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Spectral line polarization with angle-dependent partial frequency redistribution

I. A Stokes parameters decomposition for Rayleigh scattering

H. Frisch

UNS, CNRS, OCA, Laboratoire Cassiopée, BP 4229, 06304 Nice Cedex 4, France
e-mail: frisch@oca.eu

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ABSTRACT

Context. The linear polarization of a strong resonance lines observed near the solar limb is created by a multiple-scattering process. Partial frequency redistribution (PRD) effects must be accounted for to explain the polarization profiles. The redistribution matrix describing the scattering process is a sum of terms, each containing a PRD function multiplied by a Rayleigh type phase matrix. A standard approximation made in calculating the polarization is to average the PRD functions over all the scattering angles, because the numerical work needed to take the angle-dependence of the PRD functions into account is large and not always needed for reasonable evaluations of the polarization.

Aims. This paper describes a Stokes parameters decomposition method, that is applicable in plane-parallel cylindrically symmetrical media, which aims at simplifying the numerical work needed to overcome the angle-average approximation.

Methods. The decomposition method relies on an azimuthal Fourier expansion of the PRD functions associated to a decomposition of the phase matrices in terms of the Landi Degl'Innocenti irreducible spherical tensors for polarimetry $\mathcal{T}^K_Q(i,\mathbf{\Omega})$ (i Stokes parameter index, $\mathbf{\Omega}$ ray direction). The terms that depend on the azimuth of the scattering angle are retained in the phase matrices.

Results. It is shown that the Stokes parameters I and Q, which have the same cylindrical symmetry as the medium, can be expressed in terms of four cylindrically symmetrical components $\mathcal{I}^K_Q$ ($K = Q = 0$, $K = 2$, $Q = 0, 1, 2$). The components with $Q = 1, 2$ are created by the angular dependence of the PRD functions. They go to zero at disk center, ensuring that Stokes Q also goes to zero. Each component $\mathcal{I}^K_Q$ is a solution to a standard radiative transfer equation. The source term $\mathcal{S}^K_Q$ are significantly simpler than the source terms corresponding to I and Q. They satisfy a set of integral equations that can be solved by an accelerated lambda iteration (ALI) method.

Key words. line: formation – polarization – radiative transfer

1. Introduction

The solar spectrum shows some strong resonance lines such as Ca II H and K, Mg II h and k, and Ca I 4227 Å . They are formed in the upper layers of the photosphere and lower chromosphere. The main mechanism presiding over their formation is radiative excitation of the lower level of the transition followed by spontaneous radiative deexcitation of the upper level. One of the difficulties that appears when investigating these lines is that the frequencies of the incoming and scattered beams are partially correlated, a phenomenon known as partial frequency redistribution (PRD). It is quite conspicuous in the wings of the intensity profiles and in the linear polarization profiles. Created by the scattering of the photospheric radiation, this linear polarization is known as resonance polarization or Rayleigh scattering in the classical physics description (Chandrasekhar 1950). It is at a maximum in the uppermost layers of the solar atmosphere where the anisotropy of the radiation field takes its highest values. It is thus mainly observed near the solar limb. Since Ivanov (1991) coined the term, it has often been referred to as the second solar spectrum. A high spectral resolution atlas of the second solar spectrum from 3160 Å to 6995 Å has been recorded by Gandorfer (2000, 2002, 2005).

Some spectral lines like Ca I 4227 Å can be modeled, at least for the purposes of polarization, by a two-level atom with unpolarized ground level. Each scattering event is characterized by a redistribution matrix that describes the coupling between frequencies, directions, and polarization of the incoming and scattered beams. The redistribution matrix proposed in Bommier (1997a) is the last and the most sophisticated of the expressions established to describe resonance polarization with partial frequency redistribution (PRD) effects. Starting from the non-LTE semi-classical theory of Stenflo (1976), continuous progress has been made in the direction of a better description of the PRD mechanism and of the role of elastic collisions. This was made possible by the use of the density matrix approach of Omont et al. (1972), which leads to a fully quantum mechanical description of the polarized radiation field and atomic structure. In the absence of a magnetic field, as shown by Bommier (1997a) (see also Domke & Hubeny 1988), the redistribution matrix is a sum of terms, each the product of a scalar redistribution function $r(\nu,\nu',\Theta)$ by a polarization matrix $P(\mathbf{\Omega},\mathbf{\Omega}')$ (see Eq. (20)). Here ν' and ν are the frequencies of the incoming and scattered beams and Θ the scattering angle between the directions $\mathbf{\Omega}'$ and $\mathbf{\Omega}$ of the beams (see Fig. 1). The scalar redistribution functions are of the $r_{\rm II}$ or $r_{\rm III}$ type (Hummer 1962). They describe frequency coherent and frequency incoherent scattering in the atomic frame, respectively. Frequency coherent scattering plays an important role in the line wings. The dependence on the scattering angle appearing in the laboratory frame redistribution functions comes from the Doppler shifts created by the thermal velocities of the atoms. Since 1997, great progress has

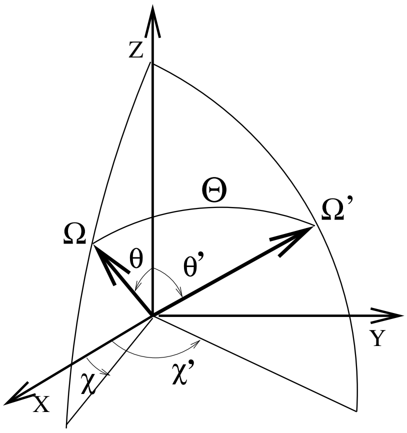

Fig. 1. Atmospheric reference system. The directions $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ are defined by their polar angles (θ,χ) and (θ',χ') and Θ is the scattering angle. The Z-axis is along the outside normal to the atmosphere.

been made to properly include the Hanle and Zeeman effects (see e.g. Bommier 1997b; Sampoorna et al. 2007). Systematic theories for multi-level atoms incorporating PRD effects are still under development.

One of the difficulties with PRD from a numerical point of view is that the dependence on the directions $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ appears in the polarization matrices and in the scalar redistribution functions. For this reason the angle-dependent redistribution functions $r_{\rm II,III}(\nu,\nu',\Theta)$ are often replaced by their angle-averaged version $\bar{r}_{\rm II,III}(\nu,\nu')$, with the averaging over all the scattering angles. The expression proposed by Rees & Saliba (1982) and the angle-averaged versions of Domke & Hubeny (1988) and Bommier (1997a) have been used quite extensively, on the whole rather successfully (see e.g Faurobert-Scholl 1996). For intensity profiles, this approximation, which is based on the isotropy of the radiation field inside the atmosphere, is indeed quite reasonable. It is definitely more questionable when one is interested in the linear polarization that is directly controlled by the anisotropy of the intensity field. A correct evaluation of the linear polarization is very important because it is a starting point for quantitative determinations of micro-turbulent magnetic fields in the solar atmosphere by the Hanle effect (e.g. Stenflo 1982; Faurobert-Scholl 1996; Trujillo Bueno et al. 2004). Comparisons between Q/I profiles calculated with angle-dependent and angle-averaged PRD can be found in Faurobert (1987, 1988) and in Nagendra et al. (2002). For strong lines (total optical thickness at line center around 10^4) differences shown in Faurobert (1988) for the ratio Q/I appear to be large enough to be measurable, but in Nagendra et al. (2002) they are essentially negligible. These two articles use a rather different numerical approach. In Faurobert (1988) it is tailored for Rayleigh scattering, and in Nagendra et al. (2002) for the Hanle effect.

In this paper we describe a new method for calculating of the linear polarization when the angle-dependence of the PRD is retained. The method is developed for a plane-parallel atmosphere. It makes use of an azimuthal Fourier decomposition of the PRD functions and of the decomposition of the elements of the polarization phase matrix in terms of the spherical tensors for polarimetry $\mathcal{T}^K_Q(i,\mathbf{\Omega})$ introduced by Landi Degl'Innocenti (1984) (see also Landi Degl'Innocenti & Landolfi 2004, henceforth LL04). Here i is a Stokes parameter index ($i=0,\ldots,3$), and K and Q are integer numbers ($K=0,1,2$ and $-K\le Q\le +K$ for each value of K). The approach is somewhat similar to a method used in Frisch (2007, henceforth HF07) and Frisch (2009, henceforth HF09) for the weak-field Hanle effect.

For simplicity, the decomposition method is presented for a redistribution matrix of the form

$$R(\nu,\mathbf{\Omega},\nu',\mathbf{\Omega}')=r(\nu,\nu',\Theta)P_{\rm R}(\mathbf{\Omega},\mathbf{\Omega}'), \tag{1}$$

where $\mathbf{P}_{\rm R}$ is the Rayleigh phase matrix (see e.g. Chandrasekhar 1950; Stenflo 1994) (see also LL04 p. 202 or the Appendix). In Sect. 5 we show how to extend the results obtained with Eq. (1) to the redistribution matrix given in Eq. (20) proposed by Bommier (1997a, henceforth VB matrix)[1]. The scattering problem considered here is simpler than the Hanle effect problem investigated in HF09, but more explicit results are obtained. Our results can be applied to pure resonance scattering, but also to the Hanle effect, if the magnetic field is microturbulent.

The organization of the paper is as follows. In Sect. 2 we show how to express the Rayleigh phase matrix elements in terms of irreducible tensors $\tilde{\mathcal{T}}^K_Q(i,\theta)$, simply related to the $\mathcal{T}^K_Q(i,\Omega)$. In Sect. 3 we show how to decompose the Stokes parameters I and Q into four irreducible components $\mathcal{I}^K_Q$ ($K=Q=0$, $K=2$, $Q=0,1,2$). The components with $Q=1,2$ come from the angle-dependence of the PRD function $r(\nu,\nu',\Theta)$. A transfer equation for the components $\mathcal{I}^K_Q$ is established in Sect. 4. In Sect. 5 we generalize the results established in the preceding sections to the full VB matrix and to the Hanle effect with a microturbulent magnetic field. We propose in Sect. 6 an approximation of the single scattering type for the polarization. Some concluding remarks on numerical aspects are presented in Sect. 7. An Appendix contains some technical details.

2. Representation of the Rayleigh phase matrix elements

We consider a plane-parallel atmosphere, cylindrically symmetrical along the normal to the surface. No incident beam or deterministic magnetic field is there to break the symmetry. The primary source of photons is unpolarized. Since there is no magnetic field, Stokes V is zero. The polarized radiation field is cylindrically symmetrical (i.e. independent of the azimuthal angle χ of the ray direction) and can be described by the two Stokes parameters I and Q[2], if the direction chosen for the measure of the linear polarization is parallel or perpendicular to the solar limb. In the Rayleigh phase matrix (see e.g. the Appendix) we retain the elements that are not cylindrically symmetrical because the redistribution function depends on the scattering angle Θ. In contrast, when dealing with monochromatic scattering, complete frequency redistribution, and angle-averaged PRD, these terms are needed only if the medium is not cylindrically symmetrical (Chandrasekhar 1950).

The elements of the full 4×4 Rayleigh phase matrix $\mathbf{P}_{\rm R}(\mathbf{\Omega},\mathbf{\Omega}')$ can be written as (see e.g. Bommier 1997a; LL04)

$$P_{ij}(\mathbf{\Omega},\mathbf{\Omega}')=\sum_K P^{(K)}_{ij}(\mathbf{\Omega},\mathbf{\Omega}'), \tag{2}$$

where

$$P^{(K)}_{ij}(\mathbf{\Omega},\mathbf{\Omega}')=\sum_Q \mathcal{T}^K_Q(i,\mathbf{\Omega})\mathcal{T}^K_Q(j,\mathbf{\Omega}')^*. \tag{3}$$

[1] The VB matrix given in Eq. (20) reduces to Eq. (1) when $r=\alpha r_{\rm II}$ and the atomic polarization coefficient W_2 is equal to one.

[2] The cylindrical symmetry of I and Q can be proven for an angle-dependent partial frequency redistribution by expanding the solution of the integral equation for the Stokes parameters source vector in a Neumann series.

The notation * stands for complex conjugate. Explicit expressions of the tensors $\mathcal{T}^K_Q$ can be found in LL04 (p. 211). They satisfy the conjugation property

$$\mathcal{T}^K_Q(i,\mathbf{\Omega})^* = (-1)^Q \mathcal{T}^K_{-Q}(i,\mathbf{\Omega}). \quad (4)$$

They can be written as

$$\mathcal{T}^K_Q(i,\theta,\chi) = \tilde{\mathcal{T}}^K_Q(i,\theta)\,\mathrm{e}^{\mathrm{i}Q\chi}, \quad (5)$$

where θ and χ are the polar angles of the direction $\mathbf{\Omega}$. We stress that $\tilde{\mathcal{T}}^K_Q(i,\theta)$ is in general a complex number. It depends on θ and i but also on the reference angle γ, which determines the reference direction for the measure of Q. Here we use $\gamma = 0$. For a given point on the solar disk, this choice corresponds to positive Stokes Q in the direction perpendicular to the solar disk at the nearest point on the limb.

As U and V are zero for the problem at hand, i takes only the values 0 and 1 and K the values 0 and 2. When in addition $\gamma = 0$, the tensors $\tilde{\mathcal{T}}^K_Q(i,\theta)$ are real. It is then easy to check that

$$\tilde{\mathcal{T}}^K_{-Q}(i,\theta) = (-1)^Q \tilde{\mathcal{T}}^K_Q(i,\theta). \quad (6)$$

Inserting Eq. (5) into Eq. (2) and using the property that the $\tilde{\mathcal{T}}^K_Q(i,\theta)$ are real, we can rewrite

$$P_{ij}(\mathbf{\Omega},\mathbf{\Omega}') = \sum_{KQ} \tilde{\mathcal{T}}^K_Q(i,\theta)\tilde{\mathcal{T}}^K_Q(j,\theta')\mathrm{e}^{\mathrm{i}Q(\chi-\chi')}, \quad i,j=0,1,\gamma=0. \quad (7)$$

Regrouping the two terms with the same value of $|Q|$ (for $Q \neq 0$) and making use of Eq. (6), we obtain

$$P_{ij}(\mathbf{\Omega},\mathbf{\Omega}') = \sum_{K,Q\geq 0} c_Q \tilde{\mathcal{T}}^K_Q(i,\theta)\tilde{\mathcal{T}}^K_Q(j,\theta')\cos[Q(\chi-\chi')], \quad (8)$$

where $c_Q = 2-\delta_{0Q}$. Here δ_{0Q} is the Kronecker symbol equal to 0 when $Q \neq 0$ and to 1 when $Q = 0$. We have four terms in the summation over K and Q corresponding to $K = Q = 0$, $K = 2$, $Q = 0,1,2$. In the Appendix we give the usual expressions of the Rayleigh phase matrix elements and the expressions of the $\tilde{\mathcal{T}}^K_Q$. It is easy to check that the Rayleigh matrix elements can indeed be written as shown in Eq. (8).

3. Decomposition of the Stokes parameters

As the radiation field is cylindrically symmetrical, the Stokes parameters only depend on the inclination θ with respect to the Z-axis. The polarized transfer equation for $I = I_0$ and $Q = I_1$ can be written in component form as

$$\mu\frac{\partial I_i}{\partial\tau} = \varphi(x)\Big[I_i(\tau,x,\mu) - S_i(\tau,x,\mu)\Big], \quad i=0,1, \quad (9)$$

with $\mu = \cos\theta$. Here, τ is the line optical depth defined by $d\tau = -k_l dz$, where k_l is the frequency-averaged line absorption coefficient and $\varphi(x)$ the normalized line absorption coefficient. Henceforth, frequencies x are measured in Doppler width units with zero at line center. The Doppler broadening is assumed to be depth-independent. Here for simplicity we ignore the continuum absorption, emission, and polarization. The source term is given by

$$S_i(\tau,x,\mu) = G_i(\tau) + \oint\!\!\int \frac{r(x,x',\Theta)}{\varphi(x)} \sum_j P_{ij}(\mathbf{\Omega},\mathbf{\Omega}') I_j(\tau,x',\mu')\,\mathrm{d}x'\,\frac{\mathrm{d}\Omega'}{4\pi}, \quad (10)$$

with $\mathrm{d}\Omega' = \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\chi'$. To avoid unnecessary complications we assume that the primary source is unpolarized; i.e., only the component $G_0(\tau)$ is non zero. The handling of a polarized primary source is described in HF07. We recall that

$$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\chi-\chi'). \quad (11)$$

Introducing Eq. (8) into Eq. (10), we observe that the components S_i can be written as

$$S_i(\tau,x,\mu) = \sum_{K,Q\geq 0} \tilde{\mathcal{T}}^K_Q(i,\mu)\mathcal{S}^K_Q(\tau,x,\mu), \quad i=0,1, \quad (12)$$

with

$$\mathcal{S}^K_Q(\tau,x,\mu) = \delta_{K0}\delta_{Q0}G_0(\tau) + \int_{-\infty}^{+\infty}\mathrm{d}x'\int_{-1}^{+1}\frac{\mathrm{d}\mu'}{2} \times \frac{\tilde{r}^{(Q)}(x,\mu,x',\mu')}{\varphi(x)}\sum_{j=0,1}\tilde{\mathcal{T}}^K_Q(j,\mu')I_j(\tau,x',\mu'), \quad (13)$$

and

$$\tilde{r}^{(Q)}(x,\mu,x',\mu') = \frac{c_Q}{2\pi}\int_0^{2\pi} r(x,\mu,x',\mu',\chi-\chi')\cos[Q(\chi-\chi')]\,\mathrm{d}(\chi-\chi'). \quad (14)$$

The functions $\tilde{r}^{(Q)}$ are the azimuthal Fourier coefficients of order 0, 1, and 2 of the PRD function $r(x,x',\Theta)$. Plots of $\tilde{r}^{(Q)}_{\mathrm{II}}(x,\mu,x',\mu')$ versus x can be found in Domke & Hubeny (1988) for some choices of μ, x', μ', and Q. The numerical method introduced in Faurobert (1987) also employs three azimuthal averages of $r(x,x',\Theta)$, but the weighting factors for $Q = 1,2$ are different. Expansions of the redistribution functions r_{II} and r_{III} in Legendre polynomials have also been employed for calculating Stokes I and Stokes Q (Mckenna 1985), although the frequency dependence of the Fourier expansion coefficients have better numerical properties than the Legendre expansion coefficients, which have sharp cusps and discontinuities in their derivatives (Domke & Hubeny 1988; see also in Milkey et al. 1975 the remark by D. Hummer).

At disk center the heliocentric angle θ is zero, therefore $\cos\Theta$ is independent of the azimuth of the scattering angle (see Eq. (11)), $r(x,x',\Theta)$ is cylindrically symmetrical, and the azimuthal Fourier components $r^{(Q)}$ with $Q = 1,2$ are zero.

The formal solution of Eq. (9) shows that the components $I_i(\tau,x,\mu)$ have a decomposition similar to Eq. (12) (for details see HF07), which can be written as

$$I_i(\tau,x,\mu) = \sum_{K,Q\geq 0}\tilde{\mathcal{T}}^K_Q(i,\mu)\mathcal{I}^K_Q(\tau,x,\mu), \quad i=0,1. \quad (15)$$

Using the expressions of the $\tilde{\mathcal{T}}^K_Q$ given in Eq. (A.6), one finds

$$\begin{aligned} I(\tau,x,\mu) &= \mathcal{I}^0_0 + \frac{1}{2\sqrt{2}}(3\mu^2-1)\mathcal{I}^2_0 - \frac{\sqrt{3}}{2}\mu\sqrt{1-\mu^2}\mathcal{I}^2_1 \\ &\quad + \frac{\sqrt{3}}{4}(1-\mu^2)\mathcal{I}^2_2, \\ Q(\tau,x,\mu) &= -\frac{3}{2\sqrt{2}}(1-\mu^2)\mathcal{I}^2_0 - \frac{\sqrt{3}}{2}\mu\sqrt{1-\mu^2}\mathcal{I}^2_1 \\ &\quad - \frac{\sqrt{3}}{4}(1+\mu^2)\mathcal{I}^2_2. \end{aligned} \quad (16)$$

The same expansion holds for the source terms S_0 and S_1 (see Eq. (12)). All the $\mathcal{I}^K_Q$ and $\mathcal{S}^K_Q$ are functions of τ, x, and μ. We show below how to calculate them. We already note here that the components $\mathcal{S}^2_Q$ and $\mathcal{I}^2_Q$, $Q = 1,2$ are zero for $\mu = 1$. This implies that Stokes Q is zero at disk center.

4. Radiative transfer equations for the irreducible components

By introducing the decompositions of S_i and I_i (see Eqs. (12) and (15)) into Eq. (9), we see that the components $\mathcal{I}_Q^K$ satisfy a transfer equation similar to Eq. (9) with a source term

$$\mathcal{S}_Q^K(\tau,x,\mu) = \delta_{K0}\delta_{Q0}G_0(\tau) + \int_{-\infty}^{+\infty} \mathrm{d}x' \int_{-1}^{+1} \frac{\mathrm{d}\mu'}{2} \times \frac{\tilde{r}^{(Q)}(x,\mu,x',\mu')}{\varphi(x)} \sum_{K',Q'\geq 0} \tilde{\Gamma}_{QQ'}^{KK'}(\mu')\mathcal{I}_{Q'}^{K'}(\tau,x',\mu'), \quad (17)$$

where

$$\tilde{\Gamma}_{QQ'}^{KK'}(\mu') = \sum_{j=0,1} \tilde{\mathcal{T}}_Q^K(j,\mu')\tilde{\mathcal{T}}_{Q'}^{K'}(j,\mu'). \quad (18)$$

The coefficients $\tilde{\Gamma}_{QQ'}^{KK'}$ are given in the Appendix. Some of them are identical to the modulus of the multipole coupling coefficients $\Gamma_{KQ,K'Q'}$ in LL04 (Appendix A.20), but differences occur because here the summation is over $j = 0, 1$. The dependence of $\mathcal{S}_Q^K$ on μ only comes from azimuthal Fourier coefficients of the redistribution function.

It is convenient to regroup the components $\mathcal{S}_Q^K$ into a 4-component vector $\mathcal{S}(\tau,x,\mu) = \{\mathcal{S}_0^0, \mathcal{S}_0^2, \mathcal{S}_1^2, \mathcal{S}_2^2\}^{\mathrm{T}}$. When combining the formal solution of the transfer equation for the $\mathcal{I}_Q^K$ with Eq. (17), we obtain for the vector $\mathcal{S}$ an integral equation which may be written as

$$\begin{aligned}\mathcal{S}(\tau,x,\mu) = \mathcal{G}(\tau) + \int_{-\infty}^{+\infty} \Bigg\{ \int_{\tau}^{\infty} \int_0^1 \mathrm{e}^{-\frac{\tau'-\tau}{\mu'}\varphi(x')} \frac{\varphi(x')}{\mu'} \\ \times \frac{\tilde{\mathcal{R}}(x,\mu,x',\mu')}{\varphi(x)} \Gamma(\mu')\mathcal{S}(\tau',x',\mu') \frac{\mathrm{d}\mu'}{2}\,\mathrm{d}\tau' \\ - \int_0^{\tau} \int_{-1}^0 \mathrm{e}^{-\frac{\tau'-\tau}{\mu'}\varphi(x')} \frac{\varphi(x')}{\mu'} \\ \times \frac{\tilde{\mathcal{R}}(x,\mu,x',\mu')}{\varphi(x)} \Gamma(\mu')\mathcal{S}(\tau',x',\mu') \frac{\mathrm{d}\mu'}{2}\,\mathrm{d}\tau' \Bigg\}\,\mathrm{d}x'. \quad (19)\end{aligned}$$

For simplicity we have assumed that we are dealing with a semi-infinite medium. Here $\mathcal{G}(\tau) = \{G_0(\tau), 0, 0, 0\}^{\mathrm{T}}$ is the primary source term, $\tilde{\mathcal{R}}$ a diagonal matrix, $\tilde{\mathcal{R}} = \{\tilde{r}^{(0)}, \tilde{r}^{(0)}, \tilde{r}^{(1)}, \tilde{r}^{(2)}\}$, and Γ a full 4×4 matrix with elements $\tilde{\Gamma}_{QQ'}^{KK'}$. For symmetry it has only 10 different elements (see the Appendix).

When the frequency redistribution function is of the form $r(x,x')$ (complete frequency redistribution or angle-averaged PRD), only $\mathcal{S}_0^0$ and $\mathcal{S}_0^2$ are non zero. Moreover, they are independent of μ. One recovers a standard decomposition in which the source vector is described by a 2-component vector depending only on optical depth and, for PRD, also on frequency. In Rees (1978) or Faurobert (1987), the starting point of the decomposition is a factorization of the Rayleigh phase matrix of the form $\mathbf{P}_{\mathrm{R}}(\mu,\mu') = \mathbf{A}(\mu)\mathbf{A}^{\mathrm{T}}(\mu')$, where T stands for transpose. This type of factorization was first proposed by Sekara (1963). The choice of the matrix $\mathbf{A}$ is not unique (see e.g. Van de Hulst 1980; Ivanov 1995). For complete frequency redistribution, Omont et al. (1973) (also Dumont et al. 1973) use an expansion of the source term in irreducible components $\mathcal{S}_Q^K$ with $K = 0, 2$ and $Q = 0$.

In the direction $\mu = 1$, the Fourier azimuthal components $\tilde{r}^{(1)}$ and $\tilde{r}^{(2)}$ are zero, as pointed out above. Therefore only the components $\mathcal{S}_0^0$, $\mathcal{I}_0^0$ and $\mathcal{S}_0^2$, $\mathcal{I}_0^2$ are non zero. Setting $\mu = 1$ in Eq. (16), one recovers Stokes $Q = 0$ at disk center, in agreement with the cylindrical symmetry of the problem. Equation (16) shows also that the contributions of $\mathcal{I}_1^2$ to Stokes I and Q go to zero when approaching the limb.

5. General PRD polarization matrix

The elements of the VB redistribution matrix for a two-level atom with unpolarized ground level, may be written as

$$R_{ij}(x,x',\Theta) = \sum_{K=0,2} \Big[\alpha r_{\mathrm{II}}(x,x',\Theta) + [\beta^{(K)} - \alpha] r_{\mathrm{III}}(x,x',\Theta)\Big] W_K P_{ij}^{(K)}(\mathbf{\Omega},\mathbf{\Omega}'),\ i,j = 0,1. \quad (20)$$

The $P_{ij}^{(K)}$ are defined in Eq. (3) and given in the Appendix. For $K = 0$, $P_{11}^{(0)} = 1$ and the other elements are zero. The coefficient W_0 is equal to one and W_2 is an atomic depolarization factor depending on the angular momenta J_{l} and J_{u} of the lower and upper levels of the transition. It is equal to one for a normal Zeeman triplet ($J_{\mathrm{l}} = 0$, $J_{\mathrm{u}} = 1$). Otherwise, it is smaller than unity. Values of W_2 for different values of J_{l} and J_{u} can be found in LL04 (p. 515). The coefficients α and $\beta^{(K)}$ are branching ratios:

$$\alpha = \frac{\Gamma_{\mathrm{R}}}{\Gamma_{\mathrm{R}} + \Gamma_{\mathrm{I}} + \Gamma_{\mathrm{E}}}, \quad (21)$$

and

$$\beta^{(K)} = \frac{\Gamma_{\mathrm{R}}}{\Gamma_{\mathrm{R}} + \Gamma_{\mathrm{I}} + D^{(K)}}, \quad (22)$$

with Γ_{R} the upper level spontaneous deexcitation radiative rate, Γ_{I} and Γ_{E} inelastic and elastic collisions rates and $D^{(K)}$ a collisional depolarization rate such that $D^{(0)} = 0$. For more detail see Bommier (1997a).

When a microturbulent magnetic field is present in the medium, the elements of the scattering phase matrix may be written as in Eq. (20) with W_K replaced by $W_K\mu_K$ (see e.g. LL04, p. 215). The coefficient μ_0 is equal to one. The coefficient μ_2 depends on the magnetic field vector probability density function. Expressions of μ_2 for a magnetic field with a single field strength value, an isotropic, or horizontal angular distribution can be found in Stenflo (1994) (see also LL04).

We easily see that the decomposition presented in the preceding sections is still valid. The matrix $\tilde{\mathcal{R}}(x,\mu,x',\mu')$ in Eq. (19) remains a diagonal matrix. Its four elements $\tilde{\mathcal{R}}_Q^K$ may be written as

$$\tilde{\mathcal{R}}_0^0 = \alpha\tilde{r}_{\mathrm{II}}^{(0)} + (\beta^{(0)} - \alpha)\tilde{r}_{\mathrm{III}}^{(0)}, \quad (23)$$

$$\tilde{\mathcal{R}}_Q^2 = W_2\mu_2\Big[\alpha\tilde{r}_{\mathrm{II}}^{(Q)} + (\beta^{(2)} - \alpha)\tilde{r}_{\mathrm{III}}^{(Q)}\Big], \quad Q = 0,1,2. \quad (24)$$

Since the Hanle effect only acts at line center, the coefficient μ_2 should be set to unity outside the line core.

6. Approximate expressions for the irreducible components

The linear polarization observed in the second solar spectrum is never more than a few percent. This implies that the components $\mathcal{I}_Q^2$, $Q = 0,1,2$ are significantly smaller than the component $\mathcal{I}_0^0$, the dominant term of Stokes I (see Eq. (16)). The same behavior holds for the components $\mathcal{S}_Q^K$. Neglecting the components $\mathcal{I}_Q^2$ ($Q = 0,1,2$) in Eq. (17), we obtain for each component

$\mathcal{S}_Q^2(\tau, x, \mu)$ the approximate expression

$$\mathcal{S}_Q^2(\tau, x, \mu) \simeq \int_{-\infty}^{+\infty} \int_{-1}^{+1} \frac{\tilde{R}_Q^2(x, \mu, x', \mu')}{\varphi(x)} \tilde{\Gamma}_{Q0}^{20}(\mu') \mathcal{I}_0^0(\tau, x', \mu') \frac{\mathrm{d}\mu'}{2} \,\mathrm{d}x', \quad (25)$$

where $\mathcal{I}_0^0$ is the solution of a non-LTE, but non-polarized, radiative transfer equation with partial frequency redistribution. For consistency, one should use $\tilde{R}_0^0$ for the redistribution function. The expression proposed in Eq. (25) is of the last scattering approximation type (see e.g. Faurobert 1987, 1988; Frisch et al. 2009; Anusha et al. 2010). The functions $\tilde{R}_Q^2(x, \mu, x', \mu')$ are defined in Eq. (24), and the coefficients $\tilde{\Gamma}_{Q0}^{20}(\mu')$ can be found in the Appendix. The approximate expressions of $\mathcal{S}_Q^2$ can then be used to calculate approximate values of the components $\mathcal{I}_Q^2$ by solving a simple transfer equation. An Eddington-Barbier approximation may give the right order of magnitude for the $\mathcal{I}_Q^2$, provided the $\mathcal{S}_Q^2$ have a variation with optical depth that is not too far from linear. For PRD, this condition may be far from being satisfied, in particular near line center.

7. Concluding remarks

In this paper we consider Rayleigh scattering with angle-dependent PRD effects in a cylindrically symmetrical medium. Because of this symmetry the Stokes parameters I and Q are also cylindrically symmetrical. That frequency redistribution depends on the scattering angle does not modify this property. We show that I and Q can be decomposed into four components $\mathcal{I}_Q^K$ ($K = Q = 0$, $K = 2, Q = 0, 1, 2$), which are also cylindrically symmetrical. They satisfy a transfer equation similar to the transfer equation for I and Q. In both equations the source terms depend on the ray inclination θ, so one may wonder what has been gained in this decomposition.

First, as shown in Eq. (16), this decomposition allows one to separate the contributions with $Q = 0$ coming from the cylindrically symmetric part of the redistribution function from the contributions corresponding to $Q \neq 0$, which are created by the departure from cylindrical symmetry. These components go to zero at disk center, but they contribute to the polarization near the limb. One can thus easily evaluate errors that are made when departures from cylindrical symmetry in the redistribution matrix are ignored.

Another interesting property of the decomposition in irreducible components is that the source term $\mathcal{S}_Q^K$ in the transfer equations for $\mathcal{I}_Q^K$ has a much simpler form than the source terms S_{I} and S_{Q} for the Stokes parameters I and Q. There is no integration over the azimuthal angle of the incident radiation, and the dependence on the inclination angle θ of the scattered radiation comes from the azimuthal Fourier coefficients $\tilde{r}^{(Q)}(x, \theta, x', \theta')$ only and not from both the frequency redistribution functions and scattering phase matrix. A consequence of this simplification is that the set of integral equations for the four components $\mathcal{S}_Q^K$ can be solved by an ALI (accelerated lambda iteration) method as will be shown in a forthcoming paper. Of course, compared to angle-averaged PRD, the dependence on the inclination angle θ introduces an additional numerical complexity since a fairly large number of values of the inclination angles of the incident and scattered beams are needed for a correct description of the azimuthal Fourier coefficients $\tilde{r}_{\mathrm{II}}^{(Q)}$ and $\tilde{r}_{\mathrm{III}}^{(Q)}$. We recall that ALI methods, which are of the operator splitting type, are quite efficient in solving polarized transfer equations. They have been developed in the past 25 years, first for complete frequency redistribution, but have then been extended to PRD problems with angle-averaged redistribution functions. They have been employed for Rayleigh scattering, but also for the Hanle effect (see the review by Nagendra & Sampoorna 2009). With the decomposition proposed in the present paper (see also HF09), it becomes possible to generalize ALI methods to angle-dependent PRD. Preliminary numerical results confirm that, for Rayleigh scattering, angle-dependent and angle-averaged PRD give essentially the same value of Stokes I. For Stokes Q there are some differences, but they seem to remain small (15% at most for the ratio Q/I). In contrast, for the Hanle effect the angle-averaged and angle-dependent PRD functions yield significantly different Stokes Q profiles (Nagendra et al. 2002).

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Appendix A: Rayleigh phase matrix, spherical tensors, and coefficients $\tilde{\Gamma}_{QQ'}^{KK'}$

The Rayleigh phase matrix can be written as (Stenflo 1994; see also LL04)

$$\begin{aligned}\mathbf{P}_{\mathrm{R}}(\mathbf{\Omega}, \mathbf{\Omega}') &= \mathbf{P}^{(0)} + \mathbf{P}_0^{(2)}(\mu, \mu') \\ &+ \frac{3}{2}\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\mu\mu' \cos(\chi - \chi')\mathbf{P}_1^{(2)}(\mu, \mu') \\ &+ \frac{3}{8}\cos 2(\chi - \chi')\mathbf{P}_2^{(2)}(\mu, \mu'), \qquad (A.1)\end{aligned}$$

where

$$\mathbf{P}^{(0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad (A.2)$$

$$\mathbf{P}_0^{(2)} = \frac{1}{8}\begin{bmatrix} (1-3\mu^2)(1-3\mu'^2) & 3(1-3\mu^2)(1-\mu'^2) \\ 3(1-\mu^2)(1-3\mu'^2) & 9(1-\mu^2)(1-\mu'^2) \end{bmatrix}, \qquad (A.3)$$

$$\mathbf{P}_1^{(2)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad (A.4)$$

$$\mathbf{P}_2^{(2)} = \begin{bmatrix} (1-\mu^2)(1-\mu'^2) & -(1-\mu^2)(1+\mu'^2) \\ -(1+\mu^2)(1-\mu'^2) & (1+\mu^2)(1+\mu'^2) \end{bmatrix}. \qquad (A.5)$$

For $i = 0, 1$ and the reference angle $\gamma = 0$, the $\tilde{\mathcal{T}}_Q^K(i, \theta)$ introduced in Sect. 2 of the text are given by the modulus of the spherical tensors $\mathcal{T}_Q^K(i, \mathbf{\Omega})$ given in LL04 (p. 211). They may be written as

$$\begin{aligned}&\tilde{\mathcal{T}}_0^0(0,\mu) = 1; \quad \tilde{\mathcal{T}}_0^2(0,\mu) = \frac{1}{2\sqrt{2}}(3\mu^2 - 1); \\ &\tilde{\mathcal{T}}_1^2(0,\mu) = -\frac{\sqrt{3}}{2}\mu\sqrt{1-\mu^2}; \quad \tilde{\mathcal{T}}_2^2(0,\mu) = \frac{\sqrt{3}}{4}(1-\mu^2); \\ &\tilde{\mathcal{T}}_0^0(1,\mu) = 0; \quad \tilde{\mathcal{T}}_0^2(1,\mu) = -\frac{3}{2\sqrt{2}}(1-\mu^2); \\ &\tilde{\mathcal{T}}_1^2(1,\mu) = -\frac{\sqrt{3}}{2}\mu\sqrt{1-\mu^2}; \\ &\tilde{\mathcal{T}}_2^2(1,\mu) = -\frac{\sqrt{3}}{4}(1+\mu^2). \qquad (A.6)\end{aligned}$$

The coefficients $\tilde{\Gamma}^{KK'}_{QQ'}(\mu)$ introduced in the text are defined by

$$\tilde{\Gamma}^{KK'}_{QQ'}(\mu) = \sum_{j=0,1} \tilde{\mathcal{T}}^{K}_{Q}(j,\mu)\tilde{\mathcal{T}}^{K'}_{Q'}(j,\mu). \quad (A.7)$$

They satisfy the symmetry $\tilde{\Gamma}^{KK'}_{QQ'} = \tilde{\Gamma}^{K'K}_{Q'Q}$, so only 10 of them are different. They are

$$\tilde{\Gamma}^{00}_{00} = 1;\ \tilde{\Gamma}^{02}_{00} = \frac{1}{2\sqrt{2}}(3\mu^2 - 1);\ \tilde{\Gamma}^{02}_{01} = -\frac{\sqrt{3}}{2}\mu\sqrt{1-\mu^2};$$
$$\tilde{\Gamma}^{02}_{02} = \frac{\sqrt{3}}{4}(1-\mu^2);\ \tilde{\Gamma}^{22}_{00} = \frac{1}{4}(5 - 12\mu^2 + 9\mu^4);$$
$$\tilde{\Gamma}^{22}_{01} = \frac{1}{2}\sqrt{\frac{3}{2}}\mu\sqrt{1-\mu^2}(2-3\mu^2);$$
$$\tilde{\Gamma}^{22}_{02} = \frac{1}{4}\sqrt{\frac{3}{2}}(1-\mu^2)(1+3\mu^2);\ \tilde{\Gamma}^{22}_{11} = \frac{3}{2}\mu^2(1-\mu^2);$$
$$\tilde{\Gamma}^{22}_{12} = \frac{3}{2}\mu^3\sqrt{1-\mu^2};\ \tilde{\Gamma}^{22}_{22} = \frac{3}{8}(1+\mu^4). \quad (A.8)$$

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