Linear force-free field of a toroidal symmetry (Research Note)

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ABSTRACT

Context. Interplanetary flux ropes are often described by linear force-free fields. To account for their curvature, toroidal configurations valid for large aspect ratios are used. However, in some cases, flux ropes need to be approximated by a toroid with a low aspect ratio. **Aims.** The aim is to find an analytical description of a linear force-free magnetic field of toroidal geometry, not restricted to large aspect ratios.

Methods. The solution is found as a superposition of fields given by linear force-free cylinders tangential to a generating toroid.

Results. The obtained solution describes toroidal flux ropes with reasonable shapes and magnetic field values. The field is exactly linear force-free for arbitrary aspect ratios. The new solution can be applied for solar and interplanetary flux ropes, astrophysical objects, and laboratory plasma.

Key words. magnetic fields – solar wind – Sun: coronal mass ejections (CMEs)

1. Introduction

Interplanetary magnetic flux ropes are commonly observed in the solar wind. The term magnetic cloud was proposed by Klein & Burlaga (1982) for such larger objects with distinct properties of higher magnetic field, smooth rotation of the magnetic field vector, and low proton temperature of the solar wind plasma. They have a loop-like shape, so they could be described by a toroidal flux rope.

The field of application of toroidal force-free fields is not only interplanetary magnetic structures or solar flux ropes, but also laboratory plasma and various astrophysical objects. For example, symbiotic stars R Aqr contain a white dwarf with an X-ray jet. The flow in the jet is curved (Ragland et al. 2008) and is governed by a strong magnetic field of the main star which is of the order of 10^5 T (Hollis & Koupelis 2000). Taking into account that temperature of the flow plasma is about 10^6 K and the density is of the order of 10^3 cm⁻³, one can infer that there is a force-free plasma in the flow.

Interplanetary magnetic clouds are usually modelled locally as straight cylinders with a linear force-free field (e.g., Burlaga 1988; Lepping et al. 1990, 2006). Globally, however, magnetic clouds as interplanetary flux ropes have a loop-like shape (Lepping et al. 1990), i.e., they are curved. In such cases, where curvature could play a role, approximation of interplanetary flux ropes by a part of a toroid would be more appropriate. Marubashi (1997) was first to discuss the implications of this global shape for the interpretation of magnetic cloud observations and approximated clouds by a toroidal flux rope. Before that, a first attempt to fit magnetic clouds with a toroidal configuration was made by Ivanov et al. (1989), who used the linear force-free

solution obtained in toroidally cylindrical coordinates by Miller & Turner (1981). This solution is restricted to a large aspect ratio of the toroid. Romashets & Vandas (2003a) have found an exact force-free solution in a toroid, which is not linear. It has been used for the interpretation of interplanetary flux rope observations by Romashets & Vandas (2003a, b) and Marubashi & Lepping (2007). But this field does not fulfill the solenoidality condition in general; it is applicable only for larger toroids and larger aspect ratios. However, in some cases, the aspect ratio may be small, e.g., in parts of interplanetary flux ropes, as MHD simulation results indicate (Vandas et al. 2002). This lead us to seek a new solution which would not be restricted to a large aspect ratio. It is presented in this paper.

2. Linear force-free solution

Force-free fields fulfill the condition $\text{rot } \mathbf{B} = \alpha \mathbf{B}$, where α is a scalar. The field is called linear when α is a constant; in this case, the field fulfills the solenoidality condition $\text{div } \mathbf{B} = 0$ automatically.

The simplest solution of an exact linear force-free configuration in cylindrical geometry is the Lundquist solution (Lundquist 1950):

$$B_\rho = 0, \quad (1)$$

$$B_\phi = B_0 J_1(\alpha \rho), \quad (2)$$

$$B_Z = B_0 J_0(\alpha \rho), \quad (3)$$

where ρ , ϕ , and Z are cylindrical coordinates, B_0 scales the magnetic field magnitude, and J_0 and J_1 are the Bessel functions. Usually α is related to the flux rope boundary $r = r_0$ by

Fig. 1. Construction of a toroidal solution.

$\alpha = 2.41/r_0$, where 2.41 stands for the first root of J_0 . Then it holds that $B_z = 0$ at the boundary. This solution will be used for the construction of an exact linear force-free solution in a toroidal geometry.

Consider a generating toroid with a small radius r_0 and a large radius R_0 (Fig. 1) described in Cartesian coordinates x , y , and z centered on the symmetry point O of the toroid and with the z -axis along the toroid's rotational axis. We assume a cylinder (with radius r_0) the symmetry axis (z') of which is lying in the xy -plane and is tangential to the central toroid's circle (with the radius R_0). The magnetic field $\tilde{\mathbf{B}}$ in the cylinder and outside it will be given by the Lundquist solution (1)–(3). Figure 1 shows one such cylinder with a local coordinate system (x' , y' , z') and another cylinder at a different location. A new field $\mathbf{B}$ will be constructed as a sum of the fields $\tilde{\mathbf{B}}$ related to all such possible cylinders, each of which with an infinitesimal contribution. Because each cylinder describes a linear force-free field (with the same α), the sum of these fields will also be a linear force-free field (and with the same α). In the local system, the field $\tilde{\mathbf{B}}$ related to the cylinder is given by Eqs. (1)–(3) in local cylindrical coordinates. Expressed in Cartesian coordinates, it reads

$$\tilde{B}_{x'} = -B_0 J_1 \left(\alpha \sqrt{x'^2 + y'^2} \right) \frac{y'}{\sqrt{x'^2 + y'^2}}, \quad (4)$$

$$\tilde{B}_{y'} = B_0 J_1 \left(\alpha \sqrt{x'^2 + y'^2} \right) \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad (5)$$

$$\tilde{B}_{z'} = B_0 J_0 \left(\alpha \sqrt{x'^2 + y'^2} \right). \quad (6)$$

The “total” magnetic field $\mathbf{B}$ will be a superposition of the magnetic fields $\tilde{\mathbf{B}}$ of all tangential cylinders along the toroid's circular axis as infinitesimal contributions proportional to $d\varphi$, where φ is the angle from the x -axis to y' -axis (in positive sense). The total magnetic field $\mathbf{B}$ is obtained by integration of $\tilde{\mathbf{B}}$ over φ :

$$B_x = B_0 \int_0^{2\pi} \left[J_1(\alpha\rho) \frac{z \cos \varphi}{\rho} - J_0(\alpha\rho) \sin \varphi \right] d\varphi, \quad (7)$$

$$B_y = B_0 \int_0^{2\pi} \left[J_1(\alpha\rho) \frac{z \sin \varphi}{\rho} + J_0(\alpha\rho) \cos \varphi \right] d\varphi, \quad (8)$$

$$B_z = -B_0 \int_0^{2\pi} J_1(\alpha\rho) \frac{x \cos \varphi + y \sin \varphi - R_0}{\rho} d\varphi, \quad (9)$$

where $\rho = \sqrt{(x \cos \varphi + y \sin \varphi - R_0)^2 + z^2}$.

It is not obvious from Eqs. (7)–(9) that there is a rotational symmetry with respect to the z axis, so the magnetic components are functions only of two arguments, $r = \sqrt{x^2 + y^2}$ and z .

Therefore we present the components in the cylindrical coordinate system r , $\phi = \arctan(y/x)$, and z (which is identical to the Cartesian axis z and is a rotational axis of the toroid). The B_r component reads

$$B_r = B_x \cos \phi + B_y \sin \phi \\ = B_0 \int_0^{2\pi} \left[J_1(\alpha\rho) \frac{z \cos(\varphi - \phi)}{\rho} - J_0(\alpha\rho) \sin(\varphi - \phi) \right] d\varphi,$$

where $\rho = \sqrt{[r \cos(\varphi - \phi) - R_0]^2 + z^2}$. Applying the substitution $\tilde{\varphi} = \varphi - \phi$, 2π periodicity of the integrand in $\tilde{\varphi}$, and the fact that the second term under the integral is an odd function of $\tilde{\varphi}$, we get the following expression for B_r (and similarly for the remaining components):

$$B_r = B_0 \int_0^{2\pi} J_1(\alpha\rho) \frac{z \cos \tilde{\varphi}}{\rho} d\tilde{\varphi}, \quad (10)$$

$$B_\phi = B_0 \int_0^{2\pi} J_0(\alpha\rho) \cos \tilde{\varphi} d\tilde{\varphi}, \quad (11)$$

$$B_z = -\frac{r}{z} B_r + B_0 \int_0^{2\pi} J_0(\alpha\rho) \frac{R_0}{\rho} d\tilde{\varphi}, \quad (12)$$

where $\rho = \sqrt{(r \cos \tilde{\varphi} - R_0)^2 + z^2}$. One can see that the components (10)–(12) are functions of two arguments only, r and z .

To integrate the formulae (10)–(12), we use the definition for the Bessel function

$$J_m(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{s}{2} \right)^{2k+m}. \quad (13)$$

Applying the binomial formula twice for each component and using the identity for even powers of cosine (it is zero for odd powers)

$$\int_0^{2\pi} \cos^{2m} \tilde{\varphi} d\tilde{\varphi} = \frac{(2m-1)! \pi}{2^{2m-2} m! (m-1)!}, \quad (14)$$

we get

$$B_r = B_0 \frac{\pi \alpha r z}{2 R_0} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \left(\frac{\alpha R_0}{2} \right)^{2k} \sum_{j=1}^k \frac{(2j)!}{j!(k-j)!} \left(\frac{z}{R_0} \right)^{2k-2j} \\ \times \sum_{i=0}^{j-1} \frac{1}{i!(i+1)!(2j-2i-1)!} \left(\frac{r}{2 R_0} \right)^{2i}, \quad (15)$$

$$B_\phi = B_0 \frac{\pi r}{R_0} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \left(\frac{\alpha R_0}{2} \right)^{2k} \sum_{j=1}^k \frac{(2j)!}{j!(k-j)!} \left(\frac{z}{R_0} \right)^{2k-2j} \\ \times \sum_{i=0}^{j-1} \frac{1}{i!(i+1)!(2j-2i-1)!} \left(\frac{r}{2 R_0} \right)^{2i}, \quad (16)$$

$$B_z = -\frac{r}{z} B_r + B_0 \pi \alpha R_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \left(\frac{\alpha R_0}{2} \right)^{2k} \left\{ \frac{1}{k!} \left(\frac{z}{R_0} \right)^{2k} \right. \right. \\ \left. \left. + \sum_{j=1}^k \frac{1}{j!(k-j)!} \left(\frac{z}{R_0} \right)^{2k-2j} \right. \right. \\ \left. \left. \times \left[1 + \sum_{i=1}^j \frac{(2j)!}{(i!)^2 (2j-2i)!} \left(\frac{r}{2 R_0} \right)^{2i} \right] \right\} \right\}. \quad (17)$$

Fig. 2. Magnetic field distribution and field lines of toroidal flux ropes. The figures show cross sections through two flux ropes. The dashed circles are cross sections of generating toroids with aspect ratios of 2 a) and 4 b).

3. Analysis of the new solution

Equations (15)–(17) represent an analytical solution for a toroidal magnetic field which is exactly solenoidal and force-free with constant α for any aspect ratio. However, for numerical calculations it is more suitable (and faster) to use expressions (7)–(9) or (10)–(12) directly; the series (13) can be reasonably used for numerical purposes only with arguments up to $|s| < 8$.

Figure 2 shows magnetic field distributions of the new solution and several magnetic field lines for two aspect ratios. The constant α in the formulae (7)–(9) was set to $\alpha = 2.41/r_0$ (see the text above on the Lundquist solution). The field has an azimuthal symmetry, so the distribution is displayed only in one plane, $y = 0$. Of course, the field is defined over all the plane, but distributions in the figures are restricted only to regions forming simple flux ropes (for clearness). So the distributions represent cross sections through the toroidal flux ropes. Magnetic field lines have helical forms and are defined by $rB_\phi = \text{const}$. Each

Fig. 3. Profiles of the magnetic field magnitude B and magnetic field components B_y and B_z through the flux ropes shown in Fig. 2. They are profiles along the x -axis, the solid line is for the flux rope with $R_0/r_0 = 2$ and the dashed line for $R_0/r_0 = 4$. The dotted line shows the Lundquist solution for comparison. The B_x component is not displayed because it is zero for all the cases. The profiles are scaled by the respective maximum value of B in the plot.

shown closed line may form a boundary of a toroidal flux rope. Flux ropes do not have a circular cross section, it is prolate and tear-drop like for larger aspect ratios. This means that the solution does not tend to the Lundquist-type solution for large aspect ratios. On the other hand, profiles of magnetic field components are similar to the Lundquist solution. This is shown in Fig. 3. The profiles are scaled by the respective maximum value B_{max} of the field in the ropes (therefore each B -plot in the top panel reaches a value of 1). The Lundquist solution is displayed for comparison by dotted lines and its profiles are shifted so that its B -maximum is close to B -maxima of the displayed flux ropes. B -maximum, which is shifted towards the toroid's hole from the flux-rope axis, becomes closer to the axis for larger aspect ratios.

4. Modification of the solution

The presented solution has one shortfall in its application to interpretation of interplanetary flux ropes, namely a prolate shape. It is known that interplanetary flux ropes are oblate (Russell et al. 2002; Hu & Sonnerup 2002; Hidalgo et al. 2002; Vandas et al. 2005). To treat this case, we apply our solution of a linear force-free field in an elliptic cylinder (Vandas & Romashets 2003) and use it in the procedure described in Sect. 2. Instead of circular cylinders, elliptic cylinders are involved with the semiminor axis ($b = r_0$) along the y' -axis (Fig. 1) and the semimajor axis (a) along the x' -axis. The linear force-free solution in elliptic cylindrical coordinates reads

$$B_u = \frac{1}{\sqrt{\epsilon(\cosh^2 u - \cos^2 v)}} \frac{\partial B_z}{\partial v}, \quad (18)$$

$$B_v = -\frac{1}{\sqrt{\varepsilon(\cosh^2 u - \cos^2 v)}} \frac{\partial B_Z}{\partial u}, \quad (19)$$

$$B_Z = B_0 \frac{\text{ce}_0(u, -\varepsilon/32) \text{ce}_0(v, -\varepsilon/32)}{\text{ce}_0^2(0, -\varepsilon/32)}, \quad (20)$$

where $u \in \langle 0, \infty \rangle$, $v \in \langle 0, 2\pi \rangle$, and Z are elliptic cylindrical coordinates defined by

$$x' = c \cosh u \cos v, \quad (21)$$

$$y' = c \sinh u \sin v, \quad (22)$$

$$z' = Z, \quad (23)$$

and $c = \sqrt{a^2 - b^2}$. The parameter $\varepsilon = (\alpha c)^2$ is related to the size and oblateness of the flux rope; ce_0 and ce_h are angular and radial even Mathieu functions of zero order. Contours with $u = \text{const.}$ are ellipses. The generating ellipse ($u = u_0$) with semimajor axis a and semiminor axis b has $\cosh u_0 = a/c$. Similarly to the Lundquist solution, α is determined by the condition $B_Z = 0$ at the generating ellipse. The details of the solution and a method of its evaluation are given in Vandas & Romashets (2003). Rotation of the generating ellipse around the z -axis creates a generating toroid with an oblate cross section.

Following the procedure described in the previous section, the field (18)–(20) is expressed in the local Cartesian coordinates (x', y', z') in analogy to Eqs. (4)–(6), then transformed into the global system (x, y, z) and integrated over φ (similarly to Eqs. (7)–(9)). The resulting field is a linear force-free field, because the generating field (18)–(20) is, also for arbitrary aspect ratio and oblateness. Due to the complexity of the transformations and the presence of special functions, the whole procedure is implemented numerically for field determination.

Figure 4 shows magnetic field distributions and several magnetic field lines in a similar way as in Fig. 2; in addition to aspect ratios, there is a new parameter, oblateness of the generating ellipse a/b . One can see how the shape of flux ropes changes with oblateness. In fact, there is a smooth transition of shapes from the one shown in Fig. 2b into those shown in Fig. 4, because the solution (1)–(3) is a limiting case of Eqs. (18)–(20) for $a/b \rightarrow 1$. Figure 4 also demonstrates that B -maxima are shifted towards the toroid's hole from the flux-rope axis, in accordance with Fig. 2.

5. Conclusions

Exact linear force-free magnetic field distributions in toroidal geometry were derived. Originally we intended them for the interpretation of interplanetary magnetic clouds and solar flux ropes, but they can be applied to laboratory plasma and astrophysical objects as well.

Fig. 4. Magnetic field distribution and field lines of toroidal flux ropes. The figures show cross sections through two flux ropes. The layout is the same as in Fig. 2. The dashed ellipses are cross sections of generating toroids with aspect ratios/oblatenesses of 4/1.25 a) and 4/2 b).

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