A&A 503, 213–223 (2009)
DOI: 10.1051/0004-6361/200912399

Astronomy & Astrophysics

On the nature of kink MHD waves in magnetic flux tubes

M. Goossens[1], J. Terradas[1], J. Andries[1,2,⋆], I. Arregui[3], and J. L. Ballester[3]

[1] Centre Plasma Astrophysics and Leuven Mathematical Modeling and Computational Science Center, Katholieke Universiteit Leuven, 3001 Leuven, Belgium
e-mail: Marcel.Goossens@wis.kuleuven.be
[2] Centre for Stellar and Planetary Astrophysics, Monash University, 3800 Victoria, Australia
e-mail: Jesse.Andries@wis.kuleuven.be
[3] Departament de Física, Universitat de les Illes Balears, Spain
e-mail: Inigo.Arregui@uib.es

Received 28 April 2009 / Accepted 4 June 2009

ABSTRACT

Context. Magnetohydrodynamic (MHD) waves are often reported in the solar atmosphere and usually classified as slow, fast, or Alfvén. The possibility that these waves have mixed properties is often ignored.

Aims. The goal of this work is to study and determine the nature of MHD kink waves.

Methods. This is done by calculating the frequency, the damping rate and the eigenfunctions of MHD kink waves for three widely different MHD waves cases: a compressible pressure-less plasma, an incompressible plasma and a compressible plasma which allows for MHD radiation.

Results. In all three cases the frequency and the damping rate are for practical purposes the same as they differ at most by terms proportional to $(k_z R)^2$. In the magnetic flux tube the kink waves are in all three cases, to a high degree of accuracy incompressible waves with negligible pressure perturbations and with mainly horizontal motions. The main restoring force of kink waves in the magnetised flux tube is the magnetic tension force. The total pressure gradient force cannot be neglected except when the frequency of the kink wave is equal or slightly differs from the local Alfvén frequency, i.e. in the resonant layer.

Conclusions. Kink waves are very robust and do not care about the details of the MHD wave environment. The adjective fast is not the correct adjective to characterise kink waves. If an adjective is to be used it should be Alfvénic. However, it is better to realize that kink waves have mixed properties and cannot be put in one single box.

Key words. magnetohydrodynamics (MHD) – waves – Sun: magnetic fields

1. Introduction

The last decade has seen an avalanche of observations of magnetohydrodynamic (MHD) waves in the solar atmosphere. It is clear now that MHD waves are ubiquitous in the solar atmosphere. This has triggered new theoretical research for explaining and interpreting the observed properties. A special point of attention is whether these MHD waves are slow, fast or Alfvén waves. Apparently, a large fraction of the solar MHD waves community favours very clear cut divisions and does not seem to appreciate the possibility of MHD waves with mixed properties. The transverse oscillations observed in coronal loops (see for example Aschwanden et al. 1999; Nakariakov et al. 1999), often triggered by a nearby solar flare, are interpreted as fast kink MHD waves. A striking property of these transverse waves is their fast damping with damping times of the order of 3–5 periods. Resonant absorption is up to today the only damping mechanism that offers a consistent explanation of this rapid damping. Resonant absorption relies on the transfer of energy from a global MHD wave to local resonant Alfvén waves. If this mechanism is indeed operational then this means that the observed transverse oscillations have Alfvénic properties in at least part of the oscillating loop. The debate on the nature of MHD waves in the solar atmosphere has gained new momentum when several groups, e.g. De Pontieu et al. (2007), Okamoto et al. (2007), Tomczyk et al. (2007) reported the detection of Alfvén waves in HINODE observations. Van Doorsselaere et al. (2008) compared fast kink MHD waves to torsional Alfvén waves and concluded that the HINODE observations can be explained in terms of fast kink MHD waves.

This paper will not try to explain the HINODE observations. Its aim is to determine the nature of kink MHD waves on magnetic flux tubes. We have no doubt about the explanation of transverse oscillation of coronal loops in terms of kink MHD waves. MHD waves with their azimuthal wave number equal to 1, i.e. $m = 1$, are the only motions that displace the axis of the loop and the loop as a whole. It is not clear to us on what arguments the use of the adjective fast is based. As far as we know there has not been any study of the forces that drive the kink waves in coronal loops. If these waves are fast, then the pressure gradient force should be, in general, the dominant force compared to the magnetic tension force. We have to admit that we also have used the adjective fast without a solid argument in favour of this classification. Our aim is to understand the spatial structure of the motions in the kink waves.

An MHD wave on an axi-symmetric 1-D cylindrical plasma equilibrium is characterised by two wave numbers, the azimuthal wave number m, and the axial wave number k_z. In addition modes can have a different number of nodes in the radial direction and this number can be used to further classify the modes.

⋆ Postdoctoral Fellow of the National Fund for Scientific Research–Flanders (Belgium) (F.W.O.-Vlaanderen).

Article published by EDP Sciences

Hence, an MHD eigenmode is characterised by three numbers. The azimuthal wave number is an integer. The modes with $m = 0$ are usually called sausage (slow and fast) or torsional (Alfvén). The modes with $m = 1$ are named kink and the modes with $m \geq 2$ are flute modes. The axial wave number k_z can be discretised as $k_z = n\frac{\pi}{L}$ with L the length of the loop and $n = 1, 2, \ldots$ Depending on the dimensions of the equilibrium model there can be more than one radial eigenmode for a given couple (m, k_z). In what follows we shall study linear MHD waves that are superimposed on a flux tube in static equilibrium with a straight and constant axial magnetic field. This equilibrium model contains the essential physics of the problem and allows a relatively straightforward mathematical analysis of the MHD waves. MHD waves have been investigated in previous studies. However, these studies almost exclusively focused on the frequencies of the MHD waves and in addition they were in most cases restricted to real frequencies. For example, the paper by Edwin & Roberts (1983), which is often referred to in the solar MHD wave community, is limited to real frequencies and does not give any information on the eigenfunctions beyond the fact that they can be expressed in terms of Bessel functions. Complex frequencies were considered by Spruit (1982) and by Cally (1985). Cally (1985) rightfully pointed out that essential physics is lost by restricting the analysis to real frequencies. Spruit (1982) is an exception to the rule in that he discussed the eigenfunctions. In short, the eigenfunctions of MHD waves are not well documented even for uniform equilibrium states and definitely not for non-uniform equilibrium states.

In order to illustrate the nature of MHD kink waves we shall study them for three widely different MHD waves cases. The first case deals with compressible MHD waves of a pressureless plasma on a high density flux tube embedded in a low density magnetic plasma exterior. The assumption of a pressureless plasma removes the slow waves from the analysis and the velocity and displacement have no axial component. The density being higher in the flux tube than in the exterior means that according to common wisdom the MHD waves are propagating (body waves) in the flux tube and evanescent (surface wave) in the exterior. The second case deals with MHD waves of an incompressible plasma. The assumption of incompressibility removes the fast waves. The waves are evanescent (surface waves) both in the interior and exterior of the flux tube and this behaviour is independent of the density being higher or lower in the interior. In the third case we consider a situation where the compressible MHD waves are propagating in the exterior. In that case the MHD waves are damped by MHD radiation, but the kink MHD wave hardly feels this wave leakage.

2. Propagating kink MHD waves on dense pressureless flux tubes

2.1. Equations for MHD waves for a pressureless plasma cylinder

The equations for linear MHD waves on a 1-dimensional pressureless cylinder with a straight field can be obtained from the more general equations by e.g. Appert et al. (1974); Sakurai et al. (1991a); Goossens et al. (1992), Goossens et al. (1995) by putting the local speed of sound v_S and the azimuthal component of the equilibrium magnetic field B_φ equal to 0. The resulting equations are (see e.g. Goossens 2008)

$$\begin{aligned}
&D\frac{\mathrm{d}(r\xi_r)}{\mathrm{d}r} = -C_2 r P',\\
&\rho\left(\omega^2 - \omega_A^2\right)\xi_r = \frac{\mathrm{d}P'}{\mathrm{d}r},\\
&\rho\left(\omega^2 - \omega_A^2\right)\xi_\varphi = \frac{\mathrm{i}m}{r}P',\\
&\xi_z = 0,\\
&\nabla\cdot\boldsymbol{\xi} = \frac{-P'}{\rho v_A^2}.
\end{aligned} \tag{1}$$

$\boldsymbol{\xi}$ is the Lagrangian displacement and P' is the Eulerian perturbation of total pressure. The coefficient functions D and C_2 in (1) are

$$\begin{aligned}
D &= \rho v_A^2\omega^2\left(\omega^2 - \omega_A^2\right),\\
C_2 &= \omega^2\left(\omega^2 - \omega_A^2 - \frac{m^2}{r^2}v_A^2\right).
\end{aligned} \tag{2}$$

ω is the frequency and m and k_z are the azimuthal and axial wave numbers, respectively. They define the dependence of the perturbed quantities on time t and on the ignorable spatial variables (φ, z) as

$$\exp[\mathrm{i}(m\varphi + k_z z - \omega t)]. \tag{3}$$

ω_A and v_A are the local Alfvén frequency and the local Alfvén velocity respectively. They are defined as

$$\omega_A^2 = k_z^2 v_A^2,\quad v_A^2 = \frac{B^2}{\mu\,\rho},\quad \omega_C^2 = \frac{v_S^2}{v_S^2+v_A^2}\omega_A^2,\quad v_S^2 = \frac{\gamma p}{\rho}. \tag{4}$$

For completeness we have also listed the expression for the local cusp frequency ω_C and the local speed of sound v_S. For a pressureless plasma these two quantities vanish and as a consequence the slow modes are removed from the system since $\xi_z = 0$.

The nature of an MHD wave is determined by the competition of the restoring forces which are the force due to the total (gas plus magnetic) pressure gradient and the magnetic tension force. In ideal MHD we can obtain the following expression for the Lorentz force, the magnetic tension force $\boldsymbol{\Pi}$ and the magnetic pressure force $\nabla_\perp P'$ in linear MHD waves on a background with a constant magnetic field

$$\begin{aligned}
\frac{1}{\mu}(\nabla\times\boldsymbol{B}')\times B\;\mathbf{1}_z &= -\left(\frac{\mathrm{d}P'}{\mathrm{d}r} + \rho\omega_A^2\xi_r\right)\mathbf{1}_r\\
&\quad -\left(\mathrm{i}\frac{m}{r}P' + \rho\omega_A^2\xi_\varphi\right)\mathbf{1}_\varphi,\\
\boldsymbol{\Pi} &= -\rho\omega_A^2\left(\xi_r\,\mathbf{1}_r + \xi_\varphi\,\mathbf{1}_\varphi\right) = -\rho\omega_A^2\boldsymbol{\xi},\\
-\nabla_\perp P' &= -\frac{\mathrm{d}P'}{\mathrm{d}r}\mathbf{1}_r - \frac{\mathrm{i}m}{r}P'\mathbf{1}_\varphi\\
&= -\rho\left(\omega^2 - \omega_A^2\right)\left(\xi_r\,\mathbf{1}_r + \xi_\varphi\,\mathbf{1}_\varphi\right)\\
&= -\rho\left(\omega^2 - \omega_A^2\right)\boldsymbol{\xi}.
\end{aligned} \tag{5}$$

$\nabla_\perp$ is the gradient operator in horizontal planes perpendicular to the constant axial magnetic field. The last equation in (5) is obtained by using the second and third equations of (1) to write the components of $\nabla_\perp P'$ in terms of the components of ξ. Note also that in the third line of (5) the first equality is general but the second equality is not. A similar thing happens between the fourth and fifth line. The second equality in both cases relies on the fact that $\xi_z = 0$ which is the case for a pressureless plasma.

The important result from (5) is that the ratio of any of the two relevant components (the radial or azimuthal) of pressure force to the corresponding component of magnetic tension force is

$$\Lambda(\omega^2) = \frac{\omega^2 - \omega_A^2}{\omega_A^2}\,. \tag{6}$$

In a non-uniform plasma with v_A^2 and hence ω_A^2 dependent on position Λ also depends on position, meaning that the nature of the MHD wave changes according to the properties of the plasma it travels through. In a uniform plasma Λ is a constant and the nature of the wave does not change as it always sees the same environment.

It is standard practice to rewrite the two first order ordinary differential equations of (1) as a second order ordinary differential equation for P':

$$\rho\left(\omega^2 - \omega_A^2\right)\frac{d}{dr}\left\{\frac{r}{\rho\left(\omega^2 - \omega_A^2\right)}\frac{dP'}{dr}\right\} = \left\{\frac{m^2}{r^2} - \Gamma(\omega^2)\right\} rP', \tag{7}$$

where $\Gamma(\omega^2)$ is the abbreviation for

$$\Gamma(\omega^2) = \frac{\omega^2 - \omega_A^2}{v_A^2} = k_z^2\Lambda(\omega^2). \tag{8}$$

Γ depends on ω^2 but also on the equilibrium through v_A^2 and for a non-uniform plasma it is a function of position. The sign of Γ, i.e. of Λ, determines the local radial behaviour of the MHD wave. For $\Gamma > 0$ the wave behaves locally in the radial direction as a propagating wave, for $\Gamma < 0$ the the wave behaves locally as an evanescent wave (this is at least valid at large distances from the tube).

2.2. Pressureless flux tubes with uniform density

For a uniform plasma we can rewrite (7) as

$$\frac{d^2P'}{dr^2} + \frac{1}{r}\frac{dP'}{dr} - \left\{\frac{m^2}{r^2} - \Gamma(\omega^2)\right\} P' = 0. \tag{9}$$

Here $\Gamma(\omega^2)$ is constant. We now specialise on uniform loops with a higher density ρ_i than the surrounding uniform plasma which has a constant density $\rho_e < \rho_i$. At the loop boundary $r = R$ the density ρ changes discontinuously from its internal value ρ_i to its external value ρ_e. Since the magnetic field is uniform with the same strength everywhere, it follows that

$$v_{Ai}^2 = \frac{B^2}{\mu\rho_i} < v_{Ae}^2 = \frac{B^2}{\mu\rho_e}, \quad \omega_{Ai}^2 = k_z^2 v_{Ai}^2 < \omega_{Ae}^2 = k_z^2 v_{Ae}^2. \tag{10}$$

Note that the quantities $\Gamma(\omega^2)$ and $\Lambda(\omega^2)$ are constant both in the interior and the exterior of the flux tube but change discontinuously at the loop boundary $r = R$. As our focus is on MHD waves that are propagating in the interior of the loop and evanescent outside the loop, we select the frequency so that

$$\Gamma_i(\omega^2) > 0, \quad \Gamma_e(\omega^2) < 0, \tag{11}$$

which means that

$$\omega_{Ai}^2 < \omega^2 < \omega_{Ae}^2. \tag{12}$$

This allows us to define radial wave numbers k_i and k_e as

$$k_i^2 = \Gamma_i(\omega^2) = \frac{\omega^2 - \omega_{Ai}^2}{v_{Ai}^2}, \quad k_e^2 = -\Gamma_e(\omega^2) = -\frac{\omega^2 - \omega_{Ae}^2}{v_{Ae}^2}\,. \tag{13}$$

Equation (9) can then be solved in terms of Bessel functions $J_m(x)$ $(x = k_i r)$ in the internal part of the flux tube and $K_m(y)$ $(y = k_e r)$ in the exterior region

$$\begin{aligned} P_i'(r) &= \alpha J_m(x), \\ \xi_{r,i}(r) &= \alpha\frac{k_i}{\rho(\omega^2 - \omega_{Ai}^2)} J_m'(x), \\ P_e'(r) &= \beta K_m(y), \\ \xi_{r,e}(r) &= \beta\frac{k_e}{\rho(\omega^2 - \omega_{Ae}^2)} K_m'(y). \end{aligned} \tag{14}$$

The prime denotes a derivative with respect to the argument x or y and α and β are constants. Continuity of total pressure and the radial component of the Lagrangian displacement leads to the dispersion relation:

$$F\,\frac{J_m'(x_0)K_m(y_0)}{J_m(x_0)K_m'(y_0)} = 1, \tag{15}$$

with the quantity F given by

$$F = \frac{k_i}{k_e}\frac{\rho_e(\omega^2 - \omega_{Ae}^2)}{\rho_i(\omega^2 - \omega_{Ai}^2)}, \tag{16}$$

being $x_0 = k_i R$ and $y_0 = k_e R$. The dispersion relation (15) can be solved numerically. This was done for real frequencies by Edwin & Roberts (1983) and for complex frequencies by e.g. Spruit (1982) and Cally (1985, 2003). However, it is instructive and also accurate to consider the so-called thin tube (TT) approximation ($k_z R \ll 1$). The Bessel functions $J_m(x)$ and $K_m(y)$ in (15) are replaced with their first order asymptotic expansions. The dispersion relation (15) is reduced to

$$1 + F\,\frac{k_e}{k_i} = 0. \tag{17}$$

The solution for the frequency is

$$\omega^2 = \frac{\rho_i\omega_{Ai}^2 + \rho_e\omega_{Ae}^2}{\rho_i + \rho_e} = \omega_k^2, \tag{18}$$

and for the radial wave numbers k_i and k_e

$$k_i^2 = k_e^2 = k_z^2\frac{\rho_i - \rho_e}{\rho_i + \rho_e}\,. \tag{19}$$

The right hand side of (18) is almost invariably called the square of the kink frequency and denoted as ω_k^2. In the thin tube approximation the frequency is independent of the wave number $m \geq 1$ as already noted Goossens et al. (1992). Hence all flute modes with $m \geq 2$ have the same frequency as the kink mode with $m = 1$. The radial wave numbers k_i and k_e depend in a simple way on the density contrast. As the density contrast decreases, radial propagation is increasingly impeded and the nature of the MHD wave becomes gradually more Alfvénic, for $\rho_i = \rho_e$ it is purely Alfvénic as the frequency of the global wave is equal to the local Alfvén frequency everywhere. This is confirmed by the value of Λ

$$\Lambda_i(\omega^2) = -\Lambda_e(\omega^2) = \frac{\rho_i - \rho_e}{\rho_i + \rho_e}\,. \tag{20}$$

$\Lambda(\omega^2)$ is constant in both the interior and exterior. It is positive in the interior meaning that the magnetic pressure force and the magnetic tension force act in the same direction. In the exterior its value is the exact opposite of that in the interior. The magnetic pressure force and the magnetic tension force now oppose each other. Equation (20) shows that the MHD waves are always dominated by magnetic tension forces and that they are predominantly Alfvénic in nature. Take as an example a density contrast $\rho_i/\rho_e = 3$ then $\Lambda_i = 1/2$, $\Lambda_e = -1/2$ so that the magnetic tension force is always twice as important as the pressure force.

The TT approximations to the eigenfunctions are

$$\begin{aligned}
\frac{\xi_{r,i}(r)}{R} &= C,\\
\frac{\xi_{\varphi,i}(r)}{R} &= \mathrm{i}\, C,\\
\frac{P_i'(r)}{(B^2/\mu)} &= C\,(k_z R)^2\,\frac{\rho_i-\rho_e}{\rho_i+\rho_e}\,\frac{r}{R},\\
\nabla\cdot\boldsymbol{\xi}_i &= -C\,(k_z R)^2\,\frac{\rho_i-\rho_e}{\rho_i+\rho_e}\,\frac{r}{R},\\
\xi_{\varphi,e}(R_>) &= -\xi_{\varphi,i}(R_<).
\end{aligned} \tag{21}$$

Note that when deriving (21) we have omitted terms of order $(k_z R)^2$ and higher unless the terms of order $(k_z R)^2$ are the first non-vanishing contribution to the expression under study. For example the expressions for $\xi_{r,i}(r)/R$ and $\xi_{\varphi,i}(r)/R$ mean that these two quantities are equal up to differences of order $(k_z R)^2$. The eigenfunctions are determined up to a multiplicative constant C which can be used to specify e.g. the radial displacement of the boundary of the loop. The radial and azimuthal components are $\pi/2$ out of phase but they have equal magnitudes and in the loop they are constant. The wave is in the propagating domain in the internal part of the loop ($k_i^2 > 0$) but there are not any spatial variations. For realistic values of $k_z R = (\pi R)/L$ the pressure perturbation and the divergence of the displacement field are zero for all practical purposes. The kink mode is to a high degree of accuracy an incompressible wave with very small magnetic pressure perturbations. Apart from ξ_φ the wave quantities are continuous at $r = R$. ξ_φ varies in a discontinuous manner at $r = R$ with opposite values at $R_<$ and $R_>$. This discontinuous behaviour is due to the change of sign of the factor $\omega^2 - \omega_A^2$ when we move from the interior to the exterior of the loop. The behaviour of ξ_φ creates strong shear layers which might undergo Kelvin-Helmholtz type instabilities as shown by Terradas et al. (2008).

So far we have described the properties of modes that involve non-zero total pressure ($P' \neq 0$). However, if the medium is uniform the system of equations given by (1) also allows pure incompressible Alfvén waves. They are only driven by magnetic tension, their eigenfrequency is simply $\omega = \omega_A$ and the total pressure, P', is equal to zero. To have such modes the displacement has to satisfy $\nabla\cdot\boldsymbol{\xi} = 0$. In cylindrical coordinates this means that,

$$\frac{\mathrm{d}\,(r\xi_r)}{\mathrm{d}r} + \mathrm{i}m\xi_\varphi = 0. \tag{22}$$

If we prescribe the radial dependence of one of the components of the displacement, we can easily calculate the other component from the previous equation. The case $m = 0$ is a particular solution that represents torsional Alfvén waves ($\xi_r = 0$, ξ_φ arbitrary), but Eq. (22) can be solved for any azimuthal wavenumber m. Let us concentrate on $m = 1$ and the homogeneous loop model. Now we can have two different incompressible Alfvén waves. One inside oscillating at the frequency ω_{Ai} and another outside oscillating at ω_{Ae}. However, it is important to note that in such a configuration the radial displacement of the internal and external Alfvén waves has to vanish at $r = R$ (otherwise the continuity of the radial component is not guaranteed), i.e. the modes are localised in regions of constant Alfvén frequency. This means that an internal incompressible Alfvén wave is unable to laterally displace the full tube (it cannot displace the tube boundary), although it is able to produce an incompressible motion of the loop axis and its surroundings (for $m = 1$). Since these pure incompressible Alfvén waves do not move the whole tube hereafter we will focus again on the kink solutions with $P' \neq 0$. Contrary to the incompressible Alfvén waves these waves (with $P' \neq 0$) are able to connect the internal and external medium and to produce a coherent motion of the system because of their mixed nature. The fact that P' is small but different from zero plays a fundamental role in the mixed properties of these kink waves.

2.3. Beyond the TT approximation for pressureless uniform flux tubes

The analytic expressions (21) ($P' \neq 0$) have been obtained in the limit $k_z R \ll 1$. It is straightforward to solve the dispersion relation (15) and calculate the spatial solutions (14). This allows us to determine how the analytical expressions are modified by effects due to a finite radius. In Fig. 1 the eigenfunctions of three loops with different radii are represented. It is clear that the spatial profile is well described by the approximated solutions in the TT limit given by Eqs. (21). The radial and azimuthal components are constant inside the loop, the azimuthal component has the expected jump at $r = R$, while the total pressure grows linearly with the radius. Increasing R results in an increase of the total pressure, and thus compressibility, since this magnitude is proportional to $(k_z R)^2$. Interestingly, for fat loops (see the case $R/L = 0.1$) the TT approximations of the eigenfunctions are still quite valid. An analysis of the forces (not shown here) indicates that, even for thick loops, the tension dominates over the magnetic pressure gradient.

2.4. Pressureless flux tubes with non-uniform density

In this subsection we remove the discontinuous variation of density from its internal value ρ_i to ρ_e by a continuous variation in a non-uniform layer $[R - l/2, R + l/2]$. A fully non-uniform equilibrium state corresponds to $l = 2R$. When the jump in ω_A is replaced by a continuous variation of ω_A new physics is introduced in the system. The continuous variation of ω_A has the important effect that the kink MHD wave, which has its frequency in the Alfvén continuum, interacts with local Alfvén continuum waves and gets damped. This resonant damping is translated in a complex frequency (and complex eigenfunction). In the present paper damped global eigenmodes that are coupled to resonant Alfvén waves in a non-uniform equilibrium state shall be computed by two methods. The first method is to use a numerical code that integrates the resistive MHD equations in the whole volume of the equilibrium state to determine a selected mode or part of the resistive spectrum of the system (see for example Van Doorsselaere et al. 2004; Arregui et al. 2005; Terradas et al. 2006). The second method was introduced by Tirry & Goossens (1996). It circumvents the numerical integration of the non-ideal MHD equations and only requires numerical integration (or closed analytical solutions) of the linear ideal MHD equations. The method relies on the fact that dissipation is important only in a narrow layer around the resonant point where the real part of the quasi-mode frequency equals the local Alfvén frequency. This makes it possible to obtain analytical solutions to simplified versions of the linear dissipative

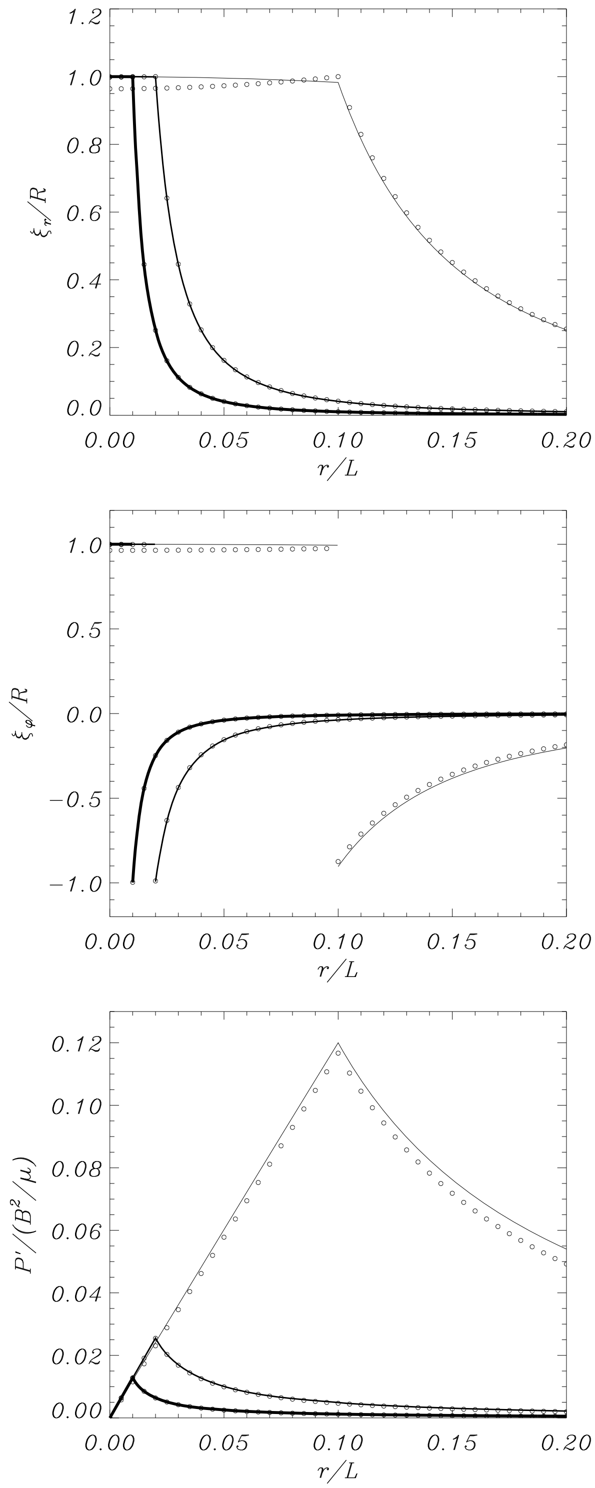

Fig. 1. Radial dependence of the normalised eigenfunctions, ξ_r (real), ξ_φ (imaginary), and P' (real) calculated using (14) after solving the dispersion relation (15) for three different loop radii, $R/L = 0.01, 0.02, 0.1$. The circles describe the incompressible eigenfunctions calculated using (40) after solving the dispersion relation given by (41). The frequencies for the three loop radii are, in the incompressible limit, $\omega L/v_{\mathrm{Ai}} = 3.846, 3.843, 3.795$. In these calculations $\rho_{\mathrm{i}}/\rho_{\mathrm{e}} = 3$ and $k_z = \pi/L$.

MHD equations which accurately describe the linear motions in the dissipative layer and in two overlap regions to the left and right of the dissipative layer. In these overlap regions both ideal MHD and dissipative MHD are valid. Asymptotic analysis of the analytical dissipative solutions allows to derive jump conditions that can be used to connect the solutions to the left and right of the dissipative layer. The jump conditions were derived by e.g. Sakurai et al. (1991a), Goossens et al. (1992), Goossens et al. (1995) and Goossens & Ruderman (1995) for the driven problem and by Tirry & Goossens (1996) for the eigenvalue problem. A schematic overview of the various regions involved in this method is shown in Fig. 1 of Stenuit et al. (1998). This method was used for computing eigenmodes of various non-uniform plasma configurations by e.g. Tirry et al. (1998a,b), Stenuit et al. (1998, 1999), Andries et al. (2000); Andries & Goossens (2001). A related method was used by Sakurai et al. (1991b) and Stenuit et al. (1995) for the computation of resonant Alfvén waves in the driven problem with a prescribed and real frequency and by Keppens et al. (1994) for computing the multiple scattering and resonant absorption of p-modes by fibril sunspots. Comparison with results of fully dissipative computations show that the method is very accurate.

A drastic variant of the method that avoids solving the non-ideal MHD equations uses the so-called thin boundary (TB) approximation. In this lazy version the ideal MHD equations are not solved in the non-uniform plasma but the plasma is treated as if it were uniform all the way up to the dissipative layer. This is definitely a very strong assumption since the thickness of the dissipative layer is measured by the quantity δ_{A}:

$$\delta_{\mathrm{A}} = \left(\frac{\omega\eta}{\mid \Delta_{\mathrm{A}} \mid}\right)^{1/3}, \quad \Delta_{\mathrm{A}} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\omega^2 - \omega_{\mathrm{A}}^2\right). \tag{23}$$

Here η is the magnetic resistivity. If we denote l the typical length scale for the variations of the equilibrium quantities then

$$\frac{l}{\delta_{\mathrm{A}}} = (R_m)^{1/3}, \tag{24}$$

where R_m is the magnetic Reynolds number ($R_m \sim \eta^{-1}$). Strictly speaking the TB approximation assumes that the nonuniform layer and the dissipative layer coincide. It is common practice not to draw attention to this assumption but instead to refer to the TB approximation as the approximation that adopts $l/R \ll 1$. This TB approximation was first used by Hollweg & Yang (1988). A discussion of the TB approximation can be found in Goossens (2008).

In the TB approximation we need to add an additional term to the dispersion relation which takes into account the jump in the radial component across the resonant layer where the real part of the kink eigenmode is equal to the local Alfvén frequency $\omega = \omega_{\mathrm{A}}(r_{\mathrm{A}})$. r_{A} is the resonant position which in the thin boundary approximation $r_{\mathrm{A}} = R$. The jump in ξ_r is

$$[\xi_r] = -\mathrm{i}\pi\frac{m^2/r_{\mathrm{A}}^2}{\rho \mid \Delta_{\mathrm{A}} \mid}P', \quad [P'] = 0. \tag{25}$$

The modified version of the ideal dispersion relation (15) is

$$F\,\frac{J_m'(x_0)K_m(y_0)}{J_m(x_0)K_m'(y_0)} - \mathrm{i}\,G\,\frac{K_m(y_0)}{K_m'(y_0)} = 1. \tag{26}$$

F is given by (16) and G is defined as

$$G = \pi\frac{m^2/r_{\mathrm{A}}^2}{\rho \mid \Delta_{\mathrm{A}} \mid}\,\frac{\rho_{\mathrm{e}}\left(\omega^2 - \omega_{\mathrm{Ae}}^2\right)}{k_{\mathrm{e}}}\,. \tag{27}$$

G contains the effect of the resonance. When we combine the thin tube (TT) approximation with the thin boundary (TB) approximation, we can simplify the dispersion relation to

$$1 + F\,\frac{k_{\mathrm{e}}}{k_{\mathrm{i}}} - \mathrm{i}\,G\,\frac{k_{\mathrm{e}}R}{m} = 0. \tag{28}$$

The zero order solution to (28), i.e. the solution when the effect of the resonance is not taken into account is of course (18). In order to take the effect of the resonance into account we write

$$\omega = \omega_{\mathrm{R}} + \mathrm{i}\gamma, \quad \omega_{\mathrm{R}} = \omega_{\mathrm{k}}, \tag{29}$$

and approximate ω^2 with $\omega_{\mathrm{k}}^2 + 2\mathrm{i}\omega_{\mathrm{k}}\gamma$. The solution for the damping decrement is

$$\frac{\gamma}{\omega_{\mathrm{k}}} = -\frac{\pi/2}{\omega_{\mathrm{k}}^2} \frac{m}{R} \frac{\rho_{\mathrm{i}}^2 \rho_{\mathrm{e}}^2}{(\rho_{\mathrm{i}} + \rho_{\mathrm{e}})^3} \frac{(\omega_{\mathrm{Ai}}^2 - \omega_{\mathrm{Ae}}^2)^2}{\rho(r_{\mathrm{A}}) \mid \Delta_{\mathrm{A}}(r_{\mathrm{A}}) \mid} \cdot \tag{30}$$

Equation (30) agrees with Eq. (77) of Goossens et al. (1992) when that equation is corrected for a typo as the factor $\left(\omega_{\mathrm{Ai}}^2 - \omega_{\mathrm{Ae}}^2\right)$ should be squared. This is surprising since that result was obtained by Goossens et al. (1992) for surface waves in incompressible plasmas. In the same section of that paper it was noted that there is no distinction between compressible and incompressible plasmas for surface waves on thin tubes.

Equation (30) shows that the damping decrement depends linearly on m. Since we are mainly interested in $m = 1$ we shall specialise to that value in the remainder of this subsection. If the variation of ω_{A}^2 is solely due to the variation of density ρ as is the case here since we have considered a constant axial magnetic field, Eq. (30) can be rewritten as

$$\frac{\gamma}{\omega_{\mathrm{k}}} = -\frac{\pi}{8} \frac{m}{R} \frac{(\rho_{\mathrm{i}} - \rho_{\mathrm{e}})^2}{\rho_{\mathrm{i}} + \rho_{\mathrm{e}}} \frac{1}{\left|\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{r_{\mathrm{A}}}} \cdot \tag{31}$$

For a linear profile of density

$$\left|\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{r_{\mathrm{A}}} = \frac{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}}{l},$$

so that

$$\frac{\gamma}{\omega_{\mathrm{k}}} = -\frac{\pi}{8} \frac{l}{R} \frac{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}}{\rho_{\mathrm{i}} + \rho_{\mathrm{e}}}, \quad \frac{\tau_{\mathrm{D}}}{T} = \frac{4}{\pi^2} \frac{1}{l/R} \frac{\rho_{\mathrm{i}} + \rho_{\mathrm{e}}}{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}} \cdot \tag{32}$$

In (32) τ_{D} is the damping time and T the period. Note that the result for $\frac{\gamma}{\omega_{\mathrm{k}}}$ of Eq. (32) agrees with Eq. (79b) of Goossens et al. (1992).

For a sinusoidal profile of density

$$\left|\frac{\mathrm{d}\rho}{\mathrm{d}r}\right|_{r_{\mathrm{A}}} = \frac{\pi}{2} \frac{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}}{l},$$

so that

$$\frac{\gamma}{\omega_{\mathrm{k}}} = -\frac{1}{4} \frac{l}{R} \frac{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}}{\rho_{\mathrm{i}} + \rho_{\mathrm{e}}}, \quad \frac{\tau_{\mathrm{D}}}{T} = \frac{2}{\pi} \frac{1}{l/R} \frac{\rho_{\mathrm{i}} + \rho_{\mathrm{e}}}{\rho_{\mathrm{i}} - \rho_{\mathrm{e}}}. \tag{33}$$

Here the results agree with those obtained by Ruderman & Roberts (2002). At this point we like to stress that the TTTB approximation turns out to be remarkably accurate far beyond its domain of applicability. This is clearly illustrated in a recent analytical seismological study by Goossens et al. (2008) which complemented a fully numerical seismology investigation by Arregui et al. (2007).

The eigenfunctions in the thin dissipative layer can be described by the functions $F(\tau)$ and $G(\tau)$ defined by Goossens et al. (1995) for the driven problem and the functions $\tilde{F}(\tau)$ and $\tilde{G}(\tau)$ defined by Ruderman et al. (1995) for the incompressible eigenvalue problem and by Tirry & Goossens (1996) for the compressible eigenvalue problem. In the dissipative layer the MHD kink waves are highly Alfvénic. This can be understood as follows. From the analysis by Sakurai et al. (1991a), Goossens et al. (1995) and Tirry & Goossens (1996) it follows that in the dissipative layer, the Eulerian perturbation of total pressure P' is constant and that $\mid \xi_r \mid \ll \mid \xi_\varphi \mid$. We do not have to worry about ξ_z since it is zero for a pressureless plasma. In addition

$$\xi_\varphi(\tau) = \frac{m\, P'}{r\, \rho} \frac{\tilde{F}(\tau)}{\delta_{\mathrm{A}} \mid \Delta_{\mathrm{A}} \mid} \cdot \tag{34}$$

Here $\tau = (r - r_{\mathrm{A}})/\delta_{\mathrm{A}}$ is the stretched independent variable used in the dissipative layer which has a typical width $[-5\delta_{\mathrm{A}}, 5\delta_{\mathrm{A}}]$. $\tilde{F}(\tau)$ is a complex function and $\xi_\varphi(\tau)$ does not suffer a discontinuous jump as in the case of a uniform plasma. It is characterised by rapid spatial variation in the dissipative layer. From (34) it follows that the ratio of the φ component of magnetic tension to that of pressure gradient is

$$\frac{-\rho\, \omega_{\mathrm{A}}^2\, \xi_\varphi}{-\mathrm{i}(m/r_{\mathrm{A}})P'} = -\mathrm{i} \frac{\omega_{\mathrm{A}}^2}{\mid \Delta_{\mathrm{A}} \mid} \frac{\tilde{F}(\tau)}{\delta_{\mathrm{A}}} \approx -\mathrm{i}\, \frac{l}{\delta_{\mathrm{A}}}\, \tilde{F}(\tau), \tag{35}$$

where we have used $\omega_{\mathrm{A}}^2 / \mid \Delta_{\mathrm{A}} \mid \approx l$. Since $\tilde{F}(\tau)$ is of order unity or bigger it follows that

$$\left|\frac{-\rho\, \omega_{\mathrm{A}}^2\, \xi_\varphi}{-\mathrm{i}(m/r_{\mathrm{A}})P'}\right| \approx (R_m)^{1/3} \gg 1. \tag{36}$$

In the dissipative layer the magnetic tension force is far bigger than the pressure gradient force. Hence the MHD kink wave is highly Alfvénic in the dissipative layer.

2.5. *Beyond the TTTB approximation for pressureless non-uniform flux tubes*

In this subsection we go beyond the thin boundary approximation and we consider thick non-uniform layers with $l/R = 0.2$ and 0.4. The eigenfunctions are numerically calculated by solving the full set of linear, resistive MHD equations described in Terradas et al. (2006) and using the PDE2D code (Sewell 1995). Figure 2 displays the obtained results. It is clear that the two inhomogeneous solutions are almost identical to the homogeneous solution, except in the non-uniform layer, where large displacements are found. This is the location where the resonance takes place.

In Fig. 3 we see that both the radial and azimuthal components of the Lorentz force are dominated by magnetic tension. The magnitude of the magnetic tension and pressure is of the same order (with the tension about twice as important as the magnetic pressure force such as corresponds to a density contrast of 3) except in the inhomogeneous layer where the tension in the azimuthal direction is clearly dominant, reflecting the strong Alfvénic nature of the solution in the dissipative layer.

3. Incompressible MHD waves on flux tubes

3.1. *Equations for incompressible MHD waves on a plasma cylinder*

The equations for incompressible MHD waves can be found in Goossens et al. (1992). Incompressibility means that we take the limit $v_{\mathrm{S}} \rightarrow \infty$ and enforce $\nabla \cdot \boldsymbol{\xi} = 0$. Note also that in the

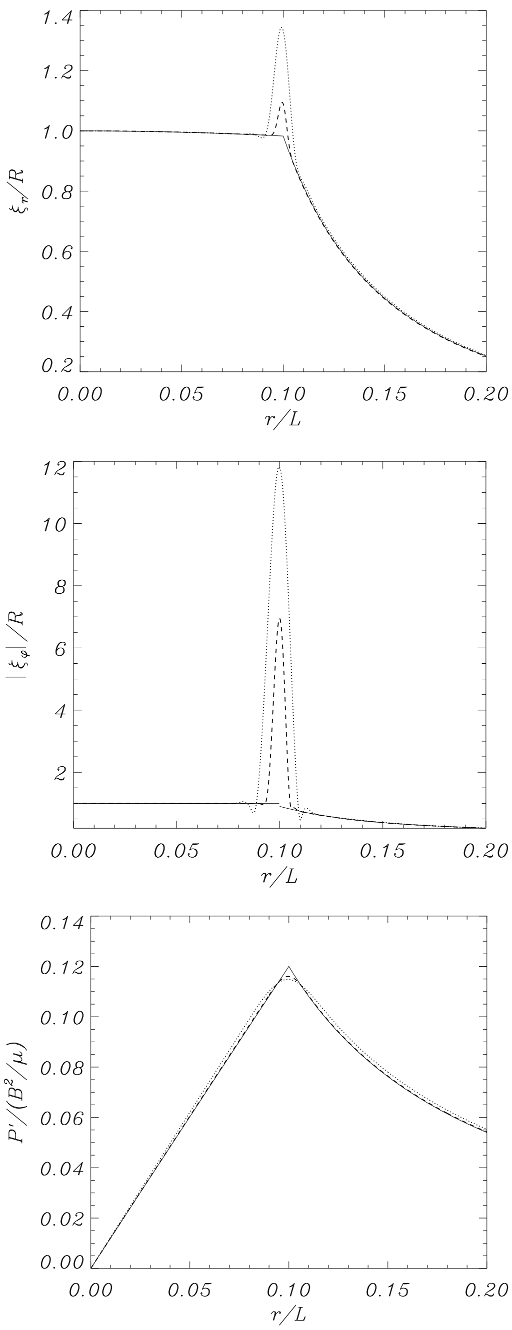

Fig. 2. Radial dependence of the moduli of the eigenfunctions, ξ_r, ξ_φ, and P' calculated numerically. The continuous lines represent the analytical solutions for $l = 0$ computed with (14) (see also Fig. 1). Dashed and dotted lines represent the numerical computed eigenfunctions ($R_m = 10^6$) for $l/R = 0.2$ and $l/R = 0.4$, respectively. The corresponding dimensionless eigenvalues are $\omega_R L/v_{Ai} = 3.847, 3.799, 3.795$ and $\omega_i L/v_{Ai} = 0, 0.092, 0.192$. In these calculations $\rho_i/\rho_e = 3$, $k_z = \pi/L$ and $R/L = 0.1$.

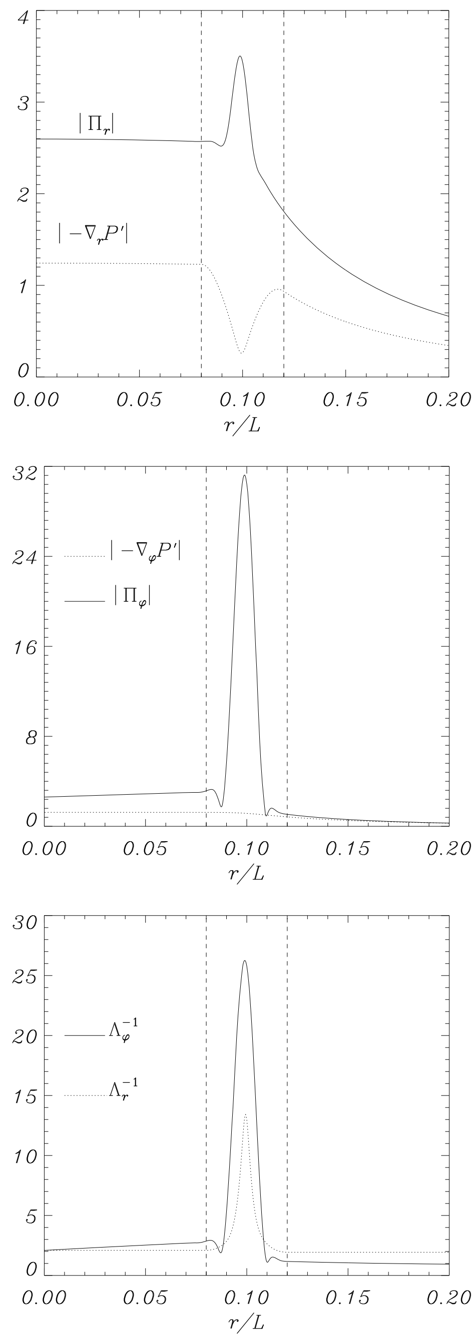

Fig. 3. *Top panel*: moduli of the radial forces. *Middle panel*: moduli of the azimuthal forces. *Bottom panel*: ratio of the modulus of the magnetic tension to the magnetic pressure gradient in the radial and the azimuthal direction. This case corresponds to the eigensolution displayed in Fig. 2 with $l/R = 0.4$. The vertical dashed lines show the limits of the non-uniform layer.

incompressible case $\omega_C = \omega_A$. The relevant equations are

$$\begin{aligned}
\rho\left(\omega^2 - \omega_A^2\right)\frac{d(r\xi_r)}{dr} &= \left(\frac{m^2}{r^2} + k_z^2\right) rP',\\
\rho\left(\omega^2 - \omega_A^2\right)\xi_r &= \frac{dP'}{dr},\\
\rho\left(\omega^2 - \omega_A^2\right)\xi_\varphi &= \frac{im}{r}P',\\
\rho\left(\omega^2 - \omega_A^2\right)\xi_z &= ik_z P'.
\end{aligned} \tag{37}$$

As before we concentrate on solutions with $P' \neq 0$ (although now we assume incompressibility) and rewrite (37) as a second order ordinary differential equation for P' (eliminating the displacements):

$$\rho\left(\omega^2 - \omega_A^2\right)\frac{d}{dr}\left\{\frac{r}{\rho\left(\omega^2 - \omega_A^2\right)}\frac{dP'}{dr}\right\} = \left\{\frac{m^2}{r^2} + k_z^2\right\} rP'. \tag{38}$$

Here we do not need to worry about propagating and/or evanescent behaviour of the solutions and the local radial wave number. The solutions are always evanescent or surface waves and the local radial wave number is k_z. Note that in our notation $\Gamma(\omega^2) = -k_z^2$.

3.2. Incompressible MHD waves on uniform flux tubes

For a uniform plasma we can rewrite (38) as

$$\frac{\mathrm{d}^2 P'}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}P'}{\mathrm{d}r} - \left\{\frac{m^2}{r^2} + k_z^2\right\} P' = 0. \tag{39}$$

Equation (39) can then be solved in terms of Bessel functions $I_m(x)$ ($x = k_z r$) in the internal part of the flux tube and $K_m(x)$ in the exterior region

$$\begin{aligned} P'_\mathrm{i}(r) &= \alpha I_m(x), \\ \xi_{r,\mathrm{i}}(r) &= \alpha \frac{k_z}{\rho_\mathrm{i}(\omega^2 - \omega_\mathrm{Ai}^2)} I'_m(x), \\ P'_\mathrm{e}(r) &= \beta K_m(x), \\ \xi_{r,\mathrm{e}}(r) &= \beta \frac{k_z}{\rho_\mathrm{e}(\omega^2 - \omega_\mathrm{Ae}^2)} K'_m(x). \end{aligned} \tag{40}$$

Continuity of total pressure and the radial component of the Lagrangian displacement leads to the dispersion relation:

$$F \; \frac{I'_m(x_0)K_m(x_0)}{I_m(x_0)K'_m(x_0)} = 1. \tag{41}$$

The incompressible version of F is

$$F = \frac{\rho_\mathrm{e}(\omega^2 - \omega_\mathrm{Ae}^2)}{\rho_\mathrm{i}(\omega^2 - \omega_\mathrm{Ai}^2)} \cdot \tag{42}$$

Now $x_0 = k_z R$. The dispersion relation (41) can be solved numerically. However, if we consider the TT approximation, the Bessel functions $I_m(x)$ and $K_m(x)$ in (41) are replaced with their first order asymptotic expansions, and the dispersion relation (41) is reduced to

$$1 + F = 0. \tag{43}$$

The solution is again given by Eq. (18), i.e. $\omega^2 = \omega_\mathrm{k}^2$. The important point to note is that we get exactly the same expression for the frequency in the incompressible limit as in the pressureless limit. As far as the frequency is concerned the MHD waves with $m \geq 1$ do not make any distinction between a pressureless plasma and an incompressible plasma at least in the thin tube approximation. Here also Λ is given by (20) $\Lambda_\mathrm{i}(\omega^2) = -\Lambda_\mathrm{e}(\omega^2) = \frac{\rho_\mathrm{i}-\rho_\mathrm{e}}{\rho_\mathrm{i}+\rho_\mathrm{e}}$. This shows that the MHD waves are always dominated by magnetic tension forces and that they are predominantly Alfvénic in nature. Since the fast waves are absent as we have imposed the condition of incompressibility, the magnetic pressure force is associated with the slow wave behaviour of the solution. The eigenfunctions are

$$\begin{aligned} \frac{\xi_{r,\mathrm{i}}(r)}{R} &= C, \\ \frac{\xi_{\varphi,\mathrm{i}}(r)}{R} &= \mathrm{i}\, C, \\ \frac{\xi_{z,\mathrm{i}}(r)}{R} &= \mathrm{i}C(k_z R)\frac{r}{R}, \\ \frac{P'_\mathrm{i}(r)}{(B^2/\mu)} &= C\,(k_z R)^2 \,\frac{\rho_\mathrm{i} - \rho_\mathrm{e}}{\rho_\mathrm{i} + \rho_\mathrm{e}}\,\frac{r}{R}, \\ \nabla \cdot \boldsymbol{\xi}_\mathrm{i} &= 0, \\ \xi_{\varphi,\mathrm{e}}(R_>) &= -\xi_{\varphi,\mathrm{i}}(R_<), \\ \xi_{z,\mathrm{e}}(R_>) &= -\xi_{z,\mathrm{i}}(R_<). \end{aligned} \tag{44}$$

As before we have omitted terms of order $(k_z R)^2$ and higher unless the terms of order $(k_z R)^2$ are the first non-vanishing contribution to the expression under study. It is instructive to compare (44) with (21). The expressions for $\xi_{r,\mathrm{i}}(r)/R$, $\xi_{\varphi,\mathrm{i}}(r)/R$ and $P'_\mathrm{i}(r)/(B^2/\mu)$ are exactly the same as in the pressureless case. There is now a small (of the order $(k_z R)$) axial component ξ_z of the Lagrangian displacement. This axial component now makes $\nabla \cdot \boldsymbol{\xi}$ exactly equal to zero while it is of order $(k_z R)^2$ for a pressureless plasma. The azimuthal component $\xi_{\varphi,\mathrm{i}}(r)$ and the axial component $\xi_{z,\mathrm{i}}(r)$ are $\pi/2$ out of phase respect to $\xi_{r,\mathrm{i}}(r)$. $\xi_{r,\mathrm{i}}(r)$ and $\xi_{\varphi,\mathrm{i}}(r)$ have equal magnitude and are constant in the flux tube. The axial component $\xi_{z,\mathrm{i}}(r)$ is of the order $(k_z R)$ and varies linearly in the loop. In the thin tube approximation $\xi_{z,\mathrm{i}}(r)$ is always small compared to $\xi_{r,\mathrm{i}}(r)$ and $\xi_{\varphi,\mathrm{i}}(r)$. The dominant motion is in horizontal planes normal to the equilibrium magnetic field. The vertical displacement and velocity are small, of the order of k_z at the tube boundary. As in the pressureless case ξ_φ is discontinuous at $r = R$ with opposite values at $R_<$ and $R_>$. This discontinuous behaviour is due to the change of sign of the factor $\omega^2 - \omega_\mathrm{A}^2$. In the incompressible limit also ξ_z is discontinuous at $r = R$ with opposite values at $R_<$ and $R_>$. This discontinuous behaviour is due to the change of sign of the factor $\omega^2 - \omega_\mathrm{C}^2$ with $\omega_\mathrm{C}^2 = \omega_\mathrm{A}^2$ in the incompressible limit.

3.3. Beyond the TT approximation for incompressible MHD waves on uniform flux tubes

As in the compressible case the analytic expressions (44) have been obtained in the limit $k_z R \ll 1$. It is easy to solve the dispersion relation (41) and determine the spatial solutions (40). In Fig. 1 the eigenfunctions of three loops with variable radii are represented with circles. Again it is clear that the spatial profile is well described by the approximated solutions in the TT limit given by Eqs. (44). As expected, the case with $R/L = 0.1$ shows the largest deviation from the TT approximation. Figure 1 also allows us a direct comparison with the eigenfunctions in the compressible approximation. As the ratio R/L decreases the eigenfunctions of the compressible and incompressible cases tend to be the same, in agreement with the analytical expressions given by (44) and (21). We arrive to the same conclusion for the forces, even for thick loops, the tension dominates over the magnetic pressure gradient.

3.4. Incompressible MHD waves on non-uniform flux tubes

We again remove the discontinuous variation of density from its internal value ρ_i to ρ_e by a continuous variation in a non-uniform. By doing so, we allow interaction of the global kink wave with local Alfvén/slow continuum waves and the discontinuous behaviour of ξ_φ and ξ_z are replaced by singular behaviour in ideal MHD and by large but finite values in non-ideal MHD. In the thin boundary approximation we need to add an additional term to the dispersion relation which takes into account the jump in the radial component across the resonant layer where the real part of the kink eigenmode is equal to the local Alfvén frequency. For the incompressible case the jump in ξ_r is (see e.g. Goossens et al. 1992)

$$\begin{aligned} [\xi_r] &= -\mathrm{i}\pi \frac{m^2/r_\mathrm{A}^2 + k_z^2}{\rho \mid \Delta_\mathrm{A} \mid} \; P', \\ [P'] &= 0. \end{aligned} \tag{45}$$

In the incompressible limit $\omega_\mathrm{A} = \omega_\mathrm{C}$ and the Alfvén resonance and slow resonance coincide. The jump in ξ_r due to the Alfvén resonance is given in (25). The jump in ξ_r due to the slow resonance is

$$[\xi_r] = -\mathrm{i}\pi \frac{k_z^2}{\rho \mid \Delta_\mathrm{C} \mid} \left\{ \frac{v_\mathrm{S}^2}{v_\mathrm{S}^2 + v_\mathrm{A}^2} \right\}^2 P',$$

$$[P'] = 0, \quad \Delta_\mathrm{C} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\omega^2 - \omega_\mathrm{C}^2\right). \tag{46}$$

For the incompressible limit $v_\mathrm{S}^2 \to \infty$ (46) becomes

$$[\xi_r] = -\mathrm{i}\pi \frac{k_z^2}{\rho \mid \Delta_\mathrm{C} \mid} P',$$

$$[P'] = 0, \quad \Delta_\mathrm{C} = \Delta_\mathrm{A} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\omega^2 - \omega_\mathrm{A}^2\right). \tag{47}$$

We see that Eq. (45) is the combination of the jump due to the Alfvén resonance (25) and the jump due to the slow resonance (47). The contribution to the jump in ξ_r due to the slow resonance is of order $(k_z R)^2$ compared to that of the Alfvén resonance and can be neglected in our thin tube approximation. Note that $(v_\mathrm{S}^2/(v_\mathrm{S}^2+v_\mathrm{A}^2))^2 \leq 1$ so that the slow resonance has its biggest effect for incompressible plasmas. Even in that case it is unimportant compared to the Alfvén resonance. This is in agreement with a result obtained for MHD waves in prominences by Soler et al. (2009). For sentimental reasons we shall keep the contribution due to the slow resonance.

The modified version of the ideal dispersion relation (41) is

$$F \frac{I_m'(x_0) K_m(x_0)}{I_m(x_0) K_m'(x_0)} - \mathrm{i}\, G \frac{K_m(x_0)}{K_m'(x_0)} = 1. \tag{48}$$

F is given by (16) and G is defined as

$$G = \pi \frac{m^2/r_\mathrm{A}^2 + k_z^2}{\rho \mid \Delta_\mathrm{A} \mid} \frac{\rho_\mathrm{e}\left(\omega^2 - \omega_\mathrm{Ae}^2\right)}{k_\mathrm{e}}. \tag{49}$$

When we combine the TT approximation with TB approximation, the dispersion relation is reduced to

$$1 + F - \mathrm{i}\, G \frac{k_z R}{m} = 0. \tag{50}$$

The zero order solution to (50) without taking into account the effect of the resonance is of course (18). The effect of the resonance is contained in G. In order to take that effect into account we proceed as before

$$\omega = \omega_\mathrm{R} + \mathrm{i}\gamma, \quad \omega_\mathrm{R} = \omega_\mathrm{k}, \tag{51}$$

and approximate ω^2 with $\omega_\mathrm{k}^2 + 2\mathrm{i}\omega_\mathrm{k}\gamma$. The solution for the damping decrement is

$$\frac{\gamma}{\omega_\mathrm{k}} = -\frac{\pi/2}{\omega_\mathrm{k}^2}\left\{\frac{m}{R} + \frac{(k_z R)^2}{mR}\right\} \frac{\rho_\mathrm{i}^2 \rho_\mathrm{e}^2}{(\rho_\mathrm{i} + \rho_\mathrm{e})^3} \frac{\left(\omega_\mathrm{Ai}^2 - \omega_\mathrm{Ae}^2\right)^2}{\rho(r_\mathrm{A}) \mid \Delta_\mathrm{A}(r_\mathrm{A}) \mid}. \tag{52}$$

If the variation of ω_A^2 is solely due to the variation of density ρ as is the case here since the axial magnetic field is constant, the equation can be rewritten as

$$\frac{\gamma}{\omega_\mathrm{k}} = -\frac{\pi}{8}\left\{\frac{m}{R} + \frac{(k_z R)^2}{mR}\right\} \frac{(\rho_\mathrm{i} - \rho_\mathrm{e})^2}{\rho_\mathrm{i} + \rho_\mathrm{e}} \frac{1}{\mid \frac{\mathrm{d}\rho}{\mathrm{d}r} \mid_{r_\mathrm{A}}}. \tag{53}$$

From here on we shall specialise to $m = 1$. For a linear profile of density

$$\frac{\gamma}{\omega_\mathrm{k}} = -\frac{\pi}{8}\left(1 + (k_z R)^2\right) \frac{l}{R} \frac{\rho_\mathrm{i} - \rho_\mathrm{e}}{\rho_\mathrm{i} + \rho_\mathrm{e}},$$

$$\frac{\tau_\mathrm{D}}{T} = \frac{4}{\pi^2} \frac{1}{l/R} \frac{\rho_\mathrm{i} + \rho_\mathrm{e}}{\rho_\mathrm{i} - \rho_\mathrm{e}}\left(1 - (k_z R)^2\right). \tag{54}$$

For a sinusoidal profile of density

$$\frac{\gamma}{\omega_\mathrm{k}} = -\frac{1}{4}\left(1 + (k_z R)^2\right) \frac{l}{R} \frac{\rho_\mathrm{i} - \rho_\mathrm{e}}{\rho_\mathrm{i} + \rho_\mathrm{e}},$$

$$\frac{\tau_\mathrm{D}}{T} = \frac{2}{\pi} \frac{1}{l/R} \frac{\rho_\mathrm{i} + \rho_\mathrm{e}}{\rho_\mathrm{i} - \rho_\mathrm{e}}\left(1 - (k_z R)^2\right). \tag{55}$$

For all practical purposes we can neglect the contribution proportional to $(k_z R)^2$ and conclude that the damping due to resonant absorption of the kink mode in an incompressible plasma is the same as that in a pressureless plasma (see Eqs. (32) and (33)). If we forget about differences proportional to $(k_z R)^2$ then the conclusion is that kink MHD waves in pressureless plasmas and incompressible plasmas are the same. In view of that conclusion it is difficult to understand why a kink mode can be called fast as fast waves are absent from incompressible plasmas.

The eigenfunctions in the thin dissipative layer can be described by the functions $\tilde{F}(\tau)$ and $\tilde{G}(\tau)$ which were first introduced by Ruderman et al. (1995) for non-stationary incompressible resonant Alfvén waves in planar plasmas. The conclusion is the same as in the previous section. The kink MHD waves are highly Alfvénic in the dissipative layer.

4. MHD kink waves in the presence of MHD radiation

4.1. Equations for compressible MHD waves on a non-zero beta plasma cylinder

So far we have seen that kink MHD waves in the thin tube approximation do not care about propagating (body wave) or evanescent (surface wave) behaviour in the internal part of the flux tube. The behaviour in the exterior plasma was until now evanescent. Here we take the next step and consider leakage of energy due to MHD radiation. MHD radiation causes the frequencies to be complex even in absence of resonant damping. MHD waves in the presence of MHD radiation were studied for uniform flux tubes by Spruit (1982) in the TT approximation and by Cally (1985, 2003) for arbitrary values of the radius. Stenuit et al. (1998) and Stenuit et al. (1999) determined MHD waves undergoing resonant absorption and/or leakage for photospheric flux tubes embedded in a non-magnetic surrounding. Stenuit et al. (1999) pointed out which Hankel function to use for leaky and non-leaky waves. We use the equations for linear MHD waves on a 1-dimensional cylinder with a straight field. Effects due to plasma pressure and compressibility are taken into account. The equations are

$$\begin{aligned}
D\frac{\mathrm{d}(r\xi_r)}{\mathrm{d}r} &= -C_2 r P', \\
\rho\left(\omega^2 - \omega_\mathrm{A}^2\right)\xi_r &= \frac{\mathrm{d}P'}{\mathrm{d}r}, \\
\rho\left(\omega^2 - \omega_\mathrm{A}^2\right)\xi_\varphi &= \frac{\mathrm{i}m}{r}P', \\
\rho\left(\omega^2 - \omega_\mathrm{C}^2\right)\xi_z &= \mathrm{i}k_z \frac{v_\mathrm{S}^2}{v_\mathrm{S}^2 + v_\mathrm{A}^2}P', \\
\nabla \cdot \boldsymbol{\xi} &= \frac{-\omega^2\, P'}{\rho\left(v_\mathrm{S}^2 + v_\mathrm{A}^2\right)\left(\omega^2 - \omega_\mathrm{C}^2\right)}.
\end{aligned} \tag{56}$$

The coefficient functions D and C_2 are now

$$D = \rho\left(v_\mathrm{S}^2 + v_\mathrm{A}^2\right)\left(\omega^2 - \omega_\mathrm{A}^2\right)\left(\omega^2 - \omega_\mathrm{C}^2\right),$$

$$C_2 = \omega^4 - \left(v_\mathrm{S}^2 + v_\mathrm{A}^2\right)\left(\omega^2 - \omega_\mathrm{C}^2\right)\left(\frac{m^2}{r^2} + k_z^2\right). \tag{57}$$

As before we rewrite the two first order differential equations of (56) as a second order ordinary differential equation for P':

$$\rho\left(\omega^2-\omega_A^2\right)\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{r}{\rho\left(\omega^2-\omega_A^2\right)}\frac{\mathrm{d}P'}{\mathrm{d}r}\right\}=\left\{\frac{m^2}{r^2}-\Gamma(\omega^2)\right\}rP', \quad (58)$$

where $\Gamma(\omega^2)$ is now defined as

$$\Gamma(\omega^2)=\frac{\left(\omega^2-k_z^2v_S^2\right)\left(\omega^2-\omega_A^2\right)}{\left(v_S^2+v_A^2\right)\left(\omega^2-\omega_C^2\right)}. \quad (59)$$

We have solved the set of Eqs. (56) under general conditions allowing for non-zero plasma pressure and compressibility, see Spruit (1982); Cally (1985) for uniform plasmas and Goossens & Hollweg (1993) for nonuniform plasmas. Here we present the results for a pressureless plasma with $v_S=0$. Equations (56) are then reduced to Eqs. (1) and (59) is reduced to Eq. (8). Sine we aim to study MHD waves that are propagating in the external medium we require that $\omega^2-\omega_{Ae}^2>0$. On the other hand, we want to show that kink waves do not need trapping and since we want to have resonant absorption present in the model we require that $\omega^2-\omega_{Ai}^2<0$. In case of a constant magnetic field then the inequalities imply that $\rho_i<\rho_e$ so that the flux tube is underdense. Note that when we allow $v_S\neq0$ and/or the equilibrium magnetic field to be non-constant then MHD radiation does not necessarily require an underdense loop.

4.2. MHD waves on uniform flux tubes

For a uniform plasma without gas pressure we recover Eq. (9). Now $\Gamma_i(\omega^2)<0$ and $\Gamma_e(\omega^2)>0$ and $\omega_{Ae}^2<\omega^2<\omega_{Ai}^2$. The radial wave numbers are now defined as (see Eq. (13))

$$k_i^2=-\Gamma_i(\omega^2)=-\frac{\omega^2-\omega_{Ai}^2}{v_{Ai}^2},\quad k_e^2=\Gamma_e(\omega^2)=\frac{\omega^2-\omega_{Ae}^2}{v_{Ae}^2}. \quad (60)$$

The solutions to Eq. (9) are now

$$\begin{aligned}P_i'(r)&=\alpha I_m(x),\\ \xi_{r,i}(r)&=\alpha\frac{1}{\rho(\omega^2-\omega_{Ai}^2)}\,k_i\,I_m'(x),\\ P_e'(r)&=\beta H_m^{(1)}(y),\\ \xi_{r,e}(r)&=\beta\frac{1}{\rho(\omega^2-\omega_{Ae}^2)}\,k_iH_m'^{(1)}(y).\end{aligned} \quad (61)$$

The formulation with the Hankel functions is convenient as it enables us to distinguish between incoming and outgoing waves. With the classic convention that $Re(\omega)>0$, and the time dependency $\exp(-\mathrm{i}\omega t)$, the function $H_m^{(1)}$ corresponds to an outgoing wave and $H_m^{(2)}$ to an incoming wave. Hence we drop a possible contribution due to $H_m^{(2)}$ since we are dealing with the eigenvalue problem and not with the driven problem where the incoming wave is prescribed. For the solutions interior in the tube we have taken the Bessel function I_m as in Goossens et al. (1992).

Continuity of total pressure and the radial component of the Lagrangian displacement leads to the dispersion relation:

$$F\,\frac{I_m'(x_0)H_m^{(1)}(y_0)}{I_m(x_0)H_m'^{(1)}(y_0)}=1, \quad (62)$$

where F is defined by (16). Now $x_0=k_iR$ and $y_0=k_eR$. The dispersion relation (62) can be solved numerically. This was done by e.g. Spruit (1982) and Cally (1985, 2003). Using the TT approximation the dispersion relation (62) is reduced to

$$1+F\,\frac{k_e}{k_i}\left\{1+\mathrm{i}\frac{\pi}{2}(k_eR)^2\right\}=0. \quad (63)$$

When we neglect in a zeroth order approximation the effect of MHD radiation the solution to Eq. (63) is again Eq. (18) for the square of the frequency and Eq. (19) for the radial wave numbers.

We can rewrite Eq. (63) correct up to second order in (k_eR) as

$$1+F\,\frac{k_e}{k_i}=\mathrm{i}\frac{\pi}{2}(k_eR)^2. \quad (64)$$

The solution is

$$\begin{aligned}\omega_R&=\omega_k,\\ \frac{\gamma}{\omega_k}&=\frac{\pi}{4}(k_eR)^2\frac{\rho_i\rho_e}{(\rho_i+\rho_e)^2}\frac{\omega_{Ae}^2-\omega_{Ai}^2}{\omega_k^2}.\end{aligned} \quad (65)$$

Equation (65) shows that in order for damping due to wave leakage to occur we need $\omega_{Ae}^2-\omega_{Ai}^2<0$. In case of a constant magnetic field this requires an underdense loop with $\rho_i<\rho_e$. Equation (65) then takes the simple form

$$\frac{\gamma}{\omega_k}=-\frac{\pi}{8}(k_zR)^2\frac{(\rho_e-\rho_i)^2}{(\rho_i+\rho_e)^2}. \quad (66)$$

The ratio of the force due to the pressure gradient to the magnetic tension force is here also given by Eq. (6).

The TT approximations to the eigenfunctions are again given by Eqs. (21). If we allow $v_S\neq0$ then we find a non-zero ξ_z with

$$\frac{\xi_{z,i}(r)}{R}\sim\frac{v_{Si}^2}{v_{Si}^2+v_{Ai}^2}\,(k_zR)\,\frac{r}{R}. \quad (67)$$

4.3. MHD radiating waves on non-uniform flux tubes

Once more we remove the discontinuous variation of density. Now the jump in ξ_r is given by (25), and the modified version of the ideal dispersion relation (62) is

$$F\,\frac{I_m'(x_0)H_m^{(1)}(y_0)}{I_m(x_0)H_m'^{(1)}(y_0)}-\mathrm{i}G\frac{H_m^{(1)}(y_0)}{H_m'^{(1)}(y_0)}=1. \quad (68)$$

The TT approximation to this equation (for $m=1$) is

$$1+F\,\frac{k_e}{k_i}\left\{1+\mathrm{i}\frac{\pi}{2}(k_eR)^2\right\}-\mathrm{i}G(k_eR)\left\{1+\mathrm{i}\frac{\pi}{2}(k_eR)^2\right\}=0. \quad (69)$$

In the third term of the left hand side of the previous equation we can drop $\mathrm{i}\frac{\pi}{2}(k_eR)^2$ for two reasons. First it produces a term of order $(k_eR)^3$, second it leads to a (small) change in the real part of the frequency ω_R which we approximate by ω_k. We can then rewrite (69) correct up to second order in (k_eR) as

$$1+F\,\frac{k_e}{k_i}=\mathrm{i}\frac{\pi}{2}(k_eR)^2+\mathrm{i}G(k_eR). \quad (70)$$

The solution of (70) is

$$\begin{aligned}\omega_R&=\omega_k,\\ \frac{\gamma}{\omega_k}&=-\frac{\pi/2}{\omega_k^2}\frac{1}{R}\frac{\rho_i^2\rho_e^2}{(\rho_i+\rho_e)^3}\frac{(\omega_{Ai}^2-\omega_{Ae}^2)^2}{\rho(r_A)\mid\Delta_A\mid}\\ &\quad+\frac{\pi}{4}(k_eR)^2\frac{\rho_i\rho_e}{(\rho_i+\rho_e)^2}\frac{\omega_{Ae}^2-\omega_{Ai}^2}{\omega_k^2}.\end{aligned} \quad (71)$$

In case of a constant magnetic field so that the variation of the local Alfvén frequency is solely due to a variation of density (71) can be further simplified to

$$\frac{\gamma}{\omega_{\mathrm{k}}} = -\frac{\pi}{8}\frac{l}{R}\frac{\rho_{\mathrm{e}}-\rho_{\mathrm{i}}}{\rho_{\mathrm{i}}+\rho_{\mathrm{e}}} - \frac{\pi}{8}(k_z R)^2 \frac{(\rho_{\mathrm{e}}-\rho_{\mathrm{i}})^2}{(\rho_{\mathrm{i}}+\rho_{\mathrm{e}})^2} < 0. \tag{72}$$

Equation (72) is derived for a linear variation of density. For a sinusoidal variation the factor $\pi/8$ is to be replaced with $1/4$. The absolute value of the ratio of the decrement due to resonant absorption (first term of Eq. (72)) and that due to MHD radiation (second term of Eq. (72)) is

$$\frac{l/R}{(k_z R)^2}\frac{\rho_{\mathrm{e}}+\rho_{\mathrm{i}}}{\rho_{\mathrm{e}}-\rho_{\mathrm{i}}}\,. \tag{73}$$

This clearly indicates that damping due to resonant absorption dominates over that due to MHD radiation (since $k_z R \ll 1$). For example, for a tube with $\rho_{\mathrm{e}}/\rho_{\mathrm{i}} = 3$ and $R/L = 0.01$ then a very tiny layer of $l/R = 5\times10^{-4}$ is enough for resonant absorption to dominate over radiation.

The remarkable result is that the frequency of the kink wave and its damping due to resonant absorption found for a compressible plasma with a non-zero plasma pressure differ by terms of order $(k_z R)^2$, even when we allow MHD radiation. If we neglect contributions proportional to $(k_z R)^2$ then the simple conclusion is that the frequency of the kink wave and its damping due to resonant absorption are the same in the three cases that we have considered.

Again, as in the two previous cases, the eigenfunctions in the thin dissipative layer can be described by the functions $\tilde{F}(\tau)$ and $\tilde{G}(\tau)$. Again the conclusion is that kink MHD waves are highly Alfvénic in the dissipative layer.

5. Conclusion

This paper has examined the nature of MHD kink waves. This was done by determining the frequency, the damping rate and in particular the eigenfunctions of MHD kink waves for three widely different MHD waves cases: a compressible pressureless plasma, an incompressible plasma and a compressible plasma which allows for MHD radiation. The overall conclusion is that kink waves are very robust and do not care about the details of the MHD wave environment. In all three cases the frequency and the damping rate are for practical purposes the same as they differ at most by terms proportional to $(k_z R)^2$. In the magnetic flux tube the kink waves are in all three cases, to a high degree of accuracy incompressible waves with negligible pressure perturbations and with mainly horizontal motions. The main restoring force of kink waves in the magnetised flux tube is the magnetic tension force. The gradient pressure force cannot be neglected except when the frequency of the kink wave is equal or slightly differs from the local Alfvén frequency, i.e. in the resonant layer. The adjective fast is not the correct adjective to characterise kink waves. If an adjective is to be used it should be Alfvénic. However, it is better to realize that kink waves have mixed properties and cannot be put in one single box.

Acknowledgements. This research was begun when M.G. was a visitor of the Solar Physics Group at the UIB. It is pleasure for M.G. to acknowledge the warm hospitality of the Solar Physics Group at the UIB and the support received from UIB. M.G. and J.T. also acknowledge support from K.U.Leuven via GOA/2009-009. J.A. was supported by an International Outgoing Marie Curie Fellowship within the 7th European Community Framework Programme. In addition, J.T., I.A. and J.L.B. acknowledge the funding provided under projects AYA2006-07637 (Spanish Ministerio de Educación y Ciencia) and PCTIB2005GC3-03 (Conselleria d'Economia, Hisenda i Innovació of the Government of the Balearic Islands).

References

Andries, J., & Goossens, M. 2001, A&A, 368, 1083
Andries, J., Tirry, W. J., & Goossens, M. 2000, ApJ, 531, 561
Appert, K., Gruber, K., & Vaclavik, J. 1974, Phys. Fluids, 17, 1471
Arregui, I., Van Doorsselaere, T., Andries, J., Goossens, M., & Kimpe, D. 2005, A&A, 441, 361
Arregui, I., Andries, J., Van Doorsselaere, T., Goossens, M., & Poedts, S. 2007, A&A, 463, 333
Aschwanden, M. J., Fletcher, L., Schrijver, C. J., & Alexander, D. 1999, ApJ, 520, 880
Cally, P. S. 1985, Sol. Phys., 103, 277
Cally, P. S. 2003, Sol. Phys., 217, 95
De Pontieu, B., McIntosh, S. W., Carlsson, M., et al. 2007, Science, 318, 1574
Edwin, P., & Roberts, B. R. 1983, Sol. Phys., 88, 179
Goossens, M. 2008, Proc. IAU Symp., 247, 228
Goossens, M., & Hollweg, J. V. 1993, Sol. Phys., 145, 19
Goossens, M., & Ruderman, M. S. 1995, Phys. Scr., T60, 171
Goossens, M., Hollweg, J. V., & Sakurai, T. 1992, Sol. Phys., 138, 233
Goossens, M., Ruderman, M. S., & Hollweg, J. V. 1995, Sol. Phys., 157, 75
Goossens, M., Arregui, I., Ballester, J. L., & Wang, T. J. 2008, A&A, 484, 851
Hollweg, J. V., & Yang, G. 1988, J. Geophys. Res., 93, 5423
Keppens, R., Bogdan., T. J., & Goossens, M. 1994, ApJ, 436, 372
Nakariakov, V. M., Ofman, L., DeLuca, E. E., Roberts, B. R., & Davila, J. M. 1999, Science, 285, 862
Okamoto, T. J., Tsuneta, S., Berger, T. E., et al. 2007, Science, 318, 1577
Ruderman, M. S., & Roberts, B. R. 2002, ApJ, 577, 475
Ruderman, M. S., Tirry, W., & Goossens, M. 1995, J. Plasma Phys., 54, 129
Sakurai, T., Goossens, M., & Hollweg, J. V. 1991a, Sol. Phys., 133, 227
Sakurai, T., Goossens, M., & Hollweg, J. V. 1991b, Sol. Phys., 133, 247
Sewell, G. 2005, The numerical Solution of Ordinary and Partial Differential Equations (Hoboken: Wiley & Sons)
Soler, R., Oliver, R., Ballester, J. L., & Goossens, M. 2009, ApJ, 695, L166
Spruit, H. 1982, Sol. Phys., 75, 3
Stenuit, H., Erdelyi, R., & Goossens, M. 1995, Sol. Phys., 161, 139
Stenuit, H., Keppens, R., & Goossens, M. 1998, A&A, 331, 392
Stenuit, H., Tirry, W. J., Keppens, R., & Goossens, M. 1999, A&A, 342, 863
Terradas, J., Oliver, R., & Ballester, J. L. 2006, ApJ, 642, 533
Terradas, J., Andries, J., Goossens, M., et al. 2008, ApJ, 687, L115
Tirry, W. J., & Goossens, M. 1996, ApJ, 471, 501
Tirry, W. J., Cadez, V. M., Erdelyi, R., & Goossens, M. 1998a, A&A, 332, 786
Tirry, W. J, Goossens, M., Pinter, B., Cadez, V., & VanLommel, P. 1998b, ApJ, 503, 422
Tomczyk, et al. 2007, Science, 318, 1192
Van Doorsselaere, T., Andries, J., Poedts, D., & Goossens, M. 2004, ApJ, 606, 1223
Van Doorsselaere, T., Nakariakov, V. M., & Verwichte, E. 2008, ApJ, 676, L73