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On the validity of nonlinear Alfvén resonance in space plasmas

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ABSTRACT

Aims. In the approximation of linear dissipative magnetohydrodynamics (MHD), it can be shown that driven MHD waves in magnetic plasmas with high Reynolds number exhibit a near resonant behaviour if the frequency of the wave becomes equal to the local Alfvén (or slow) frequency of a magnetic surface. This behaviour is confined to a thin region, known as the dissipative layer, which embraces the resonant magnetic surface. Although driven MHD waves have small dimensionless amplitude far away from the resonant surface, this near-resonant behaviour in the dissipative layer may cause a breakdown of linear theory. Our aim is to study the nonlinear effects in Alfvén dissipative layer

Methods. In the present paper, the method of simplified matched asymptotic expansions developed for nonlinear slow resonant waves is used to describe nonlinear effects inside the Alfvén dissipative layer.

Results. The nonlinear corrections to resonant waves in the Alfvén dissipative layer are derived, and it is proved that at the Alfvén resonance (with isotropic/anisotropic dissipation) wave dynamics can be described by the linear theory with great accuracy.

Key words. magnetohydrodynamics (MHD) – methods: analytical – Sun: atmosphere – Sun: oscillations

1. Introduction

Magnetic fields are ubiquitous in solar and space plasmas. For regions where plasma-beta (the ratio of the kinetic and magnetic pressures) is less than one, magnetism controls the dynamics, topology and thermal state of the plasma. The magnetic field in the solar atmosphere is not dispersed, but it tends to accumulate in thinner or thicker entities often approximated as magnetic flux tubes. These magnetic flux tubes serve as an ideal medium for guided wave propagation.

One particular aspect of the solar physics that has attracted much attention since the 1940s is the very high temperature of the solar corona compared with the much cooler lower regions of the solar atmosphere requesting the existence of some mechanism(s) that keeps the solar corona hot against the radiative cooling. One of the possible theories proposed is the transfer of omnipresent waves' energy into thermal energy by resonant absorption or resonant coupling of waves (see e.g. Poedts et al. 1990; Sakurai et al. 1991; Goossens et al. 1995).

Waves which were initially observed sporadically mainly in radio wavelengths (see e.g., Kai & Takayanagi 1973; Aschwanden et al. 1992) are now observed in abundance in all wavelengths, especially in (extreme) ultraviolet (see e.g., DeForest & Gurman 1998; Aschwanden et al. 1999; Nakariakov et al. 1999; Robbrecht et al. 2001; King et al. 2003; Erdélyi & Taroyan 2008; Mariska et al. 2008). Since the plasma is non-ideal, waves can lose their energy through transport processes, however, the time over which the waves dissipate their energy is far too long. In order to have an effective and localized energy conversion, the plasma must exhibit transversal inhomogeneities relative to the direction of the ambient magnetic field. It was recognised a long time ago that solar and space plasmas are inhomogeneous, with physical properties varying over length scales much smaller than the scales determined by the gravitational stratification. Homogenous plasmas have a spectrum of linear eigenmodes which can be divided into slow, fast and Alfvén subspectra. The slow and fast subspectra have discrete eigenmodes whereas the Alfvén subspectrum is infinitely degenerated. When an inhomogeneity is introduced the three subspectra are changed. The infinite degeneracy of the Alfvén point spectrum is lifted and replaced by the Alfvén continuum along with the possibility of discrete Alfvén modes occurring, the accumulation point of the slow magnetoacoustic eigenvalues is spread out into the slow continuum and a number of discrete slow modes may occur, and the fast magnetoacoustic point spectrum accumulates at infinity (see e.g., Goedbloed 1975, 1984).

According to the accepted wave theories, effective energy transfer between an energy carrying wave and the plasma occurs if the frequency of the wave matches one of the frequencies in the slow or Alfvén continua, i.e. at the slow or Alfvén resonances. The Alfvén resonance has been more frequently associated with heating of coronal structures given the low-β regime of the solar corona. Nevertheless, slow resonance cannot be ruled out as an additional source of energy transfer. From a mathematical point of view, a resonance is equivalent to regular singular points in the equations describing the dynamics of waves, but these singularities can be removed by, e.g., dissipation. Recently resonant absorption has acquired a new applicability when the observed damping of waves and oscillations in coronal loops has been attributed to resonant absorption. Hence, resonant absorption has become a fundamental constituent block of one of the newest branches of solar physics, called *coronal seismology* (see e.g., Nakariakov et al. 1999; Ruderman & Roberts 2002; Goossens et al. 2002; Arregui et al. 2007; Ballai et al. 2008; Goossens et al. 2008; Terradas et al. 2008) when applied to corona and solar magneto-seismology when applied to the entire coupled solar atmosphere (see e.g., Erdélyi et al. 2007; Verth 2007; Verth et al. 2007).

Given the complexity of the mathematical approach, most theories describing resonant waves are limited to the linear regime. Perturbations, in these theories, are considered to be just small deviations from an equilibrium despite the highly nonlinear character of MHD equations describing the dynamics of

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waves and the complicated interaction between waves and plasmas. Initial numerical investigations of resonant waves in a nonlinear limit (see e.g., Ofman & Davila 1995) unveiled that the account of nonlinearity introduces new physical effects which cannot be described in the linear framework.

The first attempts to describe the nonlinear resonant waves analytically appeared after the papers by Ruderman et al. (1997a,b) which were followed by further analysis by, e.g., Ballai et al. (1998); Ballai & Erdélyi (1999); Ruderman (2000); Clack & Ballai (2008); however, all these papers focused on the slow resonant waves only. These studies revealed that nonlinearity does affect the absorption of waves. In addition, the absorption of wave momentum generates a mean shear flow which can influence the stability of resonant systems.

The present paper is the first analytical study on the nonlinear resonant Alfvén wave, where we obtain governing equations using techniques made familiar from previous studies on nonlinear slow resonant MHD waves. Before embarking on the actual derivation, let us carry out a qualitative discussion. First of all we should point out that in plasmas with high Reynolds numbers (as in the solar corona) efficient dissipation only operates in a thin layer embracing the resonant surface. This layer is called *the dissipation layer*. This restriction on the effect of dissipation makes the problem more tractable from a mathematical point of view, as outside the dissipative layer the dynamics of waves is described by the ideal MHD. Dissipation is a key ingredient of the problem of resonance. As it was mentioned earlier dissipation removes singularities in mathematical solutions. From a physical point of view dissipation is important as it is the mechanism which relaxes the accumulation of energy at the resonant surface and eventually contributes to the global process of heating.

It is important to stress that the choice of dissipation has to be related to the very physics which is described as different waves are sensitive to different dissipative mechanisms. Due to the dominant role of the magnetic field in the solar corona, transport processes are highly anisotropic. Possible dissipation mechanisms acting in coronal structures can be described within the framework of Braginskii's theory (Braginskii 1965) as it was shown in applications by, e.g. Erdélyi & Goossens (1995); Ofman & Davila (1995); Mocanu et al. (2008). Alfvén waves are incompressible and transversal (in polarization), therefore, it is sensible to adopt shear viscosity and magnetic resistivity. Despite both transport processes being described by rather small coefficients, the net effect of dissipation can be increased considerably when the dissipative coefficients are multiplied by large transversal gradients.

The paper is organized as follows. In the next section we introduce the fundamental equations and discuss the main assumptions. In Sect. 3, we derive the governing equation for wave dynamics inside the Alfvén dissipative layer. Section 4 is devoted to calculating the nonlinear corrections at Alfvén resonance. Finally, in Sect. 5 we summarise and draw our conclusions, pointing out a few applications and further studies to be carried out in the future.

2. Fundamental equations and assumptions

For describing mathematically the nonlinear resonant Alfvén waves we use the visco-resistive MHD equations. In spite of the presence of dissipation we use the adiabatic equation as an approximation of the energy equation. Numerical studies by Poedts et al. (1994) in linear MHD have shown that dissipation due to viscosity and finite electrical conductivity in the energy equation does not alter significantly the behaviour of resonant MHD waves in the driven problem.

When the product of the ion (electron) gyrofrequency, $\omega_{\mathrm{i(e)}}$, and the ion (electron) collision time, $\tau_{\mathrm{i(e)}}$, is much greater than one (as in the solar corona) the viscosity and finite electrical conductivity become anisotropic and viscosity is given by the Braginskii viscosity tensor (see Appendix A). The components of the viscosity tensor that remove the Alfvén singularity are the shear components. The parallel and perpendicular components of anisotropic finite electrical conductivity only differ by a factor of 2, therefore, we will consider only one of them without loss of generality.

The dynamics of waves in our model is described by the visco-resistive MHD equations

$$\frac{\partial \bar{\rho}}{\partial t} + \nabla \cdot (\bar{\rho}\boldsymbol{v}) = 0, \quad (1)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\bar{\rho}}\nabla\overline{P} + \frac{1}{\mu_0\bar{\rho}}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} + \frac{1}{\bar{\rho}}\nabla\cdot\boldsymbol{S}, \quad (2)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) + \overline{\lambda}\nabla^2\boldsymbol{B}, \quad (3)$$

$$\frac{\partial}{\partial t}\left(\frac{\overline{p}}{\overline{\rho}^{\gamma}}\right) + \boldsymbol{v}\cdot\nabla\left(\frac{\overline{p}}{\overline{\rho}^{\gamma}}\right) = 0, \quad (4)$$

$$\overline{P} = \bar{p} + \frac{\boldsymbol{B}^2}{2\mu_0}, \quad \nabla\cdot\boldsymbol{B} = 0. \quad (5)$$

In Eqs. (1)–(5) $\boldsymbol{v}$ and $\boldsymbol{B}$ are the velocity and magnetic induction vectors, $\bar{p}$ the plasma pressure, $\bar{\rho}$ the density, $\overline{\lambda}$ the coefficient of magnetic diffusivity, γ the adiabatic exponent, and μ_0 the magnetic permeability of free space. In addition, $\nabla\cdot\boldsymbol{S}$ is the Braginskii viscosity (see Appendix A for full details). Note that even though anisotropy has been considered the Hall term has been neglected. We neglect the Hall term from the induction equation, which can be of the order of diffusion term in the solar corona, because the largest Hall terms in the perpendicular direction relative to the ambient magnetic field identically cancel. The components of the Hall term in the normal and parallel directions relative to the ambient magnetic field have no effect on the dynamics of Alfvén waves in dissipative layers, hence these too are neglected. For full details on the Hall term and the reasoning behind neglecting it, we refer to Appendix B.

We adopt Cartesian coordinates x, y, z and limit our analysis to a static background equilibrium ($\boldsymbol{v}_0 = 0$). We assume that all equilibrium quantities depend on x only. The equilibrium magnetic field, $\boldsymbol{B}_0$, is unidirectional and lies in the yz-plane. The equilibrium quantities must satisfy the condition of total pressure balance,

$$p_0 + \frac{B_0^2}{2\mu_0} = \text{const.} \quad (6)$$

For simplicity we assume that the perturbations of all quantities are independent of y ($\partial/\partial y = 0$). We note that since the magnetic field is not aligned with the z-axis, Alfvén waves still exist. In linear theory of driven waves all perturbed quantities oscillate with the same frequency, ω, which means that they can be Fourier-analysed and taken to be proportional to $\exp(\mathrm{i}[kz-\omega t])$. Solutions are sought in the form of propagating waves. All perturbations in these solutions depend on the combination $\theta = z - Vt$, rather than z and t separately, with $V = \omega/k$. In order to match linear theory as closely as possible we apply the same procedure as above. In the context of resonant absorption the phase velocity, V, must match the projection of the Alfvén velocity, v_{A}, onto the z-axis when $x = x_{\mathrm{A}}$ where x_{A} is the resonant position. To define the resonant position mathematically it is convenient to introduce the angle, α, between the z-axis and the direction of the equilibrium magnetic field, so that the components of the equilibrium magnetic field are

$$B_{0y} = B_0 \sin\alpha, \quad B_{0z} = B_0 \cos\alpha. \tag{7}$$

The definition of the resonant position can now be written mathematically as

$$V = v_A(x_A)\cos\alpha, \tag{8}$$

where v_A is the Alfvén speed defined as

$$v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}}\cdot \tag{9}$$

In addition we introduce the sound and cusp speeds as

$$c_S = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}, \quad c_T = \left(\frac{c_S^2 v_A^2}{c_S^2 + v_A^2}\right)^{1/2}. \tag{10}$$

In what follows we can take $x_A = 0$ without loss of generality. The perturbations of the physical quantities are defined by

$$\bar{\rho} = \rho_0 + \rho, \quad \bar{p} = p_0 + p, \quad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b},$$
$$P = p + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0} + \frac{\boldsymbol{b}^2}{2\mu_0}, \tag{11}$$

where P is the perturbation of total pressure.

The dominant dynamics of resonant Alfvén waves, in linear MHD, resides in the components of the perturbed magnetic field and velocity that are perpendicular to the equilibrium magnetic field and to the x-direction. This dominant behaviour is created by an x^{-1} singularity in the spatial solution of these quantities at the Alfvén resonance (Sakurai et al. 1991; Goossens & Ruderman 1995); these variables are known as *large variables*. The x-component of velocity, the components of magnetic field normal and parallel to the equilibrium magnetic field, plasma pressure and density are also singular, however, their singularity is proportional to $\ln|x|$. In addition, the quantities P and the components of $\boldsymbol{v}$ and $\boldsymbol{b}$ that are parallel to the equilibrium magnetic field are regular; all these variables are called *small variables*.

To make the mathematical analysis more concise and the physics more transparent we define the components of velocity and magnetic field that are in the yz-plane and are either parallel or perpendicular to the equilibrium magnetic field:

$$\begin{pmatrix} v_\parallel \\ b_\parallel \end{pmatrix} = \begin{pmatrix} v & w \\ b_y & b_z \end{pmatrix}\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix},$$
$$\begin{pmatrix} v_\perp \\ b_\perp \end{pmatrix} = \begin{pmatrix} v & -w \\ b_y & -b_z \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \tag{12}$$

where v, w, b_y and b_z are the y- and z-components of the velocity and perturbation of magnetic field, respectively.

Let us introduce the characteristic scale of inhomogeneity, $l_{\rm inh}$. The classical viscous Reynolds number, R_e, and the magnetic Reynolds number, R_m, are defined as

$$R_e = \frac{\widetilde{\rho}_0 V l_{\rm inh}}{\overline{\eta}}, \quad R_m = \frac{V l_{\rm inh}}{\overline{\lambda}}, \tag{13}$$

where $\widetilde{\rho}_0$ is a characteristic value of ρ_0, and $\overline{\eta} = \overline{\eta}_1$ is the shear viscosity coefficient (see Appendix A). These two numbers determine the importance of viscosity and finite electrical conductivity. We introduce the total Reynolds number as

$$\frac{1}{R} = \frac{1}{R_e} + \frac{1}{R_m}\cdot \tag{14}$$

The aim of this paper is to study the nonlinear behaviour of driven Alfvén resonant waves in the dissipative layer. We are not interested in MHD waves that have large amplitude everywhere and require a nonlinear description in the whole space. We focus on waves that have small dimensionless amplitude $\epsilon \ll 1$ far away from the ideal Alfvén resonant point $x = 0$.

In nonlinear theory, when studying resonant behaviour in the dissipative layer we must scale the dissipative coefficients (see e.g., Ruderman et al. 1997b; Ballai et al. 1998; Clack & Ballai 2008). The general scaling to be applied is

$$\overline{\eta} = R^{-1}\eta, \ \overline{\lambda} = R^{-1}\lambda. \tag{15}$$

Linear theory predicts that the characteristic thickness of the dissipative layer, $l_{\rm diss}$, is of the order of $l_{\rm inh}R^{-1/3}$ and we assume that this is true in the nonlinear regime, too. Hence, we must introduce a stretching transversal coordinate, ξ, in the dissipative layer defined as

$$\xi = R^{1/3}x. \tag{16}$$

We can rewrite Eqs. (1)–(5) in the scalar form as

$$V\frac{\partial\rho}{\partial\theta} - \frac{\partial(\rho_0 u)}{\partial x} - \rho_0\frac{\partial w}{\partial\theta} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho w)}{\partial\theta}, \tag{17}$$

$$\rho_0 V\frac{\partial u}{\partial\theta} - \frac{\partial P}{\partial x} + \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_x}{\partial\theta} = \bar{\rho}\left(u\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial\theta}\right)$$
$$-\rho V\frac{\partial u}{\partial\theta} - \frac{b_x}{\mu_0}\frac{\partial b_x}{\partial x} - \frac{b_z}{\mu_0}\frac{\partial b_x}{\partial\theta} - \overline{\eta}\frac{\partial^2 u}{\partial x^2}, \tag{18}$$

$$\frac{\partial}{\partial\theta}\left(\rho_0 V v_\perp + P\sin\alpha + \frac{B_0\cos\alpha}{\mu_0}b_\perp\right) = \bar{\rho}\left(u\frac{\partial v_\perp}{\partial x} + w\frac{\partial v_\perp}{\partial\theta}\right)$$
$$-\rho V\frac{\partial v_\perp}{\partial\theta} - \frac{b_x}{\mu_0}\frac{\partial b_\perp}{\partial x} - \frac{b_z}{\mu_0}\frac{\partial b_\perp}{\partial\theta} - \overline{\eta}\frac{\partial^2 v_\perp}{\partial x^2}, \tag{19}$$

$$\frac{\partial}{\partial\theta}\left(\rho_0 V v_\parallel - P\cos\alpha + \frac{B_0\cos\alpha}{\mu_0}b_\parallel\right) = \bar{\rho}\left(u\frac{\partial v_\parallel}{\partial x} + w\frac{\partial v_\parallel}{\partial\theta}\right)$$
$$-\frac{b_x}{\mu_0}\frac{{\rm d}B_0}{{\rm d}x} - \rho V\frac{\partial v_\parallel}{\partial\theta} - \frac{b_x}{\mu_0}\frac{\partial b_\parallel}{\partial x} - \frac{b_z}{\mu_0}\frac{\partial b_\parallel}{\partial\theta} - 4\overline{\eta}\frac{\partial^2 v_\parallel}{\partial x^2}, \tag{20}$$

$$Vb_x + B_0 u\cos\alpha = wb_x - ub_z + \overline{\lambda}\left(\frac{\partial b_x}{\partial\theta} - \frac{\partial b_z}{\partial x}\right), \tag{21}$$

$$\frac{\partial}{\partial\theta}(Vb_\perp + B_0 v_\perp\cos\alpha) = \frac{\partial(ub_\perp)}{\partial x} + \frac{\partial(wb_\perp)}{\partial\theta}$$
$$-b_x\frac{\partial v_\perp}{\partial x} - b_z\frac{\partial v_\perp}{\partial\theta} - \overline{\lambda}\nabla^2 b_\perp, \tag{22}$$

$$\frac{\partial}{\partial\theta}(Vb_\parallel + B_0 v_\parallel\cos\alpha) - \frac{\partial(B_0 u)}{\partial x} - B_0\frac{\partial w}{\partial\theta}$$
$$= \frac{\partial(ub_\parallel)}{\partial x} + \frac{\partial(wb_\parallel)}{\partial\theta} - b_x\frac{\partial v_\parallel}{\partial x} - b_z\frac{\partial v_\parallel}{\partial\theta} - \overline{\lambda}\nabla^2 b_\parallel, \tag{23}$$

$$V\left(\frac{\partial p}{\partial\theta} - c_S^2\frac{\partial\rho}{\partial\theta}\right) - u\left(\frac{{\rm d}p_0}{{\rm d}x} - c_S^2\frac{{\rm d}\rho_0}{{\rm d}x}\right)$$
$$= \frac{1}{\rho_0}\left\{V\left(\gamma p\frac{\partial\rho}{\partial\theta} - \rho\frac{\partial p}{\partial\theta}\right) - w\left[\gamma\bar{p}\frac{\partial\rho}{\partial\theta} - \bar{p}\frac{\partial p}{\partial\theta}\right]\right.$$
$$\left. + u\left[\rho\frac{{\rm d}p_0}{{\rm d}x} - \gamma p\frac{{\rm d}\rho_0}{{\rm d}x} + \bar{\rho}\frac{\partial p}{\partial x} - \gamma\bar{p}\frac{\partial\rho}{\partial x}\right]\right\} \tag{24}$$

$$P = p + \frac{1}{2\mu_0}\left(b_x^2 + b_\perp^2 + b_\parallel^2 + 2B_0 b_\parallel\right), \tag{25}$$

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_z}{\partial\theta} = 0. \tag{26}$$

In the above equations $\nabla = (\partial/\partial x, 0, \partial/\partial\theta)$ and $w = v_\parallel\cos\alpha - v_\perp\sin\alpha$.

Equations (17)–(26) will be used in the following sections to derive the governing equation for the resonant Alfvén waves inside the dissipative layer and to find the nonlinear corrections.

3. The governing equation in the dissipative layer

In order to derive the governing equation for wave motions in the Alfvén dissipative layer we employ the method of matched asymptotic expansions (Nayfeh 1981; Bender & Országh 1991). This method requires to find the so-called *outer* and *inner* expansions and then match them in the overlap regions. This nomenclature is ideal for our situation. The outer expansion corresponds to the solution outside the dissipative layer and the inner expansion corresponds to the solution inside the dissipative layer. A simplified version of the method of matched asymptotic expansions, developed by Ballai et al. (1998), is adopted here.

The typical largest *quadratic* nonlinear term in the system of MHD equations is of the form $g\partial g/\partial z$ while the typical dissipative term is of the form $\overline{\eta}\partial^2 g/\partial z^2$, where g is any "large" variable. Linear theory predicts that "large" variables have an ideal singularity x^{-1} in the vicinity of $x = 0$. This implies that the "large" variables have dimensionless amplitudes in the dissipative layer of the order of $\epsilon R^{1/3}$. It is now straightforward to estimate the ratio of a typical quadratic nonlinear and dissipative term,

$$\phi_q = \frac{g\partial g/\partial z}{\overline{\eta}\partial^2 g/\partial z^2} = \mathscr{O}(\epsilon R^{2/3}), \tag{27}$$

where the quantity ϕ_q can be considered as the *quadratic nonlinearity parameter*. If the condition $\epsilon R^{2/3} \ll 1$ is satisfied, linear theory is applicable. On the other hand, if $\epsilon R^{2/3} \gtrsim 1$ then nonlinearity has to be taken into account when studying resonant waves in dissipative layers. Using the same scalings, Ruderman et al. (1997b) showed that nonlinearity has to be considered whenever slow resonant waves are studied in the solar photosphere. For a typical dimensionless amplitude of $\epsilon \sim 10^{-2}$ linear theory can be applied if the total Reynolds number is less than 10^3. This value is much less than the resistive and shear viscosity Reynolds number (10^{10}–10^{12}). This conclusion implies that in the solar atmosphere resonant absorption should be a nonlinear phenomenon. In order to describe the role of dissipation and nonlinearity equally we assume that $\phi_q \sim 1$.

Far away from the dissipative layer the amplitudes of perturbations are small, so we use linear ideal MHD equations in order to describe the wave motion. The full set of nonlinear dissipative MHD equations are used for describing wave motion *inside* the dissipative layer where the amplitudes can be large. We, therefore, look for solutions in the form of asymptotic expansions. The equilibrium quantities change only slightly across the dissipative layer so it is possible to approximate them by the first non-vanishing term in their Taylor series expansion with respect to x. Similar to linear theory, we assume that the expansions of equilibrium quantities are valid in a region embracing the ideal resonant position, which is assumed to be much wider than the dissipative layer. This implies that there are two overlap regions, one to the left and one to the right of the dissipative layer, where both the outer (the solution to the linear ideal MHD equations) and inner (the solution to the nonlinear dissipative MHD equations) solutions are valid. Hence, both solutions must coincide in the overlap regions which provides the matching conditions.

Before deriving the nonlinear governing equation we ought to make a note. In linear theory, perturbations of physical quantities are harmonic functions of θ and their mean values over a period are zero. In nonlinear theory, however, the perturbations of variables can have non-zero mean values as a result of nonlinear interaction of different harmonics. Due to the absorption of wave momentum, a mean shear flow is generated outside the dissipative layer (Ofman & Davila 1995). This result is true for our analysis also, however, due to the length of this study we prefer to deal with this problem in a forthcoming paper.

We suppose that nonlinearity and dissipation are of the same order so we have $\epsilon R^{2/3} = \mathscr{O}(1)$, i.e. $R \sim \epsilon^{-3/2}$. We can, therefore, substitute $\epsilon^{-3/2}$ for R in Eq. (15) to rescale viscosity and finite electrical resistivity as

$$\overline{\eta} = \epsilon^{3/2}\eta, \ \overline{\lambda} = \epsilon^{3/2}\lambda. \tag{28}$$

We do not rewrite the MHD equations as they are easily obtained from Eqs. (17)–(26) by substitution of Eq. (28).

The first step in our description is the derivation of governing equations outside the dissipative layer where the dynamics is described by ideal ($\eta = \lambda = 0$) and linear MHD. The linear form of Eqs. (17)–(26) can be obtained by assuming a regular expansion of variables of the form

$$f = \epsilon f^{(1)} + \epsilon^{3/2} f^{(2)} \ldots, \tag{29}$$

and collect only terms proportional to the small parameter ϵ. This leads to a system of linear equations for the variables with superscript "1". All variables can be eliminated in favour of $u^{(1)}$ and $P^{(1)}$, leading to the system

$$V\frac{\partial P^{(1)}}{\partial \theta} = F\frac{\partial u^{(1)}}{\partial x}, \quad V\frac{\partial P^{(1)}}{\partial x} = \rho_0 A\frac{\partial u^{(1)}}{\partial \theta}, \tag{30}$$

where

$$F = \frac{\rho_0 AC}{V^4 - V^2\left(v_\mathrm{A}^2 + c_\mathrm{S}^2\right) + v_\mathrm{A}^2 c_\mathrm{S}^2\cos^2\alpha}, \tag{31}$$

$$A = V^2 - v_\mathrm{A}^2\cos^2\alpha,$$

$$C = \left(v_\mathrm{A}^2 + c_\mathrm{S}^2\right)\left(V^2 - c_\mathrm{T}^2\cos^2\alpha\right). \tag{32}$$

The quantities A and C vanish at the Alfvén and slow resonant positions, respectively. As a result these two positions are regular singular points for the system (30). The remaining variables can be expressed in terms of $u^{(1)}$ and $P^{(1)}$ as,

$$v_\perp^{(1)} = -\frac{V\sin\alpha}{\rho_0 A}P^{(1)}, \quad v_\parallel^{(1)} = \frac{Vc_\mathrm{S}^2\cos\alpha}{\rho_0 C}P^{(1)}, \tag{33}$$

$$b_x^{(1)} = -\frac{B_0\cos\alpha}{V}u^{(1)}, \quad b_\perp^{(1)} = \frac{B_0\cos\alpha\sin\alpha}{\rho_0 A}P^{(1)}, \tag{34}$$

$$\frac{\partial b_\parallel^{(1)}}{\partial\theta} = \frac{B_0\left(V^2 - c_\mathrm{S}^2\cos^2\alpha\right)}{\rho_0 C}\frac{\partial P^{(1)}}{\partial\theta} + \frac{u^{(1)}}{V}\frac{\mathrm{d}B_0}{\mathrm{d}x}, \tag{35}$$

$$\frac{\partial p^{(1)}}{\partial\theta} = \frac{V^2 c_\mathrm{S}^2}{C}\frac{\partial P^{(1)}}{\partial\theta} - \frac{u^{(1)}B_0}{\mu_0 V}\frac{\mathrm{d}B_0}{\mathrm{d}x}, \tag{36}$$

$$\frac{\partial\rho^{(1)}}{\partial\theta} = \frac{V^2}{C}\frac{\partial P^{(1)}}{\partial\theta} + \frac{u^{(1)}}{V}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}. \tag{37}$$

Since Eq. (30) has regular singular points, the solutions can be obtained in terms of Fröbenius series with respect to x (for details see, e.g., Ruderman et al. 1997b; Ballai et al. 1998) of the form

$$P^{(1)} = P_1^{(1)}(\theta) + P_2^{(1)}(\theta)x\ln|x| + P_3^{(1)}(\theta) + \ldots, \tag{38}$$

$$u^{(1)} = u_1^{(1)}(\theta)\ln|x| + u_2^{(1)}(\theta) + u_3^{(1)}(\theta)x\ln|x| + \ldots \tag{39}$$

The coefficient functions depending on θ in the above expansions are, generally, different for $x < 0$ and $x > 0$. The particular form of these series solutions indicates that the perturbation of the total pressure is regular at the ideal resonant position. From Eqs. (33)–(37), we see that the quantity $v_\parallel^{(1)}$ is also regular, while all other quantities are singular. The quantities $u^{(1)}$, $b_x^{(1)}$, $b_\parallel^{(1)}$, $p^{(1)}$ and $\rho^{(1)}$ behave as $\ln|x|$, while $v_\perp^{(1)}$ and $b_\perp^{(1)}$ behave as x^{-1}, so they are the most singular.

As the characteristic scale of dissipation is of the order of $l_{\rm inh}R^{-1/3}$ and we have assumed that $R \sim \epsilon^{3/2}$ we obtain that the thickness of the dissipative layer is $l_{\rm inh}R^{-1/3} = \mathscr{O}(\epsilon^{1/2}l_{\rm inh})$, implying the introduction of a new stretched variable to replace the transversal coordinate in the dissipative layer, which is defined as $\xi = \epsilon^{-1/2}x$. Again, for brevity, Eqs. (17)–(26) are not rewritten as they can be obtained by the substitution of

$$\frac{\partial}{\partial x} = \epsilon^{-1/2}\frac{\partial}{\partial \xi}, \tag{40}$$

for all derivatives. The equilibrium quantities still depend on x, not ξ (their expression is valid in a wider region than the characteristic thickness of the dissipative layer). All equilibrium quantities are expanded around the ideal resonant position, $x = 0$, as

$$f_0 \approx f_{0_{\rm A}} + \epsilon^{1/2}\xi\left(\frac{{\rm d}f_0}{{\rm d}x}\right)_{\rm A}, \tag{41}$$

where f_0 is any equilibrium quantity and the subscript “A” indicates that the equilibrium quantity is evaluated at the Alfvén resonant point.

We seek the solution to the set of equations obtained from Eqs. (17)–(26) by the substitution of $x = \epsilon^{1/2}\xi$ into variables in the form of power series of ϵ. These equations contain powers of $\epsilon^{1/2}$, so we use this quantity as an expansion parameter. To derive the form of the inner expansions of different quantities we have to analyze the outer solutions. First, since $v_\parallel$ and P are regular at $x = 0$ we can write their inner expansions in the form of their outer expansions Eq. (29). The amplitudes of large variables in the dissipative layer are of the order of $\epsilon^{1/2}$, so the inner expansion of the variables $v_\perp$ and $b_\perp$ is

$$g = \epsilon^{1/2}g^{(1)} + \epsilon g^{(2)} + \ldots \tag{42}$$

The quantities u, b_x, $b_\parallel$, p and ρ behave as $\ln|x|$ in the vicinity of $x = 0$, which suggests that they have expansions with terms of the order of $\epsilon\ln\epsilon$ in the dissipative layer. Strictly speaking, the inner expansions of all variables have to contain terms proportional to $\epsilon\ln\epsilon$ and $\epsilon^{3/2}\ln\epsilon$ (see e.g., Ruderman et al. 1997b). In the simplified version of matched asymptotic expansions we utilize the fact that $|\ln\epsilon| \ll \epsilon^{-\kappa}$ for any positive κ and $\epsilon \to +0$, and consider $\ln\epsilon$ as a quantity of the order of unity (Ballai et al. 1998). This enables us to write the inner expansions for u, b_x, $b_\parallel$, p and ρ in the form of Eq. (29).

We now substitute the expansion (29) for P, u, b_x, $b_\parallel$, $v_\parallel$, p and ρ and the expansion given by Eq. (42) for $v_\perp$ and $b_\perp$ into the set of equations obtained from Eqs. (17)–(26) after substitution of $x = \epsilon^{1/2}\xi$. The first order approximation (terms proportional to ϵ), yields a linear homogeneous system of equations for the terms with superscript “1”. The important result that follows from this set of equations is that

$$P^{(1)} = P^{(1)}(\theta), \tag{43}$$

that is $P^{(1)}$ does not change across the dissipative layer. This result parallels the result found in linear theory (Sakurai et al. 1991; Goossens et al. 1995) and nonlinear theories of slow resonance (see e.g., Ruderman et al. 1997b; Ballai et al. 1998; Clack & Ballai 2008). Subsequently, all remaining variables can be expressed in terms of $u^{(1)}$, $v_\perp^{(1)}$ and $P^{(1)}$ as

$$v_\parallel^{(1)} = \frac{c_{\rm S}^2}{v_{\rm A}^2}\frac{\cos\alpha}{\rho_0 V}P^{(1)}, \tag{44}$$

$$b_\perp^{(1)} = -\frac{B_0 V}{v_{\rm A}^2\cos\alpha}v_\perp^{(1)},\quad b_x^{(1)} = -\frac{B_0\cos\alpha}{V}u^{(1)}, \tag{45}$$

$$\frac{\partial b_\parallel^{(1)}}{\partial\theta} = \frac{B_0\left(v_{\rm A}^2 - c_{\rm S}^2\right)}{\rho_0 v_{\rm A}^4}\frac{{\rm d}P^{(1)}}{{\rm d}\theta} + \frac{u^{(1)}}{V}\left(\frac{{\rm d}B_0}{{\rm d}x}\right), \tag{46}$$

$$\frac{\partial p^{(1)}}{\partial\theta} = \frac{c_{\rm S}^2}{v_{\rm A}^2}\frac{{\rm d}P^{(1)}}{{\rm d}\theta} - \frac{u^{(1)}}{V}\frac{B_0}{\mu_0}\left(\frac{{\rm d}B_0}{{\rm d}x}\right), \tag{47}$$

$$\frac{\partial\rho^{(1)}}{\partial\theta} = \frac{1}{v_{\rm A}^2}\frac{{\rm d}P^{(1)}}{{\rm d}\theta} + \frac{u^{(1)}}{V}\left(\frac{{\rm d}\rho_0}{{\rm d}x}\right). \tag{48}$$

All equilibrium quantities are calculated at $x = 0$. In addition, the relation that connects the normal and perpendicular components of velocity is

$$\frac{\partial u^{(1)}}{\partial\xi} - \sin\alpha\frac{\partial v_\perp^{(1)}}{\partial\theta} = 0. \tag{49}$$

In the second order approximation we only use the expressions obtained from Eqs. (19) and (22). Employing Eqs. (43)–(49), we replace the variables in the second order approximation which have superscript “1”. The equations obtained in the second order are

$$\frac{\partial P^{(1)}}{\partial\theta}\sin\alpha + \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_\perp^{(2)}}{\partial\theta} + V\rho_0\frac{\partial v_\perp^{(2)}}{\partial\theta} = \frac{B_0 V}{\mu_0 v_{\rm A}^2}\left(\frac{{\rm d}B_0}{{\rm d}x}\right)\xi\frac{\partial v_\perp^{(1)}}{\partial\theta} - V\left(\frac{{\rm d}\rho_0}{{\rm d}x}\right)\xi\frac{\partial v_\perp^{(1)}}{\partial\theta} - \eta\frac{\partial^2 v_\perp^{(1)}}{\partial\xi^2}, \tag{50}$$

$$V\frac{\partial b_\perp^{(2)}}{\partial\theta} + B_0\cos\alpha\frac{\partial v_\perp^{(2)}}{\partial\theta} + \cos\alpha\left(\frac{{\rm d}B_0}{{\rm d}x}\right)\xi\frac{\partial v_\perp^{(1)}}{\partial\theta} = \lambda\frac{B_0 V}{v_{\rm A}^2\cos\alpha}\frac{\partial^2 v_\perp^{(1)}}{\partial\xi^2}. \tag{51}$$

Once the variables with superscript “2” have been eliminated from the above two equations, the governing equation for resonant Alfvén waves inside the dissipative layer is derived as

$$\triangle\xi\frac{\partial v_\perp^{(1)}}{\partial\theta} + \frac{V}{\rho_0}(\eta + \rho_0\lambda)\frac{\partial^2 v_\perp^{(1)}}{\partial\xi^2} = -\frac{V\sin\alpha}{\rho_0}\frac{{\rm d}P^{(1)}}{{\rm d}\theta}, \tag{52}$$

where

$$\triangle = -\left(\frac{{\rm d}v_{\rm A}^2}{{\rm d}x}\right)\cos^2\alpha. \tag{53}$$

It is clear that Eq. (52) does not contain nonlinear terms despite considering the full MHD system of equations. This result is in stark contrast with the results obtained for nonlinear slow resonance where the governing equation was found to be always nonlinear (see e.g., Ruderman et al. 1997b; Ballai et al. 1998;

Clack & Ballai 2008). The governing Eq. (52) suggests that resonant Alfvén waves can be described by the linear theory unless their amplitudes inside the dissipative layer is of the order of unity.

As the quadratic nonlinear terms cancel each other out, it is natural to take into account cubic nonlinearity (the system of MHD equations contain cubic nonlinear terms), where the nonlinearity parameter is defined as

$$\phi_c = \frac{g^2 \partial g/\partial z}{\overline{\eta}\partial^2 g/\partial x^2} \simeq \epsilon^2 R. \tag{54}$$

Despite the higher order nonlinearity the governing equation is similar to the equation derived for quadratic nonlinearity (52). These results require finding an explanation to the linear behaviour of waves inside the dissipative layer. The following section will be devoted to the study of nonlinear corrections in the Alfvén dissipative layer.

4. Nonlinear corrections in the Alfvén dissipative layer

Since we have assumed that waves have small dimensionless amplitude outside the dissipative layer, we will concentrate only on the solutions *inside* the dissipative layer.

In our analysis we use the assumptions and equations presented in Sect. 2, however, we will not impose any relation between ϵ and R. Equations (15) and (16) will be used to define the scaled dissipative coefficients and stretching transversal coordinate in the dissipative layer. For simplicity we denote $\delta = R^{-1/3}$. This means that our scaled dissipative coefficients and stretched transversal coordinate become

$$\overline{\eta} = \delta^3\eta, \ \overline{\lambda} = \delta^3\lambda, \ \xi = \delta^{-1}x. \tag{55}$$

The first step to accomplish our task is to rewrite Eqs. (17)–(26) by substituting

$$\frac{\partial}{\partial x} = \delta^{-1}\frac{\partial}{\partial \xi}, \ \frac{\partial}{\partial z} = \frac{\partial}{\partial \theta}, \ \text{and} \ \frac{\partial}{\partial t} = -V\frac{\partial}{\partial \theta}\cdot \tag{56}$$

All equilibrium quantities (which are still dependent on x, not ξ) will be approximated by the first non-vanishing term of their Taylor expansion (see, Eq. (41)).

The substitution of Eqs. (41), (55) and (56) will transform Eqs. (17)–(26) into

$$\rho_0\frac{\partial u}{\partial \xi} + \delta u\frac{\mathrm{d}\rho_0}{\mathrm{d}x} + \frac{\partial(\rho u)}{\partial \xi} - \delta\frac{\partial}{\partial \theta}\left[\overline{\rho}\,(V-w)\right] = 0, \tag{57}$$

$$\frac{1}{\overline{\rho}}\left[\frac{\partial P}{\partial \xi} - \frac{b_x}{\mu_0}\frac{\partial b_x}{\partial \xi} - \frac{\delta}{\mu_0}\left(B_0\cos\alpha + b_z\right)\frac{\partial b_x}{\partial \theta}\right] = \delta\,(V-w)\frac{\partial u}{\partial \theta} - u\frac{\partial u}{\partial \xi} + \delta^2\frac{\eta}{\rho_0}\frac{\partial^2 u}{\partial \xi^2}, \tag{58}$$

$$\frac{1}{\overline{\rho}}\left[\delta\frac{\partial P}{\partial \theta}\sin\alpha + \frac{b_x}{\mu_0}\frac{\partial b_\perp}{\partial \xi} + \frac{\delta}{\mu_0}\left(B_0\cos\alpha + b_z\right)\frac{\partial b_\perp}{\partial \theta}\right] = -\delta\,(V-w)\frac{\partial v_\perp}{\partial \theta} + u\frac{\partial v_\perp}{\partial \xi} - \delta^2\frac{\eta}{\rho_0}\frac{\partial^2 v_\perp}{\partial \xi^2}, \tag{59}$$

$$\frac{1}{\overline{\rho}}\left[\delta\frac{\partial P}{\partial \theta}\cos\alpha - \frac{b_x}{\mu_0}\frac{\partial b_\parallel}{\partial \xi} - \frac{\delta}{\mu_0}\left(B_0\cos\alpha + b_z\right)\frac{\partial b_\parallel}{\partial \theta} - \frac{\delta}{\mu_0}\frac{\mathrm{d}B_0}{\mathrm{d}x}b_x\right] = \delta\,(V-w)\frac{\partial v_\parallel}{\partial \theta} - u\frac{\partial v_\parallel}{\partial \xi} + \delta^2\frac{4\eta}{\rho_0}\frac{\partial^2 v_\parallel}{\partial \xi^2}, \tag{60}$$

$$\delta\,(V-w)\frac{\partial b_x}{\partial \theta} + \delta\left(B_0\cos\alpha + b_z\right)\frac{\partial u}{\partial \theta} + \delta^2\lambda\left(\frac{\partial^2}{\partial \xi^2} + \delta^2\frac{\partial^2}{\partial \theta^2}\right)b_x = 0, \tag{61}$$

$$\delta\,(V-w)\frac{\partial b_\perp}{\partial \theta} - u\frac{\partial b_\perp}{\partial \xi} - b_\perp\left(\frac{\partial u}{\partial \xi} + \delta\frac{\partial v_\parallel}{\partial \theta}\cos\alpha\right) + b_x\frac{\partial v_\perp}{\partial \xi} + \delta\left(B_0 + b_\parallel\right)\frac{\partial v_\perp}{\partial \theta}\cos\alpha + \delta^2\lambda\left(\frac{\partial^2}{\partial \xi^2} + \delta^2\frac{\partial^2}{\partial \theta^2}\right)b_\perp = 0, \tag{62}$$

$$\delta\,(V-w)\frac{\partial b_\parallel}{\partial \theta} - u\left(\frac{\partial b_\parallel}{\partial \xi} + \delta\frac{\mathrm{d}B_0}{\mathrm{d}x}\right) + b_x\frac{\partial v_\parallel}{\partial \xi} - \delta b_\perp\frac{\partial v_\parallel}{\partial \theta} - \left(B_0 + b_\parallel\right)\left(\frac{\partial u}{\partial \xi} - \delta\frac{\partial v_\perp}{\partial \theta}\sin\alpha\right) + \delta^2\lambda\left(\frac{\partial^2}{\partial \xi^2} + \delta^2\frac{\partial^2}{\partial \theta^2}\right)b_\parallel = 0, \tag{63}$$

$$\left[\delta\,(V-w)\frac{\partial}{\partial \theta} - u\frac{\partial}{\partial \xi}\right]\left(\frac{\overline{p}}{\overline{\rho}^\gamma}\right) = 0, \tag{64}$$

$$P = p + \frac{B_0}{\mu_0}b_\parallel + \frac{1}{2\mu_0}\left(b_x^2 + b_\perp^2 + b_\parallel^2\right). \tag{65}$$

The only condition we need to impose when deriving the nonlinear corrections to resonant Alfvén waves in the dissipative layer is imported from the linear theory which predicts that in the dissipative layer "large" variables have dimensionless amplitude of the order of $\epsilon R^{1/3}$ (see Sect. 3). We assume that the dimensionless amplitudes of the linear approximation of "large" variables ($v_\perp$ and $b_\perp$) in the dissipative layer are small, so that

$$\epsilon \ll R^{-1/3}(=\delta). \tag{66}$$

This condition ensures that the oscillation amplitude remains small inside the dissipative layer. From a naive point of view the linear theory is applicable as soon as the oscillation amplitude is small. The example of slow resonant waves clearly shows that this is not the case. The nonlinear effects become important in the slow dissipative layer as soon as $\epsilon \sim R^{-2/3}$, i.e. as soon as the oscillation amplitude in the dissipative layer, which is of the order of $\epsilon R^{1/3}$, is of the order of $R^{-1/3} \ll 1$. For example, in the corona perturbations with dimensionless amplitudes less than 10^{-4} can be considered by this theory. From Eq. (27) we would expect to see quadratic nonlinearity appear for waves with dimensionless amplitudes larger than 10^{-8} and from Eq. (54) we would expect to see cubic nonlinearity appear for waves with dimensionless amplitudes larger than 10^{-6}. If we take $\epsilon \approx R^{-1/3}$ we find that inside the dissipative layer we have dimensionless amplitudes of the order of unity. This causes a breakdown in our theory, and therefore another approach would have to be adopted. At this time we do not know of an analytical study which can carry out this task without considering the full nonlinear MHD equations throughout the domain.

We now assume that all perturbations can be written as a regular asymptotic expansion of the form

$$\overline{f} = \overline{f}_0(x) + \epsilon\overline{f}_1(\xi,\theta) + \epsilon^2\overline{f}_2(\xi,\theta) + \ldots, \tag{67}$$

where $\overline{f}_0(x)$ represents the equilibrium value. Substitution of expansion (67) into the system (57)–(65) leads to a system of equations which contains the small parameter δ. This observation inspires us to look for the solution in the form of expansions with respect to δ. In order to cast large and small variables in this description we are going to use the following expansion for small variables (u, b_x, $v_\parallel$, $b_\parallel$, ρ, p and P)

$$\overline{g}_1 = \overline{g}_1^{(1)} + \delta\overline{g}_1^{(2)} + \ldots, \tag{68}$$

while large variables ($v_\perp$ and $b_\perp$) will be expanded according to

$$\overline{h}_1 = \delta^{-1}\overline{h}_1^{(1)} + \overline{h}_1^{(2)} + \ldots \tag{69}$$

The bar notation is used here to distinguish between these expansions and the expansions used in the previous section. From this point on we drop the bar notation.

Substituting Eqs. (68), (69) into the system (57)–(65), taking terms proportional to ϵ and then only retaining terms with the lowest power of δ, results in the set of linear equations

$$\rho_0 \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta} \sin\alpha - \rho_0 \frac{\partial u_1^{(1)}}{\partial \xi} = 0, \tag{70}$$

$$\frac{\partial P_1^{(1)}}{\partial \xi} = 0, \tag{71}$$

$$\rho_0 V \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta} + \frac{B_0 \cos\alpha}{\mu_0} \frac{\partial b_{\perp 1}^{(1)}}{\partial \theta} = 0, \tag{72}$$

$$\frac{\partial P_1^{(1)}}{\partial \theta} \cos\alpha - \rho_0 V \frac{\partial v_{\parallel 1}^{(1)}}{\partial \theta} - \frac{b_{x1}^{(1)}}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right) - \frac{B_0 \cos\alpha}{\mu_0}\frac{\partial b_{\parallel 1}^{(1)}}{\partial \theta} = 0, \tag{73}$$

$$V \frac{\partial b_{x1}^{(1)}}{\partial \theta} + B_0 \cos\alpha \frac{\partial u_1^{(1)}}{\partial \theta} = 0, \tag{74}$$

$$V \frac{\partial b_{\perp 1}^{(1)}}{\partial \theta} + B_0 \cos\alpha \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta} = 0, \tag{75}$$

$$B_0 \frac{\partial u_1^{(1)}}{\partial \xi} - B_0 \sin\alpha \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta} = 0, \tag{76}$$

$$\rho_0 V \frac{\partial p_1^{(1)}}{\partial \theta} - \rho_0 c_S^2 V \frac{\partial \rho_1^{(1)}}{\partial \theta} + \rho_0 u_1^{(1)}\left[c_S^2\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right) - \left(\frac{\mathrm{d}p_0}{\mathrm{d}x}\right)\right] = 0, \tag{77}$$

$$P_1^{(1)} - p_1^{(1)} - \frac{B_0}{\mu_0} b_{\parallel 1}^{(1)} = 0. \tag{78}$$

In these equations all equilibrium quantities are calculated at $x = 0$.

Using these equations we can express all dependent variables in terms of $u_1^{(1)}$, $v_{\perp 1}^{(1)}$ and $P_1^{(1)}$,

$$v_{\parallel 1}^{(1)} = \frac{c_S^2}{v_A^2}\frac{\cos\alpha}{\rho_0 V} P_1^{(1)}, \tag{79}$$

$$b_{\perp 1}^{(1)} = -\frac{B_0 V}{v_A^2 \cos\alpha} v_{\perp 1}^{(1)},\ b_{x1}^{(1)} = -\frac{B_0\cos\alpha}{V} u_1^{(1)}, \tag{80}$$

$$\frac{\partial b_{\parallel 1}^{(1)}}{\partial \theta} = \frac{B_0\left(v_A^2 - c_S^2\right)}{\rho_0 v_A^4}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta} + \frac{u_1^{(1)}}{V}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right), \tag{81}$$

$$\frac{\partial p_1^{(1)}}{\partial \theta} = \frac{c_S^2}{v_A^2}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta} - \frac{u_1^{(1)}}{V}\frac{B_0}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right), \tag{82}$$

$$\frac{\partial \rho_1^{(1)}}{\partial \theta} = \frac{1}{v_A^2}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta} + \frac{u_1^{(1)}}{V}\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right). \tag{83}$$

It follows from Eq. (71) that

$$P_1^{(1)} = P_1^{(1)}(\theta). \tag{84}$$

Finally, we obtain the relation between $u_1^{(1)}$ and $v_{\perp 1}^{(1)}$,

$$\frac{\partial u_1^{(1)}}{\partial \xi} = \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta}\sin\alpha. \tag{85}$$

Note that Eqs. (79)–(85) are formally identical to Eqs. (43)–(49) for the linear approximation in Sect. 3. This is not surprising as both methods are designed to replicate linear theory in the first order approximation.

4.1. The second and third order nonlinear corrections

Once the first order terms are known we can proceed to derive the second and third order approximations with respect to ϵ (i.e. terms from the expansion of Eqs. (57)–(65) that are proportional to ϵ^2 and ϵ^3, respectively). First, we write out the second order approximations and substitute for all first order terms (i.e. terms of the form $f_1^{(1)}$) using Eqs. (79)–(85). Secondly, we find (by solving the inhomogeneous system) the expansions of second order terms (terms with *subscript* "2"). Thirdly, we derive the second order relations between all variables, similar to the ones obtained in the first order approximation.

The equations representing the second order approximation with respect to ϵ (with variables in the first order substituted) are

$$\rho_0\frac{\partial u_2}{\partial \xi} + \delta\left[\xi\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\frac{\partial u_2}{\partial \xi} - V\frac{\partial \rho_2}{\partial \theta} + \rho_0\left(\frac{\partial v_{\parallel 2}}{\partial \theta}\cos\alpha - \frac{\partial v_{\perp 2}}{\partial \theta}\sin\alpha\right) + \left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)u_2\right] = \frac{v_{\perp 1}^{(1)}}{v_A^2}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta}\sin\alpha + \mathscr{O}(\delta), \tag{86}$$

$$\frac{\partial P_2}{\partial \xi} - \delta\left[\rho_0 V\frac{\partial u_2}{\partial \theta} + \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_{x2}}{\partial \theta}\right] = \mathscr{O}(\delta), \tag{87}$$

$$\frac{\partial P_2}{\partial \theta}\sin\alpha + \rho_0 V\frac{\partial v_{\perp 2}}{\partial \theta} + \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_{\perp 2}}{\partial \theta} + \delta\left\{\xi\left[V\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\frac{\partial v_{\perp 2}}{\partial \theta} + \frac{\cos\alpha}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial b_{\perp 2}}{\partial \theta}\right] + \eta\frac{\partial^2 v_{\perp 2}}{\partial \xi^2}\right\} = \mathscr{O}(\delta^{-1}), \tag{88}$$

$$\frac{\partial P_2}{\partial \theta}\cos\alpha - \rho_0 V\frac{\partial v_{\parallel 2}}{\partial \theta} - \frac{b_{x2}}{\mu}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right) - \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_{\parallel 2}}{\partial \theta} - \delta\left\{4\eta\frac{\partial^2 v_{\parallel 2}}{\partial \xi^2} + \xi\left[V\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\frac{\partial v_{\parallel 2}}{\partial \theta} + \frac{\cos\alpha}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial b_{\parallel 2}}{\partial \theta}\right]\right\} = \delta^{-1}\left[\frac{\cos\alpha\sin\alpha}{V}v_{\perp 1}^{(1)}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta}\right] + \mathscr{O}(1), \tag{89}$$

$$V\frac{\partial b_{x2}}{\partial \theta} + B_0\cos\alpha\frac{\partial u_2}{\partial \theta} + \delta\left[\lambda\frac{\partial^2 b_{x2}}{\partial \xi^2} + \xi\cos\alpha\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial u_2}{\partial \theta}\right] = \mathscr{O}(1), \tag{90}$$

$$V\frac{\partial b_{\perp 2}}{\partial \theta} + B_0\cos\alpha\frac{\partial v_{\perp 2}}{\partial \theta} + \delta\left[\lambda\frac{\partial^2 b_{\perp 2}}{\partial \xi^2} + \xi\cos\alpha\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial v_{\perp 2}}{\partial \theta}\right] = \mathscr{O}(\delta^{-1}), \tag{91}$$

$$B_0\frac{\partial u_2}{\partial \xi} + \delta\left[\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)u_2 - B_0\sin\alpha\frac{\partial v_{\perp 2}}{\partial \theta} - V\frac{\partial b_{\parallel 2}}{\partial \theta} + \xi\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial u_2}{\partial \xi}\right] = \frac{B_0\sin\alpha}{\rho_0 v_A^2}v_{\perp 1}^{(1)}\frac{\mathrm{d}P_1^{(1)}}{\mathrm{d}\theta} + \mathscr{O}(\delta), \tag{92}$$

$$\rho_0 V\frac{\partial p_2}{\partial \theta} - \rho_0 c_S^2 V\frac{\partial \rho_2}{\partial \theta} + \rho_0 u_2\left[c_S^2\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right) - \left(\frac{\mathrm{d}p_0}{\mathrm{d}x}\right)\right] = \mathscr{O}(\delta^{-1}), \tag{93}$$

$$P_2 - p_2 - \frac{B_0}{\mu_0}b_{\parallel 2} + \delta\left[\frac{\xi}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)b_{\parallel 2}\right] = \delta^{-2}\left[\frac{\rho_0}{2}{v_{\perp 1}^{(1)}}^2\right] + \mathscr{O}(\delta^{-1}). \tag{94}$$

It is clear that nonlinear terms appear from this order of approximation and they are expressed in terms of variables obtained in the first order.

The analysis of the system of Eqs. (86)–(94) reveals that the expansions with respect to δ has to be written in the form

$$g_2 = \delta^{-1}g_2^{(1)} + g_2^{(2)} + \delta g_2^{(3)} + \ldots, \tag{95}$$

for u_2, b_{x2}, $v_{\perp 2}$, $b_{\perp 2}$ and P_2 and

$$h_2 = \delta^{-2} h_2^{(1)} + \delta^{-1} h_2^{(2)} + h_2^{(3)} + \ldots, \tag{96}$$

for $v_{\|2}$, $b_{\|2}$, p_2 and ρ_2.

Here we need to make a note. It follows from Eqs. (95) and (96) that the ratio of ρ_2 to ρ_1 is of the order of $\epsilon\delta^{-2}$, and the same is true for $v_{\|2}$, $b_{\|2}$ and p_2. It seems to be inconsistent with the regular perturbation method where it is assumed that the next order approximation is always smaller than the previous one. However, this problem is only apparent. To show this we need to clarify the exact mathematical meaning of the statement "in the asymptotic expansion each subsequent term is much smaller than the previous one". To do this we introduce the nine-dimensional vector $\boldsymbol{U} = (u, v_\|, v_\perp, b_x, b_\|, b_\perp, P, p, \rho)$ and consider it as an element of a Banach space. The norm in this space can be introduced in different ways. One possibility is

$$\|\boldsymbol{U}\| = \int_0^L \mathrm{d}\theta \int_{-\infty}^{\infty} |\boldsymbol{U}|\, \mathrm{d}\xi, \tag{97}$$

where L is the period. The asymptotic expansion in the dissipative layer, Eq. (68), can be rewritten as $\boldsymbol{U} = \boldsymbol{U}_0 + \epsilon \boldsymbol{U}_1 + \epsilon^2 \boldsymbol{U}_2 + \ldots$. Then the mathematical formulation of the statement "each subsequent term is much smaller than the previous one" is $\|\boldsymbol{U}_{n+1}\| \ll \|\boldsymbol{U}_n\|$, $n = 1, 2, \ldots$. It is straightforward to verify that, in accordance with Eqs. (68), (69), (95) and (96), $\|\boldsymbol{U}_2\| \ll \|\boldsymbol{U}_1\|$.

Once the expansions (95) and (96) are substituted into Eqs. (86)–(94), we can express the variables in this order of approximation as

$$v_{\|2}^{(1)} = \frac{\cos\alpha}{2V} v_{\perp 1}^{(1)2}, \quad b_{\|2}^{(1)} = -\frac{B_0}{2v_A^2} v_{\perp 1}^{(1)2}, \tag{98}$$

$$b_{\perp 2}^{(1)} = v_{\perp 2}^{(1)} = 0, \quad b_{x2}^{(1)} = -\frac{B_0 \cos\alpha}{V} u_2^{(1)}, \tag{99}$$

$$p_2^{(1)} = \rho_2^{(1)} = 0. \tag{100}$$

For the total pressure we obtain that

$$\frac{\partial P_2^{(1)}}{\partial \xi} = 0 \Longrightarrow P_2^{(1)} = P_2^{(1)}(\theta). \tag{101}$$

In addition, we obtain that the equation which determines $u_2^{(1)}$ is

$$\frac{\partial u_2^{(1)}}{\partial \xi} = -\frac{\cos^2\alpha}{V} v_{\perp 1}^{(1)} \frac{\partial v_{\perp 1}^{(1)}}{\partial \theta}. \tag{102}$$

Since the large variables in this order of approximation are $v_{\|2}$ and $b_{\|2}$, we can deduce that the linear order of approximation of resonant Alfvén waves in the dissipative layer excite *magnetoacoustic* modes in the second order of approximation. The excitation comes from the nonlinear term found in the second order approximation of the pressure equation, this drives the parallel components of the velocity and magnetic field perturbations. Since we are focussed on the Alfvén resonance only, these waves are not resonant. These waves act to cancel the very small pressure and density perturbations created by the first order approximation.

We now calculate the third order approximation with respect to ϵ. On analyzing the third order system of equations we deduce that the large variables in this order of approximation are Alfvénic ($v_{\perp 3}^{(1)}$, $b_{\perp 3}^{(1)}$), so we only need the perpendicular components of momentum and induction equations given by Eqs. (59) and (62), respectively.

First, since some of the first order approximation terms contribute to the third order approximation in integral form we must introduce a new notation

$$U_1^{(1)} = \int u_1^{(1)} \mathrm{d}\theta. \tag{103}$$

The third order approximation of the perpendicular component of momentum is

$$\begin{aligned} &\frac{\partial P_3}{\partial \theta}\sin\alpha + \rho_0 V \frac{\partial v_{\perp 3}}{\partial \theta} + \frac{B_0\cos\alpha}{\mu_0}\frac{\partial b_{\perp 3}}{\partial \theta} \\ &\quad + \delta\left[\xi\left(V\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\frac{\partial v_{\perp 3}}{\partial \theta} + \frac{\cos\alpha}{\mu_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial b_{\perp 3}}{\partial \theta}\right) + \eta\frac{\partial^2 v_{\perp 3}}{\partial \xi^2}\right] \\ &\quad = \delta^{-2}\left\{\frac{\partial v_{\perp 1}^{(1)}}{\partial \xi}\left[\frac{u_1^{(1)} P_1^{(1)}}{v_A^2} + \frac{u_1^{(1)} U_1^{(1)}}{V}\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right)\right]\right\} + \mathscr{O}(\delta^{-1}), \end{aligned} \tag{104}$$

while the perpendicular component of magnetic induction equation is

$$\begin{aligned} &V\frac{\partial b_{\perp 3}}{\partial \theta} + B_0\cos\alpha\frac{\partial v_{\perp 3}}{\partial \theta} + \delta\left[\lambda\frac{\partial^2 b_{\perp 3}}{\partial \xi^2}\right. \\ &\quad \left. + \xi\cos\alpha\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial v_{\perp 3}}{\partial \theta}\right] = \mathscr{O}(\delta^{-2}). \end{aligned} \tag{105}$$

Note that in obtaining the third order approximations we have employed all the relations we have for variables in the first and second order of approximation.

Equations (104) and (105) clearly show that the nonlinear terms on the right-hand sides do not cancel. This implies that the expansion of $v_{\perp 3}$ and $b_{\perp 3}$ should be of the form

$$h_3 = \delta^{-3} h_3^{(1)} + \delta^{-2} h_3^{(2)} + \delta^{-1} h_3^{(3)} + \ldots \tag{106}$$

We should state, for completeness, that if we derive the third order approximation for all the Eqs. (57)–(65) we obtain the expansions for u_3, b_{x3}, $v_{\|3}$, $b_{\|3}$, p_3, ρ_3 and P_3 to be

$$g_3 = \delta^{-2} g_3^{(1)} + \delta^{-1} g_3^{(2)} + g_3^{(3)} + \ldots \tag{107}$$

The expansions calculated for all the variables can now be collected together and we can write the expansions for "large" and "small" variables in the dissipative layer when studying resonant Alfvén waves. Large variables ($v_\perp$ and $b_\perp$) have the expansion

$$h = \left(\frac{\epsilon}{\delta}\right) h_1^{(1)} + \epsilon\left(\frac{\epsilon}{\delta}\right) h_2^{(1)} + \left(\frac{\epsilon}{\delta}\right)^3 h_3^{(1)} + \ldots, \tag{108}$$

and the expansion of small variables (u, b_x, $v_\|$, $b_\|$, p, ρ and P) is defined as

$$g = \epsilon g_1^{(1)} + \left(\frac{\epsilon}{\delta}\right)^2 g_2^{(1)} + \epsilon\left(\frac{\epsilon}{\delta}\right)^2 g_3^{(1)} + \ldots \tag{109}$$

Since Eq. (66) is the only condition enforced in the dissipative layer, we can state that

$$1 > \left(\frac{\epsilon}{\delta}\right) > \left(\frac{\epsilon}{\delta}\right)^2 > \left(\frac{\epsilon}{\delta}\right)^3 > \ldots \tag{110}$$

Therefore, since both Eqs. (108) and (109) contain successive higher powers of the parameter ϵ/δ we can deduce that, considering Eq. (110), higher orders of approximation of large and small variables become increasingly insignificant in comparison to the linear order of approximation, so resonant Alfvén waves in the dissipative layer can be described accurately by linear theory if condition (66) is satisfied.

5. Conclusions

In the present paper we have investigated the nonlinear behaviour of resonant Alfvén waves in the dissipative layer in one-dimensional planar geometry in plasmas with anisotropic dissipative coefficients, a situation applicable to solar coronal conditions. The plasma motion outside the dissipative layer is described by the set of linear, ideal MHD equations. The wave motion inside the dissipative layer is governed by Eq. (52). This equation is linear, despite taking into consideration (quadratic and cubic) nonlinearity. The Hall terms of the induction equation in the perpendicular direction relative to the ambient magnetic field cancel each other out.

The nonlinear corrections were calculated to explain why Eq. (52), describing the nonlinear behaviour of wave dynamics, is always linear. We found that, in the second order of approximation, magnetoacoustic modes are excited by the perturbations of the linear order of approximation. These secondary waves act to counteract the small pressure and density variations created by the first order terms. In addition, these waves are not resonant in the Alfvén dissipative layer. In the third order approximation the perturbations become Alfvénic, however, these perturbations are much smaller than those in the linear order of approximation. Equations (108) and (109) describe the expansion of large and small variables, respectively, and demonstrate that all higher order approximations of both large and small variables at the Alfvén resonance are smaller than the linear order approximation, provided condition (66) is satisfied. This condition ensures that the oscillation amplitude remains small inside the dissipative layer. From a naive point of view the linear theory is applicable as soon as the oscillation amplitude is small. The example of slow resonant waves clearly shows that this is not the case. The nonlinear effects become important in the slow dissipative layer as soon as $\epsilon \sim R^{-2/3}$, i.e. as soon as the oscillation amplitude in the dissipative layer, which is of the order of $\epsilon R^{1/3}$, is of the order of $R^{-1/3} \ll 1$. We also found that any dispersive effect due to the consideration of ions' inertial length (Hall effect) is absent from the governing equation.

This calculation of nonlinear corrections to resonant Alfvén waves in dissipative layers allows us to apply the already well-known linear theory for studying resonant Alfvén waves in the solar corona with great accuracy, where the governing equation, jump conditions and the absorption of wave energy are already derived (see e.g., Sakurai et al. 1991; Goossens et al. 1995; Erdélyi 1998).

It is interesting to note that this work can be transferred to isotropic plasma rather easily. Shear viscosity, supplied by Braginskii's viscosity tensor (see Appendix A), acts exactly as isotropic viscosity. Therefore, replacing η by $\rho_{0_a}\nu$ in Eq. (52) provides the required governing equation for resonant Alfvén waves in isotropic plasmas. Moreover, the work on the nonlinear corrections presented in this paper is also unaltered by anisotropy. This implies that we can consider resonant Alfvén waves in dissipative layers throughout the solar atmosphere and still use linear theory if condition (66) is satisfied.

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Appendix A: Braginskii's viscosity tensor and derivation of largest terms

In this Appendix, we shall derive the largest terms of Braginskii's viscosity tensor inside the Alfvén dissipative layer to be used to study the nonlinearity effects. Braginskii's viscosity tensor comprises of five terms. Its divergence can be written as (Braginskii 1965)

$$\nabla\cdot\boldsymbol{S} = \overline{\eta}_0\nabla\cdot\boldsymbol{S}_0 + \overline{\eta}_1\nabla\cdot\boldsymbol{S}_1 + \overline{\eta}_2\nabla\cdot\boldsymbol{S}_2 - \overline{\eta}_3\nabla\cdot\boldsymbol{S}_3 - \overline{\eta}_4\nabla\cdot\boldsymbol{S}_4, \quad (A.1)$$

Note that the terms proportional to $\overline{\eta}_0$, $\overline{\eta}_1$ and $\overline{\eta}_2$ in Eq. (A.1) describe viscous dissipation, while terms proportional to $\overline{\eta}_3$ and $\overline{\eta}_4$ are non-dissipative and describe the wave dispersion related to the finite ion gyroradius. They will be ignored in what follows. For simplicity in the body of the paper, we have taken $\eta = \eta_1$.

The quantities $\boldsymbol{S}_0$, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are given by

$$\boldsymbol{S}_0 = \left(\boldsymbol{b}'\otimes\boldsymbol{b}' - \frac{1}{3}\boldsymbol{I}\right)\left[3\boldsymbol{b}'\cdot\nabla(\boldsymbol{b}'\cdot\boldsymbol{v}) - \nabla\cdot\boldsymbol{v}\right], \quad (A.2)$$

$$\begin{aligned}\boldsymbol{S}_1 &= \nabla\otimes\boldsymbol{v} + (\nabla\otimes\boldsymbol{v})^T - \boldsymbol{b}'\otimes\boldsymbol{W} - \boldsymbol{W}\otimes\boldsymbol{b}' \\ &\quad + (\boldsymbol{b}'\otimes\boldsymbol{b}' - \boldsymbol{I})\,\nabla\cdot\boldsymbol{v} + (\boldsymbol{b}'\otimes\boldsymbol{b}' + \boldsymbol{I})\,\boldsymbol{b}'\cdot\nabla(\boldsymbol{b}'\cdot\boldsymbol{v}), \end{aligned} \quad (A.3)$$

$$\boldsymbol{S}_2 = \boldsymbol{b}'\otimes\boldsymbol{W} + \boldsymbol{W}\otimes\boldsymbol{b}' - 4(\boldsymbol{b}'\otimes\boldsymbol{b}')\,\boldsymbol{b}'\cdot\nabla(\boldsymbol{b}'\cdot\boldsymbol{v}), \quad (A.4)$$

$$\boldsymbol{W} = \nabla(\boldsymbol{b}'\cdot\boldsymbol{v}) + (\boldsymbol{b}'\cdot\nabla)\,\boldsymbol{v}. \quad (A.5)$$

Here $\boldsymbol{v} = (u, v, w)$ is the velocity, $\boldsymbol{b}' = \boldsymbol{B}_0/B_0$, $\boldsymbol{I}$ is the unit tensor and $\otimes$ indicates the dyadic product of two vectors. The superscript "T" denotes a transposed tensor.

The first viscosity coefficient, $\overline{\eta}_0$, (*compressional viscosity*) has the following approximate expression (see e.g., Ruderman et al. 2000)

$$\overline{\eta}_0 = \frac{\rho_0 k_B T_0 \tau_i}{m_p}, \quad (A.6)$$

where ρ_0 and T_0 are the equilibrium density and pressure, m_p is the proton mass, k_B the Boltzmann constant and τ_i the ion collision time. The other viscosity coefficients depend on the quantity $\omega_i\tau_i$, where ω_i it the ion gyrofrequency. When $\omega_i\tau_i \gg 1$ these coefficients are given by the approximate expressions

$$\overline{\eta}_1 = \frac{\overline{\eta}_0}{4(\omega_i\tau_i)^2}, \quad \overline{\eta}_2 = 4\overline{\eta}_1. \quad (A.7)$$

The viscosity described by the sum of the second and third terms in Eq. (A.1) is *the shear viscosity*. For typical coronal conditions $\omega_i\tau_i$ is of the order of 10^5–10^6, so according to Eq. (A.7) the term proportional to $\overline{\eta}_0$ in Eq. (A.1) is much larger than the second and third terms. However, it has been long understood that the compressional viscosity does not remove the Alfvén singularity (see e.g., Erdélyi & Goossens 1995; Mocanu et al. 2008) while shear viscosity does.

First, we shall calculate the components of the compressional viscosity. We will use the notation of parallel and perpendicular components as defined in the paper. It is straightforward to obtain that

$$\overline{\eta}_0(\nabla\cdot\boldsymbol{S}_0)_x = 0, \quad (A.8)$$

$$\overline{\eta}_0(\nabla\cdot\boldsymbol{S}_0)_\perp = 0, \quad (A.9)$$

$$\overline{\eta}_0(\nabla\cdot\boldsymbol{S}_0)_\parallel = \overline{\eta}_0\cos\alpha\left(2\frac{\partial^2 v_\parallel}{\partial z^2}\cos\alpha - \frac{\partial^2 u}{\partial x\partial z} + \frac{\partial^2 v_\perp}{\partial z^2}\sin\alpha\right). \quad (A.10)$$

The shear viscosity, as stated above, is the sum of the second and third terms of Eq. (A.1). To evaluate these terms we use the approximate expression for $\overline{\eta}_1$ and $\overline{\eta}_2$ given by Eq. (A.7). As a result we obtain

$$\overline{\eta}_1\left[(\nabla\cdot\boldsymbol{S}_1)_x+4\left(\nabla\cdot\boldsymbol{S}_2\right)_x\right]=\overline{\eta}_1\left[\frac{\partial^2 u}{\partial x^2}+(1+3\cos^2\alpha)\frac{\partial^2 u}{\partial z^2}+4\frac{\partial^2 v_\parallel}{\partial x\partial z}\cos\alpha\right],\tag{A.11}$$

$$\overline{\eta}_1\left[(\nabla\cdot\boldsymbol{S}_1)_\perp+4\left(\nabla\cdot\boldsymbol{S}_2\right)_\perp\right]=\overline{\eta}_1\left[\frac{\partial^2 v_\perp}{\partial x^2}+\left(4-3\sin^2\alpha-16\sin^6\alpha\right)\frac{\partial^2 v_\perp}{\partial z^2}+4\sin\alpha\cos\alpha\left(4\sin^4\alpha-1\right)\frac{\partial^2 v_\parallel}{\partial z^2}\right],\tag{A.12}$$

$$\overline{\eta}_1\left[(\nabla\cdot\boldsymbol{S}_1)_\parallel+4\left(\nabla\cdot\boldsymbol{S}_2\right)_\parallel\right]=4\overline{\eta}_1\left\{\frac{\partial^2 v_\parallel}{\partial x^2}+\left[1+\cos^2\alpha\left(4\sin^4\alpha-1\right)\right]\frac{\partial^2 v_\parallel}{\partial z^2}+\frac{\partial^2 u}{\partial x\partial z}\cos\alpha-\left(4\sin^4\alpha+1\right)\frac{\partial^2 v_\perp}{\partial z^2}\cos\alpha\sin\alpha\right\}.\tag{A.13}$$

Equations (A.8)–(A.13) are complicated, but we can simplify them further by taking the largest term only in each equation. For the viscosity in the parallel direction it would, at first, seem obvious that the largest term will be proportional to $\overline{\eta}_0$ rather than $\overline{\eta}_1$. However, some of the variables proportional to $\overline{\eta}_1$ have derivatives with respect to x which produce enormous gradients in the dissipative layer when there is a transversal inhomogeneity, hence some of the terms proportional to $\overline{\eta}_1$ are of the same order as or larger than the terms proportional to $\overline{\eta}_0$. It is also important to note that in the first order approximation the second and third terms on the right-hand side of Eq. (A.10) cancel (see Eq. (49)). For the normal and perpendicular components of viscosity, the treatment is slightly simpler. The compressional viscosity is zero, and as derivatives with respect to z are much small that derivatives with respect to x, we can select the largest term proportional to $\overline{\eta}_1$ by observation. Therefore, the viscosity tensor can be approximated by

$$(\nabla\cdot\boldsymbol{S})_x\approx\overline{\eta}_1\frac{\partial^2 u}{\partial x^2},\tag{A.14}$$

$$(\nabla\cdot\boldsymbol{S})_\perp\approx\overline{\eta}_1\frac{\partial^2 v_\perp}{\partial x^2},\tag{A.15}$$

$$(\nabla\cdot\boldsymbol{S})_\parallel\approx 4\overline{\eta}_1\frac{\partial^2 v_\parallel}{\partial x^2}.\tag{A.16}$$

Equations (A.14)–(A.16) give an appropriate approximation to Braginskii's viscosity tensor when studying nonlinear resonant Alfvén waves in dissipative layers. It is interesting to note that the terms in Eqs. (A.14)–(A.16) are identical to the largest terms when considering isotropic viscosity. Obviously, compressional viscosity cannot remove the Alfvén singularity since Eq. (A.9) is identically zero.

Appendix B: The derivation of the Hall term in the induction equation for Alfvén resonant waves

In this Appendix we will derive the components of the Hall term in the induction equation and show that neglecting the Hall effect at the Alfvén resonance is a good approximation for typical conditions throughout the solar atmosphere. The main reasons qualitatively are as follows. When we are in the lower solar atmosphere (e.g. solar photosphere) the Hall conduction is much smaller than the direct conduction since the product of the electron gyrofrequency, ω_e, and collision time, τ_e, is less than unity (see e.g., Priest 1984). For the upper atmosphere (e.g. chromosphere, corona), where the product $\omega_e\tau_e$ is greater than unity, the Hall conduction has to be considered. However, when the Hall terms are derived, the largest terms in the perpendicular direction relative to the ambient magnetic field cancel leaving only higher order approximation terms which are far smaller than the direct conduction. As the dominant dynamics of resonant Alfvén waves in dissipative layer resides in the components of velocity and magnetic field perturbation in the perpendicular direction relative to the background magnetic field we can neglect the Hall conduction completely from the analysis without affecting the governing equation.

In order to estimate the relative importance of the Hall term and resistive term in the dissipative layer we follow the sophisticated analysis which was presented by Ruderman et al. (1997b) and Clack & Ballai (2008). We do not write down all the steps of the analysis, but rather give the salient points specific to the Hall effect at the Alfvén resonance.

Equations (29) and (42) provide the following estimations in the dissipative layer:

$$u=\mathscr{O}(\epsilon),\ \ v_\perp=\mathscr{O}(\epsilon^{1/2}),\ \ v_\parallel=\mathscr{O}(\epsilon),\qquad b_x=\mathscr{O}(\epsilon),\ \ b_\perp=\mathscr{O}(\epsilon^{1/2})\ \ b_\parallel=\mathscr{O}(\epsilon),\tag{B.1}$$

where ϵ still denotes the dimensionless amplitude of oscillations far away from the dissipative layer. The thickness of the dissipative layer divided by the characteristic scale of inhomogeneity is $\delta_c/l_{\rm inh}=\mathscr{O}(\epsilon^{1/2})$. This gives rise to the estimations

$$l_{\rm inh}\frac{\partial h}{\partial x}=\mathscr{O}(\epsilon^{-1/2}h),\quad l_{\rm inh}\frac{\partial h}{\partial z}=\mathscr{O}(h),\quad l_{\rm inh}^2\frac{\partial^2 h}{\partial z^2}=\mathscr{O}(h),\tag{B.2}$$

where h denotes any of the quantities u, b_x, $b_\parallel$, $b_\perp$ or $v_\perp$. Since the first term in the expansion of $v_\parallel$ is independent of x, it follows that

$$l_{\rm inh}\frac{\partial v_\parallel}{\partial x}=\mathscr{O}(v_\parallel),\ l_{\rm inh}\frac{\partial v_\parallel}{\partial z}=\mathscr{O}(v_\parallel),\ l_{\rm inh}^2\frac{\partial^2 v_\parallel}{\partial x^2}=\mathscr{O}(\epsilon^{-1/2}v_\parallel).\tag{B.3}$$

We now need to calculate the components of the vectors of the resistive term and the Hall term from the induction equation normal to the magnetic surfaces (the x-direction) and in the magnetic surfaces parallel and perpendicular to the equilibrium magnetic field lines. We use Eqs. (B.2) and (B.3) in order to estimate all the terms and we only retain the largest. As a result we have

$$\overline{\lambda}\nabla^2 B_x=\overline{\lambda}\frac{\partial^2 b_x}{\partial x^2}+\ldots,\tag{B.4}$$

$$\overline{\lambda}\nabla^2 B_\perp=\overline{\lambda}\frac{\partial^2 b_\perp}{\partial x^2}+\ldots,\tag{B.5}$$

$$\overline{\lambda}\nabla^2 B_\parallel=\overline{\lambda}\frac{\partial^2 b_\parallel}{\partial x^2}+\ldots,\tag{B.6}$$

$$H_x=\frac{B_0\cos^2\alpha}{\mu_0 e n_e}\frac{\partial^2 b_\perp}{\partial z^2}+\ldots,\tag{B.7}$$

$$H_\perp=\frac{B_0}{\mu_0 e n_e}\left(\frac{1}{B_0}\frac{{\rm d}B_0}{{\rm d}x}\frac{\partial b_x}{\partial x}+\cos\alpha\frac{\partial^2 b_\parallel}{\partial z\partial x}\right)+\ldots,\tag{B.8}$$

$$H_\parallel=-\frac{B_0\cos\alpha}{\mu_0 e n_e}\frac{\partial^2 b_\perp}{\partial z\partial x}+\ldots,\tag{B.9}$$

where the dots indicate terms much smaller than those shown explicitly. With the aid of Eqs. (B.1)–(B.3) we obtain the ratios

$$\frac{H_x}{\bar{\lambda}\nabla^2 B_x} \sim \epsilon^{1/2}\omega_e\tau_e, \tag{B.10}$$

$$\frac{H_\perp}{\bar{\lambda}\nabla^2 B_\perp} \sim \epsilon\omega_e\tau_e, \tag{B.11}$$

$$\frac{H_\parallel}{\bar{\lambda}\nabla^2 B_\parallel} \sim \omega_e\tau_e. \tag{B.12}$$

For the Hall conduction to be significant in the direction of the dominant dynamics of resonant Alfvén waves (i.e. in the perpendicular direction) we must have $\epsilon\omega_e\tau_e \gtrsim 1$. This is plausible for the solar upper atmosphere. If this condition holds, then we must consider the Hall term in the induction equation. However, if Eq. (B.8) is expanded using Eq. (29) from Sect. 3 we obtain

$$\epsilon^{3/2}\lambda\left\{\frac{1}{B_0}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial b_x^{(1)}}{\partial\xi} + \cos\alpha\frac{\partial^2 b_\parallel^{(1)}}{\partial\theta\partial\xi}\right\} + \mathscr{O}(\epsilon^2). \tag{B.13}$$

It should be noted that in deriving Eq. (B.13) we have used the assumption that $\epsilon\omega_e\tau_e = \mathscr{O}(1)$. The terms inside the braces are of the same order as the direct conduction. Hence, they would be expected to appear in the governing equation. When substituting for $b_x^{(1)}$ and $b_\parallel^{(1)}$ using Eqs. (45) and (46), respectively, it is found that the terms inside the brackets cancel

$$\frac{\cos\alpha}{V}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial u^{(1)}}{\partial\xi} - \frac{\cos\alpha}{V}\left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)\frac{\partial u^{(1)}}{\partial\xi} = 0. \tag{B.14}$$

Equation (B.14) shows that the Hall term in the perpendicular component of induction is always smaller than the direct conduction in the solar atmosphere. The normal and parallel components of the Hall conduction are, in fact, larger than the perpendicular component. Nevertheless they play no role in derivation of the governing equation of resonant Alfvén waves in dissipative layer. The parallel component is the largest of the three components and this is to be expected as the Hall effect is strongest at right angles to the dominant wave motion. This is in complete agreement with the study on resonant slow waves by Clack & Ballai (2008) which found the largest Hall effect was in the perpendicular component of Hall conduction, which is at right angles to the dominant dynamics of resonant slow waves.

In summary, it is a good approximation to neglect the Hall term in the induction equation when studying resonant Alfvén waves in dissipative layer. This approximation holds throughout the entire solar atmosphere.

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