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The pinch-type instability of helical magnetic fields

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ABSTRACT

The current-driven instability of toroidal magnetic fields under the influence of both axial magnetic-field components and (differential) rotation is studied. The MHD equations are solved by means of a simplified model with cylindric geometry that assumes both the axial field and the axial current as uniform and the fluid as incompressible. If azimuthal and axial field components are nearly the same size, then the instability is (slightly) supported, and modes with $m > 1$ dominate. If the axial field dominates, then the most unstable modes again have $m > 1$, but the field is strongly stabilized. The modes are suppressed by a fast rigid rotation where the $m = 1$ mode maximally resists. Only this mode becomes best re-animated for $\Omega > \Omega_{\rm A}$ ($\Omega_{\rm A}$ the Alfvén frequency) if the rotation has a negative shear. The general finding of the linear theory is that the higher modes with $m > 1$ do not play an important role for the stability of rotating fluids.

If applied to dynamo-generated galactic magnetic fields influenced by the typical rotation law with constant azimuthal velocity, the result is that galactic fields should only be marginally unstable against perturbations with $m \leq 1$. Modes with higher m may not appear. The corresponding growth rates are close to the rotation period of the inner part of the galaxy.

Key words. instabilities – magnetohydrodynamics (MHD) – galaxies: magnetic fields

1. Introduction

There is still an open question whether toroidal magnetic fields are stabilized in the presence of poloidal field components. Lundquist (1951) suggested that azimuthal fields with energy exceeding the energy of the axial field component become unstable. In other words, he found that *uniform axial fields act stabilizing* what – if true – would also be a highly interesting finding for MHD experiments in the laboratory. By using a cylindric magnetic geometry, Roberts (1956) opened a new discussion when he found instability against perturbations with high azimuthal mode numbers m for all ratios of the azimuthal field component and the axial field component. Tayler (1980) discussed the adiabatic stability of stars again with mixed poloidal and toroidal fields. For poloidal and toroidal field components with amplitudes of the same order he suggested the system is stable, but the final answer to the question remained open. In his detailed paper about magnetic instabilities, Acheson (1978) only considered the stability of purely toroidal fields. In extension, one can ask that a simplified magnetic model be stable where axisymmetric and stationary toroidal fields are considered under the influence of a homogeneous axial field that itself is stable by definition. For an ideal medium, Bonanno & Urpin (2010) consider the stability of such a configuration with respect to applications in jet theory without rotation. They exclude stability for fields with a pitch $|B_z|/|B_\phi|$ close to one. Particular attention is given to the instability of nonaxisymmetric modes with azimuthal mode numbers $m > 1$. If the axial field dominates, the instability persists for rather high mode numbers (~100), found by using an almost identical model by Tayler (1960) that includes the differences in the solution for different helicities of the background field.

A more complicated model has been investigated by Braithwaite (2009) where the poloidal field component (axisymmetric as is also the toroidal one) can also be unstable alone, but the author finds stability of the combination of poloidal and toroidal field if both components are of the same order of magnitude.

For dissipative fluids we show that the configuration with $|B_z| \simeq |B_\phi|$ is (slightly) more unstable than purely toroidal fields, while an increase in $|B_z|$ stabilizes the toroidal fields more and more. The situation changes with rotation. If the rotation rate exceeds the Alfvén frequency of the toroidal field, the instability of any m is suppressed, but the mode with $m = 1$ lasts longest. If, as is almost always the case, the rotation is not rigid, then only the modes with small m survive, and the axisymmetric standard MRI starts to dominate for sufficiently fast rotation. We find that the competition between the modes $m = 0$ and $m = 1$ should be observable with galaxies. Only a few examples of nonaxisymmetric magnetic field patterns have been found by the observers (Beck et al. 1996).

Most likely the hydromagnetic jets are suitable subjects for application to magnetic field instabilities, but the galaxies containing a remarkable interstellar turbulence form better objects for those low Reynolds numbers and Hartmann numbers, which we deal with in the present paper. Galaxies possess quadrupolar-type magnetic fields with toroidal and poloidal components within the same order of magnitude (~10^{-5} Gauss) and with the phase relation $B_R B_\phi < 0$. They rotate with the characteristic rotation law

$$\Omega(R) = \frac{\text{const.}}{R} \tag{1}$$

with $R\Omega \simeq 200$ km s^{-1}. Their characteristic density is about 10^{-24} g/cm^3, and the magnetic diffusivity is about 10^{26} cm^2/s due to the action of interstellar turbulence. The resulting magnetic Reynolds number $\mathrm{Rm} \simeq UR_0/\eta$ is thus ~1000, while the Lundquist number of the toroidal field $\mathrm{S} = BR_0/\sqrt{\mu_0\rho}\eta$ reaches values of 200. For galaxies both characteristic numbers are thus the same or more precisely

$$\Omega \simeq 5\,\Omega_{\mathrm{A}}, \tag{2}$$

expressed with the rotation rate and the Alfvén frequency Ω_{A} (see Eq. (15) below). The interstellar turbulence mainly driven by the supernova explosions and stellar winds allows another simplification. For modes with a driven turbulence, the magnetic Prandtl number

$$\mathrm{Pm} = \frac{\nu}{\eta} \tag{3}$$

is close to one (Yousef et al. 2003). The stability calculations are highly simplified by the assumption Pm = 1. We showed that, without rotation, the critical magnetic Hartmann number (i.e. the magnetic amplitude) does *not* depend on Pm (Rüdiger & Schultz 2010) but the growth rates depend on Pm and also the instability map with finite rotation. Without interstellar turbulence, the interstellar gas has a very large Pm, which might also be true for the material of the various astrophysical jets that exist.

The question is whether such a magnetic configuration is stable against nonaxisymmetric disturbances with the azimuthal mode number m. The stability of a toroidal field strongly depends on its radial profile, so the current-free profile $B_\phi \propto 1/R$ is stable against disturbances with the azimuthal mode numbers $m = 0, 1, \ldots$ On the other hand, the profile $B_\phi \propto R$ is only stable against $m = 0$ but is unstable against all disturbances with $m > 0$ (Tayler 1957; Velikhov 1959). Such a profile will mainly be used in the present paper for simplicity thanks to a *homogeneous* axial electric current (cf. Roberts 1956). A strong enough toroidal field that is nearly uniform in a radial direction is also unstable against disturbances with the mode number $m > 0$ (see Fig. 8 below).

We thus model the galactic magnetic field by means of a Taylor-Couette flow periodic in the axial direction z. The axial electric current that produces the toroidal magnetic field is assumed to be homogeneous along z (cf. Foglizzo & Tagger 1995). It is obvious that such a simple cylindric model cannot describe the field geometry in a global rotating disk. But as the complete stability analysis for galactic fields is rather complicated (see Elstner et al. 2008), we start by studying a simplified model here. The model works with the rotation law (1) of a conducting fluid with Pm = 1, which contains a homogeneous axial electric current and a homogeneous axial field.

The galactic magnetic field in one hemisphere consists of toroidal fields, which come from currents that are parallel to the axial component of the poloidal field. In other words, the current helicity $\boldsymbol{B} \cdot \boldsymbol{J}$ of the background field has a definite sign in large areas of one and the same hemisphere. The amplitude of the axial field component is the free parameter in the theory. This model is simple enough to be used for a general stability analysis. We accept that the vertical structure of the background fields and also the character of the density and of the turbulence model are not reflected by the model, but we shall see that the simple model already yields a highly complex stability analysis. In the next steps the model must be reformulated so that thin disks can also be considered, including toroidal fields (Foglizzo & Tagger 1995; Ogilvie & Pringle 1996; Papaloizou & Terquem 1997; Shtemler et al. 2011).

For simplicity the cylinders that confine the fluid are highly conducting and no-slip boundary conditions are used at the cylinder walls. The tangential electric currents and the radial field components vanish at the boundaries. Because the current-driven instability transforms the energy of the electric currents, which are hardly influenced by the boundaries, their influence is very weak. We have indeed shown that the results for conducting and for insulating boundaries are not essential (Rüdiger et al. 2007b). Because we have to prescribe the rotation law at the inner and the outer boundaries, we prefer no-slip conditions. Slippery boundaries are also possible for consideration, but the rotation law will develop in this case. The magnetic background field (assumed as stationary) possesses a uniform axial field component so that the resulting field pattern forms a spiral. It is the stability of a spiral with fixed current helicity that is considered in the present paper. With respect to galactic applications, this is an oversimplification, because the current helicity always behaves antisymmetrically for dynamo-generated magnetic fields of either parity with respect to the equator.

It appears unnecessary to limit the azimuthal mode number to $|m| \le 1$ so that higher values can also be considered, as in Arlt et al. (2007) and Bonanno & Urpin (2010).

2. The equations

We are interested in the linear stability of the background field $\boldsymbol{B} = (0, B_\phi(R), B_0)$, with $B_0 = \mathrm{const}$, and the flow $\boldsymbol{U} = (0, R\Omega(R), 0)$. The perturbed system is described by

$$u_R,\ u_\phi,\ u_z,\ p,\ b_R,\ b_\phi,\ b_z. \tag{4}$$

When developing the disturbances into normal modes, the solutions of the linearized MHD equations are considered in the form

$$f = f(R)\exp(\mathrm{i}(kz + m\phi + \omega t)), \tag{5}$$

where f is any of the velocity, pressure, or magnetic field disturbances.

The resulting linear equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{U} = -\frac{1}{\rho}\nabla p + \nu\Delta\boldsymbol{u} + \frac{1}{\mu_0\rho}\mathrm{curl}\,\boldsymbol{b}\times\boldsymbol{B} + \frac{1}{\mu_0\rho}\mathrm{curl}\,\boldsymbol{B}\times\boldsymbol{b}, \tag{6}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \mathrm{curl}(\boldsymbol{u}\times\boldsymbol{B}) + \mathrm{curl}(\boldsymbol{U}\times\boldsymbol{b}) + \eta\Delta\boldsymbol{b}, \tag{7}$$

and

$$\mathrm{div}\,\boldsymbol{u} = \mathrm{div}\,\boldsymbol{b} = 0, \tag{8}$$

where $\boldsymbol{u}$ is the perturbed velocity, $\boldsymbol{b}$ the perturbed magnetic field, p the pressure perturbation, and ν and η are the kinematic viscosity and the magnetic diffusivity. The stationary background solution is

$$\Omega = a_\Omega + \frac{b_\Omega}{R^2}, \qquad B_\phi = a_B R + \frac{b_B}{R}, \tag{9}$$

where a_Ω, b_Ω, a_B, and b_B are constants defined by

$$a_\Omega = \Omega_{\mathrm{in}}\frac{\mu_\Omega - \hat{\eta}^2}{1-\hat{\eta}^2}, \qquad b_\Omega = \Omega_{\mathrm{in}}R_{\mathrm{in}}^2\frac{1-\mu_\Omega}{1-\hat{\eta}^2},$$
$$a_B = \frac{B_{\mathrm{in}}}{R_{\mathrm{in}}}\frac{\hat{\eta}(\mu_B - \hat{\eta})}{1-\hat{\eta}^2}, \qquad b_B = B_{\mathrm{in}}R_{\mathrm{in}}\frac{1-\mu_B\hat{\eta}}{1-\hat{\eta}^2} \tag{10}$$

with

$$\hat{\eta} = \frac{R_{\rm in}}{R_{\rm out}}, \qquad \mu_\Omega = \frac{\Omega_{\rm out}}{\Omega_{\rm in}}, \qquad \mu_B = \frac{B_{\rm out}}{B_{\rm in}}. \tag{11}$$

Here $R_{\rm in}$ and $R_{\rm out}$ are the radii of the inner and outer cylinders, $\Omega_{\rm in}$ and $\Omega_{\rm out}$ their rotation rates, and $B_{\rm in}$ and $B_{\rm out}$ the azimuthal magnetic fields at the inner and outer cylinders.

The outer value $B_{\rm out}$ is normalized with the uniform vertical field, i.e.,

$$\beta = \frac{B_{\rm out}}{B_0}. \tag{12}$$

As usual, the toroidal field amplitude is measured by the Hartmann number

$$\mathrm{Ha} = \frac{B_{\rm out} R_0}{\sqrt{\mu_0 \rho \nu \eta}}. \tag{13}$$

Here $R_0 = \sqrt{R_{\rm in}(R_{\rm out} - R_{\rm in})}$ is used as the unit of length, η/R_0 as the unit of velocity, and $B_{\rm in}$ the unit of the azimuthal fields. Frequencies, including the rotation Ω, are normalized with the inner rotation rate $\Omega_{\rm in}$. The ordinary Reynolds number Re and the magnetic Reynolds number Rm are defined as

$$\mathrm{Re} = \frac{\Omega_{\rm in} R_0^2}{\nu}, \qquad \mathrm{Rm} = \frac{\Omega_{\rm in} R_0^2}{\eta}, \tag{14}$$

and the Lundquist number S is defined by $\mathrm{S} = \mathrm{Ha} \cdot \sqrt{\mathrm{Pm}}$ so that it can also be understood as the magnetic Reynolds number formed with the Alfvén frequency,

$$\Omega_{\rm A} = \frac{B_{\rm out}}{\sqrt{\mu_0 \rho} R_0}, \tag{15}$$

instead of the rate of the global rotation. The normalization concerns the maximum value of the toroidal field.

The boundary conditions associated with the perturbation equations are no-slip for $\boldsymbol{u}$, i.e. $u_R = u_\phi = u_z = 0$, and perfectly conducting for $\boldsymbol{b}$, i.e. $db_\phi/dR + b_\phi/R = b_R = 0$. These boundary conditions hold for both the inner and the outer cylinders.

All our calculations refer to a container with $R_{\rm out} = 2R_{\rm in}$, i.e. $\hat{\eta} = 0.5$. For this choice a field that is current-free in the fluid is described by $\mu_B = 0.5$. A homogeneous axial electric current between the cylinders requires $\mu_B = 2$, which is the preferred value in this paper. The axial magnetic field component is assumed as uniform so that the resulting current helicity of the magnetic field is also homogeneous. This can only be true within one hemisphere of the celestial body (here the galaxy). Our model does not define an equator. We consider that hemisphere where the current helicity $\boldsymbol{B} \cdot \boldsymbol{J}$ of the background field is negative, i.e. $\beta < 0$, which is an arbitrary choice. As shown by Rüdiger et al. (2010), the resulting instability forms left spirals if the rotation is slow. The kinetic helicity $\langle \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{u} \rangle$ of the perturbations (averaged over the azimuth) proves to be positive for this field. The kinetic helicity does not change its sign if the rotation is faster, but then the magnetic pattern forms right spirals.

3. No rotation

We start by considering a nonrotating container; i.e., Re = 0. In this case the critical Hartmann number does not depend on the magnetic Prandtl number Pm for a given geometry and given vector $(0, B_\phi(R), B_0)$ of the magnetic field. The azimuthal drift of the nonaxisymmetric instability pattern vanishes (see Rüdiger & Schultz 2010). Without rotation the instability patterns do not drift in an azimuthal direction. For rapid rotation, the drift rate always grows with $\Omega_{\rm in}$ rather than with B_ϕ (see below).

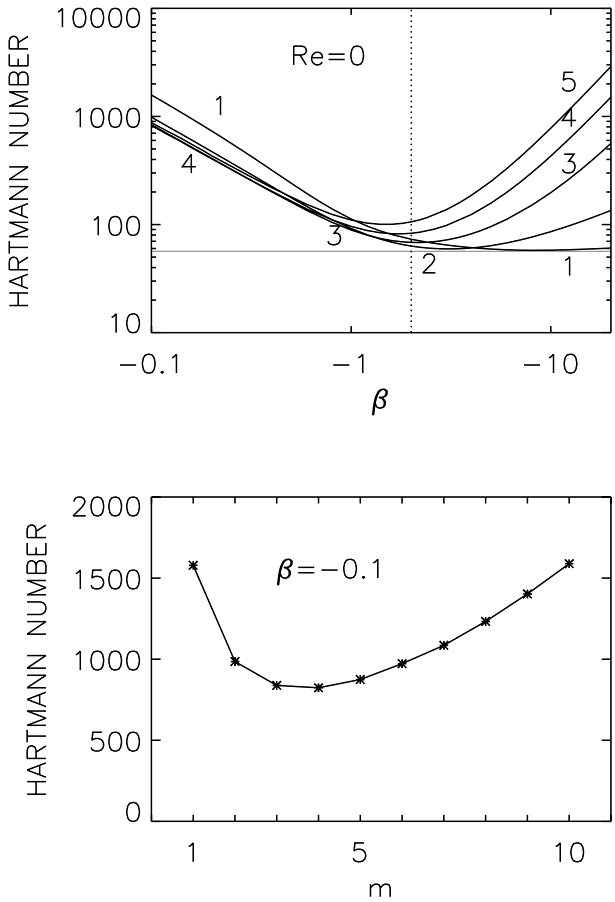

Fig. 1. Neutral stability for negative β and for $\mu_B = 2$. *Top*: the curves are marked with their mode number m. The minimum Hartmann number of the toroidal field with Ha = 57.6 exists for $|\beta| = 8$ (thin horizontal line). *Bottom*: Hartmann number as a function of the azimuthal wave number m. Note the clear minimum for medium $m > 1$. Both plots are valid for all magnetic Prandtl numbers.

The most natural stationary magnetic profile is $B_\phi \propto R$, which is the result of a homogeneous electric current flowing through the whole domain with $R < R_{\rm out}$ (see Roberts 1956). We know that, for this case and for very large $|\beta|$, the critical Hartmann number has the value of 70.6 for $m = 1$ (see Rüdiger et al. 2007a).

This value of the critical Hartmann number is slightly reduced if a small and uniform axial component of the magnetic field is added to the system. Hence, a uniform axial field supports the pinch-type instability of the toroidal field. This effect, however, is rather weak: the critical Hartmann number sinks from about 70 to about 60 (see Fig. 1). For $m > 1$ the destabilization of the toroidal field by axial fields is much stronger so that, for $|\beta|$ close to one, all the modes with different m have more or less the same critical Ha. We thus do not find a stabilizing effect on toroidal fields by axial field components compared to fields of purely toroidal fields.

For $\beta = -8$ we find Ha = 57.5 as the absolute minimum of the stability curve for $m = 1$. For stronger axial fields, the critical Hartmann number basically grows to reach values of

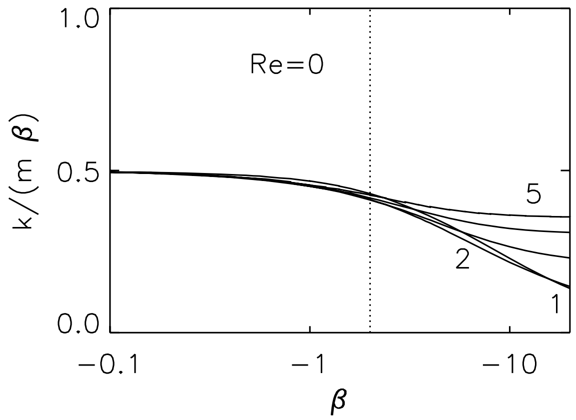

Fig. 2. The wave numbers normalized after (17) for various β. The physical wave numbers run with the value of $|\beta|$. $\mu_B = 2$, $\mathrm{Pm} = 1$.

about 1000 for $\beta \simeq -0.1$. For strong axial fields, the modes $m > 1$ possess lower critical Hartmann numbers than the mode with $m = 1$. The differences in the curves with various m are much smaller than those for weak B_0, but the Fourier component with $m = 4$ possesses the lowest critical Hartmann number for $\beta = -0.1$. Nevertheless, for a dominating axial field, the toroidal field is strongly *stabilized*, in particular the mode with $m = 1$. The modes with $m > 1$ possess somewhat lower critical Hartmann numbers, but these modes are also basically stabilized (see Fig. 1).

In summary, the pinch-type instability of toroidal fields in the presence of a uniform axial magnetic field without rotation is strongly suppressed by strong axial fields. The maximal stabilization happens for $m = 1$, so that the most unstable modes have azimuthal mode numbers $m > 1$. If B_ϕ and B_z are of the same order, then the field is (slightly) more unstable than for $B_z = 0$. We find that rather strong toroidal fields can be stored in the container with strong enough axial current-free magnetic fields. All results are invariant against the simultaneous transformation $m \to -m$ and $\beta \to -\beta$.

The wave numbers k of the unstable modes reflect the instability pattern. The shape of the cells is described by the relation

$$\frac{\delta z}{R_{\mathrm{out}} - R_{\mathrm{in}}} = \frac{\pi}{k}. \tag{16}$$

The critical wave number for purely toroidal fields with $\mu_B = 2$ is 2.8 so that the cells are almost spherical after (16). The wave numbers *with* axial field component are shown in Fig. 2. As expected, they grow linearly for growing $|m|$ and for growing $|\beta|$. For dominating axial field the cells become longer and longer. For $\beta \lesssim -1$ the simple relation

$$\frac{k}{m\beta} \simeq -0.5, \tag{17}$$

results so that the aspect ratio of the cells is given more by the pitch of the field, i.e.

$$m\frac{\delta z}{\delta R} \simeq \frac{B_z}{B_\phi}. \tag{18}$$

After Fig. 2 this relation is well-established for $|B_z| > |B_\phi|$.

The growth rates $\gamma = -\Im(\omega)$ must be given in units of the diffusion frequency η/R_0^2. We find that at least the modes with

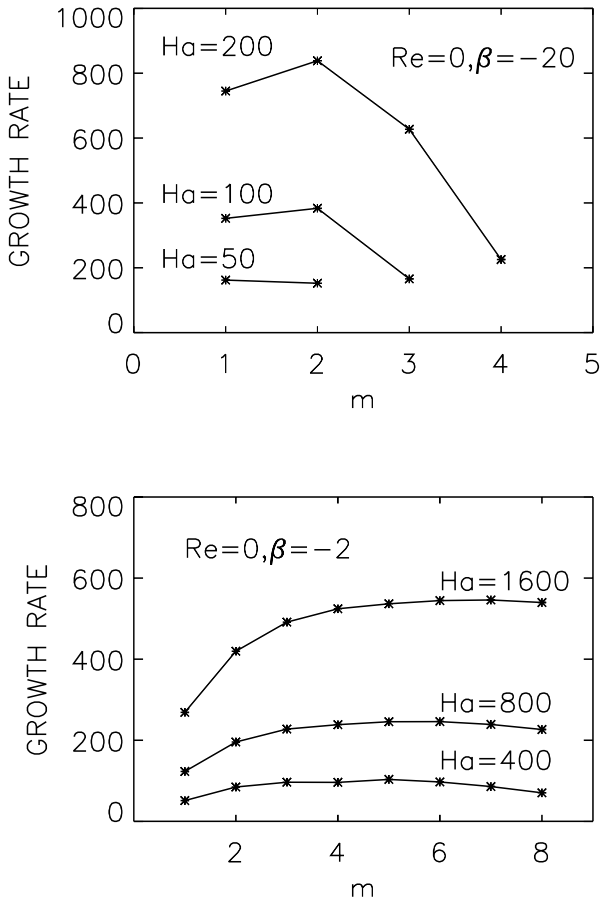

Fig. 3. No rotation. The growth rate in units of the diffusion frequency runs linearly with the Hartmann number of the toroidal field. *Top*: $\beta = -20$. The mode with $m = 2$ has the highest growth rate. *Bottom*: $\beta = -2$ (see the vertical lines in Figs. 1 and 2). The curves are marked with their mode number m. Generally, for finite β maximum growth rates occur for $m > 1$. $\mu_B = 2$, $\mathrm{Pm} = 1$.

the higher m are strongly dependent on the Reynolds number of rotation. Without rotation the growth rates for given β and Pm are plotted in Fig. 3. The used β close to the minimum where $|B_\phi| \simeq |B_z|$ will be the preferred value for many of the examples presented in this paper.

Both plots in Fig. 3 demonstrate the finding that the growth rates run with the magnetic Alfvén frequency Ω_{A}. For stronger fields, strong differences for the growth rates of various m appear. For dominating azimuthal field ($|\beta| = 20$, top), this is a weak effect, but for similar B_ϕ and B_z ($|\beta| = 2$, bottom) it is strong. Of course, there are maxima, but for higher Hartmann number, the highest growth rates belong to higher m.

The dependence of the growth rates on the magnetic Prandtl number is a complex problem. The majority of the numerical simulations concerns $\mathrm{Pm} = 1$. In Sect. 5.1 below, we show that this choice indeed forms a special case. For resting cylinders, the nonaxisymmetric mode with $m = 1$ grows fastest for $\mathrm{Pm} = 1$ if it is normalized with the geometrical average $\eta^* = \sqrt{\nu\eta}$ of both diffusivities. For any given product of ν and η, the mode for almost equal diffusivities is very unstable, while it becomes even more stabilized if the two viscosities are too different. The consequences of this finding may be dramatic if applied to numerical simulations. A field may be unstable for $\mathrm{Pm} = 1$, which

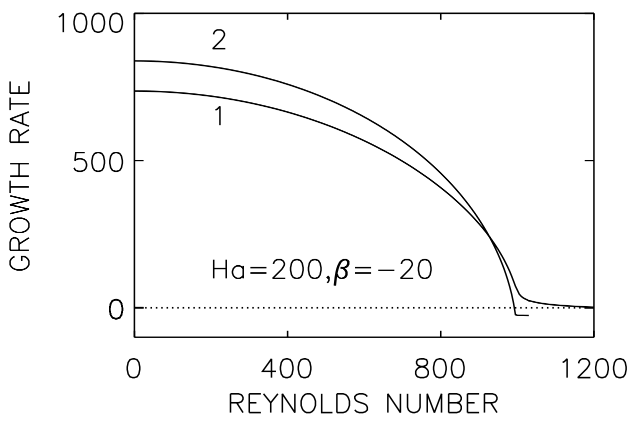

Fig. 4. Rigid rotation and nearly toroidal fields ($\beta = -20$): growth rates divided by the diffusion frequency for supercritical magnetic field with Ha = 200. The magnetic instability is stabilized for $\Omega > 5\Omega_A$ (see Eq. (2)). The kink-type mode with $m = 1$ survives rotation better than the higher modes. $\mu_B = 2, \beta = -20$, Pm = 1.

proves as stable for the more realistic very small or very large values of Pm.

4. Rigid rotation

It is known that rigid rotation stabilizes the magnetic perturbations. This effect can be realized easily with our model. For the standard model with $\mu_B = 2$ and Pm = 1, the growth rates were calculated together with the drift rates for a supercritical value of Ha.

We start with a very small pitch, i.e. with nearly toroidal fields ($\beta = -20$). Figure 4 gives the results for the growth rates normalized with the diffusion frequency. Both given modes for $m = 1$ and $m = 2$ are strongly suppressed by the basic rotation. We also find, however, that the mode with $m = 1$ survives the rotational suppression better than do the modes with higher m.

After Fig. 1 the most interesting situation should exist for magnetic fields with a pitch angle of about one. The eigenvalues for the field with $\beta = -2$ have thus been calculated. Figure 5 gives the main results. The critical Ha for $\beta = -2$ after Fig. 1 is ~70. One finds positive growth rates for slow rotation and stability for fast rotation. The instability cannot exist for Re > Ha or, in other words, for

$$\Omega > \Omega_A. \tag{19}$$

Again, the mode with $m = 1$ withstands the rotational suppression best. It is also true that the modes with the highest m are already suppressed by lower Reynolds numbers. Obviously, *the dominance of the modes with* $m > 1$ *disappears by rigid rotation*.

For fast rotation, the drift $\Re(\omega)/m\Omega$ of all modes approaches the value -1 so that, after the relation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\Re(\omega)}{m}, \tag{20}$$

an observer in the laboratory system finds the magnetic pattern to almost be corotating. If the stellar rotation can only be observed via their magnetic variation, then the rotation of such an object is easily approached by the rotation of the magnetic pattern. The observed (magnetic) rotation is slightly slower than the real plasma rotation; however, there is a jump in the curves: for slow rotation the modes with $m = 1$ and $m = 2$ rotate much faster than the container. We have already found such a jump for rotating stars with unstable toroidal fields (Rüdiger & Kitchatinov 2010).

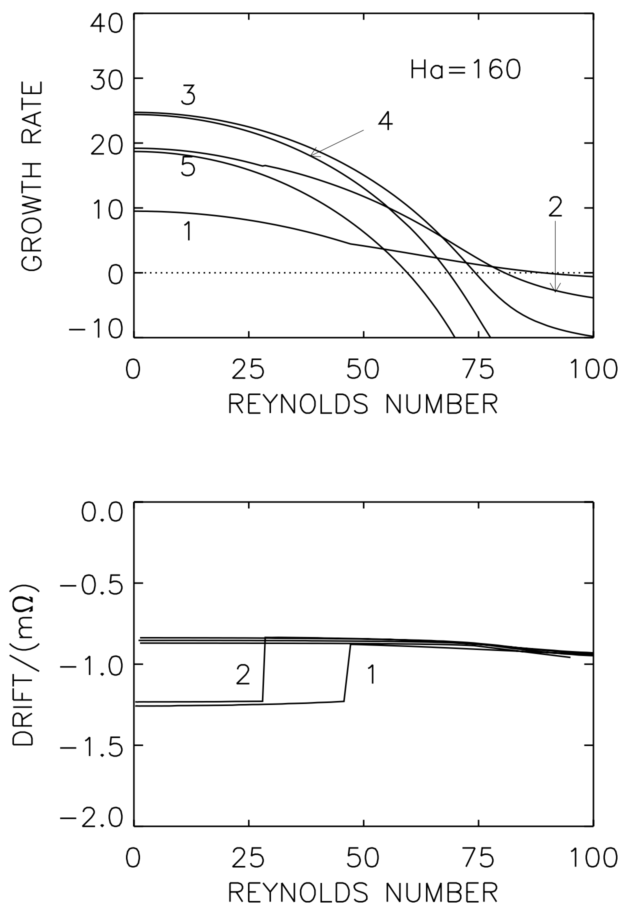

Fig. 5. Rigid rotation ($\mu_\Omega = 1$): *top*: growth rates in unit of diffusion frequency. The magnetic instability is stabilized for $\Omega > \Omega_A$. *Bottom*: the azimuthal drift after (20) of the modes. $\mu_B = 2, \beta = -2$, Ha = 160, Pm = 1.

5. Differential rotation

There is a very new situation if the outer cylinder rotates more slowly than the inner one. The simplified rotation law of our model may be the galactic one, (1), so that for $\hat{\eta} = 0.5$ the rotation ratio is $\mu_\Omega = 0.5$, which can also be considered as the normalized rotation of the outer surface of the container.

The growth rates for Ha = 160 and the mentioned differential rotation are given in Fig. 6. The plot is identical to the plot for rigid rotation (Fig. 5, top) if the rotation is slow, i.e. for $\Omega \ll \Omega_A$. All modes are rotationally stabilized. For $\Omega \gtrsim \Omega_A$, however, the magnetic instability is reanimated but at most for the lower modes. Finally the $m = 1$ mode becomes dominant, because its growth rate (in diffusion units) becomes higher and higher, finally running with the rotation frequency. This type of magnetic instability even exists for *current-free* toroidal magnetic fields so that it has been named the Azimuthal MagnetoRotational Instability (AMRI). It is basically nonaxisymmetric with low m and results from the interaction of differential rotation and toroidal fields (Ogilvie & Pringle 1996; Rüdiger et al. 2007;

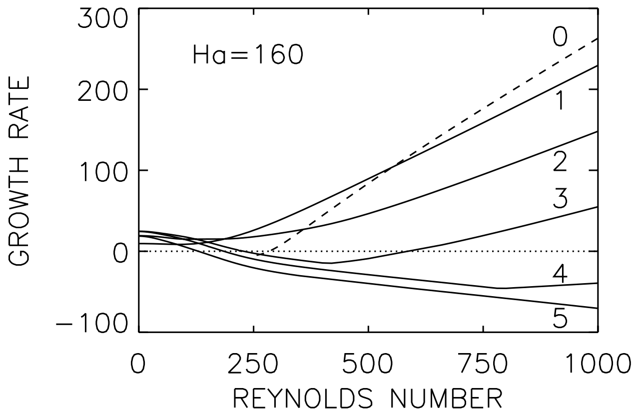

Fig. 6. Differential rotation ($\mu_\Omega = 0.5$): growth rates in unit of diffusion frequency. The kink-type instability ($m = 1$) is reanimated by fast rotation ($\Omega > \Omega_A$). $\mu_B = 2, \beta = -2$, Ha = 160, Pm = 1.

Hollerbach et al. 2010). The growth rate γ grows with growing Ω rather than Ω_A when $\Omega > \Omega_A$ (Fig. 6).

The higher modes dominate only for small Reynolds numbers. They do not contribute to the instability for high Reynolds numbers because they are damped by fast differential rotation. As shown in Sect. 3, the modes with $m > 1$ are also damped for weak and for strong extra axial magnetic field components. Where they are the most unstable (for $\beta \sim 1$), any rotation can suppress them.

5.1. Standard MRI

Axial fields can also be unstable in the presence of differential rotation. The leading mode of this standard MRI is axisymmetric (see Kitchatinov & Rüdiger 2009). In Fig. 6 we also give the growth rate of the mode $m = 0$. It possesses the highest growth rate γ if the rotation rate Ω is high enough, but in this case the growth rate also runs with Ω. We find that for $\Omega \gg \Omega_A$ the most interesting case of $|B_\phi| \simeq |B_z|$ leads to a dominance of the standard MRI. There is an intersection between the growth rates of $m = 0$ and $m = 1$. Left of this point, the nonaxisymmetric mode dominates the axisymmetric one, while to the right it is opposite. According to Eq. (2) galaxies do exist very close to that point. One should thus be aware that the stability of galactic fields should be delicate. We have shown that the negative shear of the rotation law strongly destabilizes the toroidal field. It is thus not clear, however, whether the most unstable mode is axisymmetric.

5.2. The Pm-dependence of the growth rates

By the action of the interstellar turbulence one can assume that the eddy viscosity nearly equals the turbulent magnetic-diffusivity. The current-driven instability, therefore, "feels" a conducting fluid with Pm of order unity. This is a basic simplification that is also used in the majority of numerical MHD simulations. We shall show in this section that this choice is not without consequences that are important to know. The magnetic Prandtl number of stellar plasma is much smaller than unity while the neutron star matter probably possesses very high values of Pm.

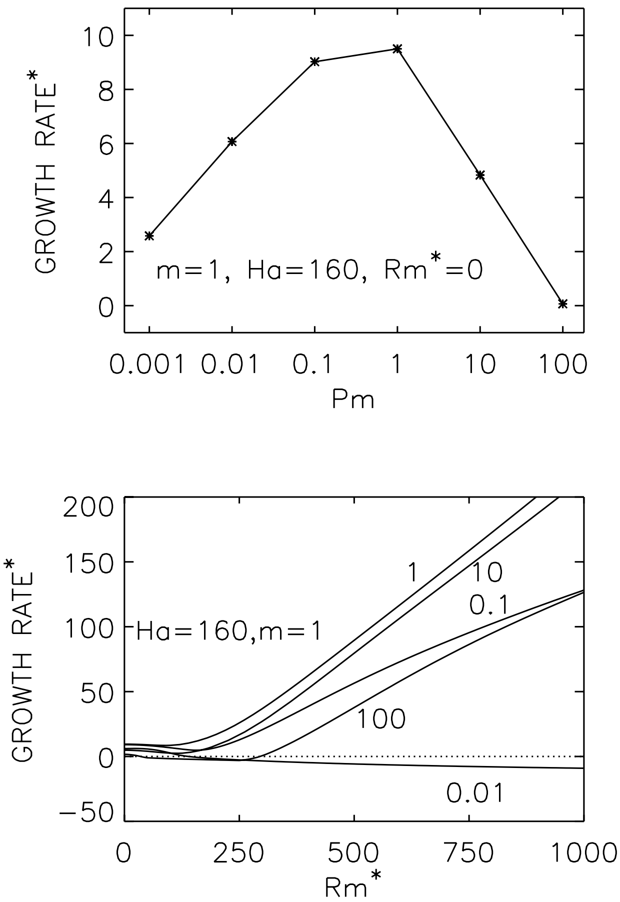

Fig. 7. The growth rates for the modes $m = 1$ normalized with η^* (see text) for fixed Hartmann number and for various Pm. Top: no rotation, Bottom: quasi-galactic differential rotation ($\mu_\Omega = 0.5$). The curves are marked with their magnetic Prandtl number. Ha = 160, $\beta = -2$, $\mu_B = 2$.

The dependence of the growth rates on Pm is not trivial. We showed earlier that for nonrotating fluids the characteristic Hartmann numbers for marginal instability do not depend on Pm (Rüdiger & Schultz 2010). This is not true for the growth rates. It is not obvious how to normalize the growth rates if both the diffusion times differ. We have also shown that the use of frequencies normalized with a geometrically averaged diffusion η^*, which is symmetrically formed with ν and η, where $\eta^* = \sqrt{\nu\eta} = \eta\sqrt{\mathrm{Pm}}$ seems to be most appropriate.

In Fig. 7 the growth rates without and with (differential) rotation are given for a fixed Hartmann number. Both the growth rates of the mode $m = 1$ and the global rotation rate are normalized with η^*, making it

$$\mathrm{Rm}^* = \frac{\Omega_{\mathrm{in}} R_{\mathrm{in}}^2}{\eta^*}. \tag{21}$$

One finds that Pm = 1, always leading to maximum growth rates for slow and fast rotation. Either small or large magnetic Prandtl numbers lead to slower growth of the instability than for Pm = 1. This effect is so strong that the considered field pattern can even be *stabilized* if the magnetic Prandtl number is too small or too high. This is indeed the case in Fig. 7 for Pm < 0.01. An instability found with numerical simulations for Pm = 1 does not automatically exist for much smaller or much larger Pm. If for

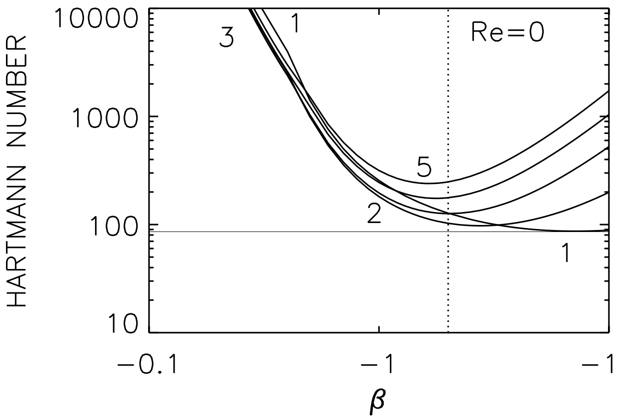

Fig. 8. The same as in Fig. 1 but for $\mu_B = 1$. The absolute minimum of the Hartmann number of Ha = 86.8 is located at $\beta = 7.25$ for $m = 1$, showing extreme stabilization of B_ϕ for increasing B_z. All Pm.

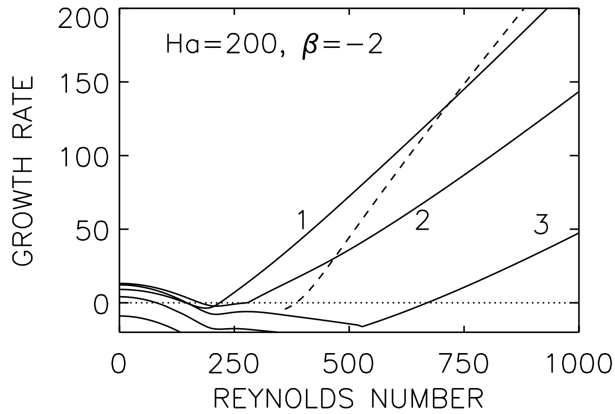

Fig. 9. The same as in Fig. 6 but for $\mu_B = 1$. The dashed line is for $m = 0$, which dominates for faster rotation. With the values characteristic of galaxies (right axis), one finds that the axisymmetric standard MRI should dominate. Pm = 1.

a given value of η^* the magnetic field is unstable for Pm $\simeq$ 1, this must not be true if the numerical values of ν and η are too different. This is an important restriction on the validity of the numerical simulations of the magnetic stability/instability that are operating with $\nu = \eta$. The stability/instability of magnetic fields strongly depends on the magnetic Prandtl number of the fluid. For resting or rotating media, the fields are very unstable for Pm = 1.

For any given Hartmann number (here Ha = 160), one finds two regimes for the rotational influence on the growth rates in Fig. 7. There is almost no influence by small Rm* on the growth rate ω^*. Figure 7 (top) shows a *maximum* growth rate (for Pm = 1) of about 10 that leads to a *minimum* growth time of 0.1 diffusion times. For galaxies with R_0 $\simeq$10 kpc and with $\eta \simeq 10^{26}$ cm^2/s, the diffusion time is then 3 Gyr. One finds, however, fast global rotation accelerating the instability. From Fig. 7 (bottom), the physical growth rate results in $\Omega/5$ so that the growth time is reduced by the rotation to about one rotation time. The rotation of the inner part of the galaxy is concerned here, with rotation times of about 50 Myr; hence, the current-driven magnetic instability is a fairly fast process.

6. Almost homogeneous toroidal field

To check the consistency of our model in more detail, Figs. 1 and 6 were modified for almost uniform toroidal fields, i.e. with $\mu_B = 1$. We find very similar results but with slight numerical differences. Figure 8 reveals the stabilizing action of axial fields to the pinch-type instability of the toroidal field to be much more effective for the case of almost homogeneous B_ϕ. The critical Hartmann number for instability grows by orders of magnitudes if the axial field only grows by a factor of five.

Also the complex influence of differential rotation on the instability of those fields with similar toroidal and poloidal magnitudes as shown by Fig. 6 exists for the case of almost homogeneous toroidal fields (Fig. 9). Slow rotation acts to stabilize but the modes with higher m dominate for a while. For fast rotation the modes with low m are re-animated as they form the new instability. For slow rotation the modes with higher m exhibit the maximum growth rates but for fast and differential rotation the mode with $m = 1$ grows fastest. Again, the axisymmetric mode with $m = 0$ is concerned, which starts to dominate beyond the given crossing point. The coordinates of the intersection are nearly the same as in Fig. 6.

Obviously, the above findings about the influence of differential rotation on the stability of magnetic fields with spiral structure basically do not depend on the radial profile of the toroidal field. It thus makes sense to call – as we did – this effect the magnetorotational instability that originally only concerned current-free toroidal fields ($\mu_B = 0.5$, in our notation).

7. Summary

In a cylindric geometry the pinch-type instability of axisymmetric and also magnetic spirals with finite current helicity $\boldsymbol{B} \cdot \boldsymbol{J}$ was considered under the influence of rotation. The field is formed by an unstable toroidal field and a uniform axial field, which is stable by definition. The pitch of the spiral is given by the inverse of $\beta = B_\phi/B_z$, which is negative for the considered lefthanded spirals. The larger the pitch angle of the background field, the higher the azimuthal Fourier number m of the mode with the highest growth rate.

The excitation of modes with low m is of particular interest here. For small $|\beta|$ and without rotation, typically a mode with $m > 1$ is excited with the highest growth rate. As stressed by Bonanno & Urpin (2010), this phenomenon could have consequences for jet theory. As we have shown, however, a growing axial field stabilizes the toroidal field more and more. The critical Hartmann number grows by orders of magnitudes if $|\beta|$ reduces from unity to 0.1 (see Figs. 1 and 8). Helical background fields with large axial field component are thus much stabler than purely toroidal fields without finite B_z.

A global rotation also stabilizes the pinch-type instability. Figure 5 shows, how the rotation quickly stabilizes the modes with $m > 1$ for a magnetic field with almost equal field components ($\beta = -2$) while the kink-instability ($m = 1$) remains unstable for a little faster rotation. The growth rates of the modes are continuously reduced by growing Reynolds numbers. Generally, the helical background fields are stable against all nonaxisymmetric perturbations if $\Omega \gg \Omega_A$.

A very new situation results for nonrigid rotation. Figure 6 uses a rotation law known from galaxies to clearly demonstrate that for $\Omega > \Omega_A$ the growth rates after a characteristic minimum at $\Omega \simeq \Omega_A$ again reach positive and high values. A rotation law with positive shear will always stabilize the nonaxisymmetric instability. It has also been shown that, in the presence of differential rotation with negative shear, the toroidal field can become unstable even if there is no electric current in the container (see Rüdiger & Schultz 2010). While the modes with higher m are the most unstable for slow rotation it is for fast rotation the mode with $m = 1$.

If the field possesses an axial component then under the influence of differential rotations with negative shear, the standard MRI appears in the form of a growing axisymmetric ($m = 0$) roll. The lines of marginal instability for $m = 0$ and $m = 1$ cross so that the axisymmetric perturbation dominates for fast enough rotation. In any case we find that a spiralic magnetic field under the influence of differential rotation with negative shear appears to be extremely unstable (see Fig. 9 for a field that is nearly uniform in radius). As our model roughly reflects the magnetic geometry in galaxies with SN-driven interstellar turbulence, one should expect their dynamo-generated magnetic fields to be rather unstable. A nonaxisymmetric global field configuration might *not* be the exception.

Our model also allows the variation in the magnetic Prandtl number. Figure 7 shows a rather clear situation. Its parameters only depend on the product of ν and η; i.e., they are invariant against an exchange of ν and η. We find strong differences in the lines for $\nu = \eta$ and $\nu \neq \eta$ for a fixed value of $\nu \cdot \eta$. The media with $\nu = \eta$ are more unstable than the media with $\nu \neq \eta$. Moreover, if the magnetic field is unstable for Pm = 1, it can even be stable for Pm $\gg$ 1 or Pm $\ll$ 1. The result requires care interpretating numerical instability calculations if the considered medium has a magnetic Prandtl number far below unity. A magnetic field configuration which turns out to be unstable for a given Hartmann number and Pm = 1 can be stable for much smaller or much larger Pm.

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