Solving the main cosmological puzzles with a generalized time varying vacuum energy

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ABSTRACT

We study the dynamics of the FLRW flat cosmological models in which the vacuum energy density varies with time, \( \Lambda(t) \). In particular, we investigate the dynamical properties of a generalized vacuum model, and we find that under certain circumstances the vacuum term in the radiation era varies as \( \Lambda(z) \propto (1+z)^{P} \), while in the matter era we have \( \Lambda(z) \propto (1+z)^{P} \) up to \( z \approx 3 \) and \( \Lambda(z) \approx \Lambda \) for \( z \leq 3 \). The confirmation of such a behavior would be of paramount importance because it could provide a solution to the cosmic coincidence problem as well as to the fine-tuning problem, without changing the well known (from the concordance \( \Lambda \)-cosmology) Hubble expansion.

Key words. cosmology: theory – methods: analytical

1. Introduction

The analysis of the available high quality cosmological data (supernovae type Ia, CMB, galaxy clustering, etc.) have converged during the last decade towards a cosmic expansion history that involves a spatial flat geometry and a recent accelerating expansion of the universe (Spergel et al. 2007; Davis et al. 2007; Kowalski et al. 2007; Komatsu et al. 2008, and references therein). This expansion has been attributed to an energy component (dark energy) with negative pressure which dominates the universe at late times and causes the observed accelerating expansion. The simplest type of dark energy corresponds to the cosmological constant (see for review Peebles & Ratra 1988; Carvalho et al. 1992; Peebles & Ratra 1998; Carvalho et al. 1992; Overduin & Cooperstock 1998; Bertolami & Martins 2000; Opher & Pellison 2004; Baurer & Clifton 2006; Montenegro & Carneiro 2007, and references therein). In this cosmological paradigm, the dark energy equation of state parameter \( w \equiv p/\rho \) is strictly equal to \(-1\), but the vacuum energy density (or \( \Lambda \)) is not a constant but varies with time. Of course, the weak point in this theory is the unknown functional form of the \( \Lambda(t) \) parameter. Also, in the \( \Lambda(t) \) cosmological model there is a coupling between the time-dependent vacuum and matter (Wang & Meng 2005; Alcaniz & Lima 2005; Carneiro et al. 2008; Basilakos 2009; Basilakos et al. 2009). Indeed, using the combination of the conservation of the total energy with the variation of the vacuum energy, one can prove that the \( \Lambda(t) \) model provides either a particle production process or that the mass of the dark matter particles increases (Basilakos 2009, and references therein). Despite the fact that most of the recent papers in dark energy studies are based on the assumption that the dark energy evolves independently of the dark matter, the unknown nature of both dark matter and dark energy implies that at the moment we cannot exclude the possibility of interactions in the dark sector (e.g., Zimdahl et al. 2001; Amendola et al. 2003; Cai & Wang 2005; Binder & Kremer 2006; Das et al. 2006; Olivares et al. 2008, and references therein).

In this work we attempt to generalize the main cosmological properties of the traditional \( \Lambda \)-cosmology by introducing a time varying vacuum energy, and specifically to investigate whether such models can yield a late accelerated phase of the
2. The time dependent vacuum in the expanding universe

In the context of a spatially flat FLRW geometry the basic cosmological equations are:

\[ \rho_{\text{tot}} = \rho_t + \rho_\Lambda = 3H^2 \]  

and

\[ \frac{d(\rho_t + \rho_\Lambda)}{dt} + 3H(\rho_t + P_t + \rho_\Lambda + P_\Lambda) = 0, \]  

where \( \rho_t \) is the density of the "cosmic" fluid:

\[ \rho_t(t) = \begin{cases} \rho_{\text{m}}(t) & \text{matter era} \\ \rho_\gamma(t) & \text{radiation era} \end{cases} \]

and

\[ P_t(t) = \beta \rho_t = \begin{cases} 0 & \text{matter era} \\ \frac{\rho_\gamma}{3} & \text{radiation era} \end{cases}, \beta = 0, 1/3 \]  

is the corresponding pressure. Also \( \rho_\Lambda \) and \( P_\Lambda \) denote the density and the pressure of the vacuum component respectively. From a cosmological point of view, at an early enough epoch, the observed generalization of the cosmic fluid behaves like radiation \( P_t = \rho_t = \rho_\Lambda/3 \) \( (\beta = 1/3) \), then behaves as matter \( P_t = P_m = 0 (\beta = 0) \) and as long as \( P_\Lambda = -\rho_\Lambda \) it creates an accelerated phase of the cosmic expansion (see below). Notice that in order to simplify our formalism we use geometrical units \( (8\pi G = c = 1) \) in which \( \rho_\Lambda = \Lambda. \) In the present work, we would like to investigate the potential of a time varying \( \Lambda = \Lambda(t) \) parameter to account for the observed acceleration of the expansion of the universe. Within this framework it is interesting to mention that the equation of state takes the usual form of the Friedmann-Lemaître-Robertson-Walker (FLRW) geometry it is interesting to mention that the equation of observed acceleration of the expansion of the universe. Within the framework the idea of the time-dependent vacuum, it is required, in the classical Friedmann-Lemaître-Robertson-Walker (FLRW) geometry) the basic cosmological equations. In these sections we prove further generalizations which treat the time-dependent vacuum models with the traditional \( \Lambda \) cosmology. In this section we treat analytically, the basic cosmological puzzles (the fine-tuning and the cosmic coincidence problem) with the aid of the time varying \( \Lambda(t) \) parameter. Finally, we draw our conclusions in Sect. 6.

The above equation is reduced to \( H(t) = 2(\beta + 1)^{-1}/3t. \) Therefore, in the case of \( \beta = 0 \) (matter era) we get the Einstein-de Sitter model as we should, \( H(t) = 2/3t, \) while for \( \beta = 1/3 \) we trace the radiation phase of the universe i.e., \( H(t) = 1/2t. \) On the other hand, if we consider the case of \( \Lambda(t) \neq 0 \) it becomes evident (see Eq. (5)) that there is a coupling between the time-dependent vacuum and matter (or radiation) component.

Of course, in order to solve the above differential equation we need to define explicitly the functional form of the \( \Lambda(t) \) component. Note that the traditional \( \Lambda = \text{const}. \) cosmology can be described directly by the integration of the Eq. (6) (for more details see Sect. 3.1).

It is worth noting that the \( \Lambda(t) \) scenario has the caveat of its unknown exact functional form, which however is also the case for the vast majority of the dark energy models. In the literature there have been different phenomenological parametrizations which treat the time-dependent \( \Lambda(t) \) function. In particular, Freese et al. (1987) considered that \( \Lambda(t) = 3c_1H^2, \) with the constant \( c_1, \) being the ratio of the vacuum to the sum of vacuum and matter density (see also Arcuri & Wag 1994). Chen & Wu (1990) proposed a different ansatz in which \( \Lambda(t) \propto a^{-2}. \)

Recently, many authors (see for example Ray et al. 2007; Sil & Som 2008, and references therein) have investigated the global dynamical properties of the universe considering that the vacuum energy density decreases linearly either with the energy density or with the square Hubble parameter. Attempts to provide a theoretical explanation for the \( \Lambda(t) \) have also been presented in the literature (see Shapiro & Solá 2000; Babić et al. 2002; Grande et al. 2006; Solá 2008, and references therein). There it was found that a time dependent vacuum could arise from the renormalization group (RG) in quantum field theory. The corresponding solution for a running vacuum is found to be \( \Lambda(t) = c_0 + c_1H^2(t) \) (where \( c_0 \) and \( c_1 \) are constants; Grande et al. 2006) and it can mimic the quintessence or phantom behavior and a smooth transition between the two. Alternatively, Schutzhold (2002) used a different pattern in which the vacuum term is proportional to the Hubble parameter, \( \Lambda(a) \propto H(a) \) (see also Carneiro et al. 2008), while Basilakos (2009) considered a power series form in \( H. \) Note that the linear pattern, \( \Lambda(a) \propto H(a), \) has been motivated theoretically through a possible connection of cosmology with the QCD scale of strong interactions (Schutzhold 2002). In this context it has also been proposed that the vacuum energy density can be defined from a possible link of dark energy with QCD and the topological structure of the universe (Urban & Zhitnitsky 2009a–c).

In this paper we have phenomenologically identified a functional form of \( \Lambda(a) \) for which we can solve the main differential equation (see Eq. (6)) analytically. This is:

\[ \Lambda_{ym}(t) = 3\gamma H^2(t) + 2mH(t) + 3(\beta + 1 - \gamma)e^{2mt} \]  

where the constants \( m \) and \( n \) are included for the consistency of units (see below). Although the above functional form was not motivated by some physical theory but rather phenomenologically by the fact that it provides analytical solutions to the Friedmann equation, its exact form can be physically justified a posteriori within the framework of the previously mentioned theoretical models (see Appendix A).

Using now Eq. (7), the generalized Friedmann’s equation (see Eq. (6)) becomes

\[ H = -\frac{3(\beta + 1 - \gamma)}{2}H^2 + mH + \frac{3n(\beta + 1 - \gamma)}{2}e^{2mt} \]  

where the over-dot denotes derivatives with respect to time. If the vacuum term is negligible, \( \Lambda(t) \longrightarrow 0, \) the solution of the above equation is reduced to \( H(t) = 2(\beta + 1)^{-1}/3t. \) Therefore, in the case of \( \beta = 0 \) (matter era) we get the Einstein-de Sitter model as we should, \( H(t) = 2/3t, \) while for \( \beta = 1/3 \) we trace the radiation phase of the universe i.e., \( H(t) = 1/2t. \) On the other hand, if we consider the case of \( \Lambda(t) \neq 0 \) it becomes evident (see Eq. (5)) that there is a coupling between the time-dependent vacuum and matter (or radiation) component.
and indeed, it is routine to perform the integration of Eq. (8) to obtain (see Appendix B):

$$H(t) = \sqrt{n} e^{\omega t} \coth \left( \frac{3(\beta + 1 - \gamma) \sqrt{n}}{2} S(t) \right)$$  

(9)

where

$$S(t) = \left\{ \begin{array}{ll} (e^{\omega t} - 1)/m & m \neq 0 \\ m & m = 0 \end{array} \right.$$

(10)

while the range of values for which the above integration is valid is $n \in (0, +\infty)$ (for negative $n$ values see the Appendix B). Using now the definition of the Hubble parameter $H \equiv \dot{a}/a$, the scale factor of the universe $a(t)$ evolves with time as

$$a(t) = a_1 \sinh \frac{3(\beta + 1 - \gamma) \sqrt{n}}{2} S(t).$$

(11)

The relevant units of $m \neq 0$ should correspond to time$^{-1}$, which implies that $m \propto H_0$. The parameter $a_1$ is the constant of integration given by

$$a_1 = \left( \frac{\rho_{0\Lambda}}{\rho_0} \right)^{\frac{1}{3\gamma - 1}}$$

(12)

where $\rho_0$ and $\rho_{0\Lambda}$ are the corresponding densities at the present time (for which $a(t_0) \equiv 1$).

In this context, the density of the cosmic fluid evolves with time (see Eq. (1)) as:

$$\rho(t) = 3H^2(t) - \Lambda m(t)$$

(13)

or

$$\rho(t) = 3(1 - \gamma)H^2(t) - 2mH(t) - 3n(\beta + 1 - \gamma)e^{\omega t}.$$

(14)

In the following sections, we investigate thoroughly whether such a generalized vacuum component in an expanding universe allows for a late accelerated phase of the universe, and under which circumstances such an approach provides a viable solution to the fine-tuning problem as well as to the cosmic coincidence problem.

3. The matter + vacuum scenario

In a matter + vacuum expanding universe ($\rho_I \equiv \rho_m$), we attempt to investigate the correspondence of the $a(t)$ pattern with the traditional $\Lambda$-cosmology in order to show the extent to which they compare. In particular, we will prove that the Hubble expansion, provided by the current time-dependent vacuum, is a generalization of the traditional $\Lambda$ cosmology. Note that in the present formalism the matter era corresponds to $\beta = 0$.

3.1. The standard $\Lambda$-cosmology

Let us first investigate the solution for $(\gamma, m) = (0, 0)$ (hereafter $\Lambda_{ym}$ model). The Hubble expansion and the corresponding evolution of the scale factor are

$$H(t) = \sqrt{\Omega_\Lambda H_0} e^{\omega t} \coth \left( \frac{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0}{2m} (e^{\omega t} - 1) \right)$$

(22)

and

$$a(t) = a_1 \sinh \frac{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0}{2m} (e^{\omega t} - 1)$$

(23)

or

$$t(a) = \frac{1}{m} \ln \left[ 1 + \frac{2m}{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0} \sinh^2 \left( \frac{a}{a_1} \right)^{3(1 - \gamma)/2} \right].$$

(24)

Obviously, if $(\gamma, m) \rightarrow (0, 0)$ (or $e^{\omega t} - 1 \approx mt$) then the $\Lambda_{ym}$ model tends to the traditional $\Lambda$ cosmology, which implies that the latter should be considered as a particular solution of the general $\Lambda_{ym}$ model. Thus this limit together with Eq. (12) provides that

$$a_1 = \left( \frac{\Omega_\Lambda}{\Omega_\Lambda} \right)^{\frac{1}{3\gamma - 1}}.$$

(25)

the scale factor of the universe is given by

$$a_I(t) = a_I \sinh \frac{3H_0 \sqrt{\Omega_\Lambda}}{2}$$

(17)

where (see Eq. (12))

$$a_I = \left( \frac{\rho_{0\Lambda}}{\rho_0} \right)^{1/3} = \left( \frac{\rho_m}{\Omega_\Lambda} \right)^{1/3}.$$

(18)

The cosmic time is related with the scale factor as

$$t_I(a) = \frac{2}{3 \sqrt{\Omega_\Lambda} H_0} \sinh^{-1} \left( \frac{\Omega_\Lambda}{\Omega_m} a^{3/2} \right).$$

(19)

Combining the above equations we can define the Hubble expansion as a function of the scale factor:

$$H_I(a) = H_0 \left[ \Omega_\Lambda + \Omega_m a^{-3} \right]^{1/2}.$$

(20)

In principle, $H_0$ and $\Omega_m$ are constrained by the recent WMAP data combined with the distance measurements from the type Ia supernovae (SNIa) and the Baryonic Acoustic Oscillations (BAOs) in the distribution of galaxies. Following the recent cosmological results provided by Komatsu et al. (2009), we fix the current cosmological parameters as $H_0 = 70.5$ km s$^{-1}$ Mpc$^{-1}$ and $\Omega_m = 1 - \Omega_\Lambda = 0.27$. The current age of the universe ($a_I = 1$) is $t_{0I} = 13.77$ Gyr, while the inflection point takes place at

$$t_{\Lambda} = \frac{2}{3 \sqrt{\Omega_\Lambda} H_0} \sinh^{-1} \left( \frac{1}{\sqrt{2}} \right), \quad a_{\Lambda} = \left( \frac{\Omega_m}{2 \Omega_\Lambda} \right)^{1/3}.$$

(21)

Therefore, we estimate $t_{\Lambda/I} \approx 0.51 t_{0I}$ and $a_{\Lambda/I} \approx 0.56$.

Finally, due to the fact that the traditional $\Lambda$ cosmology is a particular solution of the current time varying vacuum models with $(\gamma, m)$ strictly equal to $(0, 0)$, the constant value $n$ is always defined by Eq. (16). That is why all relevant cosmological quantities are parametrized according to $n = \Omega_\Lambda H_0^2$ throughout the paper.

3.2. “The general” $\Lambda(t)$ model

In this section, we examine a more general class of vacuum models with $(\gamma, m) \neq (0, 0)$ (hereafter $\Lambda_{ym}$ model). The Hubble expansion and the corresponding evolution of the scale factor are

$$H(t) = \sqrt{\Omega_\Lambda H_0} e^{\omega t} \coth \left( \frac{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0}{2m} (e^{\omega t} - 1) \right)$$

(22)

and

$$a(t) = a_1 \sinh \frac{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0}{2m} (e^{\omega t} - 1)$$

(23)

or

$$t(a) = \frac{1}{m} \ln \left[ 1 + \frac{2m}{3(1 - \gamma) \sqrt{\Omega_\Lambda} H_0} \sinh^2 \left( \frac{a}{a_1} \right)^{3(1 - \gamma)/2} \right].$$

(24)

Obviously, if $(\gamma, m) \rightarrow (0, 0)$ (or $e^{\omega t} - 1 \approx mt$) then the $\Lambda_{ym}$ model tends to the traditional $\Lambda$ cosmology, which implies that the latter should be considered as a particular solution of the general $\Lambda_{ym}$ model. Thus this limit together with Eq. (12) provides that

$$a_1 = \left( \frac{\Omega_\Lambda}{\Omega_\Lambda} \right)^{\frac{1}{3\gamma - 1}}.$$

(25)
Taking the above expressions into account, the basic cosmological quantities as a function of the scale factor become

\[ H(a) = H_0 \left[ 1 + g(a) \right] \left[ \Omega_\Lambda + \Omega_m a^{-3(1-\gamma)} \right]^{1/2} \]  

(26)

and

\[ \Lambda_{\text{ym}}(a) = 3yH^2 + 2mH + 3H^2_0\Omega_\Lambda(1 - \gamma)[1 + g(a)]^2 \]  

(27)

where

\[ g(a) = \frac{2m}{3 \sqrt{(1 - \gamma)\Omega_\Lambda H_0}} \sinh^{-1} \left( \frac{\Omega_\Lambda}{\Omega_m} a^{3(1 - \gamma)/2} \right). \]  

(28)

If we take \((\gamma, m) = (0, m)\) with \(m \neq 0\) (hereafter mild vacuum model or \(\Lambda_{0m}\)), the corresponding Hubble flow becomes:

\[ H(a) = \left[ 1 + g(a) \right] H_0(a) \]  

(29)

which means that as long as the function \(g(a)\) takes small values \((g(a) \ll 1)\), the \(\Lambda_{0m}\) model has exactly the constant vacuum feature due to \(H(a) \approx H_0(a)\). In this context, utilizing Eq. (27) we simply have

\[ \Lambda_{0m}(a) = 2mH(a) + 3H_0^2\Omega_\Lambda[1 + g(a)]^2. \]  

(30)

Finally, the fact that the vacuum term has units of time\(^{-2}\) implies that the vacuum term is proportional to \(H_0^2\) or the constant \(m\) has to satisfy the following scaling relation: \(m \approx H_0\) (see also Sect. 2). Therefore, in the far future the condition \(m \approx H_0 \neq 0\) represents a super-accelerated expansion of the universe because \(a(t) \propto \exp\left(\frac{\ln H_0}{m}\right)\).

3.3. “The modified” \(\Lambda\) model

Now we consider \((\gamma, m) = (\gamma, 0)\) with \(\gamma \neq 0\) (hereafter \(\Lambda_{0y}\) model). From Eq. (9) we can easily write the corresponding Hubble flow as a function of time

\[ H(t) = \sqrt{\Omega_\Lambda} H_0 \coth \left[ \frac{3(1 - \gamma)\sqrt{\Omega_\Lambda} H_0}{2} t \right]. \]  

(31)

Using now Eqs. ((10), (11)), the scale factor of the universe \(a(t)\) evolves with time as

\[ a(t) = a_1 \sinh \frac{3(1 - \gamma)\sqrt{\Omega_\Lambda} H_0}{2} t \]  

(32)

where

\[ a_1 = \left( \frac{\Omega_m}{\Omega_\Lambda} \right)^{1/(3(1 - \gamma))}. \]  

(33)

Inverting Eq. (32) we estimate the cosmic time:

\[ t(a) = \frac{2}{3(1 - \gamma)\sqrt{\Omega_\Lambda} H_0} \sinh^{-1} \left( \frac{\Omega_\Lambda}{\Omega_m} a^{3(1 - \gamma)/2} \right). \]  

(34)

The corresponding inflection point \((\dot{a}(t) = 0)\) is found to be

\[ t_I = \frac{2}{3(1 - \gamma)\sqrt{\Omega_\Lambda} H_0} \sinh^{-1} \left( \frac{1 - 3\gamma}{2} \right) \]  

(35)

or

\[ a_I = \left[ \frac{3(1 - 3\gamma)\Omega_\Lambda}{2\Omega_m} \right]^{1/(3(1 - \gamma))}. \]  

(36)

which implies that the condition for which an inflection point is present in the evolution of the scale factor is \(\gamma < 1/3\).

As expected, for \(\gamma \ll 1\) the above solution tends to the concordance model, \(a_{0y}(t) \rightarrow a_0(t).\) Now from Eqs. ((31), (32)), using the well known hyperbolic formula \(\sinh^2 x - 1 = \sinh^2 x\), we arrive after some algebra:

\[ H(a) = H_0 \left[ \Omega_\Lambda + \Omega_m a^{3(1 - \gamma)} \right]^{1/2}. \]  

(37)

From this analysis it becomes clear that the Hubble expansion predicted by the \(\Lambda_{0y}\) model extends well beyond that of the usual \(\Lambda\) cosmology. To this end, utilizing Eq. (27) we can obtain the vacuum energy density

\[ \Lambda_{0y}(a) = 3yH^2(a) + 3\Lambda_{0y}^2(1 - \gamma). \]  

(38)

As we have previously mentioned in Sect. 2, the above phenomenological functional form (see Eq. (38)) is motivated theoretically by the renormalization group (RG) in the quantum field theory (Shapiro & Solá 2000; Babić et al. 2002; Solá 2008). Moreover, recent studies (see Grande et al. 2006; and Grande et al. 2009) find that this solution alleviates the cosmic coincidence problem (see Sect. 5.1). Obviously, at late enough times \((a \gg 1)\) the above solution asymptotically reaches the de Sitter regime \(\Lambda \sim H^2\).

4. The radiation+vacuum scenario

In this section, we consider a universe that is spatially flat but contains both radiation and a time varying vacuum term. This crucial period in the cosmic history corresponds to \(\beta = 1/3\). For clarity reasons we re-formulate our approach by using \(\rho_{\text{r}} \equiv \rho_{\text{r}}\) and \(\rho_{\text{v}} \equiv \rho_{\text{v}}\) in the following sections. These restrictions imply that

\[ \frac{\rho_{\text{w0}}}{\rho_{\text{x0}}} = \frac{\rho_{\text{r0}}}{\rho_{\text{x0}}} = \frac{\Omega_{\text{r}}}{\Omega_\Lambda}, \]  

where, \(\Omega_{\text{r}} \approx 10^{-4}\) is the radiation density parameter at the current epoch derived by the CMB data (see Komatsu et al. 2009). Within this context, based on Eqs. (7), (11), and (12) we present briefly the following cosmological situations:

- **radiation+constant vacuum** \((\gamma, m) = (0, 0)\): The scale factor is

\[ a(t) = \left( \frac{\Omega_\Lambda}{\Omega_\Lambda} \right)^{\frac{1}{3}} \sinh \left( \frac{\sqrt{\Omega_\Lambda} H_0 t}{\Omega_\Lambda} \right). \]  

(39)

Owing to the fact that in this period \(t \ll 1\), the above solution reduces to the following simple analytic approximation:

\[ a(t) \approx (2\sqrt{\Omega_\Lambda} H_0)^{1/2} \text{ with } H(t) \equiv \frac{\dot{a}}{a} \approx \frac{1}{2t}. \]  

(40)

- **radiation+general vacuum** \((\gamma, m) \neq (0, 0)\): this general scenario provides

\[ a(t) = \left( \frac{\Omega_\Lambda}{\Omega_\Lambda} \right)^{\frac{1}{3}} \sinh \left( \frac{2\gamma_1 \sqrt{\Omega_\Lambda} H_0 (e^{mt} - 1)}{m} \right). \]  

(41)

where \(\gamma_1 = 1 - 3\gamma/4\). The vacuum component as a function of time (see Eq. (7)) is

\[ \Lambda_{\text{ym}}(t) \approx \frac{4(1 - \gamma_1)}{4\gamma_1^2 t^2} + \frac{m}{\gamma_1 t}. \]  

(42)
or
\[
\Lambda_{\gamma m}(a) \approx \frac{4(1 - \gamma) \Omega_\Lambda H_0^2}{a^{3\gamma}} + \frac{2m \sqrt{\Omega_m H_0^2}}{a^{3\gamma}}.
\] (43)

It is very interesting that during the radiation epoch \(\Lambda_{\gamma m}(a) \propto a^{-3\gamma}\). For small values of \(\gamma\) or \(\gamma_1 = O(1)\), the latter relation implies that as long as the scale factor tends to zero the vacuum term moves rapidly to infinity (see Sect. 6). In the case of \((\gamma, m) = (0, m)\) (or \(\gamma_1 = 1\), the vacuum term (see Eqs. (42) and (43)) varies with time as
\[
\Lambda_{\gamma m}(a) \approx \frac{2m \sqrt{\Omega_m H_0^2}}{a^{3\gamma}}. \quad (44)
\]

Now the vacuum component evolves as \(\Lambda_{\gamma 0}(a) \propto a^{-2}\), in agreement with the Chen & Wu (1990) model.

- **radiation+modified vacuum**: \((\gamma, m) = (0, 0)\), \(\gamma \neq 0\): in this cosmological model we have
\[
a(t) = \left(\frac{\Omega_\Lambda}{\Omega_\Lambda(t)}\right)^{\frac{1}{2\gamma}} \sinh \left(\frac{2\gamma_1 \sqrt{\Omega_\Lambda H_0 t}}{a}\right)
\] (45)

where \(\gamma_1 = 1 - 3\gamma/4\). The approximate solution now becomes
\[
a(t) \approx (2\gamma_1 \sqrt{\Omega_\Lambda H_0 t})^{1/2\gamma_1} \quad \text{with} \quad H(t) \approx \frac{1}{2\gamma_1 t}.
\] (46)

The vacuum component (see Eq. (7)) evolves with time as
\[
\Lambda_{\gamma 0}(t) \approx \frac{4(1 - \gamma_1)}{4\gamma_1} \frac{\Omega_\Lambda H_0^2}{a^{3\gamma_1}} \quad \text{or}
\]
\[
\Lambda_{\gamma 0}(a) \approx \frac{4(1 - \gamma_1) \Omega_\Lambda H_0^2}{a^{3\gamma_1}} \approx \Lambda_{\gamma m}(a).
\] (48)

Obviously, for \(a \to 0 (\gamma_1 \approx O(1))\) the vacuum energy density goes rapidly to infinity.

### 5. Tackling the cosmological puzzles

As we have stated already in the introduction, there is a possibility for the vacuum energy to be a function of time rather than having a constant value. Therefore, in this section we compare the cosmic phases of the \(\Lambda(t)\) scenarios (described in the previous sections) and the concordance \(\Lambda\)-cosmology. The aim here is to investigate the consequences of such a comparison on the basic cosmological puzzles, namely the cosmic coincidence problem and fine-tuning problem.

#### 5.1. The coincidence problem

In order to investigate the coincidence problem we define the time-dependent proximity parameter of \(\rho_{\Lambda 0}(a)\) (see Eq. (14)) and \(\rho_{\Lambda 0}(a)\) (see Egan & Lineweaver 2008, and references therein):
\[
r(a) \equiv \min \left(\frac{\rho_{\Lambda 0}(a)}{\rho_{\Lambda 0}(a)}, \frac{\rho_{\Lambda 0}(a)}{\rho_{\Lambda 0}(a)} \right) \quad (49)
\]

where in this work we use \(\rho_{\Lambda 0}(a) \equiv \Lambda(a)\) (see Eq. (7)). If the two densities differ by many orders of magnitude then \(r \approx 0\). If on the other hand the two densities are equal the proximity parameter is \(r \approx 1\). The current observational data shows that the proximity parameter at the present time \((a = 1)\) is \(r_0 = \rho_{\Lambda 0}(1) = \frac{2.5}{H_0} \approx 0.37\). A cosmological model may therefore suffer from the so called coincidence problem if its proximity parameter is close to zero before the inflection point, \(r(a < a_1) \approx 0\). As an example, for the traditional \(\Lambda\)-cosmology we have \(r(a < a_1) \approx O(1)\) then this model possibly does not suffer from the cosmic coincidence problem.

In particular, suppose that we have a cosmological model which accommodates a late time accelerated expansion and contains \(n\)-free parameters, described by the vector \(\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_n)\). The main question that we should address here is the following: “what is the range of input \((\epsilon_1, \epsilon_2, ..., \epsilon_n)\) parameters for which the coincidence problem can be avoided?” Below we implement the following tests.

(i) We find the range of the free parameters of the considered cosmological model that implies \(r \approx r_0\) for at least two different epochs, one of which is precisely the present epoch.

(ii) We know that for epochs between the inflection point and the present time \(a_1 \leq a \leq 1\), the proximity parameter is \(r(a) \geq r_0\). As an example, for the traditional \(\Lambda\)-cosmology we have \(r(a) \approx 0.37\). Thus, the goal here is to define the range of the free parameters in which at least a second region with \(r(a < a_1) \geq r_0\) occurs before the inflection point \((a < a_1)\).

(ii) Once steps (i) and (ii) are accomplished, we finally check whether the remaining parameters fit the recent Smnla data by performing a standard \(\chi^2\) minimization. In this work, we use the so called Union08 sample of 307 supernovae of Kowalski et al. (2008). In particular, the \(\chi^2\) function can be written as:
\[
\chi^2(\epsilon) = \sum_{j=1}^{307} \frac{[\mu^0_j(\epsilon_j, \epsilon) - \mu^0_j(\epsilon_j, \epsilon)]^2}{\sigma_j^2}.
\] (50)

where \(a_j = (1 + z_j)^{-1}\) is the observed scale factor of the universe, \(z_j\) is the observed redshift, \(\mu\) is the distance modulus \(\mu = m - M = 5\log d_L + 25\) and \(d_L(\epsilon, \epsilon)\) is the luminosity distance, given by
\[
d_L(\epsilon, \epsilon) = \frac{c}{H_0} \int_0^1 \frac{dx}{x^2E(x)}.
\] (51)

where \(\epsilon\) is the vector containing the unknown free parameters and \(c\) is the speed of light \((=1\) here). A cosmological model for which the present tests are successfully passed should not suffer from the coincidence problem. Below we apply our tests to the current \(\Lambda(t)\) cosmological models (see also Table 1).

#### 5.2. The modified vacuum model with \(\epsilon = (\gamma, 0, ..., 0)\): We sample the unknown \(\gamma\) parameter as follows: \(\gamma \in [-1, 1/3]\) in steps of \(10^{-4}\). We confirm that in the range of \(\gamma \in [0.004, 0.03]\) the \(\Lambda_{\gamma 0}\) model satisfies both the criteria (i); and (ii) respectively. Also, we verify that this range of values fits the Smnla data, \(\chi^2_{\text{min}}/\text{d.o.f.} = 1.01\) very well. Notice that for \(\gamma > 0.03\) the criterion (i) is not satisfied. As an example, in the upper panel of Fig. 1 we present the evolution of the proximity parameter for \(\gamma = 0.004\) (solid line) and 0.03 (dashed line). It becomes

\footnote{Note that from a theoretical viewpoint the predicted value of the \(\gamma\) parameter is \(\gamma|_{\text{theory}} = \frac{1}{1 + 2m}\), where \(M_P\) is the Planck mass and \(M\) is an effective mass parameter representing the average mass of the heavy particles of the Grand Unified Theory (GUT) near the Planck scale, after taking into account their multiplicities. In the case of \(M \approx M_P\), we can derive an upper limit of \(|\gamma| \leq 1/12m\) (for more details see Basilakos et al. 2009).}
clear is that for $0.1 \leq a \leq 0.34$ (or $2 \leq z \leq 10$) the vacuum density is low enough ($\rho \sim 0$) to allow galaxies and galaxy clusters to form (Garriga et al. 1999; Basilakos et al. 2009).

From now on, we will utilize $\gamma \approx 0.004$ that corresponds to the best fit parameter. Thus it becomes clear that the $\Lambda_0$ model passes the above criteria and does not suffer from the cosmic coincidence problem.

- The mild vacuum model with $\epsilon = (0, m, \ldots, 0)$: In this cosmological model we find that for $m \geq 0.17H_0$, the corresponding age of the universe is $t_0 \leq 12.7$ Gyr. The latter appears to be ruled out by the ages of the oldest known globular clusters (Krauss 2003; Hansen et al. 2004). Using this constraint the unknown $m$ parameter has an upper limit of 0.17$H_0$, and we perform the following sampling: $m \in [5 \times 10^{-3}H_0, 0.17H_0]$ in steps of $5 \times 10^{-4}H_0$. Within this range, we find that the required (i) and (ii) criteria are not satisfied. Thus, the $\Lambda_0$ cosmological model suffers from the coincidence problem. The resulting minimization provides: $m = 2.4 \times 10^{-3}H_0$ with $x_{\text{min}}^2 / \text{d.o.f.} \approx 1.01$. That the errors of the fitted parameters represent $1\sigma$ uncertainties.

- The general vacuum model with $\epsilon = (\gamma, m, \ldots, 0)$: This cosmological model contains 2 free parameters. Using the sampling mentioned previously, we obtain that our main criteria for the $\Lambda_v$ scenario are fulfilled for $\gamma \in [0.004, 0.02]$, $m \in [1.4 \times 10^{-3}H_0, 9 \times 10^{-3}H_0]$ with $x_{\text{min}}^2 / \text{d.o.f.} \in [1.01, 1.02]$. Throughout the rest of the paper we will use the best fit parameters. These are: $m = 2.8 \times 10^{-3}H_0$ and $\gamma = 0.004$.

In addition to the SnIa data, we further check our statistical results using the dimensionless distance to the surface of the last scattering $R = 1.71 \pm 0.019$ (Komatsu et al. 2009), and the baryon acoustic oscillation (BAO) distance at $z = 0.35$, $\Lambda = 0.469 \pm 0.017$ (Eisenstein et al. 2005; Padmanabhan et al. 2007). We find that the above results remain unaltered.

5.2. The cosmic evolution – fine-tuning problem

Using now our best fit parameters for the different kind of vacuums, we present in Fig. 1 the corresponding normalized energy densities, vacuum $\Lambda(a)/H_0^2$, matter $\rho_m(a)/H_0^2$ and radiation $\rho_b(a)/H_0^2$ as a function of the scale factor. We verify that both the $\Lambda_0$ (solid line) and $\Lambda_m$ (open stars) solutions are models that provide large values for the vacuum energy density at early epochs, in contrast with the usual $\Lambda$ cosmology (open circles) in which the vacuum energy density remains constant everywhere. Also, within a Hubble time ($0 < a \leq 1$) and for each ($\gamma, m$) pair we find the well known cosmic behavior for the matter density $\rho_m(a) \propto a^{-3}$ and the radiation density $\rho_b(a) \propto a^{-4}$ respectively. As an example, in Fig. 1 we present the density evolution of the cosmic fluid for the $\Lambda_0$ cosmological model: matter (long dashed line) and radiation (dashed line). For a comparison we also plot the predictions of the traditional $\Lambda$ cosmology: matter (open squares) and radiation (open triangles). From Fig. 1 it becomes clear that the radiation-matter equality takes place close to $a_{eq} \approx 3.7 \times 10^{-4} \sim \Omega_b/\Omega_m$. For these vacuum models where $m \neq 0$ ($\Lambda_m$ and $\Lambda_v$), we verify that the behavior of their cosmic fluid (matter+radiation) deviates from the $\Lambda_0$ solution in the far future ($t \gg t_0$), since the exponential term $e^{\epsilon t}$ in Eq. (14) plays an important role in the global dynamics (see Sect. 3.4 and below).

In particular, for the $\Lambda_0$, vacuum scenario (the same behavior holds for $\Lambda_m$) we have revealed the following phases: (a) at early enough times ($a < a_{eq}$) the scale factor of the universe tends to its minimum value, $a \rightarrow 0$, which means that the vacuum energy density initially moves quickly to infinity. So, as long as the scale factor increases the vacuum energy roll down rapidly as $\Lambda_0(a) \propto a^{-4\gamma_1}$ (where $\gamma_1 \sim 0(1)$). This evolution may solve the fine-tuning problem. Indeed, for $\gamma \in (0, 1/3)$, we find that prior to the inflation point ($a_{inf} \approx 10^{-32}$ s), the vacuum energy density divided by its present value is $\Lambda(t_{inf})/\Lambda(t_0) \approx 10^{102}$. Finally, if we consider that the functional form of $\Lambda(a) \propto a^{-4\gamma_1}$ is still valid during the Planck time ($t_{pl} \approx 10^{-43}$ s), then $\Lambda(t_{pl})/\Lambda(t_0) \approx 10^{124}$ (see the last rows in Table 1); and (b) in the early era the vacuum density continues to roll down but with a different power law $\Lambda_0(a) \propto a^{-(3\gamma_1 - \gamma)}$ and it tends to a constant value close to $a \sim 0.25$ (close to 3). Finally, for $a \geq 0.25$ the vacuum energy density is effectively frozen to the nominal value, $\Lambda_0(a) \approx \Lambda = 3\Omega_m H_0^2$, which implies that the considered time varying vacuum model explains why the matter energy density and the dark energy density are
of the same order prior to the present epoch. The moment of
radiation-vacuum equality occurs at \( a_r \approx 0.1 \approx (\Omega_b/\Omega_{\Lambda})^{1/4} \).
Similarly, the moment of matter-vacuum equality takes place at
\( a_{\text{meq}} \approx 0.72 \approx (\Omega_m/\Omega_{\Lambda})^{1/3} \). From the observational viewpoint,
in order to investigate whether the vacuum energy density
follows the above evolution, we need a robust cosmological probe
at redshifts \( z \geq 3 \). In a recent paper (Basilakos et al. 2009) we
have investigated how realistic it would be to detect differences
among the vacuum models. In particular, we have found that the
Sunayev-Zeldovich cluster number-counts (as expected from the
survey of the South Pole Telescope, Staniszewski et al. 2009,
and the Atacama Cosmology Telescope, Hincks et al. 2009)
indicate that we may be able to detect significant differences among
the vacuum models in the redshift range \( 2.5 \leq z \leq 3 \) at a level of
\( \sim 6\% - 12\% \), which translates in number count differences over the
whole sky of \( \sim 100 \) clusters (see Fig. 6 in Basilakos et al. 2009).

Finally, in Fig. 1 we also show the evolution of the mild vac-
uum model \( \Lambda_{0\text{m}}(a) \) (dot line), in which \( \gamma = 0 \). Briefly, we get the follow-
ing dependence: (a) \( \Lambda_{0\text{m}} \propto a^{-2\gamma} \) for \( a \leq a_{\text{meq}} \), while we
estimate that \( \Lambda_{0\text{m}}(t_{\text{int}})/\Lambda_{0\text{m}}(t_0) \sim 10^{51} \) and \( \Lambda_{0\text{m}}(t_0)/\Lambda_{0\text{m}}(t_0) \sim 10^{63} \); (b) between \( a_{\text{meq}} \leq a \leq 0.08 \) we have \( \Lambda_{0\text{m}} \propto a^{-3/2} \); and (c) for \( a \geq 0.08 \) the \( \Lambda_{0\text{m}} \) becomes constant.

We would like to end this section with a discussion of the evolution
of the scale factor. In particular, our approach provides an
account of how the scale factor in the \( \Lambda_{0\text{m}} \) model appears in the upper
panel of Fig. 2 as the solid line, which mimics the corresponding
scale factor of the \( \Lambda \) cosmological model (open points), despite
the fact that they describe the vacuum term differ-
cently. On the other hand, in the bottom panel of Fig. 2 we present the corre-
sponding deviation \( (\Delta a_{\text{m}} - a_{\Lambda_0})/a_{\Lambda_0})\% \), of the growth factors. It
becomes evident that within the range \( 0 < H_d < 5 \) the evolu-
tion of the scale factor provided by the \( \Lambda_{0\text{m}} \) model closely re-
sembles, the corresponding scale factor of the \( \Lambda_{0\text{m}} \) model (the
same result holds also for the \( \Lambda \) cosmology). However, for mod-
els where \( m \neq 0 \) the situation is somewhat different in the far
future. Indeed, for \( H_d \geq 5 \) the \( \Lambda_{0\text{m}} \) (or \( \Lambda_{0\text{m}} \)) cosmological scenari-
ods deviates from the \( \Lambda_{0\text{m}} \) model by \( \sim 5\% - 10\% \). Thus,
we conclude that the models with \( m \neq 0 \) give a super-accelerated
expansion of the universe in the far future with respect to those
vacuum models where \( m = 0 \).

6. Conclusions

The reason why a cosmological constant leads to a late cosmic
acceleration is because it introduces in Friedmann’s equation a
component which has an equation of state with negative pres-
sure, \( P_\Lambda = -\rho_\Lambda \). In the last decade the so called concordance
\( \Lambda \)-cosmology is considered to be the model which describes
the cosmological properties of the observed universe because it
fits the current observational data accurately. However, the
traditional \( \Lambda \) cosmology suffers from two fundamental puzzles.
These are the fine-tuning and the cosmic coincidence problems.
An avenue through which the above cosmological problems
could be solved is via the time varying vacuum energy which has
the same equation of state as the traditional \( \Lambda \)-cosmology.

Below we wish to present the basic assumptions and conclu-
sions of our analysis.

– We are assuming a time varying vacuum pattern in which
the specific functional form is: \( \Lambda(t) = 3\gamma H^2(t) + 2mH(t) +
3\beta (1 + 1 - \gamma)e^{2mt} \), where \( \beta = 0 \) (matter era) or \( \beta = 1/3 \)
(radiusation era), \( n = 3\Omega_\Lambda H^2_0 \), while the pair \( (\gamma, m) \) character-
izes the different types of vacuum. Note that the above func-
tional form includes the effect of the quantum field theory
(for \( m = 0 \)) (Shapiro & Solá 2000; Babić et al. 2002; Grande
et al. 2006; Solá 2008) and it also extents recent studies (see
for example Ray et al. 2007; Carneiro et al. 2008; Sil & Som
2008; Basilakos 2009). In this context we can easily prove that
the cosmological constant is a particular solution of the
general vacuum, that \( (\gamma, m) = (0, 0) \). We have also investi-
gated the following models: (a) modified vacuum in which
\( (\gamma, m) = (0, 0) \), mild vacuum with \( (\gamma, m) = (0, m) \) and general
vacuum in which \( (\gamma, m) \neq (0, 0) \). In this framework we find that
the time evolution of the basic cosmological func-
tions (scale factor and Hubble flow) is described in terms of
hyperbolic functions which can accommodate a late time ac-
celerated expansion equivalent to the standard \( \Lambda \) model.

– We find that within the framework of either the modified or
general vacuum models the corresponding vacuum term in
the radiation era varies as \( \Lambda(a) \propto a^{-3} \) while in the
matter-dominated era we have \( \Lambda(a) \sim a^{-1} \), up to \( z \sim a^{-1} \) while
\( \Lambda(a) \approx \Lambda = 3\Omega_\Lambda H^2_0 \) for \( z \leq 3 \). This vacuum mech-
anism simultaneously sets (a) the value of \( \Lambda \) at the present
time to its observed value; and (b) at the Planck time to
a value which is \( 10^{24} \) at its present value \( (\Lambda(t_0)/\Lambda(t_0)) \sim
10^{24} \). Additionally, we verify that our models appear to
overcome the cosmic coincidence problem. Finally, in or-
der to confirm the above results, we need to define a robust
cosmological probe at high redshifts (\( z \geq 3 \)). In Basilakos
et al. (2009) we propose that the future cluster surveys based
on the Sunayev-Zeldovich detection method will possibly
distinguish the closely resembling vacuum models at high
redshifts.

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Appendix A

In this appendix we provide a physical justification of the func-
tional form of \( \Lambda(a) \) used in our paper. As we have already
mentioned in section two the vacuum energy density can take
several forms, depending on the theoretical approach. Briefly,
the renormalization group from the quantum field theory introduces only even powers of $H$ out of which the $H^2$ is the leading term (Grande et al. 2006; Solá 2008, and references therein). In another vein, the aforementioned possibility that the vacuum energy could be evolving linearly with $H$ has been motivated theoretically through a possible connection of cosmology with the QCD scale of strong interactions (see Schutzhold 2002; Carneiro et al. 2008). In this framework it has also been proposed a possible link of dark energy with QCD and the topological structure of the universe (Urban & Zhitnitsky 2009). The simplest approach therefore to introduce the effects of the DE is to consider a potential $V(\phi) \approx V_0 + m^2 \phi^2/2$, where the homogeneous scalar field $\phi$ obeys the Klein-Gordon equation. It is well known that for $H \approx \text{const.}$ the corresponding $\phi$ evolves with time as $\phi(t) = \phi_0 e^{\omega t}$ (where in general $m$ is a complex number). In this context, one would expect that the functional form of the $\Lambda(t)$ should contain also an additional term of $\phi^2(t) \propto e^{2\omega t}$ in order to take into account the possible link between dark energy and QCD.

All the above options have merits and demerits. In the current paper the functional form of $\Lambda(t)$ is motivated by a combination of the above possibilities, namely $H^2(t)$ [RG], $H(t)$ [QCD] and $e^{2\omega t}$ (dark energy). In particular, the linear combination reads as follows:

$$\Lambda(t) = n_1 H^2(t) + n_2 H(t) + n_3 e^{2\omega t}$$

which obviously is very similar to the original (phenomenologically selected) form of $\Lambda(t)$ (Eq. (7)). Finally, from a mathematical point of view we can select the constants $n_1$, $n_2$ and $n_3$ to match those presented in the original Eq. (7).

### Appendix B

With the aid of the differential equation theory we present solutions that are relevant to our Eq. (8). If we have a Riccati differential equation which is given by the following special form

$$\frac{dy}{dx} = f(x)y^2(x) + my(x) - ne^{mx}f(x)$$

then the general solution of Eq. (52) for $n > 0$ is

$$y(x) = \sqrt{n}e^{mx}\coth\left[-\sqrt{n} \int_0^x e^{mu}f(u)\,du\right].$$

On the other hand, if $n < 0$ then the solution of Eq. (52) is

$$y(x) = \sqrt{|n|}e^{mx}\cot\left[-\sqrt{|n|} \int_0^x e^{mu}f(u)\,du\right].$$

Note that in our formulation the function $f(x)$ is a constant: $f(x) = -3(\beta + 1 - \gamma)/2$. Also, $n < 0$ implies that $\Omega_m > 1$ (or $\Lambda < 0$).