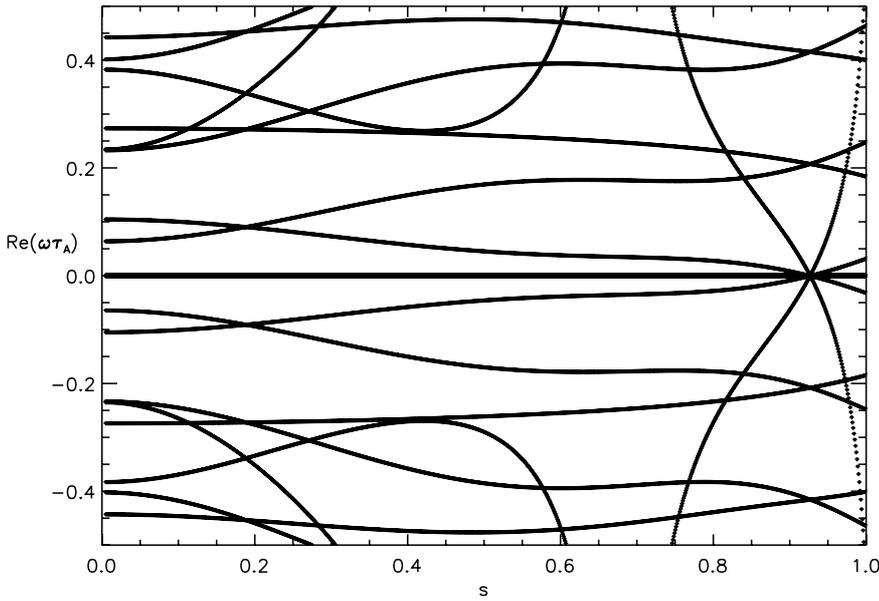


Fig. 10. A part of the continuous MHD spectrum of double layered solar prominence with gravity $g = 0.001$ computed by PHOENIX as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$.

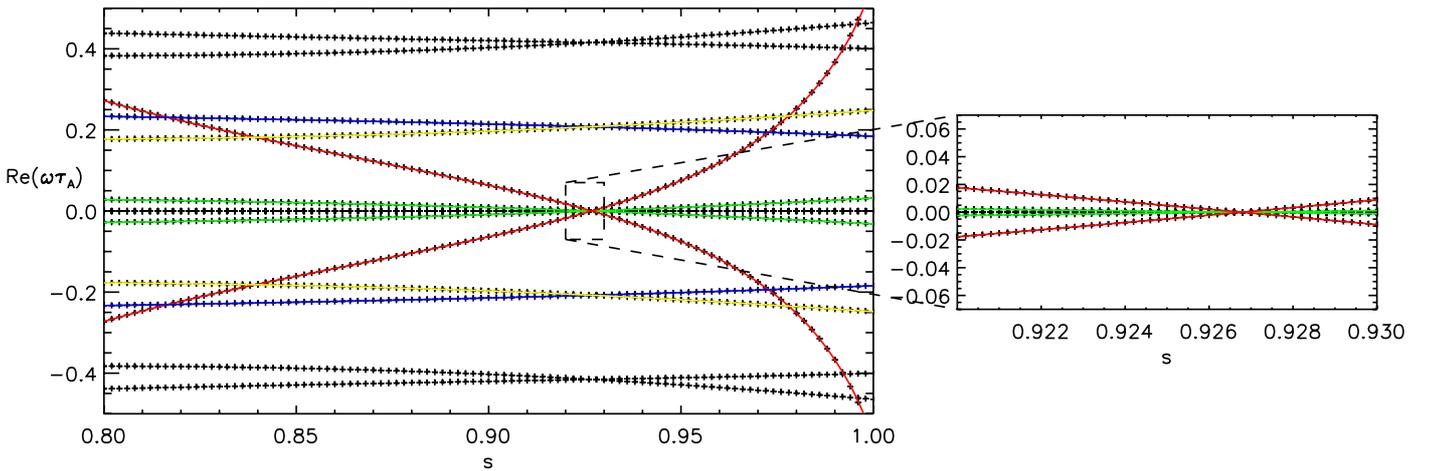


Fig. 11. The real part of the sub-spectrum of the MHD continua of a double layered solar prominence equilibrium with small gravity ($g = 0.001$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Here, the temperature is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

7.1. Continua for increasing gravity

The first equilibrium that was analyzed with PHOENIX is the one where the temperature is assumed to be a flux function and the gravity is low at $g = 0.001$. Part of the MHD spectrum is shown in Fig. 10. The individual branches near the magnetic axis can be labeled using the analytical expressions for the Alfvén continuum in Eq. (42) and slow continuum in Eq. (43). The associated radial flux coordinate s of the $q = 1$ surface is approximately 0.927. At exactly this location, the $m = 1$ Alfvén and $m = 1$ slow continuum are zero, as expected, and the $m = 0$ and $m = 2$ slow continuum have the same frequency.

The plot shown in Fig. 11 contains the PHOENIX results (crosses) together with the solutions of the four-mode coupling scheme represented by Eq. (61) (colored lines), and illustrates the excellent agreement between the two methods. As for the cool prominences, the $m = 1$ Alfvén continuum (red) intersects the $m = 0$ (blue) and $m = 2$ slow continuum (yellow). This may in turn lead to mode coupling, creating a high frequency $\Delta m = 1$ gap. The analysis of these $\Delta m = 1$ gaps was not performed here, but left to future work. We note that the crosses with zero frequency correspond to the Eulerian entropy continuum.

Figure 12 contains the plot for an equilibrium with intermediate gravity $g = 0.500$. The temperature is again assumed to be a flux function. Near the $q = 1$ surface ($s \approx 0.927$), the results of PHOENIX and the four-mode coupling scheme show excellent agreement for the $m = 1$ Alfvén and slow continuum. Owing to the non-zero gravity, a gap has been created in the $m = 1$ Alfvén continuum. However, $m = 0$ and $m = 2$ slow continua of the coupling scheme slightly mismatch the PHOENIX results. This mismatch is because by our neglect of the coupling with $m = -1$ and $m = 3$ slow continua. In the plot, these continua are represented by the crosses with non-zero frequency in combination without a solid line. At four different radial flux coordinates, $s \approx 0.82$, $s \approx 0.837$,

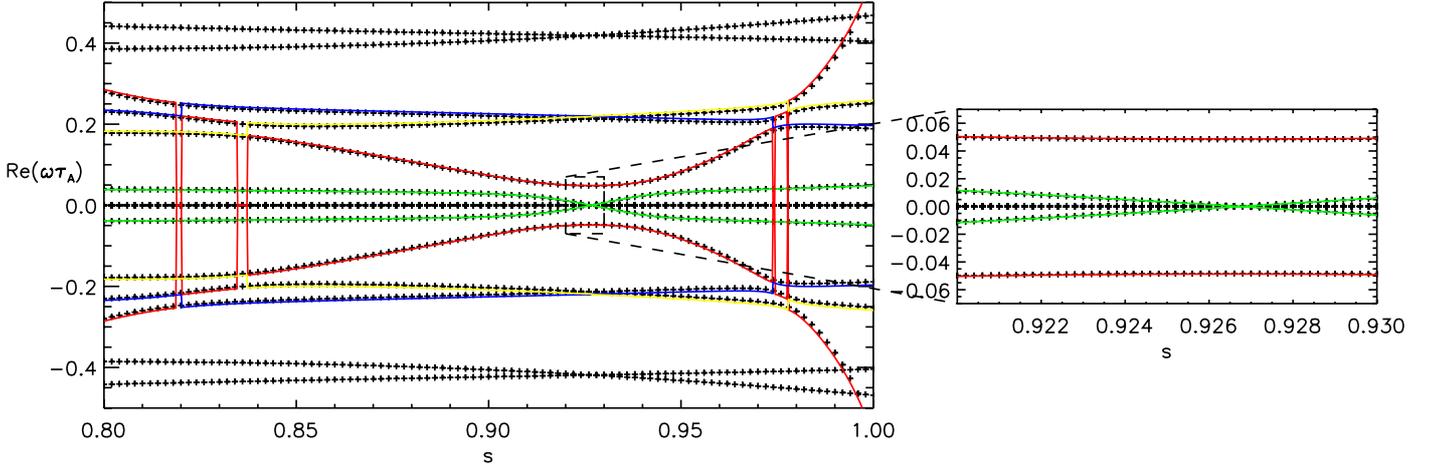


Fig. 12. The real part of the sub-spectrum of the MHD continua of a double layered solar prominence equilibrium with strong gravity ($g = 0.500$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Here, the temperature is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

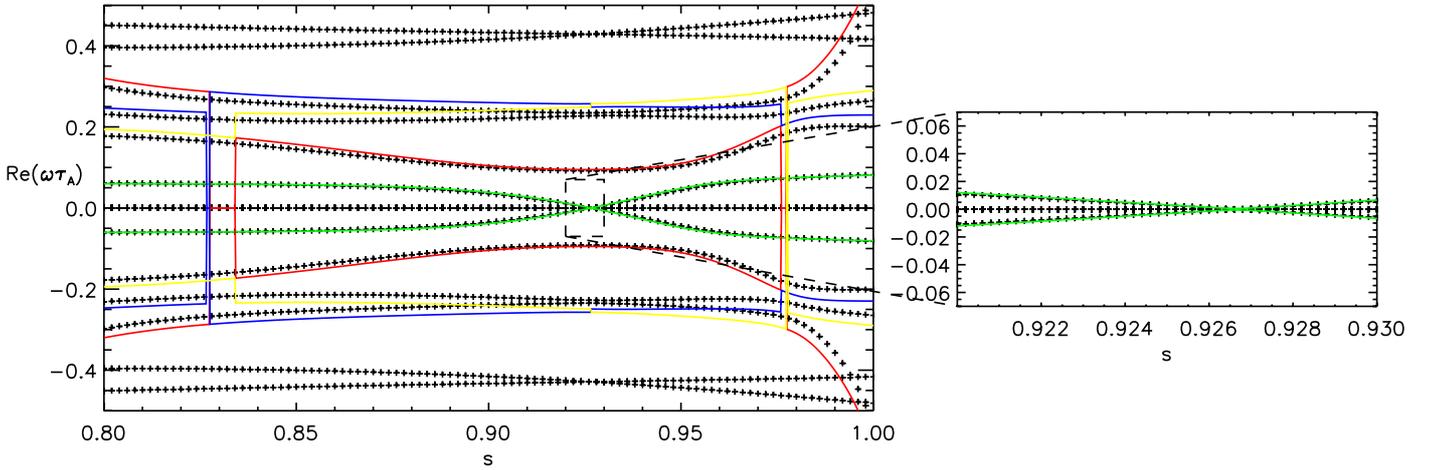


Fig. 13. The real part of the sub-spectrum of the MHD continua of a double layered solar prominence equilibrium with strong gravity ($g = 1.000$) as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$ are shown for axial wavenumber $k = -1$ and poloidal mode numbers $m = \{-2, \dots, 4\}$. Here, the temperature is assumed to be a flux function. The crosses are the PHOENIX results, while the colored lines are the results of the four-mode coupling scheme. The color red, green, blue, and yellow correspond to the dominant $\bar{\eta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_0$, and $\bar{\zeta}_2$ component of the eigenvector.

$s \approx 0.973$, and $s \approx 0.978$ a $\Delta m = 1$ gap was produced (note the color change of the solid lines about these radii). As expected, the four-mode coupling scheme only partly captures this kind of coupling. Therefore, about these $\Delta m = 1$ gaps there is a mismatch between the results of both methods, and a separate detailed analysis of the $\Delta m = 1$ coupling schemes appears to be required.

The last equilibrium of this subsection that we discuss is the one with strong gravity $g = 1.000$. The temperature is assumed to be a flux function. For this equilibrium, one observes the onset of the double layered structure in the pressure and density as demonstrated in our accompanying paper (Blokland & Keppens 2011). The results for the $m = 1$ Alfvén and slow continua show excellent agreement near the $q = 1$ surface ($s \approx 0.927$). Owing to the stronger gravity compared to the previous case, the gap in $m = 1$ Alfvén continuum is larger. Near this surface, the previously mentioned mismatch of both the $m = 0$ and $m = 2$ slow continuum becomes larger because of the neglect of the coupling with both the $m = -1$ and $m = 3$ slow continuum. This coupling becomes stronger if the gravity is increased. The four-mode coupling scheme is also unable to capture the coupling at the radial flux coordinates $s \in [0.828, 0.833]$ and $s \in [0.975, 0.978]$. This is again because we neglected the coupling with both the $m = -1$ and $m = 3$ slow continuum. Therefore, close to these radii there is a large discrepancy between PHOENIX and the coupling scheme in Eq. (61). Furthermore, we note the coupling between these two slow continua at the $q = 1$ surface. This kind of coupling creates a $\Delta m = 2$ gap that was neglected in our theoretical analysis.

As for the equilibrium class of cool prominences, we observe that when the gravity is increased, the gap in the $m = 1$ Alfvén continuum increases. This gravity dependence of the gap was investigated in detail by computing the Alfvén frequency at the $q = 1$ surface for different values of the gravity. The results of this investigation are presented in Fig. 14. The results of PHOENIX and the four-mode coupling scheme show excellent agreement with each other. Furthermore, we note that the frequency varies almost linearly with the gravity parameter g . This linear dependence is very different from the clear non-linear dependence found for cool

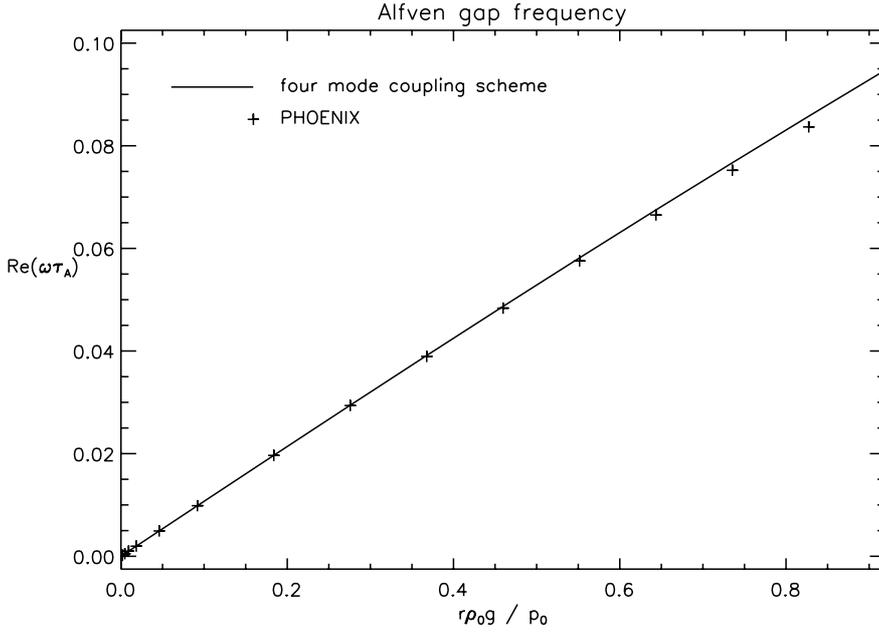


Fig. 14. The frequency of the Alfvén continuum mode $\bar{\eta}_1$ at the resonant $q = 1$ surface as a function of the gravity. The temperature is assumed to be a flux function.

prominences. This excellent agreement is again only valid for the Alfvén continuum. As demonstrated in Figs. 12 and 13, this does not automatically mean that it also holds for the slow continua. The dimensionless gravity parameter $g = 0.001$, $g = 0.500$, and $g = 1.000$ correspond to $r\rho_0g/p_0 = 0.0009$, $r\rho_0g/p_0 = 0.4597$, and $r\rho_0g/p_0 = 0.9192$, respectively.

7.2. Continua for strong gravity, with axial flow

We now discuss the last equilibrium of the second class. For this equilibrium, we assume that the temperature is a flux function, we adopt an extra strong gravity parameter value ($g = 5.000$), and we include an axial, shear flow. The derivation in our accompanying paper (Blokland & Keppens 2011) shows that the axial flow must be a flux function. The flow itself is described by the profile

$$v_z(\psi) = A_4(1 - \psi), \quad (68)$$

where the constant $A_4 = 0.11$. This corresponds to a maximum Alfvén Mach number of 0.105. Using the typical Alfvén speed of 218 km s^{-1} , as mentioned in our accompanying paper (Blokland & Keppens 2011), this translates into a filament-aligned flow of at most 22.9 km s^{-1} . Part of the resulting continuous MHD spectrum computed by PHOENIX is shown in Fig. 15. Instead of using 7 poloidal harmonics, we used 15 harmonics to obtain an accurate result (i.e., one that no longer changed when adding even more poloidal harmonics). This is a clear indication that the strong gravity causes strong coupling between different continua. Owing the extra strong gravity, there is no meaningful comparison between the PHOENIX results and those predicted from our theoretical analysis, which is only appropriate if the gravity is weak. In the figure, we indicate the non-zero Eulerian entropy continuum (where $\omega = kv_z$ labeled E) that breaks the up-down symmetry about the zero frequency. There is instead now symmetry about this Eulerian entropy continuum. Furthermore, PHOENIX finds modes with zero frequency, which correspond to the solution of the generalized eigenvalue equation in combination with the perturbed magnetic field being divergence free. If one carefully examines the plot, one notices that one continuum branch intersects the Eulerian entropy continuum twice, namely at $s \approx 0.808$ and at $s \approx 0.917$. At these locations exactly, the safety factor q is one. Figure 14 of our accompanying paper (Blokland & Keppens 2011) clearly shows the $q = 1$ surface locations, as seen in a plot of safety factor q across the vertical direction. Using the radial flux coordinate s as a flux surface label where we go from the magnetic axis to the flux rope edge, there will be two $q = 1$ surfaces. We note that the clear coupling effects at other radial locations as well, creating gaps that may define preferred frequency ranges for global eigenoscillations.

8. Conclusions

We presented the equations describing the complete MHD spectrum. These equations have been specialized for the continuous MHD spectrum using techniques frequently used in research on fusion plasma oscillations. From the equations for the continuous spectrum, a stability criterion was derived, showing that prominence equilibria for which the temperature or the entropy is a flux function have stable continuum modes. However, when the density is a flux function, the continua may turn unstable resulting in the convective continuum instability. Whether this instability is related to the intricate dynamics witnessed in actual prominence plasmas, or explains why certain filaments disappear abruptly, remains to be investigated by follow-up nonlinear computations. We have demonstrated how the onset of CCI modes in flux ropes, which carry embedded prominences, will manifest itself as truly flux-surface localized fluctuations. Their growth rate increases when a flux rope expands (the gravity parameter g increases because of both rope widening and the associated magnetic field weakening) or when it becomes more mass-loaded. We note that this route

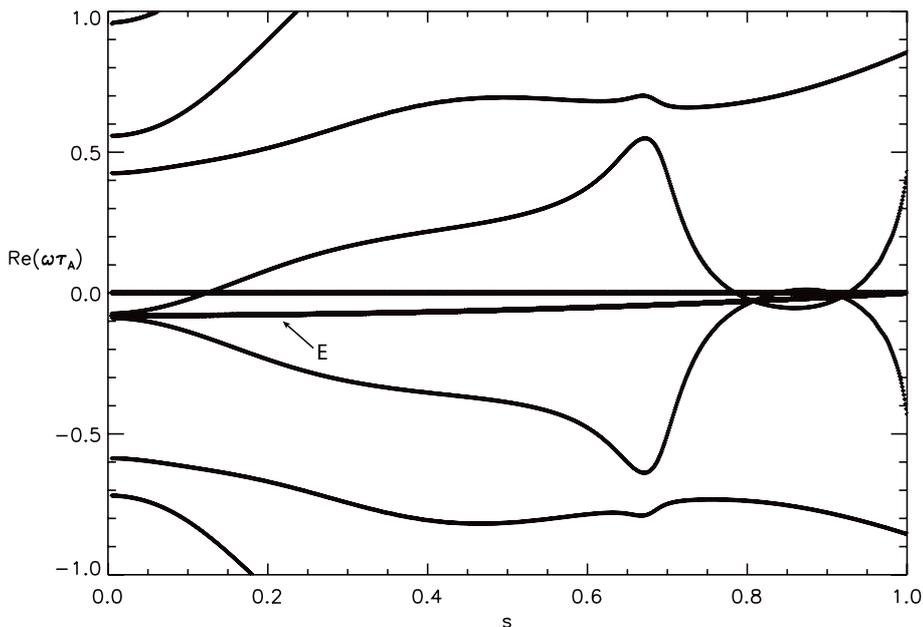


Fig. 15. A part of the continuous MHD spectrum of a double layered solar prominence with strong gravity $g = 5.000$ computed by PHOENIX as a function of the radial flux coordinate $s \equiv \sqrt{\psi}$.

to potentially violent prominence dynamics is here related to internal prominence properties. This contrasts with studies (see e.g., MacTaggart & Hood 2009) that emphasize the role of external influences, studies that emphasize the response to changes in the overlying field, or the associated pressure buildup in the surroundings by the shearing of its footpoints. We note that the nonlinear simulations that may ultimately decide between these competing or possibly cooperating aspects, will need to go beyond the zero beta, and/or no gravity assumption typically employed in these studies (Aulanier et al. 2002; Amari et al. 1999). Moreover, to capture the details of flux-surface localized continuum modes with varying MHD character (slow versus Alfvén) from one flux surface to another, the requirements of the 3D resolution will likely only be achievable by using effectively high resolution, grid-adaptive simulations. In addition, guidance on the actual magnetic field structure in prominence surroundings will be needed (such as that obtained in Schmieder et al. 2007), to distinguish between the theoretical freedom in flux profiles.

The equations for the MHD continuous spectrum have been specialized for small gravity, using an expansion in poloidal Fourier harmonics. The resulting equations have been applied to a low frequency $\Delta m = 0$ gap near a resonant q surface. Near this surface, a four-mode coupling scheme has been identified. This scheme couples one Alfvén mode and three slow modes, and can cause gaps (avoided crossings) between continuum branches as these are traced across all nested flux surfaces. These gaps appear at low frequency, and form the ideal locations for global discrete eigenoscillations. It remains to be shown which kind of coupling occurs at finite frequency, $\Delta m = 1$ gaps, and to determine whether discrete global modes exist within these frequency ranges, under realistic prominence conditions. These global modes, as well as their coupling to flux localized surface modes (with resonant damping), need further study, to address whether previously established prominence seismology results are consistent with these more realistic prominences.

The MHD spectral code PHOENIX has been used to accurately compute the continuous MHD spectrum of prominence equilibria. Two classes of equilibria have been considered, cool prominences embedded in hot medium and prominences with a double layered pressure and density structure for different values of the dimensionless gravity parameter as discussed in our accompanying paper (Blokl and Keppens 2011). The results of PHOENIX are compared with analytic results of the four-mode coupling scheme, and we have found excellent agreement for small gravity (as expected), while for strong gravity, we have identified progressive discrepancies, emphasizing the need for full numerical diagnosis. For cool prominences, this discrepancy manifested itself in the slow continuum with the same poloidal mode number as the Alfvén continuum, while for double layered prominences, differences occurred in the slow continuum with a poloidal mode number ± 1 with respect to the Alfvénic one. For both classes, when temperature is a flux function, the results demonstrate that a gap in the Alfvén continuum can appear due to the gravity. For cool prominences, the Alfvén frequency associated with the gap depends non-linearly on the gravity parameter, while for double layered prominences it scales linearly with it.

For cool prominences for which the density or the entropy is a flux function, the continuous MHD spectrum has also been computed. In the case of the density, the continuous spectrum becomes unstable leading to convective continuum instability. Our analysis reveals that close to the $q = 1$ surface, the CCI mode has a dominant Alfvén character, which changes away from this surface to a dominant slow type. Furthermore, no gap in the Alfvén continuum appears, in contrast to the case when the temperature is a flux function. For the entropy case, the continuous spectrum is stable and again no gap appears in the Alfvén continuum. In the context of prominence configurations, long-lived equilibria almost necessitate the temperature flux function dependence, but for active region prominences, the findings about CCI possibilities may provide important information about the onset of turbulent dynamics.

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