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The Paczyński-Wiita potential. A step-by-step "derivation"

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Paczyński realized that a properly chosen gravitational potential may accurately model (in a "pseudo Newtonian" theory) general relativistic effects that determine the motion of matter near a nonrotating black hole. Paczyński's choice, known today as the "Paczyński-Wiita potential", proved to be very practical. It was used by numerous researchers in the black-hole accretion theory and became a standard tool in relativistic astrophysics. The model is an example of Paczyński's admired ability to invent simple ideas that were brilliant, deep, and useful "out of nowhere". Paczyński has intuitively guessed the form of the potential. However, it could be also derived by a step-by-step formal procedure. I show the derivation here that is based on a standard definition of the relativistic "effective potential" in the Schwarzschild spacetime. The relativistic effective potential may be uniquely divided into its "gravitational" and "centrifugal" part. The gravitational part only differs from the Paczyński-Wiita potential by a constant.

Abramowicz et al. (1978), working in Paczyński's research group in Warsaw, found a practical mathematical scheme to construct fully relativistic models of thick accretion disks, known today as "Polish doughnuts". The models displayed several astrophysically interesting features, among them seriously super-Eddington luminosities, long and narrow funnels that could collimate jets, and a self-crossing "Roche lobe" equipotential. The scheme developed in Warsaw was suitable for both analytic and numerical calculations.

Some properties of the thick disks were obviously connected to the strong-field effects of general relativity. Our leader, Bohdan Paczyński, who was not familiar with the technicalities of general relativity, asked me to find a Newtonian way to describe these effects. I was rather unhappy about Paczyński's request, because initially I wrongly imagined that the only method adequate for the task should be the post-Newtonian scheme. It uses tedious, long, and boring expansions. I was desperately working, producing longer and longer formulae, when one day Paczyński came to my office, and said "Stop working on that. I found the solution." And he showed me his solution – a Newtonian[1] potential,

$$\Phi(r) = -\frac{GM}{r - r_{\rm G}}, \qquad r_{\rm G} = \frac{2GM}{c^2}, \tag{1}$$

where r is the spherical radius, M the mass of the black hole, and $r_{\rm G}$ its gravitational radius. Paczyński had checked that the two most important radii that characterize circular Keplerian orbits, the radius of a marginally stable orbit $r_{\rm ms}$ (i.e. ISCO) and the radius of a marginally bound orbit $r_{\rm mb}$, have the same values in Newton's gravity with his potential (1) as in Einstein's gravity in the Schwarzschild metric,

$$r_{\rm ms} = 3\, r_{\rm G}, \quad r_{\rm mb} = 2\, r_{\rm G}. \tag{2}$$

This was a brilliant display of the qualities of Paczyński's mind: he just *guessed* the right, simple, and powerful solution to the problem. His solution immediately proved to be very practical. Shortly afterwards, Paczyński & Wiita (1980) used (1) to numerically calculate the models of thick disks. The models differed from those calculated with the full strength of general relativity by only a few percent. Later, this opened a flood gate when numerous authors used the Paczyński-Wiita potential in their calculations of black-hole accretion flows. The potential is so remarkably successful that some researchers use it even outside its obvious limits of applicability: (a) for rotating black holes, which is wrong because (1) does not include the Lense-Thirring effect; and (b) for self-gravitating fluids, which is wrong because $\nabla^2\Phi \neq 0$.

Why is the Newtonian Paczyński-Wiita potential (1) such an accurate model of the strong relativistic effects? Should this be considered a fortunate, unexpected coincidence, or could one "derive" the potential from the first principles of Einstein's general relativity? I remember discussing this question briefly with Thibault Damour in late 1970. Although we were convinced that the "effective potential approach" should provide such a derivation, we had not completed the relevant calculations. I summarize them here.

In Newtonian theory, let E be energy, L angular momentum, $\Phi(r)$ gravitational potential, and V radial velocity. The orbital motion is often described in terms of the effective potential $U(r, L) = \Phi(r) + L^2/2r^2$,

$$\frac{1}{2}\, V^2 = E - U(r, L), \tag{3}$$

circular orbits located at the effective potential extrema,

$$\left(\frac{\partial U}{\partial r}\right)_L = 0, \tag{4}$$

or in terms of the gravitational potential Φ,

$$\left(\frac{{\rm d}\Phi}{{\rm d}r}\right) - L^2\left(\frac{1}{r^3}\right) = 0. \tag{5}$$

[1] It is often called "pseudo Newtonian" to stress that it does not obey the Poisson equation. However, when the external gravity is fixed (as in the black-hole accretion theory), "pseudo Newtonian" is practically equivalent to "Newtonian". This is why I am using both terms here.

Let us consider almost circular motion of particles on the $\theta = \pi/2$ "equatorial" plane in the Schwarzschild spacetime. From the radial component of the four-velocity u^r, let us construct a positive small quantity $V^2 \equiv (u_r)^2 g^{rr} \ll 1$. It is known that u_t and u_ϕ are constants of motion, therefore $L \equiv -u_\phi/u_t$ is also a constant of motion (the specific angular momentum). The $1 = u_i u_k g^{ik} = (u_t)^2 g^{tt} + (u_\phi)^2 g^{\phi\phi} + (u_r)^2 g^{rr}$ condition may be written in the form,

$$\frac{1}{2}\ln(1 + V^2) = \ln(u_t) + \frac{1}{2}\ln\left[g^{tt} + L^2 g^{\phi\phi}\right]. \tag{6}$$

Expansion of the lefthand side yields $V^2/2$. One also defines $E \equiv \ln(u_t)$, and

$$U(r, L) \equiv -\frac{1}{2}\ln\left[g^{tt} + L^2 g^{\phi\phi}\right]. \tag{7}$$

This brings Eq. (6) into a form identical with the Newtonian formula (3). Thus, the Newtonian condition (4) for the vanishing derivative of the effective potential may be applied to the relativistic effective potential (7), which gives

$$\left(\frac{\mathrm{d}g^{tt}}{\mathrm{d}r}\right) + L^2\left(\frac{\mathrm{d}g^{\phi\phi}}{\mathrm{d}r}\right) = 0. \tag{8}$$

Because at the equatorial plane $g^{\phi\phi} = -1/r^2$ and $g^{tt} = r/(r - r_G)$, this may be written in the form,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{GM}{r - r_G}\right) - L^2\left(\frac{1}{r^3}\right) = 0. \tag{9}$$

Comparing Newton's Eq. (5) with Einstein's Eq. (9), we see that the gravitational potential in both equations has to have the same Paczyński-Wiita form (1). In deriving Eq. (9) we used

$$g^{tt} = \frac{r}{r - r_G} = \frac{r_G}{r - r_G} + 1 = 2\,\Phi(r) + 1. \tag{10}$$

Thus, the Keplerian angular momentum derived (in the Schwarzschild spacetime) according to Einstein's theory, and derived with the Paczyński-Wiita potential, are both given by the same formula

$$L_K^2 = \frac{GMr^3}{(r - r_G)^2}\,. \tag{11}$$

In Newton's theory the angular momentum L and angular velocity Ω are connected by $L = \Omega r^2$, but in Schwarzschild geometry by $L = \Omega r^2/(1 - r_G/r)$. Therefore, the Keplerian angular velocity calculated in Schwarzschild geometry and in the Paczyński-Wiita potential are *not* the same,

$$\Omega_{\rm SCH} = \left(\frac{GM}{r^3}\right)^{1/2} \neq \left(\frac{GM}{r^3}\right)^{1/2}\left(\frac{r}{r - r_G}\right) = \Omega_{\rm PW}. \tag{12}$$

Nowak & Wagoner (1991) found that the potential given by a fitting formula

$$\Phi_{\rm NW} = -\left(\frac{GM}{r}\right)\left[1 - 3\frac{GM}{r\,c^2} + 12\left(\frac{GM}{r\,c^2}\right)^2\right] \tag{13}$$

reproduces the angular velocity $\Omega(r)$ and the radial epicyclic frequency $\kappa(r)$ *better* than the Paczyński-Wiita potential. The fitting formula used by Kluźniak & Lee (2002) reproduces the ratio $\kappa(r)/\Omega(r)$ of these frequencies *exactly*:

$$\Phi_{\rm KL} = \left(\frac{GM}{3r_G}\right)\left[1 - e^{3r_G/r}\right]. \tag{14}$$

Semerák & Karas (1999) discuss the Newtonian potential suitable for modeling the gravity of the Kerr black hole, including the Lense-Thirring effect, and Stuchlík & Kovář (2008) for the Schwarzschild-de Sitter spacetime.

Neither these four potentials nor a few other potentials introduced by some other authors have become popular. Nowadays many more astrophysicists know Einstein's general relativity than in the late 1970s, but quotations of Paczyński-Wiita potential show no sign of declining[2].

Velocities of matter calculated with the Paczyński-Wiita potential could exceed the light speed. This creates a serious problem when one calculates the observed appearance of matter (e.g. spectra) by the method of ray tracing. Abramowicz et al. (1996) found a solution to this problem by showing how to incorporate the effects of special relativity into the Paczyński-Wiita scheme. One should interpret the "true" physical velocities in terms of the calculated ones by $V_{\rm cal} = V_{\rm tru}/(1 - V_{\rm tru}^2/c^2)$. Here $V_{(\dots)}$ denotes each of the three components of the velocity, i.e. V_r, V_θ, V_ϕ.

The Paczyński-Wiita potential (1) accurately models general relativistic effects in the Newtonian theory that determines the motion of matter near a nonrotating black hole. The Paczyński-Wiita potential is neither an approximation of relativistic gravity nor a fitting formula. Instead, it is a unique ("pseudo") Newtonian model of the gravity of a nonrotating black hole. It reproduces *exactly*:

– the location of the marginally stable orbit $r_{\rm ms}$ = ISCO,
– the location of the marginally bound orbit $r_{\rm mb}$,
– the form of the Keplerian angular momentum $L(r)$.

It also reproduces accurately, but not exactly, the form of the Keplerian angular velocity $\Omega(r)$ and of the radial epicyclic frequency $\kappa(r)$.

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References

Abramowicz, M. A., Jaroszyński, M., & Sikora, M. 1978, A&A, 63, 221
Abramowicz, M. A., Beloborodov, A. M., Chen, X.-M., & Igumenshchev, I. V. 1996, A&A, 313, 334
Kluźniak, W., & Lee, W. H. 2002, MNRAS, 335, L29
Nowak, M. A., & Wagoner, R. V. 1991, ApJ, 378, 656
Paczyński, B., & Wiita, P. J. 1980, A&A, 88, 23
Semerák, O., & Karas, V. 1999, A&A, 343, 325
Stuchlík, Z., & Kovář, J. 2008, Int. J. Mod. Phys. D, 17, 2089

[2] Number of quotes in years 2000–2008, according to ADS: 21, 40, 32, 37, 45, 39, 30, 37, 46.