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Luminosity of a quark star undergoing torsional oscillations and the problem of γ-ray bursts

J. Heyvaerts[3], S. Bonazzola[1], M. Bejger[1,2], and P. Haensel[2]

[1] LUTh, UMR 8102 du CNRS, Pl. Jules Janssen, 92195 Meudon, France
[2] N. Copernicus Astronomical Center, Polish Academy of Sciences, Bartycka 18, 00-716 Warszawa, Poland
[3] Observatoire Astronomique de Strasbourg, 11 rue de l'Université, 67000 Strasbourg, France
e-mail: heyvaerts@newb6.u-strasbg.fr

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ABSTRACT

Aims. We discuss whether the winding-up of the magnetic field by differential rotation in a new-born quark star can produce a sufficiently-high, energy, emission rate of sufficiently long duration to explain long gamma-ray bursts.
Methods. In the context of magnetohydrodynamics, we study the torsional oscillations and energy extraction from a new-born, hot, differentially-rotating quark star.
Results. The new-born compact star is a rapid rotator that produces a relativistic, leptonic wind. The star's torsional oscillation modulates this wind emission considerably when it is odd and of sufficient amplitude, which is relatively easy to reach. Odd oscillations may occur just after the formation of a quark star. Other asymmetries can cause similar effects. The buoyancy of wound-up magnetic fields is inhibited, or its effects are limited, by a variety of different mechanisms. Direct electromagnetic emission by the torsional oscillation in either an outside vacuum or the leptonic wind surrounding the compact object is found to be insignificant. In contrast, the twist given to the outer magnetic field by an odd torsional oscillation is generally sufficient to open the star's magnetosphere. The Poynting emission of the star in its leptonic environment is then radiated from all of its surface and is enhanced considerably during these open episodes, tapping at the bulk rotational energy of the star. This results in intense energy shedding in the first tens of minutes after the collapse of magnetized quark stars with an initial poloidal field of order of 10^{14} Gauss, sufficient to explain long gamma-ray bursts.

Key words. gamma rays: bursts – elementary particles – stars: general – magnetic fields – magnetohydrodynamics (MHD) – pulsars: general

1. Introduction and motivation

Understanding the physical nature of γ-ray bursts (GRBs) in a way that is consistent with observations of their entire evolution remains a challenging mystery. A vast literature on the subject exists and a large number of models have been proposed to explain this phenomenon (for a review see e.g., Zhang & Mészáros 2004; and Mészáros 2006). In this paper, we limit ourselves to the case of long GRBs, of duration of between about 10 s and 1000 s. There are several basic facts to be explained. First, the release of about 10^{49}–10^{51} erg in γ-rays, of mean power higher than 10^{48} erg s^{-1}. The violent energy outflow, which eventually transforms into a GRB, must originate in a compact volume of linear size $\sim 10^6$ cm, because of the observed millisecond variability of the GRBs. To achieve the observed bulk Lorentz factor $\Gamma = 100$–1000, an energy outflow of 10^{51} erg should have a rest-mass load of only 10^{-5}–10^{-6} $M_\odot$. In other words, the baryon wind associated with long GRBs is only 10^{-8}–10^{-6} $M_\odot$ s^{-1}. Therefore, within the inner engine of GRBs the separation of light from the matter is realized, producing the most luminous electromagnetic explosions in the Universe.

Quark stars are hypothetical stars that consist of deconfined quarks (the structure of these stars was studied in detail by Haensel et al. 1986; and Alcock et al. 1986a). They are presumably born in special supernovae, such as SNIc, from the collapse of very massive Wolf Rayet stars (Paczyński & Haensel 2005). The quark star itself forms in a second collapse a few minutes after the new-born proto-neutron star simultaneously deleptonizes and spins up. Alternatively, a quark star could result from the collapse of accreting neutron stars in X-ray binaries (Cheng & Dai 1996). The collapse of stars of an initial mass less than 30 $M_\odot$ is expected to result in the formation of a compact object rather than a black hole (Fryer & Kalogera 2001). Some authors claim that this limit is in fact 50 $M_\odot$ (Gaensler et al. 2005), which implies that a large fraction of the progenitors of SNIc's should eventually evolve into compact stars. Among those, a possibly non-negligible fraction could be quark stars (Paczyński & Haensel 2005).

Bare quark stars differ from the normal, nucleonic, neutron stars in that the surface of a bare quark star (a strange star) plays the role of a membrane from which only leptons and photons can escape. This property was noted already in the early papers about quark stars (Alcock et al. 1986a; Haensel et al. 1991). The quark star surface then effectively separates baryonic from both leptonic matter and radiation. The exterior of a quark star is free of baryons, since the latter would be accreted onto the star and converted into deconfined quarks, resulting in a release of energy. The close environment of a newly-born quark star is therefore expected to be baryon-free.

If quark stars do indeed exist, they would be prime candidates for emitting relativistic winds with small baryonic pollution. The importance of small baryonic pollution in the

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context of GRB fireball models was already emphasized by Haensel et al. (1991), when discussing the *collision* of quark stars as an inner engine of short gamma-ray bursts. Models of the emission of gamma radiation in a GRB (De Rújula 1987; Zhang & Mészáros 2004; Dar & De Rújula 2004) require that the central engine emits a bulk relativistic outflow with a Lorentz factor Γ ranging from 100 to 1000 (Mészáros 2006; Dar 2006). Such a high Lorentz factor can only be reached if the baryon content of the outflow is very low. For example Paczyński (1990) has shown that a radiation-driven wind can reach a Lorentz factor of 100 only if the luminosity injected in the wind exceeds by a factor 10^2 the rest mass energy blown away per second. Bucciantini et al. (2006) indicated (using the results of 1D calculation of relativistic winds by Michel 1969) that low baryonic pollution is necessary to obtain high Lorentz factors in a centrifugally-driven magnetized wind. Dessart et al. (2007) reached the same conclusion. For this reason, the *quark star* model is preferable because the low baryonic pollution of a quark star's environment ensures that energy can be deposited cleanly outside the star in the form of accelerated electron-positron pairs and γ ray radiation without unnecessarily accelerating baryons. This is why we strongly favour a quark star model and consider that the central engine of a long GRB may be such a star. The fact that quark stars could be the source of GRBs was first suggested by Alcock et al. (1986b). Our calculations are however not specific to a quark star, except in Sects. 3.1 and 3.2. They would also apply to a strongly magnetized neutron star.

The way in which the relativistic wind energy from the quark star is eventually released as gamma ray radiation is model-dependent and could be due to a range of environments at a distance far larger than the light-cylinder radius from the star. Low baryonic pollution is needed only within a few light-cylinder radii from the central engine, a few thousand km, where the magnetized relativistic wind is expected to be accelerated to the required high Lorentz factors. Since the collapse from proto-neutron star to quark star is slightly delayed, no thick envelope is expected to be present in the rather small region where the leptonic wind is accelerated.

Newly formed quark stars could be at the origin of GRBs because of the sudden transformation of hadronic matter into deconfined quark-matter when a neutron star or a proto-neutron star collapses to a quark star. Different ways in which such a collapse could be triggered have been described by a number of authors (Cheng & Dai 1996, 1998a; Dai & Lu 1998a; Cheng & Dai 1998b; Bombaci & Datta 2000; Wang et al. 2000; Ouyed et al. 2002; Lugones et al. 2002; Ouyed & Sannino 2002; Berezhiani et al. 2002, 2003; Drago et al. 2004a,b; Paczyński & Haensel 2005; Drago et al. 2006, 2007; Haensel & Zdunik 2007).

Another promising subclass of models for a GRB central engine is based on the existence of a rapidly (millisecond period) and differentially rotating compact star endowed with a strong (10^{15} to 10^{17} Gauss) surface magnetic field, or spontaneously developing such a field from differential rotation. This millisecond magnetar model of the GRB central engine was first proposed by Usov (1992), who suggested that the GRB energy release derived its origin from the pulsar activity of a millisecond-period compact object with a dipole field of 10^{15} Gauss. The idea of a millisecond magnetar has been revisited and discussed by many authors (Thompson 1994; Zhang & Mészáros 2001, 2002). Following Kluźniak & Ruderman (1998), a number of authors proposed differential rotation as a mechanism for strengthening the toroidal magnetic field in the interior of a newly-born neutron star (Ruderman et al. 2000) or in an accreting neutron star that develops differential rotation as a result of an r-mode instability (Spruit 1999). Amplified magnetic fields, of the order of 10^{17} Gauss, would then be brought to the star's surface by buoyancy forces where the energy would be emitted in the form of bursts and Poynting flux. Ruderman et al. (2000) suggested that the emergence of strong fields would generate episodic, pulsar activity from the open magnetosphere of the object. Dai & Lu (1998b) considered this model for quark stars. However, it would be interesting to identify mechanisms that could generate the GRB phenomenon in objects with fields weaker than 10^{17} Gauss. In this paper, we propose such a mechanism.

A newly-born, compact star is entirely fluid. It supports differential rotation and internal oscillations of large amplitude that act on its internal magnetic field. In particular, differential rotation in the star would create a toroidal field from the poloidal component. This process is often referred to as an ω process, or winding-up of the magnetic field. The wound-up field and the differential rotation velocity constitute an energy reservoir into which electromagnetic emission could tap. More generally, the star's internal motions may have some effect on the Poynting power radiated. The main problem of the quark-magnetar model is to explain how the existence of the wound-up magnetic field could affect the star's emission, either by direct extraction of the associated energy or by any other means.

Our aim in this paper is to discuss the efficiency of the various ways in which a differentially-rotating, magnetized, compact star could shed energy in its environment.

We restrict our attention to the case of aligned rotators in which the magnetic dipole axis is parallel to the rotation axis. The magnetic-field distribution in the star, although it may be locally structured, is regarded as smoothly distributed on a larger scale.

We first show that field winding-up by differential rotation is not a steady, but oscillating process (Sect. 2). When much of the available kinetic energy of the differential rotation has been transformed into toroidal magnetic energy, the magnetic tension forces react back and reverse the motion. By means of this mechanism, any initial differential rotation develops into a torsional, standing, Alfvén wave in the star. In the absence of losses, the star would oscillate, in the rest frame accompanying the average rotation, like a torsion pendulum (see for instance, Bonazzola et al. 2007). In Sect. 2, we determine the amplitude of the wave and in Sect. 3.1 we discuss its damping in a quark star as a result of second viscosity. This damping is found to be small.

We then discuss various mechanisms by which the energy of the torsional oscillation could emerge from the star. Buoyancy is a possibility. It may however be quenched if, while the buoyant matter moves, some of the weak reactions between quarks remain frozen, which occurs if the matter is in a colour-superconducting state with a large enough gap, and the strange quark is sufficiently massive (Sect. 3). Alternatively, buoyancy may be inhibited by magnetic stratification or, if it develops, it could only redistribute flux in the star if the latter is magnetized in bulk.

Since the internal magnetic field is time-variable, it could conceivably act as an antenna and radiate a large-amplitude, electromagnetic wave in an external vacuum. We calculate this radiation in (Sect. 4.1) and find that the emitted power is insufficient to produce a GRB. Radiation in a leptonic medium surrounding the star is shown to be equally inefficient (Sect. 4.2).

We next consider the modulation by differential rotation of the DC Poynting emission of the fast, aligned magnetic rotator (Sect. 5). We find that an even oscillation, in which the southern hemisphere oscillates in phase with respect to the northern hemisphere and the magnetic structure has a dipolar-type of symmetry causes a negligible modulation of the energy output. However, an odd torsional oscillation, in which the southern hemisphere oscillates in phase opposition with respect to the northern hemisphere, easily causes the star's magnetosphere to be blown open in a time-dependent way. An even oscillation acting on a magnetospheric structure that would not be strictly symmetrical with respect to the equator has a similar effect. These openings of the magnetosphere cause a modulation of the power emitted in the relativistic wind blown by the fast rotator that is large enough to meet the requirements of energy and time scale necessary to explain the GRB phenomenon, even for moderately magnetized stars, with a field of order of a few 10^{14} Gauss (Sects. 5.3, 5.4). A collapse that is strictly symmetrical with respect to the equator would not excite odd oscillations. However, the existence of a kick received by neutron stars at their birth indicates that a supernova collapse is in reality not strictly symmetrical. We show that even a weak, odd oscillation is sufficient to open the star's magnetosphere during several tens of minutes after the collapse.

We use a Gaussian CGS system of units throughout the paper and spherical polar coordinates based on the rotation axis, r, θ, and ϕ, where θ is the colatitude. The corresponding unit vectors are $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$.

2. Torsional oscillation in the star

Differential rotation necessarily induces, in an highly magnetized and conducting star, a torsional oscillation. Such Alfvénic oscillations in fluid, magnetized, compact stars were studied by, e. g., Bastrukov & Podgainy (1996), Rincon & Rieutord (2003), and Bonazzola et al. (2007). We assume that inside the star the poloidal part of the magnetic field is time-independent. This is reasonable because the strange matter is but weakly compressible. We also assume that perfect MHD is valid.

2.1. Magnetic diffusion time scale

Perfect MHD is a good approximation when the magnetic diffusion time $\tau_{\rm B}$ is longer than the timescale of the considered phenomenon. The time $\tau_{\rm B}$ depends on the electrical conductivity $\sigma_{\rm e}$ and the gradient lengthscale of the field which we assume to equal the star's radius R, such that $\tau_{\rm B} \approx 4\pi\sigma_{\rm e} R^2/c^2$. The electric conductivity is the sum of the electronic and quark conductivity, $\sigma_{\rm e}(e)$ and $\sigma_{\rm e}(q)$. The electron fraction in quark matter depends considerably on the physical conditions. The quark conductivity was calculated for normal quarks by Heiselberg & Pethick (1993). Dynamical screening of transverse interactions by the Landau damping of the exchanged gluons is important in determining quark mobility. The result can be expressed, for normal, massless quarks and a strong coupling constant supposedly equal to 0.1, as:

$$\sigma_{\rm e}(q) \approx 5.6\times 10^{18}\, T_{10}^{-5/3}\, (n_{\rm b}/n_0)^{8/9}\ {\rm s}^{-1}, \tag{1}$$

where $n_{\rm b}$ is the baryonic number density, n_0 the nuclear density (0.17 fm^{-3}), and $T_{10} = T/10^{10}$ K. If quarks are colour-superconducting in a colour-flavour locked (CFL) state, charge neutrality of the quark component alone is enforced because pairing is maximized when $n_d = n_u = n_s$. This happens when the mass $m_{\rm s}$ of the strange quark is insufficiently large compared to the gap Δ. In the absence of electrons, the CFL colour-superconductor is an electric insulator (Alford et al. 2007). Electron suppression results in a difference between the chemical potentials of s and d quarks, which must remain limited. Electrons are suppressed (Rajagopal & Wilczek 2001) only if:

$$\left|\frac{m_{\rm s}c^2}{4\overline{\mu}} - \frac{\mu_s-\mu_d}{2}\right| < \Delta, \tag{2}$$

where $\overline{\mu}$ is a mean quark chemical potential. When the mass of the strange quark is so large that the inequality (2) is not satisfied, electrons remain present in the colour-superconducting quark matter and this matter is then a conductor with a conductivity which may be far larger than Eq. (1). If quarks are in a two-flavour colour-superconductor (2SC) state, only d and u quarks of two colours are paired, and electrons are present in quark matter. Then $\sigma_{\rm e}(q)$ differs only by numerical factors from Eq. (1) and $\sigma_{\rm e}(e)$ does not vanish. The magnetic diffusion timescale $\tau_{\rm B}$ associated with the scale length R and the conductivity (1) is about 8×10^{10} s for $n_{\rm b} = n_0$ and $T_{10} = 1$. This is a lower bound to the true magnetic diffusion timescale, which is longer than this value when the electronic conductivity does not vanish. This time is long enough for perfect MHD to be valid on the timescale of a burst. The exception to this general conclusion is when quarks are in a colour CFL-superconductor state in most of the star, and the s quark is so light that inequality (2) holds true, in which case quark matter is an insulator. The star's interior is then electrically inactive and none of the phenomena described in Sects. 2.2 and 5.3 occurs. In this paper, we assume that the star's interior is a good electrical conductor, which implies that for the electrical conductivities implied by Eq. (1) perfect MHD holds true.

2.2. Period and amplitude of the torsional oscillation

In perfect MHD, the evolution equations for the velocity $\boldsymbol{V}$ and for the magnetic field $\boldsymbol{B}$ inside the star can be written, in a Galilean rest frame, as:

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} - \frac{1}{4\pi\rho}(\boldsymbol{\nabla}\wedge\boldsymbol{B})\wedge\boldsymbol{B} + \boldsymbol{\nabla}(\epsilon + U) = 0, \tag{3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\wedge(\boldsymbol{V}\wedge\boldsymbol{B}) = 0, \tag{4}$$

where ρ, ϵ, and U are the baryonic mass density, the enthalpy (the fluid is supposed to be barotropic), and the gravitational potential, respectively. The equations for the toroidal components of $\boldsymbol{V}$ and $\boldsymbol{B}$ can be written in the form:

$$\frac{\partial V_\phi}{\partial t} = \frac{1}{4\pi\rho\, r\sin\theta}\, \boldsymbol{B}_{\rm P}\cdot\boldsymbol{\nabla}\left(r\sin\theta\, B_\phi\right) \tag{5}$$

$$\frac{\partial B_\phi}{\partial t} = r\sin\theta\, \boldsymbol{B}_{\rm P}\cdot\boldsymbol{\nabla}\left(\frac{V_\phi}{r\sin\theta}\right). \tag{6}$$

We note that for rigid rotation, i.e. for $V_\phi(r,\theta) = \Omega_* r\sin\theta$ with constant Ω_*, the r.h.s. of Eq. (6) vanishes, so that no winding-up of the field in this case occurs. Winding-up results from differential rotation with depth or latitude, or both. The Proudman-Taylor theorem does not apply when non-potential forces, such as the magnetic-tension force, are exerted onto the fluid and the motions are non-stationary. It is however possible that when non-potential forces are small compared to pressure forces and the timescale of the internal motions is long compared to the star rotation period, these motions organize themselves such that V_ϕ becomes a function of the axial distance $r\sin\theta$ only. In that case, the field winding is only possible when the poloidal magnetic field has a component perpendicular to the rotation axis.

The buildup of the toroidal field B_ϕ causes a reaction magnetic-tension force (the terms on the right hand side of Eq. (5)) to grow. A torsional oscillation then develops. The system of Eqs. (5) and (6) for the azimuthal velocity and field inside the star has been solved numerically with appropriate boundary conditions (Bonazzola et al. 2007).

We define R to be the star's radius and $R_{10} = R/(10\text{ km})$. The period P_T of the torsional oscillation is esssentially that of an Alfvén wave with a node at both poles, that is, with a wavelength $2\pi R$ in a poloidal field $B_\mathrm{P} = B_\mathrm{P14}\,10^{14}$ Gauss. We adopt 10^{14} Gauss as a reference value since fields of order of between 5×10^{13} and 10^{14} Gauss are commonplace in isolated neutron stars (Haberl 2007) and fields of several 10^{14} Gauss are reported to be typical of anomalous X-ray pulsars and soft gamma-ray bursters (Ziolkowski 2002). We assume that the field is initially rooted deep inside the star. This view is supported by simulations of collapse (Obergaulinger et al. 2006a), which indicate that the magnetic field after collapse is concentrated in the inner core. The density of the medium inside the star is of the order of the mean star density $\rho_* = M/V$, M being the mass of the star and $V = 4\pi R^3/3$ its volume. The period P_T of the torsional oscillation is that of an Alfvén wave of wavelength $2\pi R$ in a medium of density ρ_*:

$$P_\mathrm{T} = \sqrt{\frac{12\pi^2 M}{B_\mathrm{P}^2 R}} = 4.9\text{ s}\,\sqrt{\frac{M/M_\odot}{B_\mathrm{P14}^2\,R_{10}}}\,. \tag{7}$$

This result may also be obtained from a linearization of Eqs. (5), (6). The magnetic field, instead of pervading all of the star, could conceivably be present only in some superficial layer where the density is $\overline{\rho} < \rho_*$, although the aforementioned simulations do not support this. In this case, P_T would be smaller than given by the expression in Eq. (7) by a factor $\sqrt{\overline{\rho}/\rho_*}$. The surface density of a quark star is $4\mathcal{B}/c^2$, where $\mathcal{B}$ is the bag constant. For $\mathcal{B} = 40\text{ MeV fm}^{-3}$, this surface density is $2.8\times 10^{14}\text{ g cm}^{-3}$, which is about 0.6 times the average density ρ_*. For comparable poloidal fields, the period of a superficial torsional wave is only slightly less than the period given by Eq. (7) of a global torsional wave.

The wave amplitude is set by the amount of energy W_D initially stored in the collapse as kinetic energy of the differential rotation. We assume W_D to be a fraction α_D of the order of a few percent of the star's total rotational energy, W_*. Simulations by Obergaulinger et al. (2006a,b) indicated that α_D does not exceed 10% after the collapse to a neutron star. Burrows et al. (2007) found that the value of α_D is less constrained. We adopt $\alpha_\mathrm{D} = 0.1$ as a representative upper limit. We define I_* to be the moment of inertia of the star and $I_{45} = I_*/(10^{45}\text{ g cm}^2)$, P_* to be the star's average rotation period, and $P_*(\text{ms})$ its value in milliseconds. Obergaulinger et al. (2006a,b) found that when the collapsed core reaches quasi-equilibrium, P_* most often ranges in value between 5 and 40 millisec. Burrows et al. (2007) found that 2 millisec is a lower bound. Since the quark star forms from the hot neutron star after a second collapse, its rotation accelerates by a factor ~1.5, according to the moments of inertia calculated by Bejger & Haensel (2002). Thus, $P_* = 3$ ms would be a representative value of its spin period. The rotational energy of the star being $W_* = I_*\Omega_*^2/2$, the kinetic energy available from the differential rotation is $W_\mathrm{D} = \alpha_\mathrm{D} W_*$. This is the energy of the torsional oscillation, if it is global. For $\alpha_\mathrm{D} = 0.025$, $P_* = 3$ ms, and $I_{45} = 1$, $W_\mathrm{D} \approx 6\times 10^{49}$ erg.

The oscillating toroidal field in the star is at its maximum amplitude B_T when the energy of the torsional oscillation W_D is entirely in magnetic form, which implies that, for a global oscillation of a star of volume V, $VB_\mathrm{T}^2 = 8\pi W_\mathrm{D}$. Thus:

$$B_\mathrm{T} \approx 10^{17}\left(\frac{10\,\alpha_\mathrm{D}\,I_{45}}{P_*^2(\text{ms})\,R_{10}^3}\right)^{1/2}\text{ Gauss.} \tag{8}$$

For $P_* = 3$ ms, $\alpha_\mathrm{D} = 0.025$, and the other parameters equal to unity, $B_\mathrm{T} \approx 1.7\times 10^{16}$ Gauss. Thus, for a poloidal magnetic field of order of a few 10^{14} Gauss, the torsional Alfvénic oscillation is non-linear. Equation (8) indicates the typical amplitude of the toroidal field inside the star. The field distribution has however a certain profile with depth and latitude, so that Eq. (8) is not a precise estimate of the sub-surface toroidal field.

The matter's angular velocity in the star is $\Omega_* + \delta\Omega$, where $\delta\Omega$ varies with position and time and has a null time average value. Indicating time averaging by brackets, we have

$$W_* + W_\mathrm{D} = \frac{I_*}{2}\left(\Omega_*^2 + \langle\delta\Omega^2\rangle\right). \tag{9}$$

From $W_\mathrm{D} = \alpha_\mathrm{D} W_*$, it is found that the rms value of the time-varying angular velocity equals $\sqrt{\alpha_\mathrm{D}}\,\Omega_*$ and the velocity amplitude of the torsional wave close to the surface is about a factor of $\sqrt{\alpha_\mathrm{D}}$ times the rotational velocity:

$$\delta V_\phi = \sqrt{\alpha_\mathrm{D}}\,R\Omega_*. \tag{10}$$

When magnetic flux is present only in a superficial layer of mass m, only that part of the kinetic energy of differential rotation that develops in this layer feeds the energy of the torsional oscillation. If this energy is distributed proportionally to mass, the estimate given by Eq. (10) of the oscillation's amplitude remains valid. The radial component of the current in the star is then:

$$j_\mathrm{r} = \frac{c}{4\pi}\,\boldsymbol{e}_\mathrm{r}\cdot\mathrm{rot}\boldsymbol{B} = \frac{c}{4\pi r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta B_\phi\right). \tag{11}$$

If this component of the current does not vanish, a DC current could flow from the star to the magnetospheric lepton plasma. In Sect. 5.3, we consider the consequences of these currents.

3. Energy-extraction mechanisms

The energy of the torsional oscillation could leak out of the star by a number of different mechanisms. We consider these mechanisms in turn and discuss whether they could represent the origin of long GRBs.

3.1. Viscous damping of the torsional oscillation

The oscillation could be damped in the star by viscous friction or Ohmic dissipation and then escape by means of heat conduction and thermal radiation from the surface. The surface of a quark star at temperatures $T \sim 10^{10}$ °K is an efficient source of γ photons and e^+e^- pairs (see e.g. Aksenov et al. (2003) and references therein). However, Ohmic dissipation is negligible under the conditions assumed in Sect. 2.1. The shear viscosity of normal quark matter has been calculated by Heiselberg & Pethick (1993). The viscosity of quark matter with unpaired components is of a comparable order of magnitude. The Reynolds number for a scale R and a velocity δv_ϕ given by Eq. (10) is found to be of order 10^{14}. Thus, shear-viscous dissipation is negligible.

Even superfluid quark-matter suffers bulk viscosity (Madsen 1992). An Alfvén wave, being non-compressive, is not damped by bulk viscosity at the linear approximation, but the Alfvénic torsional oscillation is non-linear. By non-resonant coupling, its magnetic-pressure gradients generate a compressive oscillation. Bulk viscosity acting on this compressive part of the non-linear oscillation causes a damping which can be calculated by solving the MHD equations perturbatively to second order. We simplified the calculation of this damping by considering an homogeneous medium of mass density ρ_0 contained in a Cartesian box with unperturbed density ρ_0 and magnetic field $B_0\,\boldsymbol{e}_z$ and an extension $L = \pi/k$ in the z-direction. The solution to first order is the non-compressive standing Alfvén wave. We define c_{A0} and $c_{\mathrm{S0}} \gg c_{\mathrm{A0}}$ to be the Alfvén and the sound speed in the unperturbed medium respectively, B_{T} the toroidal magnetic amplitude of the wave and ζ the coefficient of bulk viscosity. The second-order solution brings in the following damping time:

$$\tau_{\mathrm{bv}} = \frac{32\rho_0}{\zeta k^2}\,\frac{c_{\mathrm{S0}}^4}{c_{\mathrm{A0}}^4}\,\frac{B_0^2}{B_{\mathrm{T}}^2}\,. \tag{12}$$

The bulk viscosity in quark-matter depends on the finite time required by quarks to return to the weak-interaction equilibrium after the flavour equilibrium is disturbed by the leptonless strangeness-changing reaction

$$u + d \leftrightarrow s + u. \tag{13}$$

Any compression of the medium causes such a disturbance, because the s quark is more massive than the u and d quarks. For colour-superconducting quark matter, the existence of a gap Δ drastically reduces the reaction rate when $k_{\mathrm{B}}T \ll \Delta$. The gap energy Δ, which is uncertain, is between 1 and 50 MeV (Madsen 2000). According to Madsen (1992), the bulk viscosity coefficient ζ of quark matter experiencing an harmonic density perturbation of pulsation ω is:

$$\zeta \approx \frac{\alpha T^2}{\omega^2 + \beta T^4}, \tag{14}$$

where T is the temperature of the medium. Equation (14) is valid only when the Fermi energies of the s and d quarks differ by less than $2\pi k_{\mathrm{B}}T$. The coefficients α and β are:

$$\alpha = \left(4\pi^2/9\right)\,K_{\mathrm{F}}\,m_{\mathrm{s}}^4 c^8\,k_{\mathrm{B}}^2\,\mu_d^3, \tag{15}$$

$$\beta = (\pi^2/9)\,K_{\mathrm{F}}^2\,k_{\mathrm{B}}^4\,\mu_d^6\,h^6 c^6\left(1 + m_{\mathrm{s}}^2 c^4/(4\mu_d^2)\right)^2. \tag{16}$$

In Eqs. (15), (16), h is the Planck's constant, c the speed of light, k_{B} the Boltzmann's constant, m_{s} the mass of the strange quark, and μ_d the Fermi energy of the d quarks, which in a 1 $M_\odot$ quark star of radius 10 km is 196 MeV. The mass m_{s} is expressed in units of 100 MeV by

$$m_{\mathrm{s}}c^2 = m_{\mathrm{s100}} \times 100\ \mathrm{MeV}. \tag{17}$$

The rate of the reactions represented by Eq. (13) depends on the weak-coupling parameter (Madsen 1992):

$$K_{\mathrm{F}} = 3.0781\,10^{74}\ \mathrm{g^{-8}\,cm^{-19}\,s^{15}}. \tag{18}$$

The time needed to re-establish the equality of the chemical potentials of the s and d quarks after a perturbation (the strangeness equilibration time) is $\tau_{\mathrm{sg}} = (\beta T^4)^{-1/2}$. Writing $T = T_9 10^9$ °K, the strangeness equilibration time is:

$$\tau_{\mathrm{sg}} = 6.9 \times 10^{-4} T_9^{-2}\ \mathrm{s}. \tag{19}$$

This is far shorter than the period of the torsional wave, which means that s and d quarks always remain close to the equilibrium of the reaction given by Eq. (13) when the quarks are in a normal state. For these representative numbers, $(\mu_s - \mu_d)/2\pi k_{\mathrm{B}}T$ remains small. This legitimates Eq. (14), in which ω^2 can also be neglected so that:

$$\zeta = 2.45 \times 10^{29}\ m_{\mathrm{s100}}^4\ T_9^{-2}\ \ \mathrm{g\,cm^{-1}s^{-1}}. \tag{20}$$

If the quarks are in a colour-superconducting state with a gap Δ, K_{F} is reduced by $\exp(-2\Delta/k_{\mathrm{B}}T)$ (Madsen 2000). For $\Delta = 1$ MeV and $T > 2.2 \times 10^9$ °K, the relaxation time of reactions (13) remains less than the period of the torsional wave. For lower temperatures, it rapidly becomes much longer. For example, at 10^9 K with a gap of 1 MeV, this time is 8×10^6 s, which is so long compared to the period of the wave that bulk viscosity is quenched. The damping time τ_{bv} in a 10^{14} Gauss poloidal field with a toroidal magnetic field amplitude of 10^{16} Gauss is given by Eq. (12). For normal quarks this time is:

$$\tau_{\mathrm{bv}} = 4.8 \times 10^{10}\ \ T_9^2\ m_{\mathrm{s100}}^{-4}\ \ \mathrm{s}. \tag{21}$$

This is much longer than the torsional wave period, owing to the fact that the compression in this wave is small, so that bulk-viscous damping is negligible.

3.2. Flux emergence by magnetic buoyancy

The internal magnetic field can emerge through the surface of the star and expand into the quasi-vacuum outside as an electromagnetic signal. This point of view is adopted in the models of a number of authors such as Kluźniak & Ruderman (1998), Ruderman et al. (2000), Spruit (1999), and Dai & Lu (1998b): amplified magnetic fields, supposedly of order 10^{17} Gauss, would be brought to the surface of a neutron star by buoyancy forces and the energy will be emitted in the form of bursts and Poynting flux.

We estimate the flux emergence time for a given magnetic field, assuming that nothing opposes buoyancy. Since the field in the wound-up flux tubes is essentially toroidal, these tubes may be regarded as thin circular annuli centred on the axis. The rapid rotation of the star inhibits their expansion or contraction perpendicular to the axis, so that the flux tubes move parallel to it towards the closest pole. Their length l remains constant. Consider a flux tube of a small cross section S and length l, threaded by a field B. Under perfect MHD conditions, it conserves its magnetic flux and its baryonic content. For subsonic motions, it also remains in pressure equilibrium with its environment. We define $P_{\mathrm{ext}}(z)$ to be the material pressure, $B_{\mathrm{ext}}(z)$ to be the magnetic field in this environment at an altitude z along the polar axis, and $P_{\mathrm{in}}(z)$ to be the material pressure and $B_{\mathrm{in}}(z)$ the magnetic field in the tube when it reaches the altitude z. Total pressure equilibrium implies that:

$$P_{\mathrm{ext}}(z) + \frac{B_{\mathrm{ext}}^2(z)}{8\pi} = P_{\mathrm{in}}(z) + \frac{B_{\mathrm{in}}^2(z)}{8\pi}\,. \tag{22}$$

The pressure P_c in the inner regions of a quark star is about 10^{35} erg cm^{-3}. Since the magnetic pressure of a field of 10^{16} Gauss is far lower, the difference between the matter densities ρ_{in} and ρ_{ext} in the tube and in its environment can be calculated perturbatively:

$$\rho_{\mathrm{ext}}(z) - \rho_{\mathrm{in}}(z) \approx \frac{3\left(B_{\mathrm{in}}^2(z) - B_{\mathrm{ext}}^2(z)\right)}{8\pi c^2}\,. \tag{23}$$

where we have used $\mathrm{d}P/\mathrm{d}\rho = c^2/3$. The difference in magnetic-energy density should be added to derive the difference in total energy density. Denoting by $(-g_z)$ the component of gravity parallel to the rotation axis, the vertical motion of the flux tube is described by the equation

$$\left(\rho_{\rm in}(z) + \frac{B_{\rm in}^2(z)}{8\pi c^2}\right)\ddot{z} = -g_z\,\frac{B_{\rm ext}^2(z) - B_{\rm in}^2(z)}{4\pi c^2}\,. \tag{24}$$

The existence of the buoyancy instability depends on the distribution of the magnetic field in the star. If the field intensity increases with altitude such that $B_{\rm in}(z)$ is always less than $B_{\rm ext}(z)$ at a higher level, the distribution of flux is stable. By contrast, if the flux tube moves in an unmagnetized environment, buoyancy can only be inhibited by density differences between the magnetized and non-magnetized medium. These differences may result from a situation of non-equilibrium of weak interactions in the moving fluid, as described below. For $B_{\rm ext} = 0$, if we assume that $B_{\rm in}$ remains almost constant during the motion and that $g_z \sim 0.5\ (GM/R^2)$ and $\rho_{\rm in} \approx \rho_* = M/V$, we find from Eq. (24) that the buoyancy time $\tau_{\rm b}$ needed to raise the tube by about a stellar radius is:

$$\tau_{\rm b} \approx \sqrt{\frac{6c^2}{GB^2}} = 2.8\times10^{-2}\ {\rm s}\ \left(\frac{10^{16}\,{\rm G}}{B_{\rm in}}\right)\cdot \tag{25}$$

This is the time required to bring an isolated flux tube to the surface when the toroidal field has reached the value $B_{\rm in}$. After the sudden formation of the quark star, the differential rotation causes this toroidal field to increase in strength as $B(t) = B_{\rm T}\sin(t/P_{\rm T})$, where $B_{\rm T}$ is the maximum toroidal field developed in the torsional oscillation represented by Eq. (8). Buoyancy starts to be effective only when the growing toroidal field has reached a value such that its buoyancy time in Eq. (25) calculated for $B(t)$, has become shorter than the age t of the new-born quark star. For standard stellar parameters and $B_{\rm P14} = 1$, $P_* = 3$ ms and $\alpha_{\rm D} = 0.025$, this occurs at the buoyancy starting time, $t_{\rm st} \approx 0.5$ s.

The ratio of the buoyancy time $\tau_{\rm b}$ (Eq. (25)) to the period of the torsional oscillation $P_{\rm T}$ (Eq. (7)) is:

$$\frac{\tau_{\rm b}}{P_{\rm T}} = \frac{1}{\pi\sqrt{2}}\,\frac{B_{\rm P}}{B_{\rm in}}\left(\frac{GM}{Rc^2}\right)^{-1/2} \tag{26}$$

where $B_{\rm P}$ is the poloidal field and $B_{\rm in}$ is the total field. When the wound-up magnetic field strength is close to its maximum value (Eq. (8)), $B_{\rm in} \approx 100B_{\rm P}$. This implies that when nothing opposes buoyancy, wound-up fields in isolated flux tubes float to the star's surface in a time $\tau_{\rm b}$ shorter than the period $P_{\rm T}$ of the oscillation.

However, buoyancy motions are reduced or quenched when the ascending magnetized matter becomes denser, at the same total pressure, than matter in its neighbourhood. This may happen when the magnetic field pervades the entire volume of the star and the field intensity increases with the altitude z. Another effect opposing buoyancy is when reactions such as Eq. (13) or the β-decay reactions

$$d \leftrightarrow u + e \tag{27}$$

cannot reach equilibrium in the buoyancy time $\tau_{\rm b}$. The relaxation time $\tau_{\rm sg}$ of the strangeness-changing reaction (13) in normal quark matter is about $7\times10^{-6}\ T_{10}^{-2}$ s (Sect. 3.1). For colour-superconducting quark-matter with a gap energy Δ, the time to achieve equilibrium of the same reactions is lengthened by a factor $\exp(2\Delta/k_{\rm B}T)$ (Madsen 2000). It may become longer than both $\tau_{\rm b}$ and $P_{\rm T}$ if the gap Δ is sufficiently large. Similarly, the relaxation time τ_β of the quark β-decay reactions in Eq. (27) is, for normal quarks, $\tau_\beta \approx 2.7\times10^{-2}\ T_{10}^{-4}$ s (Iwamoto 1983). For colour-superconducting quark-matter, this time is lengthened by a factor $\exp(\Delta/k_{\rm B}T)$.

The temperature is the parameter controlling whether chemical equilibrium of the reactions in Eqs. (13) and (27) can be achieved on a given timescale. When a proto-neutron star collapses into a quark star, an energy of about 10^{53} erg is released, which is reflected in the initial temperature of the new-born object of $\sim3\times10^{11}$ K. The star is then opaque to neutrinos (Steiner et al. 2001). It cools by emitting thermal $\nu_{\rm e}$ and $\overline{\nu}_{\rm e}$'s from a neutrino-sphere, thermal photons of frequency higher than the plasma frequency and lepton pairs. According to Usov (2001), thermal, photon emission dominates over lepton emission at $T > 5\times10^{10}$ K. At 10^{11} K, the photon emissivity is barely smaller than that of the black-body. Adding neutrino and antineutrino thermal emission, the net effective emissivity at this temperature is a little less than a factor of two higher than the black-body emissivity. The star then cools to about 3×10^{10} K in 0.5 s.

Does the non-equilibrium of the strangeness-changing reactions or the β-reactions suppress the ascent of magnetized matter to the surface? If quarks are in a colour-superconducting state with a gap Δ, the matter in the buoyant tube retains its original strangeness during its ascent if $\tau_{\rm b} < \tau_{\rm sg}\exp(2\Delta/k_{\rm B}T)$, where $\tau_{\rm sg}$ is the relaxation time of the reactions in Eq. (13) in normal, quark matter (Eq. (19)). This condition is satisfied when:

$$\Delta({\rm MeV}) > T_{10}\,(2\log_{10}T_{10} + 3.6 - \log_{10}B_{16}). \tag{28}$$

The time $\tau_{\rm b}$ is definitely longer than $\tau_{\rm sg}$ at 3×10^{10} K, the temperature when buoyancy starts, meaning that in the absence of a gap, the reactions in Eq. (13) remain in equilibrium. At this temperature and for $B_{16} = 1$, the inequality of Eq. (28) holds true when Δ is larger than about 14 MeV, which is plausible because the gap could be as high as 50 MeV (Madsen 2000). For a gap larger than 14 MeV, the reactions in Eq. (13) will remain out of equilibrium during the buoyancy motion. Similarly, the β-decay reactions in Eq. (27) remain frozen during buoyancy motions if $\tau_{\rm b} < \tau_\beta\exp(\Delta/k_{\rm B}T)$. This inequality is satisfied without the need for a gap when $T_{10} < B_{16}^{0.25}$, which is not quite the case for $T = 3\times10^{10}$ K and $B_{16} = 1$; this implies that, at this temperature and in the absence of a gap, β-decay reactions remain more or less in equilibrium during buoyancy. In the presence of a gap, the inequality $\tau_{\rm b} < \tau_\beta\exp(\Delta/k_{\rm B}T)$ is satisfied if the gap Δ is such that:

$$\Delta({\rm MeV}) > 2\,T_{10}\,(4\log_{10}T_{10} - \log_{10}B_{16}). \tag{29}$$

At 3×10^{10} K and for $B_{16} = 1$, the inequality (29) holds true when Δ is larger than 12 MeV. For this or a larger gap, the reactions in Eq. (27) keep out of equilibrium during the buoyancy motion. It is surprizing that the gap values which freeze the reactions in Eqs. (13) and (27) on the timescale $\tau_{\rm b}$ are so close. This results from the fact that the relaxation time of reactions in Eq. (13) lengthens more rapidly with gap energy than that for the β-decay reactions in Eq. (27).

We therefore have two situations. The gap is either less than 10 MeV and both reactions in Eqs. (13) and (27) reach equilibrium on a timescale shorter than the buoyancy timescale when buoyancy starts. In this case, chemical non-equilibrium has no role in limiting buoyancy. Otherwise, the gap exceeds 14 MeV and both reactions remain frozen on the buoyancy timescale. We disregard any intermediate situation.

Depressurized, non-equilibrated, quark matter weighs more than equilibrated matter at the same pressure because its energy density is not minimal. For a gap larger than 14 MeV, buoyancy is quenched when the total energy density $\epsilon_{\rm in}$ in the rising flux tube (including its magnetic energy density) exceeds the total energy density $\epsilon_{\rm ext}$ in the ambient medium. To illustrate this, we consider a flux tube reaching a region in the star, at an altitude of z, where its material pressure is less than at the altitude z_1 where it started its ascent. We assume that during its motion the weak reactions remain frozen. The difference between the mass density $\rho_{\rm eq}$ of equilibrated matter at this pressure and the mass density $\rho_{\rm fr}$ of the frozen matter at the same pressure is expressed by Haensel & Zdunik (2007) as:

$$\rho_{\rm fr} - \rho_{\rm eq} = f_{\rm ab}\,\rho_{\rm eq}. \tag{30}$$

They calculated $f_{\rm ab}$ for cold quark-matter in which only the β-decay reactions (27) are frozen. However we are interested in a situation where the reactions in Eqs. (13) and (27) are both frozen. We find that in this case:

$$f_{\rm ab} = 6.4 \times 10^{-4}\, m_{\rm s100}^{3.7} \tag{31}$$

where $m_{\rm s100}$ is defined by Eq. (17). The field $B_{\rm in}$ of a flux tube that is still buoyant at an altitude z where the ambient magnetic field is $B_{\rm ext}$ must be such that

$$B_{\rm in}^2 > B_{\rm ext}^2 + 4\pi c^2 f_{\rm ab}\,\rho_{\rm eq}(P_{\rm in}), \tag{32}$$

where $P_{\rm in}$ is the material pressure in the rising flux tube at this altitude. Since $f_{\rm ab} \ll 1$, we can assume $P_{\rm in}$ to be almost equal to the total external pressure. For isolated flux tubes moving through an unmagnetized medium, Eq. (32) becomes:

$$B_{\rm in}^2 > 4\pi c^2 f_{\rm ab}\,\rho_{\rm ext}, \tag{33}$$

where $\rho_{\rm ext}$ is the mass density in the equilibrated unmagnetized environment. At the star's surface $\rho_{\rm ext}\, c^2 = 4\mathcal{B}$, $\mathcal{B}$ being the bag constant. We define $\mathcal{B}_{50} = \mathcal{B}/$ (50 MeV fm^{-3}). We assume that $m_{\rm s} = 100$ MeV. If quark matter is to be more stable than nucleonic matter, $\mathcal{B}_{50}$ should not exceed 1.6 for free quarks confined in the bag, and 1.4 for the QCD coupling constant 0.2 (see, e.g., Fig. 8.2 in Haensel et al. (2007)). With $f_{\rm ab}$ given by Eq. (31), the minimum value of a field that would be buoyant near the surface, at a level z, is:

$$B_{\rm min,\,top} = 4 \times 10^{16}\, m_{\rm s100}^{1.85}\, \mathcal{B}_{50}^{1/2}\ \text{Gauss}\,\cdot \tag{34}$$

Deeper inside the star, at a level z_1, the field in this same flux tube had a value $B_{\rm min}$ due to the conservation of its magnetic flux and quark number content. From Eq. (22), $B_{\rm min} = B_{\rm min,\,top}\,(P_{\rm ext}(z_1)/P_{\rm ext}(z))^{3/4}$. The pressure deep inside the star is taken to be $\rho_* c^2/3 \approx Mc^2/4\pi R^3$. The pressure close to the star's surface is the bag constant $\mathcal{B}$. The minimum buoyant field deep inside the star, $B_{\rm min}$, is then:

$$B_{\rm min} = 8.5 \times 10^{16}\, m_{\rm s100}^{1.85}\, R_{10}^{-9/4}\, \mathcal{B}_{50}^{-1/4} \left(\frac{M}{M_\odot}\right)^{3/4}\ \text{Gauss}. \tag{35}$$

Taking the moment of inertia of the star to be $I_* = 0.4\, MR^2$, the ratio of the toroidal field $B_{\rm T}$ generated by the torsional oscillation (Eq. (8)) to $B_{\rm min}$ is:

$$\frac{B_{\rm T}}{B_{\rm min}} \approx 0.95\, \frac{(\alpha_{\rm D}/0.1)^{1/2}}{P_*({\rm ms})} \left(\frac{M_\odot}{M}\right)^{1/4} R_{10}^{7/4}\, m_{\rm s100}^{-1.85}\, \mathcal{B}_{50}^{1/4}. \tag{36}$$

Buoyancy is quenched when $B_{\rm T}/B_{\rm min} < 1$. For $P_* = 3$ ms, $\alpha_{\rm D} = 0.025$, $M = 1\ M_\odot$, and $\mathcal{B}_{50} = 4/5$, this occurs (provided that the colour-superconductivity gap exceeds 14 MeV) when

$$m_{\rm s} c^2 > 38\ R_{10}\ \text{MeV}. \tag{37}$$

An important question is whether the buoyant flux is entirely expelled out of the star with the leptonic wind or whether, although the magnetic field partly emerges, it remains rooted in the subsurface layers.

This depends on how rapidly the magnetic field can diffuse through quark matter, which itself depends on its electrical conductivity $\sigma_{\rm e}$ and the gradient lengthscale l_M of the field. Since the magnetic field decreases in the flux tube during its ascent, its cross section at the surface cannot be smaller than when it started. Since rapid rotation prevents radial motions perpendicular to the rotation axis, the field scale length l_M of buoyant flux tubes cannot diminish. The fact that the magnetic diffusion timescale is about 10^{11} s for conducting quark matter implies that during the first few minutes after the formation of the strange star, the magnetic flux emerging through the star's surface as a result of buoyancy remains rooted in quark matter at starspots. In this case, magnetic activity from the torsional oscillations, as described below, continues after the first burst of magnetic buoyancy has brought the inner magnetic field closer to the surface.

If matter is magnetized in bulk, buoyancy assumes the form of a convective instability. When it develops, the more magnetized material is brought to the star's surface, while the less magnetized material sinks deeper into the star. This results in a redistribution of magnetic field in the star, not in a net loss of flux. The end result of the field redistribution should be close to a state of marginal buoyancy instability. A fraction of the surface magnetic-flux tubes should emerge from the star, baryonic matter draining down along the field as it emerges. However, since this matter cannot diffuse out of the field, the emerging magnetic loops remain connected to the subsurface flux. The magnetized volume experiences little change in this process, so that a substantial part of the star's volume remains magnetized, if it was initially, and magnetic activity from the torsional oscillation persists after the flux redistribution.

In the following, we consider cases when the initial field stratification in the star is stable against buoyancy or when the colour-superconductivity gap is larger than 14 MeV and the mass of the strange quark sufficiently high to inhibit the buoyancy of wound-up magnetic fields. Our results also apply to when the star was initially magnetized throughout a substantial fraction of its volume and remained so after a short, first episode of buoyancy.

3.3. Direct magnetic dipole radiation

The time-dependent, internal, stellar magnetic field may be a source of electromagnetic emission from the star's environment, whether a vacuum or a leptonic plasma. For example, if the newborn star is an oblique rotator (Usov 1992), it will emit electromagnetic radiation due to the rotation of its magnetic dipole. We define χ to be the angle between the magnetic and rotation axes, $B_{\rm p}$ the polar field, and Ω_* the star's rotation rate. The power emitted in vacuo by the magnetic dipole rotation is (Landau & Lifshitz 1975):

$$\mathcal{P}_{\rm MD} = \frac{8\sin^2\chi}{3c^3}\, B_{\rm p}^2 R^6 \Omega_*^4. \tag{38}$$

An orthogonal rotator with a polar field of 10^{15} Gauss and a rotation period of 3 ms would radiate a flux of 1.9×10^{48} erg s^{-1}, similar to that required to explain the high-energy photon emission of the gamma-ray burst. For a polar field of only 10^{14} Gauss the emitted power should decline to 1.9×10^{46} erg s^{-1}, which is insufficient to account for a GRB. We note that this emission taps the rotational energy of the compact star, which is about $E_{\rm rot} = I_* \Omega_*^2/2 \approx 2.2\times10^{51}$ erg for a rotation period of 3 ms. Even at this high rate, the radiation of a 10^{15} Gauss millisec magnetar should last for about 10^3 s. In the next section, we discuss whether the internal star's torsional oscillation could somehow act as a substitute for a rotating, magnetic dipole.

4. Radiation by torsional oscillation

The collapse leads to a state of differential rotation in the star, the angular velocity varying either with depth or with latitude or both. In an aligned rotator, somewhat analogously to the rotating oblique dipole, the oscillating internal toroidal magnetic field may act as an antenna generating a large-scale electromagnetic wave in the star's environment at the period $P_{\rm T}$ of the torsional star's oscillation. We calculate in Sect. 4.1 the power emitted in a vacuum environment. The radiation in a leptonic wind is considered in Sect. 4.2.

4.1. Radiation in a vacuum

To study the electromagnetic emission from the compact star driven by the torsional oscillation, we begin by calculating the electromagnetic field in an external vacuum. Maxwell's equations are solved outside the star under the boundary conditions that $B_{\rm r}$ and the tangential components of the electric field $\boldsymbol{E}$ are continuous at the star's surface. The matching of the conditions at the star's surface requires neither B_ϕ nor B_θ to be continuous, since a surface current could support a sharp discontinuity in these components. We denote by a superscript $<$ (or $>$) quantities relevant to the inside (or the outside) of the star. The electric field just below the star's surface is given by the law of perfect conductivity:

$$c\,\boldsymbol{E}^< + \boldsymbol{v}^< \wedge \boldsymbol{B}^< = 0. \qquad (39)$$

Since the velocity of the fluid in the star is assumed to be azimuthal only, the condition that the tangential components of the electric field are continuous reduces to $E_\phi^> = 0$ and:

$$c\,E_\theta^> = -v_\phi^<\, B_{\rm r}^<. \qquad (40)$$

In the presence of a torsional oscillation in the rotating star, Eq. (40) has both a time-varying and a constant component. The latter determines the time-independent, external electric field, while the former determines the outside radiation caused by the torsional oscillation. The boundary condition in Eq. (40) determines completely the solution in the vacuum outside the star. We calculate the electromagnetic field radiated out of the star by the internal torsional oscillation of pulsation $\Omega_{\rm T} = 2\pi/P_{\rm T}$, assuming axisymmetry. The toroidal field is then the only time-dependent component of the outer magnetic field. Omitting for simplicity the superscripts $>$ which refer to the outside region, the equations for the electromagnetic fields in this region are Ampere's and Faraday's equations in a vacuum, given by:

$$\frac{\partial \boldsymbol{E}}{\partial t} - c\boldsymbol{\nabla}\wedge\boldsymbol{B} = 0, \qquad (41)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + c\boldsymbol{\nabla}\wedge\boldsymbol{E} = 0. \qquad (42)$$

The only non-vanishing and time-dependent component of the magnetic field is the azimuthal one. It is useful to introduce an angular potential $\mu(r,\theta,t)$ such that

$$B_\phi = \frac{\partial \mu}{\partial \theta}\,\cdot \qquad (43)$$

The system (41), (42) reduces to an equation for B_ϕ alone, which, for harmonic time-dependance in the form of $\exp(-{\rm i}\Omega_{\rm T} t)$, translates into the Helmholtz equation for μ:

$$c^2\Delta\mu + \Omega_{\rm T}^2\,\mu = 0. \qquad (44)$$

The operator Δ is the ordinary scalar Laplacian. We expand $\mu(r,\theta,t)$ in spherical axisymmetric harmonics. The solution for each harmonic component of degree ℓ of μ is, to an arbitrary multiplicative factor:

$$\mu_\ell(r,\theta,t) = \frac{H^{(1)}_{\ell+1/2}(k_{\rm o} r)}{\sqrt{k_{\rm o} r}}\ \exp(-{\rm i}\Omega_{\rm T}\, t)\, P_\ell^0(\cos\theta) \qquad (45)$$

where $k_{\rm o}$ is given by the dispersion law of free-space electromagnetic waves, that is $k_{\rm o} = \Omega_{\rm T}/c$. In Eq. (45), $H^{(1)}_{\ell+1/2}(x)$ is a semi-integer Hankel function and P_ℓ^0 is the Legendre polynomial of order ℓ. The value of $\ell = 2$ is the lowest value of ℓ pertinent to our problem. In fact Eq. (6) illustrates that $B_\phi^<$ inside the star is generated by $\boldsymbol{V}_\phi \wedge \boldsymbol{B}_{\rm P}$ terms. The dipole components of the poloidal magnetic field $\boldsymbol{B}_{\rm P}^<$ correspond to the lowest value of ℓ, ($\ell = 1$), since $B_{\rm r}^< = B_p^< \cos\theta$ and $B_\theta^< = -B_p^< \sin\theta$, where $B_{\rm p}$ is the polar field. Similarly, the lowest value of ℓ for the time-dependent rotation velocity at the star's surface is $\ell = 1$, when $\delta V^\phi = \delta v_{\rm eq}^\phi \sin\theta$, $\delta v_{\rm eq}^\phi$ being the velocity of the time-dependent part of the rotation at the equator. This angular variation in δV^ϕ corresponds to a constant-amplitude modulation of the rigid-body rotation-rate at the star's surface: $\delta\Omega(R,\theta) = \delta\Omega_{\rm eq}$. Such an oscillation would be induced by variations in the fluid's angular velocity Ω with depth. Differential rotation in latitude corresponds to $\ell > 2$. Since the vector product of $\delta\boldsymbol{V}^\phi$ and the poloidal magnetic field is expanded in spherical harmonics with $\ell \geq 2$, we now restrict our attention to emission in the $\ell = 2$ mode. The semi-integer Hankel functions can be expressed as a sum of a finite number of simple terms. For $\ell = 2$, the solution for μ which behaves as an outgoing wave at infinity is:

$$\mu = m_0 R^3 \left(\frac{k_{\rm o}^2}{3r} + {\rm i}\frac{k_{\rm o}}{r^2} - \frac{1}{r^3}\right)(1 - 3\cos^2\theta)\,{\rm e}^{{\rm i}(k_{\rm o} r - \Omega_{\rm T} t)}, \qquad (46)$$

where m_0 is a complex factor. The complete solution can be derived from Eqs. (41)–(43) and is written in the following form, where B_0 is a complex amplitude:

$$B_\phi^> = \frac{R^3 B_0}{2}\left(\frac{k_{\rm o}^2}{3r} + {\rm i}\frac{k_{\rm o}}{r^2} - \frac{1}{r^3}\right)\sin 2\theta\,{\rm e}^{{\rm i}(k_{\rm o} r - \Omega_{\rm T} t)}, \qquad (47)$$

$$E_r^> = \frac{R^3 B_0 c}{{\rm i}\Omega_{\rm T}}\left(\frac{k_{\rm o}^2}{3r^2} + {\rm i}\frac{k_{\rm o}}{r^3} - \frac{1}{r^4}\right)(1 - 3\cos^2\theta)\,{\rm e}^{{\rm i}(k_{\rm o} r - \Omega_{\rm T} t)}, \qquad (48)$$

$$E_\theta^> = \frac{R^3 B_0 c}{2\,{\rm i}\Omega_{\rm T}}\left[\left(\frac{2}{r^4} - \frac{k_{\rm o}^2}{r^2}\right) + {\rm i}k_{\rm o}\left(\frac{k_{\rm o}^2}{3r} - \frac{2}{r^3}\right)\right]\sin 2\theta\,{\rm e}^{{\rm i}(k_{\rm o} r - \Omega_{\rm T} t)}. \qquad (49)$$

The relations (47)–(49) solve the system of Eqs. (41), (42). The complex amplitude B_0 is determined from the boundary conditions, such that the θ component of the electric field is continuous at the star's surface (Eq. (40)). The time-dependent field component $E_\theta^>$ of Eq. (49) must match the corresponding $\ell = 2$ time-dependent part of the field component $E_\theta^<$ just below the star's surface, which requires that:

$$c\,E_\theta^>\,|_{r=R} = B_{\rm p}\,\delta v_{\phi\,{\rm eq}}\ \sin\theta\cos\theta, \qquad (50)$$

where $B_{\rm p}$ is the polar field and $\delta v_{\phi\,{\rm eq}}$ is the time-dependent part of the $\ell = 1$ component of the azimutal velocity at the equator.

By matching the two members of Eq. (50) we derive the complex wave amplitude B_0. Since the star's radius is far smaller than the wavelength of the emitted wave, Eq. (50) should be evaluated to the dominant order in the small parameter $k_o R \approx 2 \times 10^{-5}$, providing:

$$B_0 = \frac{i\,k_o R}{2}\,B_p\,\frac{\delta v_{\phi\,\mathrm{eq}}}{c}\,. \tag{51}$$

The velocity amplitude of the torsional oscillation is given by Eq. (10) and $k_o = \Omega_T/c$. The modulus of B_0 is then:

$$|\,B_0\,| = 6.94 \times 10^7 \sqrt{\frac{\alpha_D}{10^{-1}}}\left(\frac{10^{-3}\,\mathrm{s}}{P_*}\right)\left(\frac{10\,\mathrm{s}}{P_T}\right) B_{p14}\,R_{10}^2\ \mathrm{Gauss.} \tag{52}$$

The magnetic amplitude of the wave is far smaller than the sub-surface magnetic field because of the significant impedance mismatch between the star's interior and the outside vacuum. Denoting by $v_A^<$ the Alfvén speed inside the star, the ratio of these impedances is $v_A^</c \approx 4 \times 10^{-5}\,B_{p14}$. The peculiarities of the spherical wave solution for E_θ are also responsible for the smallness of this amplitude, which is not set by assuming continuity of B_ϕ. The correct boundary condition is Eq. (50) and its fulfilment implies that B_ϕ is discontinuous at the star's surface.

It is interesting to evaluate the Poynting power radiated off the star's surface. If the low-frequency wave emission can be represented by radiation in vacuo, the solution of Eqs. (47)–(49) provides an upper bound to the power that may be dissipated in the star's environment and radiated away as X and γ photons. The radial component Φ_P^r of the Poynting vector associated with the low-frequency radiation is

$$\Phi_P^r = \frac{1}{2}\left(\frac{c}{4\pi}\,\mathcal{R}e\,(E_\theta^* B_\phi)\right), \tag{53}$$

where the superscript $*$ designates the complex conjugate and $\mathcal{R}e$ the real part of a complex number. The power $\mathcal{P}_{\mathrm{vac}}$ radiated as Poynting flux by the torsional oscillation in its supposedly vacuum environment can be calculated from Eq. (53) by integrating Φ_P^r over the star's surface, taking Eqs. (47)–(49) into account. A number of simplifications occur in this calculation, which finally infers that:

$$\mathcal{P}_{\mathrm{vac}} = \frac{1}{135\,c^3}\left(|\,B_0\,|\,R^3\Omega_T^2\right)^2. \tag{54}$$

The magnetic amplitude B_0 of the wave is given by Eq. (51) and $\delta v_{\phi\,\mathrm{eq}}$ is given by Eq. (10). We then have:

$$\mathcal{P}_{\mathrm{vac}} = \frac{\alpha_D}{540\,c^7}\,B_p^2\,\Omega_T^6\,\Omega_*^2\,R^{10}. \tag{55}$$

Numerically, the power $\mathcal{P}_{\mathrm{vac}}$ amounts to:

$$\mathcal{P}_{\mathrm{vac}} = 2.06 \times 10^{17}\,B_{p14}^2\,R_{10}^{10}\left(\frac{\alpha_D}{10^{-1}}\right)\left(\frac{10^{-3}\,\mathrm{s}}{P_*}\right)^2\left(\frac{10\,\mathrm{s}}{P_T}\right)^6\ \mathrm{erg\,s^{-1}}. \tag{56}$$

A glance at Eqs. (38) and (55) indicates that much less energy is radiated away in an outside vacuum by the torsional oscillation than by an oblique, rotating, magnetic dipole. The Poynting power radiated by the torsional wave is smaller than the emission of a rotating dipole for several reasons. First, the radiation is quadrupolar instead of dipolar. Then, the field $|B_0|$ is only of the order 10^8 Gauss for the values of parameters adopted as representative (Eq. (52)), much less than the polar field of an ordinary pulsar or magnetar. Finally, the period P_T is of the order of a few seconds, much longer than the rotation period of the new-born compact star. As a result, the power in Eq. (56) falls short by many orders of magnitude of the observed power of γ radiation in a long GRB!

4.2. Radiation in a leptonic wind

Could the presence of a circumstellar, leptonic plasma drastically change the power radiated by the torsional oscillation? This plasma originates in charges, electrons, and positrons that have passed the bag of the quark star. We note that at a distance from the star of larger than the light-cylinder radius, the plasma cannot be in rigid corotation, but must flow outward (Goldreich & Julian 1969). Since the wavelength associated with the frequency of the torsional oscillation is much larger than the light-cylinder radius of the rapidly spinning star, the wave emitted by this oscillation propagates into the wind driven by the rapid global rotation. We have to determine the amount of energy of the torsional oscillation radiated per second in these conditions. We assume that the wind has already reached its terminal velocity at the surface of a sphere of radius comparable to that of the light-cylinder, c/Ω_*. Since the ratio of the torsional oscillation period to the spin rotation period is large, the torsional oscillation appears, on both of the scales of the spin period and the light-cylinder radius, as a quasistatic perturbation. Its effect is not only to emit a signal that assumes the character of a wave at distances larger than c/Ω_T, but it also modulates the wind in which it propagates as a result of the variations imposed on the conditions of its lauching. These modulational effects are distinct from emission of low-frequency radiation in a given wind. Much of the action causing wind modulation occurs below or close to the light-cylinder and will be discussed in Sect. 5.

We now calculate the emission by the torsional oscillation in an expanding, possibly resistive, leptonic wind. The conductivity σ of the medium is assumed to be real. This is because the wave would be highly non-linear if the gyrofrequency ω_B of leptons in the wave's magnetic field was much larger than the wave frequency. As a result, the effect of the plasma current on the real part of the index of refraction would become negligible and the wave would force its way non-linearly through the leptonic environment (Asseo et al. 1975; Salvati 1978). We may then restrict our consideration to resistive effects. The modulus of the wind speed $\boldsymbol{w}$ is assumed to be constant, both in time and space, and oriented radially outwards: $\boldsymbol{w} = w\,\boldsymbol{e}_r$. We assume the wind to be ultra-relativistic and to have a velocity equal to the speed of light. When taking the wind to be radial, we assume that corotation is lost at distances of the order or larger than c/Ω_T. The background, magnetic field in the wind is severely wound up by the star's rotation. At distances much larger than the light-cylinder radius, its azimuthal component dominates over the poloidal component and declines proportionally to $1/r$. We thus neglect the poloidal field component and assume the unperturbed magnetic field $\boldsymbol{B}_0$ to be azimuthal, so that:

$$\boldsymbol{B}_0 = \hat{B}_0\,\frac{R}{r}\,f(\theta)\,\boldsymbol{e}_\phi. \tag{57}$$

Specifically, we consider $f(\theta) = \cos\theta/(|\cos\theta|\,\sin\theta)$. The electric current associated with Eq. (57) is radial. For the adopted angular profile it reduces to zero almost everywhere, except at both the polar axis and the equatorial plane. Heyvaerts & Norman (2003) demonstrated that the magnetic field in a perfect MHD wind asymptotically becomes potential almost everywhere, the current being confined to boundary layers about the polar axis and at surfaces where the poloidal polarity reverses. Our choice of $f(\theta)$ complies with this. The singularity at the polar axis represents the current carried by a jet, while the change in the direction of the field at the crossing of the equator is caused by the change in polarity of the poloidal field at $\theta = \pi/2$. Ohm's law infers that:

$$\boldsymbol{j} = \sigma\left(\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{w} \wedge \boldsymbol{B}\right) + \rho_{\mathrm{e}}\boldsymbol{w}. \tag{58}$$

By considering the divergence of Eq. (58) and solving the resulting differential equation for the charge density ρ_{e}, it is found that the latter vanishes in the unperturbed wind. The unperturbed electric field then vanishes too. We define $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{j}$, ρ_{e}, and $\boldsymbol{u}$ to be the perturbations of the electric and magnetic field, current density, charge density, and leptonic fluid velocity, respectively. It is sufficient to describe the lepton's dynamics at the inertia-less (also called force-free) approximation. The magnetic-field perturbation is toroidal in the considered geometry and the electric force $\rho_{\mathrm{e}}\boldsymbol{E}$ is a negligible second-order term. The perturbed Maxwell equations, Ohm's law, and the dynamical equation in the zero-inertia limit can be written (using the compact notation ∂_{t} for time derivatives) as:

$$c\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\partial_{\mathrm{t}}\boldsymbol{B}, \tag{59}$$

$$c\boldsymbol{\nabla} \wedge \boldsymbol{B} = 4\pi\boldsymbol{j} + \partial_{\mathrm{t}}\boldsymbol{E}, \tag{60}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{61}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho_{\mathrm{e}}, \tag{62}$$

$$c\boldsymbol{j} = \sigma\left(c\boldsymbol{E} + \boldsymbol{w} \wedge \boldsymbol{B} + \boldsymbol{u} \wedge \boldsymbol{B}_0\right) + c\rho_{\mathrm{e}}\boldsymbol{w}, \tag{63}$$

$$\boldsymbol{j} \wedge \boldsymbol{B}_0 = 0. \tag{64}$$

The toroidal components of Eqs. (59) and (63) imply that the electric field is only poloidal. The other two components of Eq. (63) infer the fluid velocity once the solution for the other unknowns has been found. The current $\boldsymbol{j}$ is eliminated by taking the vector product of Eq. (60) and $\boldsymbol{B}_0$. As a result the conductivity is eliminated from the equations describing the perturbation. With Eq. (59), this infers an equation for the magnetic perturbation, which is expressed most accurately in terms of the angular potential μ of this perturbation (Eq. (43)). For harmonic time-dependence $\exp(-\mathrm{i}\Omega_{\mathrm{T}}t)$, it is found that μ satisfies the Helmholtz equation (Eq. (44)). Regardless of the conductivity, the perturbations propagate in this geometry as electromagnetic waves, provided the inertia-less limit is considered. The complete solution is identical to the vacuum result (Eqs. (47)–(49)) as is of course the Poynting flux at the star's surface and the emitted Poynting power (Eq. (55)).

5. Modulation by the torsional oscillation of the energy emitted in the rotator's wind

A rapidly-spinning aligned rotator emits a wind carrying power in electromagnetic, potential, thermal and kinetic energy form. The contributions of these different forms of energy depend on the distance to the star. Some forms of energy may dissipate en route or at terminal shocks, producing observable X and γ radiation. Close to the compact star, much of this flux is in Poynting form because the kinetic energy remains low while the wind has not yet been effectively accelerated. The thermal and gravitational energy fluxes often constitute but a little part of the total energy flux. The energy output of the star in its wind environment then enters the latter as DC Poynting flux, the radial component of which is given in terms of the field components just above the star's surface by:

$$\Phi_{\mathrm{DC}} = \frac{c}{4\pi}\,E_\theta^{>}B_\phi^{>}. \tag{65}$$

The boundary condition at the star's surface implies that $E_\theta^{>} = E_\theta^{<}$. We define v_ϕ to be the subsurface fluid velocity and B_{r} the radial field component, which is continuous across the star's surface. We then find that:

$$\Phi_{\mathrm{DC}} = -\frac{1}{4\pi}\;v_\phi(R)\;B_{\mathrm{r}}(R)\;B_\phi^{>}(R). \tag{66}$$

The value of $B_\phi^{>}$ at the base of a relativistic wind is given approximately by Eq. (67) below. This can be sketchily explained as follows: due to the rotation of the star and the effect of flux freezing, a toroidal field is generated on open field lines from the poloidal field. The knowledge that the foot point of a field line is rotating is propagated along this line at a finite velocity v_{prop}, by means of convective transport and propagation as an Alfvénic signal. The field line thus curves away from the sense of rotation at an angle of ξ to the radial direction, such that $\tan\xi = \Omega_* R\sin\theta/v_{\mathrm{prop}}$. Since the wind is relativistic and the Alfvén speed in the tenuous external plasma is close to the speed of light, $v_{\mathrm{prop}} \sim c$ and the toroidal field just above the star's surface is:

$$B_\phi^{>}(R) \approx -\frac{R\,\Omega_*\sin\theta}{c}\;B_{\mathrm{r}}(R). \tag{67}$$

A more precise justification of the approximate relation in Eq. (67) is omitted for conciseness. Then, from Eq. (66):

$$\Phi_{\mathrm{DC}} \approx \frac{v_\phi^2 B_{\mathrm{r}}^2}{4\pi c}\,\cdot \tag{68}$$

This flux is only emitted from the polar caps, the regions on the stellar surface connected to open, field lines. The magnetosphere is closed, where the apex of the local field line is at a distance D smaller than the light-cylinder radius c/Ω_*. When field lines are dipolar, the polar caps extend to a colatitude θ_{pc}, which we refer to as classical, and described by:

$$\sin^2\theta_{\mathrm{pc}} = \frac{\Omega_* R}{c}\,\cdot \tag{69}$$

Under certain conditions however, the magnetospheric field may depart considerably from dipolarity (Sect. 5.3). When the flux is distributed on the star as a dipolar field, the radial, field component varies with θ as $B_{\mathrm{r}} = B_{\mathrm{p}}\cos\theta$, B_{p} being the field at the pole. By considering $v_\phi(\theta)$ to be the solid body rotation velocity at the angular speed Ω_*, we obtain, by integrating over the colatitudes corresponding to the two polar caps, the DC Poynting power emitted by the star under these conditions:

$$\mathcal{P}_{\mathrm{sol}} = \frac{B_{\mathrm{p}}^2 R^6\Omega_*^4}{4c^3}\,\cdot \tag{70}$$

The power represented by Eq. (70) is comparable to the power emitted by an oblique, rotating dipole (Eq. (38)). This is a classical result (see for example Michel 1991). With a rotation period of 3 ms, the power $\mathcal{P}_{\mathrm{sol}} \approx 1.78\times10^{45}B_{p14}^2$ erg s^{-1}. This is insufficient to match the high luminosity of a GRB unless fields in excess of 10^{15} Gauss are involved (Usov 1992). Could differential rotation drastically change this result?

5.1. Quasistatic modulation of the wind

We assume that the star experiences a torsional oscillation. The velocity $v_\phi(\theta)$ then differs from the solid-body rotation velocity and varies with time. Because the period of the oscillation is much longer than the mean spin period, this causes a quasistatic change in both the structure of the magnetosphere and the polar cap angle. It even causes a change, which we neglect, in the shape of the light cylinder. The structure of the magnetosphere and the energy output of the wind adjust to equilibrium values corresponding to the instantaneous velocity profile on the star's surface.

This profile may be even with respect to the equator, or odd, or a mixture of both. An even oscillation is one in which the azimuthal velocity perturbation is symmetric with respect to the equator, i.e. where the time-varying azimuthal velocity is in phase at two points positioned symmetrically with respect to the equator. An odd oscillation is one in which it is antisymmetrical, i.e. where the velocity is in phase-opposition at two points symmetric with respect to the equator. It may appear that nature should provide only even profiles, by a principle, or rather a postulate, of north-south symmetry. However, this symmetry is broken in the case of the collapse of supernovae. It is indeed well known that new-born neutron stars receive a kick, that is, a net thrust from the collapse. This would be forbidden by the principle of north-south symmetry. Therefore, it cannot be excluded that, similarly, odd modes of torsional oscillation are present in the initial excitation of a new-born compact star. The amplitude of these modes is expected to be small, but we show below that it need not be large to produce important effects. The kick received by a new-born neutron star produces a velocity of the order of 200–500 $\mathrm{km\,s^{-1}}$. This corresponds to an asymmetry in the momentum emission of the order of 1 $M_\odot \times 300\ \mathrm{km\,s^{-1}}$. The supernova explosion emits a momentum per steradian of about 10 $M_\odot \times 30\,000\ \mathrm{km\,s^{-1}}/4\,\pi$. The asymmetry in the momentum emission appears to be a fraction of between a few 10^{-4} and a few 10^{-3} of the total. A similar fraction of the total star's rotational energy may appear after the collapse in the form of odd differential rotation.

5.2. Modulation of the wind by an even oscillation

An even torsional oscillation has but little effect on the structure of the magnetosphere and wind because the two footpoints of a closed field line follow the same motion exactly if, as assumed in this subsection, the poloidal field lines are strictly symmetric with respect to the equator. Otherwise, these field lines would undergo a twist in the presence of an even torsional oscillation, because their footpoints would not be at exactly opposite latitudes, and would be carried in the azimuthal direction at different angular velocities. It is difficult to realistically anticipate the degree af asymmetry in poloidal field lines. It could vary from very little to a complete absence of symmetry. The degree of asymmetry necessary for the magnetosphere to open will be estimated in Sect. 5.3. Strictly symmetric field lines whose footpoints are moved by an even torsional oscillation are, however, not twisted and, as a result, no poloidal electric current is driven in the magnetosphere. Nevertheless, because the rotation rate on the star varies with colatitude, the Goldreich-Julian charge distribution in the magnetosphere differs slightly from the case of solid body rotation, as well as the DC Poynting flux (Eq. (68)). Using as a model the following even differential rotation:

$$v_\phi = R \sin\theta \, (\Omega_* + \delta\Omega_* \sin\theta \cos\Omega_{\rm T} t)\,, \tag{71}$$

we calculate $\Phi_{\rm DC}$ from Eq. (68) and integrate over the classical polar caps to derive the emitted power $\mathcal{P}_{\rm even}$:

$$\mathcal{P}_{\rm even} = \mathcal{P}_{\rm sol} \left(1 + \varepsilon_1 \cos\Omega_{\rm T} t + \varepsilon_2 \cos^2\Omega_{\rm T} t\right), \tag{72}$$

where

$$\varepsilon_1 = \frac{8}{5}\sqrt{\frac{\Omega_* R}{c}}\left(\frac{\delta\Omega_*}{\Omega_*}\right), \quad \varepsilon_2 = \frac{2}{3}\left(\frac{\Omega_* R}{c}\right)\left(\frac{\delta\Omega_*}{\Omega_*}\right)^2. \tag{73}$$

The wind power is modulated slightly at a level of ε_1, which is about 10 % for a period of 3 ms and $\delta\Omega_*/\Omega_* = \sqrt{\alpha_{\rm D}} \approx 1/3$. The change in the time-averaged power is at the 0.1% level for the same figures. These small changes cannot account for the existence of a gamma-ray burst.

5.3. Magnetosphere opening by an odd oscillation

An odd oscillation differs from an even one in that the two footpoints of a closed field line experience differential motion in longitude, introducing a twist in this field line. This causes a poloidal current to flow in the closed magnetosphere and drastically changes its structure. When the twist exceeds a threshold of order π, the magnetosphere opens. Field expansion by the shearing of the footpoints of field lines was first discussed in the context of solar flares (Heyvaerts et al. 1982; Aly 1985; Low 1990). It was established that it occurs in Cartesian geometry with a direction of invariance by theorems constraining the properties of line-tied force-free fields (Aly 1985, 1990) and by numerical simulations (Biskamp & Welter 1989). The same process has been considered also in the case of axisymmetric structures extending above a spherical surface on which the field lines are tied. In general, it was demonstrated that rapid expansion occurs when a finite shear is reached (Aly 1995). This is also supported by numerical simulations (Mikić & Linker 1994). Full opening occurs for a finite twist (of order π) in some specific examples (Lynden-Bell & Boily 1994; Wolfson 1995). In the present context, the light-cylinder radius imposes a limit on the distance to the apex of closed field lines, such that field opening is even easier when magnetospheric inflation proceeds. When the field opening becomes almost complete, it causes a growth in the polar caps and the emitted wind power. In this case, the role of the torsional oscillation is not to modulate the energy output by the addition of its own electromagnetic emission but to open the door for a more significant wind emission from the central object. This enhanced wind emission would acquire its energy directly from the rotational kinetic energy of the star, not only from the energy of the differential rotation, and lasts for as long as the torsional oscillation survives with sufficient amplitude.

The magnetosphere opens if the difference in longitude ψ between the two conjugate footpoints of a field line exceeds typically half a turn ($\psi > \pi$). To justify this statement, one should try to solve for the structure of the magnetosphere as a function of the difference in longitude between the footpoints of field lines. By assuming axisymmetry, the poloidal field is represented by a flux function $a(r,\theta)$, so that the total magnetic field can be written as:

$$\boldsymbol{B} = \frac{\boldsymbol{\nabla} a \wedge \boldsymbol{e}_\phi}{r\sin\theta} + B_\phi(r,\theta)\,\boldsymbol{e}_\phi. \tag{74}$$

Any field line follows a surface of constant a (a magnetic surface) because the magnetic flux being transmitted through a circle perpendicular to and centred on the polar axis passing at (r,θ), is $\Phi_{\rm M} = 2\pi a(r,\theta)$. The perturbations of the closed magnetosphere of the star are of quasi-static nature (Sect. 5.1). Neglecting the particle's inertia, the force equation for the instantaneous equilibrium is the force-free equation:

$$c\,\rho_{\rm e}\boldsymbol{E} + \boldsymbol{j} \wedge \boldsymbol{B} = 0, \tag{75}$$

The electromagnetic state of the magnetosphere is not described by the magnetic field alone but also by the electric potential $U(r,\theta)$. Equation (75) is supplemented by the time-independent Maxwell's equations:

$$\boldsymbol{\nabla}\wedge\boldsymbol{B} = 4\pi\boldsymbol{j}/c, \tag{76}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho_{\rm e}, \tag{77}$$

$$\boldsymbol{E} = -\boldsymbol{\nabla}U. \tag{78}$$

The components of Eq. (75) can be expressed in terms of the functions $a(r,\theta)$, $U(r,\theta)$, and $I(r,\theta)$, the latter being defined by:

$$I(r,\theta) = c\, r\sin\theta\, B_\phi. \tag{79}$$

The current through the circle perpendicular to and centred on the polar axis passing at (r,θ) is $J = I/2$. We refer to I as the poloidal current. In an axisymmetric state, the electric field is poloidal. The toroidal component of Eq. (75) then shows that the gradients of I and a are everywhere parallel, which implies that I is a function of a: $I(r,\theta) = I(a(r,\theta))$. The poloidal part of Eq. (75) can then be written as:

$$r^2\sin^2\theta\,\Delta U\;\;\boldsymbol{\nabla}U = \left(\Delta^* a + \frac{I}{c^2}\frac{{\rm d}I}{{\rm d}a}\right)\boldsymbol{\nabla}a, \tag{80}$$

where Δ is the scalar Laplacian and

$$\Delta^* a = \frac{\partial^2 a}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial a}{\partial\theta}\right). \tag{81}$$

Equation (80) indicates that the gradients of U and a are everywhere parallel, which implies that U is a function of a, $U(r,\theta) = U(a(r,\theta))$. The rotation rate of the matter, $\Omega(a)$, is given by the electric drift velocity of particles and is found to be:

$$\Omega(a) = c\,U'(a)\cdot \tag{82}$$

The projection of Eq. (80) onto $\boldsymbol{\nabla}a$ provides the so-called pulsar equation (Michel 1991):

$$\left(1-\frac{r^2\Omega^2\sin^2\theta}{c^2}\right)\Delta a - \frac{r^2\sin^2\theta}{c^2}\,\Omega\,\Omega' \\ -\frac{2}{r}\frac{\partial a}{\partial r} - \frac{2\cos\theta}{r^2\sin\theta}\frac{\partial a}{\partial\theta} + \frac{I\,I'(a)}{c^2} = 0. \tag{83}$$

This equation has a singularity at the light-cylinder, which causes any field line reaching this limit to diverge (Contopoulos et al. 1999). To determine approximately which field lines become open, it suffices to solve Eq. (83) for $r\Omega\sin\theta \ll c$ and confirm which field lines reach a distance larger than c/Ω_*. In this limit, Eq. (83) reduces to:

$$\frac{\partial^2 a}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial a}{\partial\theta}\right) + \frac{I\,I'(a)}{c^2} = 0. \tag{84}$$

It is shown in Appendix A that, for a self-similar model of the magnetospheric field, there is no solution to Eq. (84) with closed field lines when the twist exceeds π.

We represent odd differential rotation by the following simple model for surface differential rotation:

$$\delta\Omega(\theta) = \delta\Omega_*\;\sin 2\theta\cos\Omega_{\rm T}t. \tag{85}$$

The magnetic flux distribution on the surface of the star is a function of the colatitude θ, so that θ is a known function of a, $\theta(a)$. The twist at time t associated with Eq. (85) is:

$$\psi(a) = \frac{2\delta\Omega_*}{\Omega_{\rm T}}\;\sin 2\theta(a)\,\sin\Omega_{\rm T}t. \tag{86}$$

It is implied here that the footpoint P_2 is at the colatitude θ and that P_1 and P_2 are at the same longitude at $t = 0$. In the model presented in Appendix A, we regard $\langle\psi\rangle$ at time t as being half the maximum twist implied by Eq. (86). The limit value of π for the twist is reached for a rather small amplitude of the torsional oscillation because the period of the latter is long compared to the spin period. Similarly to Eq. (10), we parametrize $\delta\Omega_*$ as $\delta\Omega_* = \sqrt{\alpha_{\rm odd}}\;\Omega^*$ in terms of the fraction $\alpha_{\rm odd}$ of the star's rotational energy available in this odd oscillation mode. The magnetosphere is in an open magnetic configuration when:

$$\sin\Omega_{\rm T}t > 10^{-2}\left(\frac{P_*}{10^{-3}\,{\rm s}}\right)\left(\frac{10\,{\rm s}}{P_{\rm T}}\right)\left(\frac{10^{-4}}{\alpha_{\rm odd}}\right)^{1/2}\cdot \tag{87}$$

As explained in Sect. 5.1, a fraction 10^{-4} of the star's rotational energy may be stored in odd torsional oscillation modes. At this level of excitation of odd modes, the magnetosphere would be open during a large fraction of the oscillation period. There would be no opening only when $\alpha_{\rm odd}$ is very small, i.e.:

$$\alpha_{\rm odd} < 10^{-8}\left(\frac{P_*}{10^{-3}\,{\rm s}}\right)^2\left(\frac{10\,{\rm s}}{P_{\rm T}}\right)^2\cdot \tag{88}$$

We assume that odd oscillation modes are initially excited to a level higher than the limit indicated by Eq. (88).

A similar result is obtained when an even oscillation is considered (with an amplitude given by the larger value indicated in Eq. (10)) but the dipolar-like magnetic-field lines are not strictly symmetric with respect to the equator. This would happen if, for example, the field is a non-centred dipole. We define $\delta\theta$ to be the difference of the absolute values of the latitudes of two conjugate footpoints. The twist experienced by these footpoints will be larger than π, and thus the magnetosphere will open, when $\sqrt{\alpha_{\rm D}}\,\Omega_*\;(\delta\theta/(\pi/2))\;P_{\rm T} > \pi$, which translates into the condition:

$$\frac{\delta\theta}{\pi/2} > 1.5\times 10^{-4}\left(\frac{P_*}{10^{-3}\,{\rm s}}\right)\left(\frac{10\,{\rm s}}{P_{\rm T}}\right)\left(\frac{10^{-1}}{\alpha_{\rm D}}\right)^{1/2}\cdot \tag{89}$$

For Eq. (89) to be satisfied for typical values of P_*, $P_{\rm T}$, and $\alpha_{\rm D}$, it suffices that the dipole field be decentred by a fraction of a few 10^{-3} of the stellar radius. Since the odd torsional oscillation has an amplitude larger than indicated by Eq. (88) or the magnetic geometry is north-south asymmetric to a degree larger than indicated by Eq. (89), the magnetosphere will alternate during the oscillation cycle between a classical state, which we refer to as closed, and an open state.

For an odd torsional oscillation, the configuration is closed when the condition of Eq. (87) is not satisfied. It becomes an open state, where all field lines are open and carry winds, when the twist is sufficiently large for Eq. (87) to be satisfied. During closed episodes, the polar caps opening is limited to $\theta_{\rm pc} = (\Omega_* R/c)^{1/2}$, and during open episodes $\theta_{\rm pc} = \pi/2$. There is a transitory state which we neglect because it lasts much less than a wave period. In an open state, the power fed by the compact star into its relativistic wind is much larger than the classical value given by Eq. (70). This may be the reason why the emitted power is enhanced considerably in the first moments after the collapse, an enhancement that should decline as the star's rotation decelerates and last at most until the odd mode amplitude has decreased below the limit fixed by Eq. (88). We calculate the lifetime of odd oscillations and their associated emission in Sect. 5.4. The idea that a GRB would be the result of pulsar-type emission from a compact star with an entirely open magnetosphere was considered by Ruderman et al. (2000), who however regard the expansion of the magnetosphere as being caused by magnetic buoyancy rather than by twisting, as we suggest in this paper.

The power emitted at time t is calculated by integrating the Poynting flux given in Eq. (68) over the wind-emitting star surface, where v_ϕ is given by (see Eq. (85)):

$$v_\phi = R\Omega_* \sin\theta \,(\Omega_* + \delta\Omega_* \sin 2\theta \cos\Omega_{\rm T} t)\,. \tag{90}$$

When the magnetoshere is in a closed state, the emitted power is, neglecting terms of order $(\delta\Omega_*/\Omega_*)^2$:

$$\mathcal{P}_{\rm odd/cl} = \frac{B_{\rm p}^2 R^6 \Omega_*^4}{4c^3}\,. \tag{91}$$

Similarly, when the magnetosphere is open, it is given by:

$$\mathcal{P}_{\rm odd/op} = \frac{2}{15}\,\frac{B_{\rm p}^2 R^4 \Omega_*^2}{c}\,. \tag{92}$$

Numerically, with $B_{p14} = B_{\rm p}/10^{14}$ Gauss and $R = 10$ km:

$$\mathcal{P}_{\rm odd/op} = 1.8\times 10^{48}\, B_{p14}^2 \left(\frac{10^{-3}\,{\rm s}}{P_*}\right)^2 {\rm erg\,s^{-1}}\,. \tag{93}$$

The power emitted during the open episodes is larger than that emitted during the closed episodes by a factor of $8c^2/(15\Omega_*^2 R^2)$. For $P_* = 3$ ms, this factor is about 110. For the same rotation period and $B_{p14} = 3$, $\mathcal{P}_{\rm odd/op}$ reaches 1.8×10^{48} erg s^{-1}, which is enough to explain the GRB emission, allowing for a conversion factor from kinetic wind energy to radiation that is smaller than unity.

5.4. Damping of rotation and odd oscillation

Neither the amplitude of the odd torsional oscillation nor the rapid stellar rotation will last long in the presence of such large losses. A rapidly spinning star with an odd oscillation is characterized by two parameters, the average star-rotation rate Ω_* and the amplitude of the differential rotation $\delta\Omega_*$. Due to wind losses, both decrease in time. To calculate their evolution, a model of the internal magnetic field of the star is required. Although this is not an accurate model for the winding-up of the field when the rotation depends only on the distance to the axis, we shall assume for simplicity that the unperturbed, magnetic field is uniform in the star and parallel to the rotation axis, i.e. that $\boldsymbol{B}_< = B_{\rm p}\,\boldsymbol{e}_z$. The normal component of this field on the star's surface equals that of a dipolar field and the inner field $B_{\rm p}$ equals the outer field at the $z > 0$ pole. It is convenient for us to use the cylindrical coordinates D, ϕ, and z, the parameter r representing the spherical distance to the star's centre, R the radius of the star, and θ the colatitude. Since the fluid is almost incompressible with a uniform mass density ρ, we assume as in Sect. 2 that its velocity is azimuthal and can be written as:

$$\boldsymbol{V} = (\Omega_* D + v_\phi(D,z,t))\;\boldsymbol{e}_\phi. \tag{94}$$

The velocity v_ϕ is supposedly odd in z. When writing the even part of the rotation velocity as $\Omega_* D$, we neglect the even torsional modes, which play no role in the magnetosphere opening in the case of a north-south, symmetrical, magnetic structure. The magnetic field in the presence of the perturbation develops an azimuthal component equal to:

$$\boldsymbol{B} = B_{\rm p}\,\boldsymbol{e}_z + B_\phi(D,z,t)\,\boldsymbol{e}_\phi. \tag{95}$$

For the assumed, uniform, unperturbed field, Eqs. (5), (6) can be written as:

$$\frac{\partial v_\phi}{\partial t} = \frac{B_{\rm p}}{4\pi\rho}\,\frac{\partial B_\phi}{\partial z}, \tag{96}$$

$$\frac{\partial B_\phi}{\partial t} = B_{\rm p}\,\frac{\partial v_\phi}{\partial z}\cdot \tag{97}$$

If v_ϕ were, at any given time, structured according to the Proudman-Taylor theorem, it would be cylindrical, that is, it would depend only on the distance to the rotation axis. Then, for a completely axial field as assumed above, the term on the right-hand side of Eq. (97) would vanish. The assumption of a uniform axial field, however, is only applied to simplify the following calculations. Nothing constrains the structure of the field inside the star, especially when it is not dynamically significant. For non-axial $\boldsymbol{B}_{\rm P}$, Eqs. (96), (97) still provide a sketchy representation of the torsional oscillation: the operator $B_{\rm p}\,\partial/\partial z$ must be assumed to represent $(\boldsymbol{B}_{\rm P}\cdot\nabla)$ (see Eqs. (5), (6)) and the variations in $r\sin\theta$ along a field line should be ignored, although they exist. Using Eqs. (96), (97) is equivalent here to replacing a curved poloidal magnetic field in a cylindrical velocity field with a uniform axial field in a z-dependent velocity field. An odd torsional oscillation in a cylindrical velocity field would correspond to north-south asymmetric poloidal field lines. The simple approach implied by Eqs. (96), (97) should be sufficient for our purposes of estimating the damping time of the torsional oscillation. These equations hold for a linear as well as for a non-linear axisymmetric perturbation of an incompressible medium. The velocity perturbation v_ϕ satisfies the Alfvén propagation equation:

$$\frac{\partial^2 v_\phi}{\partial t^2} = \frac{B_{\rm p}^2}{4\pi\rho}\,\frac{\partial^2 v_\phi}{\partial z^2}\,. \tag{98}$$

A standing wave solution of Eq. (98) is

$$v_\phi(D,z,t) = \hat{v}_\phi(D)\,\sin kz \cos\Omega_{\rm T} t, \tag{99}$$

where k and $\Omega_{\rm T}$ are related by the dispersion relation $\Omega_{\rm T}^2 = k^2 v_{\rm A}^2$ and $v_{\rm A}^2 = B_{\rm p}^2/(4\pi\rho)$. The wavenumber k could depend on the distance D to the axis, each cylindrical magnetic surface then having its own oscillation period. To keep things simple, we assume that k is a constant equal to π/R, $\Omega_{\rm T} = \pi v_{\rm A}/R$, and v_ϕ is an odd function of z. The proper choice of $\hat{v}_\phi(D)$ in Eq. (99) ensures that the velocity field of the perturbation at the star's surface coincides with Eq. (90), that is:

$$\hat{v}_\phi(R\sin\theta) = \frac{2R\,\delta\Omega_*\,\sin^2\theta\cos\theta}{\sin(\pi\cos\theta)}\,. \tag{100}$$

Equations (97) and (99) infer the magnetic perturbation to be:

$$B_\phi = B_{\rm p}\,\frac{\hat{v}_\phi}{v_{\rm A}}\,\cos kz\,\sin\Omega_{\rm T} t\,\cdot \tag{101}$$

To obtain the angular-momentum balance equation for both the $z > 0$ and $z < 0$ hemispheres, the torques acting on each must be calculated. Each hemisphere experiences volume torques exerted by magnetic tension and surface torques caused by the drag produced by the emission of Poynting energy at the star's surface. The angular momentum J_+ of the $z > 0$ hemisphere and the angular momentum J_- of the $z < 0$ hemisphere can be calculated from Eqs. (94), (99), and (100). They are:

$$J_+ = I_1\,\Omega_* + I_2\,\delta\Omega_*\cos\Omega_{\rm T} t, \tag{102}$$

$$J_- = I_1\,\Omega_* - I_2\,\delta\Omega_*\cos\Omega_{\rm T} t. \tag{103}$$

The moments of inertia I_1 and I_2 are:

$$I_1 = \frac{4\pi}{15}\rho R^5, \tag{104}$$

$$I_2 = 4\rho R^5 \int_0^{\pi/2} \sin^4\theta\,\cos^2\theta\,\tan\left(\frac{\pi}{2}\cos\theta\right)\,\mathrm{d}\theta. \tag{105}$$

The numerical value of the integral on the right-hand side of Eq. (105) is 0.187. The moment of inertia I_* of the entire star with respect to the rotation axis is $I_* = 2I_1$. The torque T_{B+} exerted by magnetic tension on the $z > 0$ hemisphere can be calculated from the Lorentz force density (Eq. (96)). If B_ϕ is given by Eq. (101), this implies that:

$$T_{B+} = -\frac{B_\mathrm{p}^2 R^4}{v_\mathrm{A}}\,\delta\Omega_*\,\sin\Omega_\mathrm{T} t \int_0^{\pi/2} \sin^4\theta\cos^2\theta\tan\left(\frac{\pi}{2}\cos\theta\right)\mathrm{d}\theta. \tag{106}$$

The magnetic tension torque T_{B-} exerted on the $z < 0$ hemisphere is $T_{B-} = -T_{B+}$. The Poynting torque $\mathrm{d}T_{\mathrm{P}+}$ exerted on the strip of the star's surface between colatitudes θ and $\theta + \mathrm{d}\theta$ in the $z > 0$ hemisphere is related to the Poynting power $\mathrm{d}\mathcal{P}_+$ emanating from that strip by $\mathrm{d}\mathcal{P}_+ = -\dot{\phi}\,\mathrm{d}T_{\mathrm{P}+}$, where $\dot{\phi}$ is the angular velocity of the fluid at that colatitude and at that time (see Eqs. (90), (68) and (99), (100)). The torque $\mathrm{d}T_{\mathrm{P}-}$ on the strip between θ and $\theta + \mathrm{d}\theta$ in the $z < 0$ hemisphere is similarly calculated:

$$\mathrm{d}T_{\mathrm{P}+} = -\frac{B_\mathrm{p}^2 R^4}{2c}(\Omega_* + \delta\Omega_*\sin 2\theta\cos\Omega_\mathrm{T} t)\sin^3\theta\cos^2\theta\,\mathrm{d}\theta, \tag{107}$$

$$\mathrm{d}T_{\mathrm{P}-} = -\frac{B_\mathrm{p}^2 R^4}{2c}(\Omega_* - \delta\Omega_*\sin 2\theta\cos\Omega_\mathrm{T} t)\sin^3\theta\cos^2\theta\,\mathrm{d}\theta. \tag{108}$$

The total Poynting torques $T_{\mathrm{P}+}$ and $T_{\mathrm{P}-}$ on the $z > 0$ and $z < 0$ hemispheres are derived by integrating Eqs. (107) and (108), respectively, over colatitudes from zero to the polar cap angle θ_pc. When the magnetosphere is closed, $\theta_\mathrm{pc} \approx (\Omega_* R/c)^{1/2}$. When it is entirely open, $\theta_\mathrm{pc} = \pi/2$. The angular-momentum balance equation for the $z > 0$ and $z < 0$ hemisphere are given respectively by:

$$\frac{\mathrm{d}J_+}{\mathrm{d}t} = T_{B+} + T_{\mathrm{P}+}, \tag{109}$$

$$\frac{\mathrm{d}J_-}{\mathrm{d}t} = T_{B-} + T_{\mathrm{P}-}. \tag{110}$$

By adding the expressions in Eqs. (109) and (110), we derive an equation for the time evolution of the global rotation Ω_*:

$$\frac{\mathrm{d}\Omega_*}{\mathrm{d}t} = -\left(\frac{B_\mathrm{p}^2 R^4}{I_* c}\int_0^{\theta_\mathrm{pc}} \sin^3\theta\,\cos^2\theta\,\mathrm{d}\theta\right)\Omega_*. \tag{111}$$

By substracting Eqs. (110) from (109) we derive after some algebra:

$$\frac{\mathrm{d}\delta\Omega_*}{\mathrm{d}t} = -\left(\frac{B_\mathrm{p}^2 R^4}{I_2 c}\int_0^{\theta_\mathrm{pc}} \sin^4\theta\,\cos^3\theta\,\mathrm{d}\theta\right)\delta\Omega_*. \tag{112}$$

A characteristic damping time τ appears to be defined by:

$$\tau = \frac{cI_*}{B_\mathrm{p}^2 R^4} = 3\times 10^3\ I_{45} B_{p14}^{-2} R_{10}^{-4}\ \mathrm{s}. \tag{113}$$

When the magnetosphere is completely open, Eqs. (111), (112) reduce to:

$$\frac{\mathrm{d}\Omega_*}{\mathrm{d}t} = -\frac{2}{15\tau}\,\Omega_*, \tag{114}$$

$$\frac{\mathrm{d}\delta\Omega_*}{\mathrm{d}t} = -\frac{2I_*}{35\,I_2\,\tau}\,\delta\Omega_*. \tag{115}$$

From Eqs. (104), (105), we find that the damping times for Ω_* and $\delta\Omega_*$ in the open regime are, respectively:

$$\tau_\mathrm{brake} = \frac{15\tau}{2} = 7.5\,\tau, \tag{116}$$

$$\tau_\mathrm{odd} = \frac{15\tau}{2\pi}\,\frac{\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,\tan\left(\frac{\pi}{2}\cos\theta\right)\,\mathrm{d}\theta}{\int_0^{\pi/2}\sin^4\theta\cos^3\theta\,\mathrm{d}\theta} = 7.8\,\tau. \tag{117}$$

When the fast-spinning aligned rotator experiences episodes of magnetospheric opening, its evolution consists of a succession of open (or high) states and closed, classical (low) states. During open periods, Eqs. (114), (115) apply and the energy output, of the order indicated by Eq. (92), is considerable. During these periods, the open field should occasionally reconnect, attempting to return to a closed structure, but, once reformed, the latter is again blown open after the very short time needed to build a twist again of approximately half a turn. Large irregular variability is then expected during these open periods, down to the millisecond timescale, which is the time to cross through a light-cylinder size, expected to be representative of the equatorial current sheet, at the speed of light. We note that the closed episodes are initially short in duration when α_odd is of the order of 10^{-4} as supected. When a total time of order of τ_odd has been spent in the open state, the oscillation has damped to an amplititude insufficient to open the magnetosphere, and the average rotation has been substantially reduced. Neglecting the weak damping experienced during the closed episodes, the spin rate $\Omega_*(t)$ and the largest amplitude of the differential rotation $\delta\Omega_*(t)$ vary as

$$\Omega_*(t) = \Omega_{*0}\ \mathrm{e}^{-t/\tau_\mathrm{brake}}, \tag{118}$$

$$\delta\Omega_*(t) = \delta\Omega_{*0}\ \mathrm{e}^{-t/\tau_\mathrm{odd}} \tag{119}$$

where t is the cumulated time spent in the open state. The emitted power scales as Ω_*^2, and declines in a time of approximately $15\tau/4$. The GRB would disappear from view in about this time, which, for a polar field of 3×10^{14} Gauss, is about 20 min. The openings completely cease when the maximum twist falls below half a turn, that is, from Eq.(86), when $\delta\Omega_*/\Omega_\mathrm{T} \sim \pi$. This happens after a time t_stop such that

$$t_\mathrm{stop} = \tau_\mathrm{odd}\ \ln\left(\frac{\delta\Omega_{*0}}{\pi\,\Omega_\mathrm{T}}\right) = \tau_\mathrm{odd}\ \ln\left(\frac{\sqrt{\alpha_\mathrm{odd}}\,P_\mathrm{T}}{\pi\,P_{*0}}\right). \tag{120}$$

The mean rotation rate $\Omega_{*\mathrm{stop}}$ at time t_stop is given by Eq. (118). We define $\xi = \tau_\mathrm{odd}/\tau_\mathrm{brake}$, which, from Eqs. (116), (117), is nearly unity. Relating $\delta\Omega_{*0}$ to α_odd as in Eq. (87), we obtain, for $\xi = 1$, $P_{*\mathrm{stop}} \approx \sqrt{\alpha_\mathrm{odd}}\,P_\mathrm{T}/\pi$. Numerically:

$$P_{*\mathrm{stop}} \approx 30\ \mathrm{ms}\left(\frac{\alpha_\mathrm{odd}}{10^{-4}}\right)^{1/2}\left(\frac{P_\mathrm{T}}{10\ \mathrm{s}}\right). \tag{121}$$

For $\alpha_\mathrm{odd} = 3\times 10^{-4}$ and $P_\mathrm{T} = 0.8$ s (i.e. half the period (7) for $B_\mathrm{p} = 3\times 10^{14}$ Gauss) $P_{*\mathrm{stop}} \approx 4$ ms. After spending a time t_stop in the open state, the high episodes cease and the object becomes a pulsar with a period of the order of 5 millisec. As indicated above, the time of high activity is about $3.5\,\tau$ (Eq. (113)), which is comparable to the observed timescale of long duration GRBs when the field of the compact star is somewhat higher than 10^{14} Gauss. For high fields, of order 10^{15} Gauss, this timescale is about 100 s.

6. Conclusion

It is natural to consider that a new-born quark star experiences differential rotation, causing its internal wound-up toroidal field to increase in strength to about 10^{16} Gauss. This motion then develops into a magnetic torsional oscillation, which could be the origin of long-duration γ-ray bursts. We have indeed shown that an odd oscillation of small amplitude, which should be easily reached, is sufficient to open the star's magnetosphere. A similar effect would also result from other causes of north-south asymmetries. The rapid rotation then drives a relativistic wind from the entire stellar surface. When the star is a quark star, this wind is entirely leptonic. We have calculated the Poynting power released and the timescale of this phenomenon, which meet the observational constraints if the polar field of the quark star is of the order of a few 10^{14} Gauss and its initial rotational angular velocity is of the order of 300 Hz. Large amplitude variations in the light curve on timescales ranging from minutes to milliseconds is a natural outcome of this process.

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Appendix A: Magnetosphere opening: an example

We describe the asymptotic properties of the solutions to Eq. (84) in the context of a self-similar model and show that there is a limit twisting for closed solutions to exist. We define A to be the equatorial value of the flux function a (Eq. (74)). When a field line on the magnetic surface a is twisted, there is a relation between its twist $\psi(a)$, which is the difference in longitude between its footpoints, and the poloidal current $I(a)$. The differential equation of a field line is indeed:

$$\frac{\mathrm{d}r}{B_{\mathrm{r}}} = \frac{r\,\mathrm{d}\theta}{B_\theta} = \frac{r\sin\theta\,\mathrm{d}\phi}{B_\phi}\cdot \tag{A.1}$$

Using Eqs. (74) and (79), we evaluate the change in longitude ψ accumulated following a field line on the magnetic surface a from one of its footpoints P_1 to the other P_2:

$$\psi(a) = I(a)\int_{\mathrm{P}_1}^{P_2} \frac{\mathrm{d}\ell_{\mathrm{P}}}{r^2\sin\theta|B_{\mathrm{P}}|}, \tag{A.2}$$

where $\mathrm{d}\ell_{\mathrm{P}}$ is the line element along the poloidal field line a. It is the twist ψ, not the poloidal current I, which is known. The relation Eq. (A.2) does not infer $I(a)$ directly when $\psi(a)$ is known, because the line of constant a on which the integration in Eq. (A.2) is to be carried is unknown before the problem expressed in Eq. (84) has been solved. To determine under which conditions a dipolar field line closing at a few stellar radii would be inflated sufficiently by the magnetospheric current to reach the light-cylinder, we attempt to identify separable solutions to Eq. (84) (Lynden-Bell & Boily 1994; Wolfson 1995; Bardou & Heyvaerts 1996) of the form:

$$a(r,\theta) = A\left(\frac{R}{r}\right)^p g(\theta). \tag{A.3}$$

Wolfson (1995) numerically studied the solutions of Eq. (84) under the ansatz (A.3). We show here that the solutions must open when the twist reaches a finite value. The largest value of a being the total star flux A, $g(\theta) \le 1$, and $g(\pi/2) = 1$. Similarly $g(0) = 0$, since there is no flux through a circle of zero radius centred on the polar axis. The apex of field line a is at a distance $D(a)$, such that $D(a) = R\,(A/a)^{1/p}$: the more inflated the magnetosphere, the smaller the parameter p. We will then deal with the small p limit. A solution of the form of Eq. (A.3) cannot match any given flux distribution on the star, nor any given twist $\psi(a)$. The constraint of Eq. (A.2) can only be satisfied in an average sense. Using Eq. (A.3) in Eq. (84) the following equation is obtained:

$$\frac{p(p+1)}{g^{2/p}} + \frac{\sin\theta}{g^{1+2/p}}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial g}{\partial\theta}\right) = -\frac{A^{2/p}}{c^2/R^2}\frac{I(a)I'(a)}{a^{1+2/p}}\cdot \tag{A.4}$$

The left-hand side is a function of θ, while the right-hand side is a function of a. Both then equal a common constant, $-K$. This means that for the ansatz of Eq. (A.3) to be satisfied when the similarity exponent is p, the current function $I(a)$ must be:

$$\frac{I(a)}{c} = \sqrt{\frac{Kp}{p+1}}\,\frac{a^{1+1/p}}{A^{1/p}R}\cdot \tag{A.5}$$

The dipolar angular function $g = \sin^2\theta$ is recovered for $p = 1$ and $I = K = 0$. For non-vanishing K, the constraint of Eq. (A.2) becomes, for solutions of the form Eq. (A.3):

$$\frac{I(a)}{c} = \psi(a)\,\frac{a^{1+1/p}}{A^{1/p}R}\,\frac{p}{\int_0^\pi g^{1/p}(\theta)\,\mathrm{d}\theta}\cdot \tag{A.6}$$

The limits on the integral at the denominator reflects the fact that all field lines span the interval $[0,\pi]$ in θ (Eq. (A.3)). Equation (A.6) is consistent with Eq. (A.5) only when $\psi(a)$ is independent of a, which corresponds to the peculiar twist profile in which one hemisphere rotates like a solid body and the other in an opposite sense. We define $\langle\psi\rangle$ to be this twist. We may think of $\langle\psi\rangle$ as being some average on one hemisphere of a more realistic twist profile. K is related to $\langle\psi\rangle$ by:

$$\sqrt{\frac{Kp}{p+1}} = \frac{p\,\langle\psi\rangle}{\int_0^\pi g^{1/p}(\theta)\,\mathrm{d}\theta}\cdot \tag{A.7}$$

Having chosen a value of $\langle\psi\rangle$ this relation infers K for a given p. Given this relation, the value of p itself results from the need for the solution of the angular function $g(\theta)$ to satisfy the three requirements $g(\pi/2) = 1$, $g(0) = 0$, and $g'(\pi/2) = 0$, the last one resulting from the symmetry of magnetic surfaces with respect to the equator. A second-order differential equation accepting only two boundary conditions, the extra condition eventually determines the value of p. To express this condition explicitly, Eq. (A.4) for the angular function g must be solved in the limit of small p. In terms of the variable $x = \cos\theta$, Eq. (A.4) can be written:

$$(1-x^2)\,\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + p(p+1)g + Kg^{1+2/p} = 0. \tag{A.8}$$

For small p the second term of Eq. (A.8) is negligible. The exponent $(1+2/p)$ being very large, the third term of (A.8) essentially vanishes wherever $g < 1$. It remains non-negligible only in the vicinity of the equator ($x = 0$), where g reaches unity. The solution $g(x)$ is then almost a linear function for all x, except in a small region about $x = 0$. This allows us to simplify Eq. (A.8) in the small p limit as:

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + Kg^{1+2/p} = 0. \tag{A.9}$$

This equation has a first integral. The condition that $g'(0) = 0$ at $x = 0$, where g must equal unity, can be satisfied by an appropriate choice of the integration constant, giving:

$$g'^2 = \frac{Kp}{p+1}\left(1 - g^{2+2/p}\right). \tag{A.10}$$

Wherever g is sufficiently less than unity, $g'^2 \approx Kp$. However, we know that in these regions the modulus of the slope should be unity, because g has already been recognized to be a linear function varying from $g = 0$ at $x = 1$ to very nearly $g = 1$ at $x = 0$. The relation between K and p is then, in the small p limit:

$$Kp = 1. \tag{A.11}$$

We should now establish the relation between $\langle\psi\rangle$ and p resulting from Eq. (A.7). To calculate the angular integral at the denominator, we must solve Eq. (A.10) when $p \ll 1$. Taking Eq. (A.11) into account, changing the unknown function g for h such that $g = (1 - ph)$ and making use of the fact that the limit for p approaching 0 of $(1 + py)^{1/p}$ is $\exp(y)$, it can be shown that the solution of Eq. (A.10) is in this limit:

$$g = 1 - p \ln\left(\cosh\left(\frac{x}{p}\right)\right)\cdot \tag{A.12}$$

Since p is small, $g^{1/p} \approx 1/\cosh(x/p)$. The integral that appears in Eq. (A.7) can then be calculated, resulting in:

$$\lim_{p\to 0} \langle\psi\rangle = \pi. \tag{A.13}$$

Thus, the exponent p approaches zero as the twist approaches π and the magnetosphere swells boundlessly in this limit. For our purpose, it is sufficient that a field line extends farther than the light-cylinder for it to open. We can then safely adopt the limit of a twist of a half a turn in causing an almost complete opening.

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