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Spatial and observational homogeneities of the galaxy distribution in standard cosmologies

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ABSTRACT

Context. An important aim of standard relativistic cosmology is the empirical verification of its geometrical concept of homogeneity by considering various definitions of distance and astronomical observations occurring along the past light cone.

Aims. We analyze the physical consequences of distinguishing between *spatial homogeneity* (SH), defined by the Cosmological Principle, and *observational homogeneity* (OH). We argue that OH is falsifiable by means of astronomical observations, whereas SH can be verified only indirectly.

Methods. We simulate observational counts of cosmological sources, such as galaxies, by means of a generalized number-distance expression that can be specialized to produce either the counts of the Einstein-de Sitter (EdS) cosmology, which has SH by construction, or other types of counts, which do, or do not, have OH by construction. Expressions for observational volumes are derived using the various cosmological-distance definitions in the EdS cosmological model. The observational volumes and simulated counts are then used to derive differential densities. We present the behavior of these densities for increasing redshift values.

Results. Simulated counts that have OH by construction do not always exhibit SH features. The reverse situation is also true. In addition, simulated counts with no OH features at low redshift begin to show OH characteristics at high redshift. The comoving distance appears to be the only distance definition for which both SH and OH are applicable simultaneously, even though with limitations.

Conclusions. We demonstrate that observations indicative of a possible absence of OH do *not* necessarily falsify the standard Friedmannian cosmology, which implies that this cosmology does not always produce observable homogeneous densities. We conclude that using different cosmological distances in the characterization of the galaxy distribution can produce significant ambiguities in reaching conclusions about the large-scale galaxy distribution in the Universe.

Key words. cosmology: theory – cosmology: large-scale structure of the Universe – cosmology: observations – galaxies: statistics – X-rays: galaxies: clusters

1. Introduction

The determination of whether or not the large-scale distribution of matter in the Universe reaches a homogeneous distribution on some redshift scale has been a disputed topic in observational cosmology for many decades. With the availability of increasingly larger galaxy redshift survey databases stemming from increasingly more complete and deeper galaxy samples, there have been renewed efforts to solve this problem by means of statistical techniques of growing sophistication. However, this issue has still not been settled because contradictory results have been reported by various authors who support opposite claims (Ribeiro & Miguelote 1998; Sylos Labini et al. 1998; Joyce et al. 1999, 2000, 2005; Gabrielli et al. 2005; Hogg et al. 2005; Jones et al. 2005; Yadav et al. 2005).

The conventional wisdom on this topic claims that the solution to the problem lies in our ability to acquire more and better data, that is, as more complete galaxy samples at both lower and higher redshift ranges become available the statistical techniques currently applied to the analysis of these data should suffice to settle the controversy. In other words, this view assumes implicitly or explicitly that more complete galaxy redshift survey samples will eventually clarify this point once and for all. The difficulty with the conventional wisdom is that we have witnessed an enormous improvement in the technological methods for astronomical data acquisition and, as a direct result of those technological advances, have indeed obtained higher quality and more complete galaxy redshift survey data sets. Despite these technical advances, the controversy, however, continues to resurface (see Ribeiro 1994, for a brief historical account of this debate in the past century). Ribeiro (1992) proposed that the source of the controversy was not the observations themselves, but the conceptual tools used to analyze the observations. The initial ideas were gradually refined and an alternative perspective developed (Ribeiro 2001b, 2005; Albani et al. 2007). This perspective suggests that improvements in observational techniques and the acquisition of larger and larger galaxy data samples is not the path that will shed light and clarify this debate.

This alternative conceptual framework grew out of various works (Ribeiro 1992a,b, 1993, 1994, 1995, 2001a; Abdalla et al. 2001) and was comprehensively analyzed in Ribeiro (2001b; hereafter R01b). Albani et al. (2007; hereafter A07) further developed this alternative perspective in a somewhat condensed version. First principles were used to point out that General Relativity allows us to define two different concepts of homogeneity perfectly applicable to cosmological models: *spatial homogeneity* (SH) and *observational homogeneity* (OH). Furthermore, it was argued that the Cosmological Principle is based on the concept of SH, whereas the astronomical search for the possible homogeneity of the Universe occurs mostly in

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the context of OH: this is because astronomical observations are carried out where OH is defined, that is, along the backward null cone. The Cosmological Principle implies that SH is not directly observable on space-like surfaces of constant time defined in the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies. On the other hand, the geometrical locus of OH is not along those space-like surfaces of constant time, but along the past null cone, and so OH will only occur if a density measured directly from observations, usually galaxy number counts, remains constant along these null surfaces. These two concepts of homogeneity do overlap, but they are *not* the same[1]. Reports of the searches for the homogenization of the matter distribution in the Universe have not, for the most part, acknowledged this important difference since the use of the generic, but ambiguous, term "homogeneity" has become commonplace (see Sect. 7 below). As discussed in R01b and A07, such a distinction arises only if one takes an entirely relativistic perspective of this problem.

Nevertheless, acknowledging the existence of these two types of homogeneities is not enough to clarify the controversial points outlined above. One has to go a step further because the only way to discriminate SH from OH is by building observational densities using different distance measures with the same number count data. Bearing in mind the conceptual framework summarized above, R01b and Ribeiro (2005; hereafter R05) showed that although the Einstein-de Sitter (EdS) cosmology has SH by construction, it may, or may not exhibit OH, since the possible presence of OH depends on the distance measure chosen in the statistical analysis of its EdS theoretically derived number count expression as a function of the redshift. Such a result was confirmed by A07 using observations of number counts extracted from the Canadian Network Observational Cosmology 2 (CNOC2) galaxy redshift survey and applied to two types of standard cosmologies, EdS and FLRW with $\Omega_{m_0} = 0.3$, $\Omega_{\Lambda_0} = 0.7$.

When these two types of homogeneities are discussed, a question which also arises is how deep the observations must be to be able to distinguish SH from OH. In other words, we must determine whether or not the redshift ranges of current galaxy surveys are sufficiently deep to be able to detect this difference. It is important to point out that previous work (Ribeiro 1995) indicated that due to the high nonlinearity of General Relativity the distinction between SH and OH effects may occur, in theory, at redshifts as low as $z \lesssim 0.1$, depending on the chosen relativistic cosmological model and the observational quantity under study.

The aim of this paper is to analyze further the issues discussed above. Our goal here is to extend the studies presented in R01b, R05, and A07. These papers started with SH by construction and sought to show whether or not OH was also featured in the models. R01b and R05 initiated their analysis from the theoretical EdS number counts, whereas A07 used observed number counts extracted from the luminosity function. We aim here to investigate the opposite situation, i.e., start with OH by construction and then investigate whether or not the models show SH. Instead of using actual observations, our intention here is to *simulate observations* by means of a generalized number-distance relation which may, or may not, produce OH from the start and then search for possible SH in the resulting model. In addition, we shall analyze our results in two redshift ranges, namely $0.001 < z < 0.1$ and $z > 0.1$, since most direct measurements of galaxy correlations have been limited to $z \sim 0.1$. This aims to try to answer the question posed above about the redshift depth which these effects begin to manifest themselves, as well as probing which quantities could possibly offer observational results from which we can attempt to obtain observational evidence allowing us to discriminate between SH and OH[2].

Our results show that a model with no OH by construction does not always remain that way at higher values of redshift. In fact, with a specific distance definition we may have a model with OH at low redshifts, but no OH at higher redshifts. We also found that if we start a model with OH, it may or may not become SH. These results imply that the use of different distance measures to calculate cosmological densities produces significant ambiguities in reaching conclusions about the behavior of the large-scale distribution of galaxies in the Universe due to the impossibility of uniquely characterizing observational densities from galaxy distribution data. Conclusions on this matter reached by means of the use of just one cosmological distance, usually comoving, should therefore be seen as applicable to this distance measure only and are most likely not valid generally. Therefore, the proposal of R05 that *observers should utilize all possible distance measures in their data analysis* is reinforced here. It is the view of these authors that *this is the path towards clarifying the controversy discussed above*.

The plan of the paper is as follows. In Sect. 2 we derive the basic equations and definitions of observational distances, areas, volumes, number counts, and densities in cosmology. Section 3 reviews the results of R01b, R05, and A07, where those definitions are applied to FLRW cosmological models, showing that although the EdS cosmology is spatially homogeneous by construction, it has observational homogeneity only when the comoving distance is adopted in calculating observational densities. In Sect. 4 we simulate models that do, or do not, have observationally-homogeneous features, concluding that even a model without OH at low redshifts may become observationally homogeneous at higher values of z. Section 5 studies the asymptotic behavior of the fractal dimension D of the EdS model at $z \to 0$ and at the Big Bang singularity hypersurface where $z \to \infty$, showing that the former case leads to $D = 3$ for all distance measures. In contrast, the latter case implies that $D = 0$ for all distances, except the comoving distance where the value $D = 3$ remains unchanged for all z. Section 6 discusses the relationship between number counts N and magnitudes by means of the distance modulus μ. We show that the same ambiguous results obtained in Sect. 4 for the $N \times z$ functions, constructed with the various distance definitions, are also present in the $N \times \mu$ functions. Section 7 provides a conceptual discussion about the caveats of the use of the generic term "homogeneity" in cosmological models, arguing that observational homogeneity is a relative concept entailed by the relativity of time intervals. Finally, Sect. 8 summarizes the results obtained in this paper.

2. Distances, volumes, densities and number counts

As has been extensively argued elsewhere, measuring distances in cosmology depends on circumstances, that is, on the method

[1] In earlier works (Ribeiro 1992a,b, 1993, 1994, 1995), the term "apparent homogeneity" was used instead of the currently adopted term "observational homogeneity".

[2] It should be mentioned here that if one uses the galaxy luminosity function data derived from various surveys with a relativistic cosmology number count theory (Ribeiro & Stoeger 2003), one is able to indirectly obtain measurements of galaxy number counts at $z \approx 1$, or at far higher redshifts, where the distinction between SH and OH is easily detected. See A07 and Iribarrem et al. (2008, in preparation).

of measurement (McVittie 1974; Sandage 1988; R01b; R05, and references therein). This does not imply that distances cannot be compared with each another. These are true, physical distances to an object and they can indeed be compared simply because they are distances to cosmological sources, mostly galaxies, for which intrinsic physical characteristics can be determined (intrinsic measurement, intrinsic luminosity, etc.) *independently* of a cosmological model. In addition, the reciprocity theorem (Ellis 2007) relates the various distances to each another and allows conversions between them.

That is the theory. In practice, however, due to technological limitations and our incomplete knowledge of the physical processes occurring in the evolution of galaxies, we are presently unable to find those intrinsic measurements, that is, standard candles and standard rods, for *every* galaxy in a redshift survey. Therefore, we are left to measure their redshifts only and, by using a cosmological model, to relate those redshifts to some distance. Textbooks and reviews discussing cosmological distance definitions offer a plethora of names for the distance measures, a fact which only adds confusion to, not infrequently, poorly understood concepts about what is a distance to a cosmological object, the definition that should be chosen, and the context for choosing a certain definition and not another one. As soon as one delves into this problem and reads how it is dealt with in the literature, it becomes clear that familiar Newtonian concepts slipped into a subject that can only be understood properly by means of relativistic ideas. Thus, to avoid those Newtonian concepts of absolute and unique definitions, which are not applicable to the relativistic discussion proposed here, we must follow along the wise footsteps of others and accept that *there is no such a thing as an unique cosmological distance: all are correct, and all can be compared with each another.* To argue otherwise is to allow Newtonian ideas to slip into a subject that is entirely relativistic.

The difficulties described above have, of course, been previously perceived by others, such as observational cosmologists, who resorted to the convenient convention of using, for the most part, only one distance measure, the *comoving distance*. There are, however, three caveats to this practice. Firstly, not all practitioners follow this convention and this means that there are still those who are misled into treating different distance definitions as if they were the same, when they are not, and, worst of all, results derived from those different distance measures are then improperly compared with each other. This can obviously add even more confusion to an already-confused subject. As we show below, the second caveat is that the comoving distance implies that the task of distinguishing spatial from observational homogeneity becomes very difficult. Thirdly, by adopting just one distance as convention some may still unconsciously fall into the familiar, but in this context very misleading, *Newtonian trap of believing that in cosmology the distance to an object could be uniquely defined, when General Relativity does not allow such a conclusion.*

2.1. Cosmological distances

Our proposed solution to these difficulties is therefore to use in our analysis below all observational distances, the *luminosity distance* $d_{\rm L}$, the *area distance* $d_{\rm A}$, the *galaxy area distance* $d_{\rm G}$, and the *redshift distance* $d_{\rm Z}$[3]. These are quantities that can in principle, be directly measured (R05; Ellis 2007). However, since we need to assume a cosmological model to start with, other distances like the comoving distance, proper distance, etc, can all be written in terms of these four above (see below). The advantage of starting with these observational four, which to simplify the notation from now on shall be referred as d_i (i = A, G, L, Z), is to know beforehand that they are defined along the past light cone, since they are observational distances. Relating them to, for example, comoving distance, implies that the solution of the null geodesic equation has therefore to be included in the expression of the comoving distance. This is standard practice, rarely mentioned, but, for the purposes of this work, it must be stated explicitly to avoid confusion.

These observational distances are related to each other by an important result called the *reciprocity theorem*, or *Etherington reciprocity law*, proven long ago by Etherington (1933), which reads as follows[4],

$$(1+z)^2 d_{\rm A} = (1+z)\, d_{\rm G} = d_{\rm L}. \tag{1}$$

This theorem is valid for *all* cosmologies (Ellis 1971, 2007; Plebański & Krasiński 2006). In addition, R01b and R05 also defined a distance by redshift as follows,

$$d_{\rm Z} = \frac{cz}{H_0}, \tag{2}$$

where c is the light speed and H_0 is the Hubble constant. This is not a distance in the sense of the other distance measures; since it is often used in observational cosmology, it is, however, useful to adopt it here and assume Eq. (2) to be the definition of redshift distance $d_{\rm Z}$ for all z.

2.2. Observational areas and volumes

We now define observational areas and volumes. Following R05, the area of the observed spherical shell of radius d_i may be written as,

$$S_i = 4\pi (d_i)^2, \tag{3}$$

and the observational volume at this same radius can be straightforwardly written as,

$$V_i = \frac{4}{3}\pi\,(d_i)^3. \tag{4}$$

These equations imply that the observed volume element is given by,

$$\mathrm{d}V_i = S_i\,\mathrm{d}(d_i). \tag{5}$$

Here $\mathrm{d}(d_i)$ is the elementary shell thickness and, since cosmological distances are function of the redshift, the shell thickness can be written approximately as $\Delta d_i = [\mathrm{d}(d_i)/\mathrm{d}z]\,\Delta z$, for a certain observed redshift interval Δz.

These expressions are completely general and Eq. (4) agrees with the usual volume definitions. We note that these areas and volumes are observational, that is, they are defined along the past light cone.

[3] $d_{\rm A}$ is also known as "angular diameter distance", "corrected luminosity distance" and "observer area distance". $d_{\rm G}$ is also known as "effective distance", "angular size distance", "transverse comoving distance" and "proper motion distance" (see details and references in R01b and R05). It is also possible to define another distance, the *parallax distance* $d_{\rm P}$ due to galaxy parallaxes (Ellis 1971). This distance is not often mentioned since galaxy parallaxes cannot yet be measured.

[4] See Ellis (2007) for an appraisal of how fundamental this theorem is in every aspect of modern cosmology.

2.3. Generalized number-distance relation

We aim to simulate number counts as if they were actual observations. To do so in a general manner, it is convenient to adopt the following expression for the *number of observed cosmological sources N* at a certain observational distance d_i,

$$ {}^{\scriptscriptstyle D}\!N_i = (B\, d_i)^D. \tag{6}$$

Here B is an as yet unspecified constant and D is the fractal dimension (Pietronero 1987; Ribeiro & Miguelote 1998; Sylos Labini et al. 1998). This expression provides a general way of simulating galaxy counts, since, for $D = 3$ we have an *observationally*-homogeneous distribution (see below), whereas for $D < 3$ we have an *observationally-in*homogeneous galaxy counting. The subscript index i and superscript index D therefore label respectively the choice of distance and how far the simulated counting differs from OH. We note that it is *not* our intention to either prove or disprove the possible fractality of the galaxy distribution. The adoption of the equation above for N is just a matter of convenience due to its generality.

De Vaucouleurs (1970) and Wertz (1970, 1971) long ago proposed similar relations to Eq. (6), which were written instead in terms of density. They appear to be the first authors to use it in the context of galaxy distributions. Pietronero (1987) independently advanced an expression virtually identical to the above. As discussed in Ribeiro & Miguelote (1998; see also Ribeiro 1994), the models of Wertz (1970, 1971) and Pietronero (1987) shared more similarities than differences, both conceptually and analytically. Wertz did not use the word "fractal", although self-similar ideas can be found in his discussion. Pietronero (1987) named his expression the "generalized mass-length relation"; here we instead choose to refer to Eq. (6) as the *generalized Pietronero-Wertz number-distance relation*, or simply *generalized number-distance relation*, since we believe the emphasis on number-distance, rather than mass-length, is more appropriate to observational cosmology.

2.4. Observational densities

As discussed in R05 and A07, the *differential density* ${}^{D}\gamma_i$ at a certain distance d_i, and with a specific choice of the dimension D, is defined by the following expression,

$$ {}^{D}\gamma_i = \frac{1}{S_i}\frac{\mathrm{d}\left({}^{\scriptscriptstyle D}\!N_i\right)}{\mathrm{d}\,(d_i)} \cdot \tag{7}$$

This equation provides a measure of the rate of growth in the number density as one moves down the past light cone along the observable distance d_i. Obviously, the behavior of the differential density depends heavily on the distance employed in Eq. (7). The *integral differential density* ${}^{D}\gamma_i^*$ is defined to be the integration of ${}^{D}\gamma_i$ over the observational volume V_i, corresponding to,

$$ {}^{D}\gamma_i^* = \frac{1}{V_i}\int_{V_i} {}^{D}\gamma_i \,\mathrm{d}V_i. \tag{8}$$

If we now define ${}^{\scriptscriptstyle D}[n]_i$ to be the *radial number density* for a given distance measure d_i, the following result clearly holds, once we consider Eqs. (3), (4), (7), and (8),

$$ {}^{\scriptscriptstyle D}[n]_i = \frac{{}^{\scriptscriptstyle D}\!N_i}{V_i} = {}^{D}\gamma_i^* . \tag{9}$$

At this point some important remarks are necessary. The radial number density ${}^{\scriptscriptstyle D}[n]_i$ considers the number counts of objects as a function of distance from a single point. In the context of FLRW cosmology, this single point can be any point in a 4-dimensional Riemannian manifold, since this spacetime assumes a maximal spatial isotropy (see Sect. 7 below). Therefore, this is not an average quantity, or ensemble average, obtained as an average made over many realizations of a stationary stochastic process, where the ensemble average can be replaced by a volume average. Confusion arises when Eq. (6) is viewed in the framework of fractal geometry, where the number count of this equation is interpreted as an average quantity. Here Eq. (6) is simply the radial matter distribution which *could* be given by a perfect fluid approximation of cosmological solutions of Einstein's field equations. As discussed in R05, the relationship between these two quantities, the radial number density and the average number density, remains an open question although it appears to be reasonable to assume that they are related. The present paper is not concerned with proving or disproving the fractal hypothesis for the galaxy distribution, but aims to investigate the limitations of the standard concept of "homogeneity", in particular its application to interpreting real astronomical observations.

It is useful to write the differential densities in terms of the redshift z. The results are as follows,

$$ {}^{D}\gamma_i(z) = \left[\frac{\mathrm{d}}{\mathrm{d}z}\left({}^{\scriptscriptstyle D}\!N_i\right)\right]\left[S_i\frac{\mathrm{d}}{\mathrm{d}z}\,(d_i)\right]^{-1}, \tag{10}$$

$$ {}^{D}\gamma_i^*(z) = \frac{1}{V_i}\int_0^z {}^{D}\gamma_i\left(\frac{\mathrm{d}V_i}{\mathrm{d}z'}\right)\mathrm{d}z' = \frac{{}^{\scriptscriptstyle D}\!N_i(z)}{V_i(z)} \cdot \tag{11}$$

In both of the above equations one can clearly identify two distinct parts. The first term is the geometrical term, determined by the spacetime geometry of the chosen metric; this is the case of the functions $d_i(z)$, $S_i(z)$ and $V_i(z)$ and their derivatives. The second term is given by the number count ${}^{\scriptscriptstyle D}\!N_i(z)$ and the differential number count $\mathrm{d}\left[{}^{\scriptscriptstyle D}\!N_i(z)\right]/\mathrm{d}z$ and are determined either by theory or observationally. Various tests of cosmological models rely, in one way or another, on the comparison of number counting, determined observationally, with its theoretical prediction. We note that, in Eq. (10), this division between the geometrical and theoretical/observational parts are clearly visible, since they are represented by each term inside the brackets on the right hand side.

As mentioned above, the geometrical part was determined entirely by assuming special cases of the FLRW metric, EdS cosmology in R01b, R05, and A07 and the FLRW open model in A07. The number count, however, was either theoretically determined from the cosmological model, that is, by taking the expression for $N(z)$ as given by the matter distribution in these cosmologies, which meant assuming a spatially-homogeneous matter distribution from the start (R01b, R05), or by using differential counts $\mathrm{d}N/\mathrm{d}z$ obtained from the luminosity function of galaxy surveys, as demonstrated by A07[5]. We determine the geometrical part by choosing a spacetime metric, since this is unavoidable; instead of restricting ourselves however to obtaining the number count solely from the chosen cosmology, we use

[5] It must be noted that when comparing densities constructed from observationally-derived number counts with the theoretical predictions of a FLRW model with $\Omega_{m_0} = 0.3$, $\Omega_{\Lambda_0} = 0.7$, that is, having SH by construction, A07 found deviations from pure SH. However, it was not clear if these deviations were due to possible incompleteness of the sample, the use of an inappropriate evolution function when deriving the luminosity function parameters, or if they were true deviations from SH. See details in the caption Figs. 7 and 8 of A07.

Eq. (6) to simulate number counts, *as if* they were real observations. This methodology attempts to make possible the investigation of the model behavior, when OH is assumed from the outset; it configures an approach that is opposite to that investigated in R01b, R05, and A07. The advantage of this methodology is to depart from pure spatially-homogeneous cosmologies.

3. Observational inhomogeneity of the Einstein-de Sitter spacetime

We shall assume this spacetime metric for analytical simplicity only, but this choice does not affect our main results (see below). The methodology that we present can be extended easily to other FLRW spacetime metrics. The EdS expressions shown next were derived in R05 (see also R01b).

In the EdS model, the four observational distances discussed above are given by the following set of equations[6],

$$d_A(z) = \frac{2c}{H_0}\left[\frac{1+z-\sqrt{1+z}}{(1+z)^2}\right], \tag{12}$$

$$d_G(z) = \frac{2c}{H_0}\left(\frac{1+z-\sqrt{1+z}}{1+z}\right), \tag{13}$$

$$d_L(z) = \frac{2c}{H_0}\left(1+z-\sqrt{1+z}\right), \tag{14}$$

$$d_Z(z) = \frac{2c}{H_0}\left(\frac{z}{2}\right). \tag{15}$$

Equations (12)–(14) have Taylor-series expansions in terms of redshift that are given by,

$$d_A(z) = \frac{c}{H_0}\left(z - \frac{7}{4}z^2 + \frac{19}{8}z^3 + \ldots\right), \tag{16}$$

$$d_G(z) = \frac{c}{H_0}\left(z - \frac{3}{4}z^2 + \frac{5}{8}z^3 + \ldots\right), \tag{17}$$

$$d_L(z) = \frac{c}{H_0}\left(z + \frac{1}{4}z^2 - \frac{1}{8}z^3 + \ldots\right), \tag{18}$$

which shows that all four distances above reduce to the same expression to first order.

The number count in this cosmology is well known, producing the following expression,

$$N^{\rm EdS} = \alpha\left(\frac{1+z-\sqrt{1+z}}{1+z}\right)^3, \tag{19}$$

where the dimensionless constant α is defined to be,

$$\alpha = \frac{4c^3}{H_0 M_g G}. \tag{20}$$

Here M_g is the average galactic rest mass ($\sim 10^{11}\ M_\odot$) and G is the gravitational constant. The differential densities in this cosmology are given by the following equations,

$$\gamma_A^{\rm EdS} = \mu_0\left[\frac{(1+z)^3}{(3-2\sqrt{1+z})}\right], \tag{21}$$

$$\gamma_G^{\rm EdS} = \mu_0, \tag{22}$$

$$\gamma_L^{\rm EdS} = \mu_0\left[\frac{1}{(2\sqrt{1+z}-1)(1+z)^3}\right], \tag{23}$$

$$\gamma_Z^{\rm EdS} = \mu_0\left[\frac{4\,(1+z-\sqrt{1+z})^2}{z^2\,(1+z)^{7/2}}\right], \tag{24}$$

and the integral differential densities yield,

$$\gamma_A^{*\rm EdS} = \mu_0(1+z)^3, \tag{25}$$

$$\gamma_G^{*\rm EdS} = \mu_0, \tag{26}$$

$$\gamma_L^{*\rm EdS} = \mu_0(1+z)^{-3}, \tag{27}$$

$$\gamma_Z^{*\rm EdS} = \mu_0\left[\frac{2(1+z-\sqrt{1+z})}{z(1+z)}\right]^3, \tag{28}$$

where the constant μ_0 is given as below,

$$\mu_0 = \frac{3{H_0}^2}{8\pi M_g G}. \tag{29}$$

Taylor series expansions for the differential densities above are given as follows,

$$\gamma_A^{\rm EdS} = \mu_0\left(1 + 4z + \frac{27}{4}z^2 + \frac{55}{8}z^3 + \ldots\right) \tag{30}$$

$$\gamma_L^{\rm EdS} = \mu_0\left(1 - 4z + \frac{41}{4}z^2 - \frac{171}{8}z^3 + \ldots\right) \tag{31}$$

$$\gamma_Z^{\rm EdS} = \mu_0\left(1 - 3z + \frac{95}{16}z^2 - \frac{39}{4}z^3 + \ldots\right) \tag{32}$$

$$\gamma_A^{*\rm EdS} = \mu_0\left(1 + 3z + 3z^2 + z^3\right) \tag{33}$$

$$\gamma_L^{*\rm EdS} = \mu_0\left(1 - 3z + 6z^2 - 10z^3 + \ldots\right) \tag{34}$$

$$\gamma_Z^{*\rm EdS} = \mu_0\left(1 - \frac{9}{4}z + \frac{57}{16}z^2 - \frac{39}{8}z^3 + \ldots\right). \tag{35}$$

We note that all densities above have a non-vanishing zeroth order term, whereas the series expansions for the cosmological distances, given by Eqs. (16)–(18), do not contain this term. This implies that, according to Eqs. (30)–(35), deviations from a constant density value, that is, from OH (see below), occur in the first-order terms of the series, whereas Eqs. (16)–(18) imply that deviations from the Hubble law occur in the second-order terms. In other words, Hubble-law deviations occur in higher redshift ranges than deviations from OH. This was first noticed in Ribeiro (1995), discussed again in R01b and explored further in Abdalla et al. (2001) by means of a simple perturbed model.

As discussed at length in R01b, OH corresponds to a constant value of observational average density, that is, when this average density is calculated along the chosen spacetime's past null cone. This requirement was operationally defined in R05 and A07 to mean the following condition,

$$\gamma_i^* = \text{const.}, \qquad \text{(OH definition)}. \tag{36}$$

[6] In EdS cosmology, the comoving distance is equal to the galaxy area distance d_G multiplied by a constant factor (see R05).

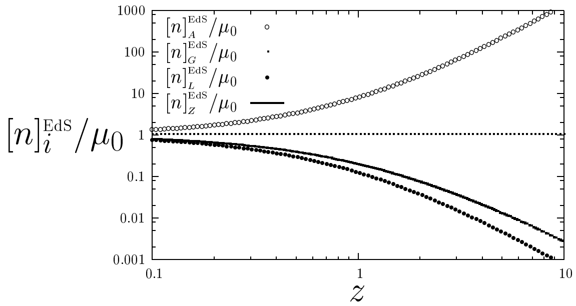

Fig. 1. Plot similar to the one appearing in R05 showing the normalized radial number densities $[n]_i$, or integral differential densities γ_i^* (see Eq. (9)), along the past light cone versus the redshift z in the EdS cosmology. Note that although all cases are spatially-homogeneous by construction, only $[n]_{\rm G}^{\rm EdS}$ is observationally homogeneous as well. The other three radial number densities are spatially homogeneous, but observationally *in*homogeneous. The asymptotic limits for these densities are also different (see R01b), yielding, $\lim\limits_{z\to\infty}[n]_{\rm L}^{\rm EdS}=0$, $\lim\limits_{z\to\infty}[n]_{\rm A}^{\rm EdS}=\infty$, $\lim\limits_{z\to\infty}[n]_{\rm G}^{\rm EdS}=\mu_0$, $\lim\limits_{z\to\infty}[n]_z^{\rm EdS}=0$.

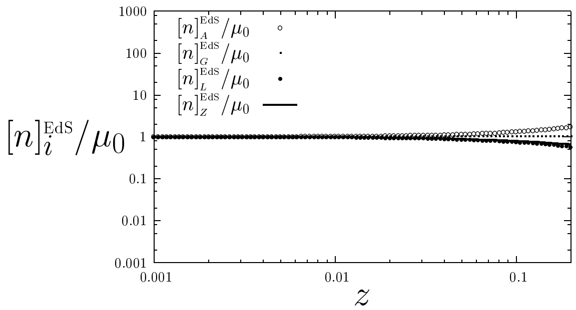

Fig. 2. This figure shows the normalized radial number densities $[n]_i$ of the previous graph, but at a redshift range of up to $z=0.2$. As noted in Ribeiro (1995), the distinction between the densities built with different distance measures can be seen before the redshift reaches $z=0.1$.

It is clear from a simple inspection of Eqs. (25)–(28) that although EdS cosmology is SH by construction for *all* distance measures, according to the definition provided by Eq. (36) the conditions of OH are met only when one builds an average density using the galaxy area distance $d_{\rm G}$ (see Eq. (26)). The three other distance definitions, $d_{\rm A}$, $d_{\rm L}$, and $d_{\rm Z}$ are apparently unsuitable as tools for searching the possible OH in the galaxy redshift data, if one adopts the EdS cosmology. This was the main conclusion reached by R05, which was also found to be valid for open FLRW cosmology in A07. This conclusion, reproduced here to compare with the analysis below, is graphically summarized in Figs. 1 and 2.

4. Models with or without observational homogeneity

We now adopt the generalized number-distance relation provided by Eq. (6) to obtain more general models, whose main properties have or do not have OH. As discussed above, we shall use Eq. (6) to simulate possible matter distributions and test whether or not an assumed OH number count distribution produces SH, or if a distribution with no OH produces, or not, models with no SH.

Considering the definitions given by Eqs. (3) and (4), it is straightforward to conclude that the generalized number-distance relation provided by Eq. (6) corresponds to the following expressions for the differential density given by Eq. (7) and the integral differential density given by Eq. (8),

$$ {}^D\gamma_i\,(d_i) = \frac{DB^D}{4\pi} d_i^{D-3}, \tag{37}$$

$$ {}^D\gamma_i^*\,(d_i) = \frac{3B^D}{4\pi} d_i^{D-3}. \tag{38}$$

From these results, it becomes obvious that these two densities are related by the following expression,

$$ {}^D\gamma_i^* = \frac{3}{D}\,{}^D\gamma_i. \tag{39}$$

To proceed with our analysis, it is unavoidable at this stage to choose a cosmological model (see Sect. 2 above). Our choice then is to continue using an EdS cosmology, due to its simplicity, for everything besides number counts. However, our results can be extended to other FLRW spacetimes, or even to non FLRW cosmologies.

We substitute the four EdS distance definitions, given by Eqs. (12)–(15) into Eq. (6). The results may be written as follows,

$$ {}^D N_{\rm A}(z) = \left(\frac{2cB}{H_0}\right)^D \left[\frac{1+z-\sqrt{1+z}}{(1+z)^2}\right]^D, \tag{40}$$

$$ {}^D N_{\rm G}(z) = \left(\frac{2cB}{H_0}\right)^D \left(\frac{1+z-\sqrt{1+z}}{1+z}\right)^D, \tag{41}$$

$$ {}^D N_{\rm L}(z) = \left(\frac{2cB}{H_0}\right)^D \left(1+z-\sqrt{1+z}\right)^D, \tag{42}$$

$$ {}^D N_{\rm Z}(z) = \left(\frac{2cB}{H_0}\right)^D \left(\frac{z}{2}\right)^D. \tag{43}$$

If we now substitute these EdS distance measures into Eq. (37), we may write the expressions for the differential densities as shown below,

$$ {}^D\gamma_{\rm A}(z) = \left(\frac{DH_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left[\frac{1+z-\sqrt{1+z}}{(1+z)^2}\right]^{D-3}, \tag{44}$$

$$ {}^D\gamma_{\rm G}(z) = \left(\frac{DH_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(\frac{1+z-\sqrt{1+z}}{1+z}\right)^{D-3}, \tag{45}$$

$$ {}^D\gamma_{\rm L}(z) = \left(\frac{DH_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(1+z-\sqrt{1+z}\right)^{D-3}, \tag{46}$$

$$ {}^D\gamma_{\rm Z}(z) = \left(\frac{DH_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(\frac{z}{2}\right)^{D-3}. \tag{47}$$

For the integral differential density given by Eq. (38), the expression in Eq. (39) allows us to write the results as below,

$$ {}^D\gamma_{\rm A}^*\,(z) = \left(\frac{3H_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left[\frac{1+z-\sqrt{1+z}}{(1+z)^2}\right]^{D-3}, \tag{48}$$

$$ {}^D\gamma_{\rm G}^*\,(z) = \left(\frac{3H_0{}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(\frac{1+z-\sqrt{1+z}}{1+z}\right)^{D-3}, \tag{49}$$

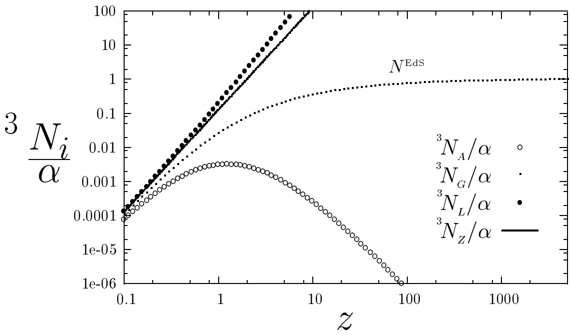

Fig. 3. Number counts for the case $D = 3$ in Eqs. (40)–(43). By definition these counts are *all* observationally-homogeneous in *all* values of z, as can be seen from Eq. (38). The four expressions start similarly, that is, close to the spatially-homogeneous case ${}^3N_{\mathrm{G}} = N^{\mathrm{EdS}}$, but as z increases deviations start to occur. At $z = 0.5$ these deviations are significant. The only expression that exhibits both OH and SH is the one obtained from the galaxy area distance d_{G}. Since ${}^3N_{\mathrm{G}}(z)$ and ${}^3N_{\mathrm{L}}(z)$ have higher counts than N^{EdS} they should be spatially *in*homogeneous. The same is also true for ${}^3N_{\mathrm{A}}(z)$, although in this case the counts are less than the EdS one with SH. One can also easily verify that in the asymptotic limit of the big bang singularity hypersurface the following results hold, $\lim_{z\to\infty} {}^3N_{\mathrm{A}} = 0$, $\lim_{z\to\infty} {}^3N_{\mathrm{G}} = \alpha$, $\lim_{z\to\infty} {}^3N_{\mathrm{L}} = \infty$, $\lim_{z\to\infty} {}^3N_{\mathrm{Z}} = \infty$.

$$ {}^D\gamma^*_{\mathrm{L}}\,(z) = \left(\frac{3{H_0}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(1 + z - \sqrt{1+z}\right)^{D-3}, \tag{50} $$

$$ {}^D\gamma^*_{\mathrm{Z}}\,(z) = \left(\frac{3{H_0}^3}{32\pi c^3}\right)\left(\frac{2cB}{H_0}\right)^D \left(\frac{z}{2}\right)^{D-3}. \tag{51} $$

When $D = 3$, the definition (36) is fulfilled in view of Eqs. (37) and (38) above and, therefore, this choice of dimension clearly corresponds to OH. From this follows an interesting result. We saw in Sect. 3 that densities constructed using the galaxy area distance d_{G} in an EdS cosmology produces a model having both SH and OH. This property allows us to find the constant B. For $D = 3$, we can equate Eq. (26) to Eq. (49) and, considering Eq. (29), we obtain the following result,

$$ B = \left(\frac{{H_0}^2}{2M_{\mathrm{g}}G}\right)^{1/3}. \tag{52} $$

This is valid as long as the geometrical part of the model is given by EdS spacetime. Thus, each adopted metric corresponds to a different value of the constant B, whose dimension is [length unit]$^{-1}$.

4.1. Case of D = 3

As seen above, this is the condition for the existence of OH and holds for *all* distance definitions (see Eq. (38)). This case reduces Eq. (39) to the following simple expression,

$$ {}^3\gamma^*_i \;=\; {}^3\gamma_i = \frac{3{H_0}^2}{8\pi M_{\mathrm{g}}G}. \tag{53} $$

Expressions for the number counts ${}^3N_{\mathrm{A}}(z)$, ${}^3N_{\mathrm{G}}(z)$, ${}^3N_{\mathrm{L}}(z)$, and ${}^3N_{\mathrm{Z}}(z)$ are respectively obtained from Eqs. (40)–(43). Recalling Eq. (19), we verify that ${}^3N_{\mathrm{G}}(z) = N^{\mathrm{EdS}}(z)$. These functions are plotted in Figs. 3 and 4, where the caption of the former Figure discusses that only ${}^3N_{\mathrm{G}}$ shows both SH and OH, although all number-counting functions are observationally-homogeneous by construction.

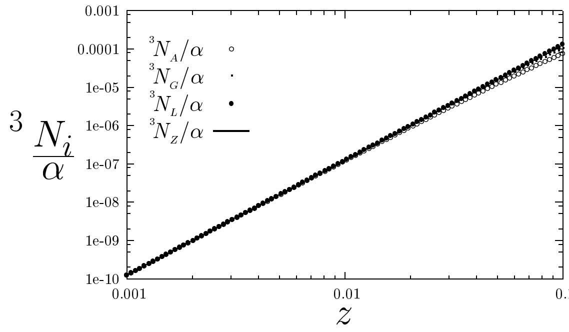

Fig. 4. Graph also showing the number counts for the case $D = 3$ in Eqs. (40)–(43), but at small redshifts compared to the previous figure. Clearly only at $z \approx 0.1$, the differences between the number counts constructed with the various distance definitions can be seen.

4.2. Case of D = 2

We have observational *in*homogeneity by construction for any $D < 3$. Since Sylos Labini et al. (1998) reported a value close to $D = 2$ for the fractal dimension of the distribution of galaxies, this however provides a suitable choice for our toy model. By definition, this choice implies no SH and, therefore, it might be argued that the Friedmann models, as well as their distance definitions, are not valid for any $D < 3$. We emphasize that our aim is not to validate completely the standard cosmology, but to determine an unambiguous answer to the following question. *If* the number counting produced by assuming $D = 2$ in Eq. (6) *corresponded to true observations*, can we conclude, by applying a FLRW framework, that the galaxy distribution does not follow the standard cosmology? In other words, using galaxy counts simulated by Eq. (6) with $D = 2$, is it possible by employing procedures based on the standard cosmological model to conclude with certainty that this number count could not possibly be an observationally-homogeneous galaxy distribution? This is a very important question as this approach was taken by many studies found in the literature, that is, the framework given by the standard cosmology was used to determine if the observed data was consistent with this model. Although most who carry out these studies implicitly assume that this is possible, as we show below, our results indicate that ambiguities in interpreting observations within the standard model framework still remain. As noted by Joyce et al. (2000), *straightforward interpretations of FLRW standard cosmologies are problematic; they include the interpretation that the isotropy of microwave background radiation implies that observationally-inhomogeneous matter distributions are impossible.*

We proceed and assume that $D = 2$ in Eqs. (40) to (43). Considering the definitions provided in Eqs. (20) and (52), we obtain the following results,

$$ {}^2N_{\mathrm{A}}(z) = \alpha^{2/3}\left[\frac{1 + z - \sqrt{1+z}}{(1+z)^2}\right]^2, \tag{54} $$

$$ {}^2N_{\mathrm{G}}(z) = \alpha^{2/3}\left(\frac{1 + z - \sqrt{1+z}}{1+z}\right)^2, \tag{55} $$

$$ {}^2N_{\mathrm{L}}(z) = \alpha^{2/3}\left(1 + z - \sqrt{1+z}\right)^2, \tag{56} $$

$$ {}^2N_{\mathrm{Z}}(z) = \alpha^{2/3}\left(\frac{z}{2}\right)^2. \tag{57} $$

As can be seen in Fig. 5, the curves are similar to those in Fig. 3, apart from the scale. Although ${}^3N_{\mathrm{G}}(z) \neq {}^2N_{\mathrm{G}}(z)$, both functions

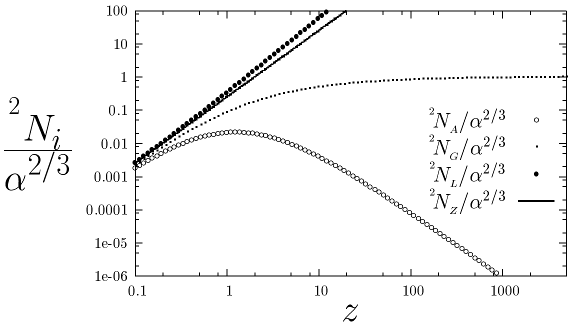

Fig. 5. Number counts for the case of $D = 2$. The functions show a behavior similar to those shown in Fig. 3. We note that $^2N_{\rm G}$ tends to a constant value at its asymptotic limit. Indeed, the limits of the four functions are as follows, $\lim_{z\to\infty} {}^2N_{\rm A} = 0$, $\lim_{z\to\infty} {}^2N_{\rm G} = \alpha^{2/3}$, $\lim_{z\to\infty} {}^2N_{\rm L} = \infty$, and $\lim_{z\to\infty} {}^2N_{\rm Z} = \infty$.

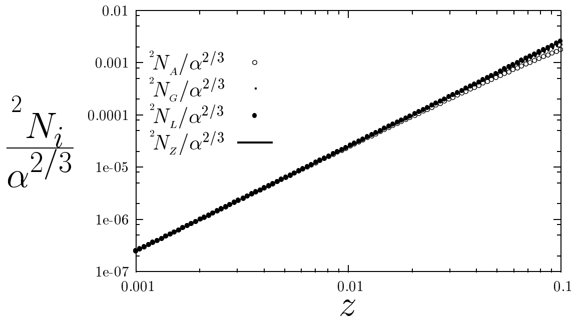

Fig. 6. Number counts for the case where $D = 2$ at very small redshifts. In a similar way to the graph shown in Fig. 4, only at $z \approx 0.1$ the various distance measures begin to affect the number counts. This range is, nevertheless, dependent on the cosmological model, which means that if another cosmology is adopted, deviations caused by the use of different distance measures could possibly occur at redshifts smaller than $z = 0.1$.

converge to a constant value in their asymptotic limits. As we show below, this result has an interesting consequence.

The integral differential densities are calculated using Eqs. (48) to (52), to be,

$$^2\gamma^*_{\rm A}(z) = \mu_1 \left[\frac{1+z-\sqrt{1+z}}{(1+z)^2} \right]^{-1}, \quad (58)$$

$$^2\gamma^*_{\rm G}(z) = \mu_1 \left(\frac{1+z-\sqrt{1+z}}{1+z} \right)^{-1}, \quad (59)$$

$$^2\gamma^*_{\rm L}(z) = \mu_1 \left(1+z-\sqrt{1+z} \right)^{-1}, \quad (60)$$

$$^2\gamma^*_{\rm Z}(z) = \mu_1 \left(\frac{z}{2} \right)^{-1}. \quad (61)$$

where,

$$\mu_1 = \frac{3{H_0}^{7/3}}{8\pi c (2 M_{\rm g} G)^{2/3}}. \quad (62)$$

We obtain the following power-series expansions of these expressions,

$$^2\gamma^*_{\rm A}(z) = \mu_1 \left(\frac{2}{z} + \frac{7}{2} + \frac{11}{8}z - \frac{1}{16}z^2 + \ldots \right), \quad (63)$$

$$^2\gamma^*_{\rm G}(z) = \mu_1 \left(\frac{2}{z} + \frac{3}{2} - \frac{1}{8}z + \frac{1}{16}z^2 - \ldots \right), \quad (64)$$

$$^2\gamma^*_{\rm L}(z) = \mu_1 \left(\frac{2}{z} - \frac{1}{2} + \frac{3}{8}z - \frac{5}{16}z^2 + \ldots \right). \quad (65)$$

We note that for small redshifts the first terms of the series above dominate. Considering the result provided by Eq. (61), we therefore have that at small redshifts $^2\gamma^*_{\rm A}$, $^2\gamma^*_{\rm G}$ and $^2\gamma^*_{\rm L}$ become equal to $^2\gamma^*_{\rm Z}$.

The functions above are plotted versus the redshift in Figs. 7 and 8. We can see clearly in Fig. 7 that, although we started with an observationally-*in*homogeneous model, if we use the galaxy area distance $d_{\rm G}$, the density, as a function of redshift, is constant for $z > 10$. In other words, *even a model that is observationally-inhomogeneous by construction appears to become observationally-homogeneous at higher z, if the density is built using the appropriate distance measure, in this case* $d_{\rm G}$. Both $d_{\rm L}$ and $d_{\rm Z}$ produce observational inhomogeneity for any z, which is reproduced as a power law decay. The density constructed with the area distance $d_{\rm A}$ has an odd behavior while starting to decay as a power law at small redshifts, but at $z \approx 1$ this decay turns into an increase.

It is clear from these results that developing an observationally-inhomogeneous density that decreases as z increases is no guarantee that it will remain so for all z. Clearly the use of the various distance measures creates too many ambiguities, which prevent definitive conclusions being made about the behavior of the large-scale galaxy distribution in the Universe[7].

5. Behavior of the dimension *D* in a spatially homogeneous case

We have so far assumed constant values for the dimension D. However, it is interesting to study the behavior of this dimension as one approaches the Big Bang singularity hypersurface. Inasmuch as it is well known that at those very early times curvature effects become negligible, studying the behavior of D as $z \to \infty$ in the EdS model should shed some light on its general behavior. To do so, we proceed as follows.

In the EdS model, the generalized number-distance relation given by Eq. (6) may be written as below if we consider the number counting given by Eq. (19),

$$(B\, d_i)^D = \alpha \left(\frac{1+z-\sqrt{1+z}}{1+z} \right)^3. \quad (66)$$

These are in fact four equations, one for each distance definition indicated by the index $i = $ A, G, L, Z. We define the EdS distance measures as follows,

$$d_i(z) = \frac{2c}{H_0} f_i(z), \quad (67)$$

where $f_i(z)$ is a function given by each distance definition provided in Eqs. (12)–(15). Considering the definitions provided

[7] We note that a further source of ambiguity in defining OH is that here it is defined in terms of a density that remains unchanged when the generic *distance* d_i changes, whereas we might possibly conceive a definition of OH in terms of densities that remain unchanged for different values of the *redshift*.

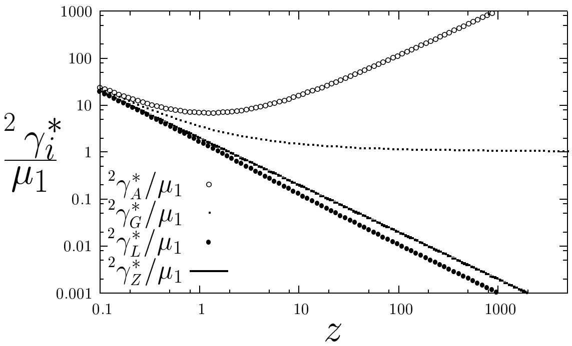

Fig. 7. Graph of the integral differential densities versus the redshift for the case of $D = 2$. It is easy to show that the following asymptotic limits hold, $\lim_{z\to\infty} {}^2\gamma^*_{\rm A} = \infty$, $\lim_{z\to\infty} {}^2\gamma^*_{\rm G} = \mu_1$, $\lim_{z\to\infty} {}^2\gamma^*_{\rm L} = 0$, and $\lim_{z\to\infty} {}^2\gamma^*_{Z} = 0$. All densities have a power-law decay at small redshifts. However, at $z \approx 1$ the densities constructed with the redshift and luminosity distances continue to decay, whereas ${}^2\gamma^*_{\rm A}$ begins to change from a decay to an increasing behavior. More interestingly, the density constructed with the galaxy area distance ${}^2\gamma^*_{\rm G}$ begins to change from a power-law decay to a constant value at $z \approx 2.5$. This is a consequence of the fact that the number counting constructed with $d_{\rm G}$ becomes constant at higher values of z (see Fig. 5). Since ${}^3N_{\rm G}$ tends to a constant value as the redshift increases, and since ${}^3N_{\rm G} = N^{\rm EdS}$ (see Fig. 3), this suggests that ${}^2\gamma^*_{\rm G}$ becomes spatially *homogeneous* for $z > 10$. This occurs despite the fact that this density is observationally *in*homogeneous by construction. This result therefore suggests that for $D = 2$ the integral differential density constructed with the galaxy area distance $d_{\rm G}$ is *not* observationally and spatially homogeneous for $z < 10$, but it seems to turn into both for $z > 10$. This is a simple and clear example showing that the use of different distance measures in the characterization of cosmological densities may lead to significant ambiguities in reaching conclusions about the behavior of the large-scale galaxy distribution.

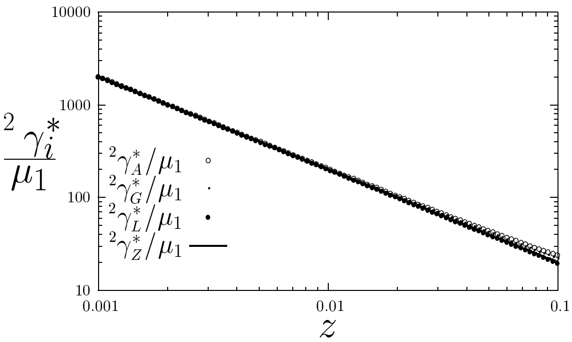

Fig. 8. Graph of the integral differential densities versus the redshift for the case of $D = 2$ at very small redshifts. Only when $z \approx 0.1$, this density starts to be affected by the different cosmological distance definitions.

by Eqs. (20) and (52), the four equations given by the expression (66) may be rewritten in terms of the dimension D, as can be seen below,

$$D_i = 3\left(\frac{\ln\alpha^{1/3} + \ln f_{\rm G}}{\ln\alpha^{1/3} + \ln f_i}\right). \tag{68}$$

The following result comes directly from this Eq. (68),

$$D_{\rm G} = 3. \tag{69}$$

This value for the fractal dimension should not come as a surprise since the density defined with the galaxy area distance $d_{\rm G}$ remains constant, that is, observationally homogeneous for all z in the EdS cosmology (see Fig. 1). The other three expressions for the dimension D in each distance measure yield,

$$D_{\rm A} = 3\left\{\frac{\ln\alpha^{1/3} + \ln\left(\dfrac{1+z-\sqrt{1+z}}{1+z}\right)}{\ln\alpha^{1/3} + \ln\left[\dfrac{1+z-\sqrt{1+z}}{(1+z)^2}\right]}\right\}, \tag{70}$$

$$D_{\rm L} = 3\left[\frac{\ln\alpha^{1/3} + \ln\left(\dfrac{1+z-\sqrt{1+z}}{1+z}\right)}{\ln\alpha^{1/3} + \ln\left(1+z-\sqrt{1+z}\right)}\right], \tag{71}$$

$$D_{Z} = 3\left[\frac{\ln\alpha^{1/3} + \ln\left(\dfrac{1+z-\sqrt{1+z}}{1+z}\right)}{\ln\alpha^{1/3} + \ln\left(\dfrac{z}{2}\right)}\right]. \tag{72}$$

Series expansions for the expressions above may be written as follows,

$$D_{\rm A} = \left[3 + \left(\frac{3z}{\ln\alpha^{1/3} - \ln 2 + \ln z}\right) + \ldots\right], \tag{73}$$

$$D_{\rm L} = \left[3 - \left(\frac{3z}{\ln\alpha^{1/3} - \ln 2 + \ln z}\right) + \ldots\right], \tag{74}$$

$$D_{Z} = \left[3 - \left(\frac{3}{4}\right)\left(\frac{3z}{\ln\alpha^{1/3} - \ln 2 + \ln z}\right) + \ldots\right]. \tag{75}$$

Clearly these results are valid for small, but nonzero, values of the redshift. However, we show that functions given by the Eqs. (70)–(72) converge as the redshift vanishes. In fact, we have the following results,

$$\lim_{z\to 0} D_{\rm A} = 3, \tag{76}$$

$$\lim_{z\to 0} D_{\rm L} = 3, \tag{77}$$

$$\lim_{z\to 0} D_{Z} = 3. \tag{78}$$

Again, these results are not surprising since, according to Fig. 1, all densities tend to OH at very small redshifts. The asymptotic limits of these functions at the Big Bang are also found easily, yielding,

$$\lim_{z\to\infty} D_{\rm A} = 0, \tag{79}$$

$$\lim_{z\to\infty} D_{\rm L} = 0, \tag{80}$$

$$\lim_{z\to\infty} D_{Z} = 0. \tag{81}$$

We compare these results with Eq. (69) and conclude that the dimension D constructed with the area distance $d_{\rm A}$, luminosity distance $d_{\rm L}$, and redshift distance d_Z corresponds to a vanishing fractal dimension at the Big Bang, whereas that constructed with the galaxy area distance $d_{\rm G}$ (or comoving distance) produce a finite nonzero dimension at the Big Bang singularity hypersurface. That indicates that the ambiguities arise when one considers all distance measures in cosmology, in view of the fact that the value of the fractal dimension at the Big Bang depends on the chosen cosmological distance definition. These results are graphically shown in Figs. 9 and 10.

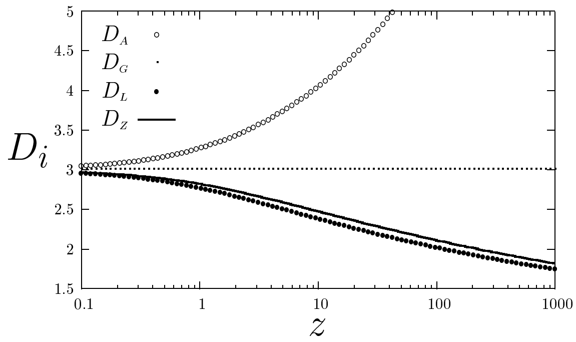

Fig. 9. Graph of the dimension D defined in terms of the four distance measures d_A, d_G, d_L, and d_Z and plotted against the redshift in the spatially-homogeneous EdS cosmological model. Assuming $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $M_g = 10^{11}\ M_\odot$, we have $\ln \alpha^{1/3} = 9.6353$. For $z \ll 1$, all dimensions are equal to 3 (see Fig. 10 and Eqs. (69), (76)–(78)). For higher values of z, both D_L and D_Z decrease steadily and vanish as $z \to \infty$. The dimension D_A constructed with the area distance d_A shows an odd behavior, initially increasing very rapidly well above 3 for $z > 0.1$. However, according to Eq. (79), it eventually vanishes at the Big Bang singularity hypersurface which implies that this function must experience dramatic changes. Indeed, it is discontinuous at $z \approx 15\,170$, changing to negative values that increase towards zero. D_G remains constant for all redshifts. This plot is a different way of presenting the results in Fig. 1. It is clear from this graph that although the EdS cosmology is spatially homogeneous, it may or may not be observationally homogeneous depending on the distance measure adopted for analyzing the behavior of the density in this model.

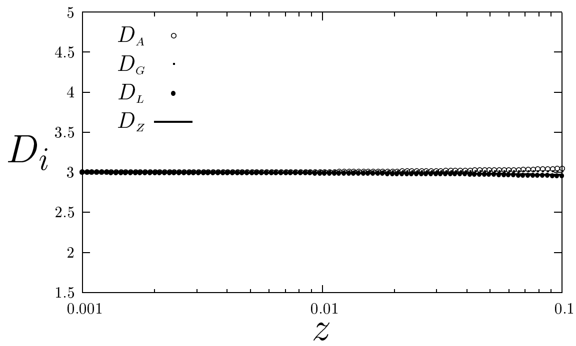

Fig. 10. Graph of the dimension D versus the redshift at very small values of z. Note that only when $z \approx 0.1$ that deviations from the constant values $D = 3$ start to appear due to the definition of $D(z)$ in terms of the four distance measures.

6. Number counts and magnitudes

The results discussed in the previous sections show that if we use the framework of the EdS cosmology, only when we have $z > 0.1$ it does become theoretically possible to detect the distinction between SH and OH. If we depart from EdS cosmology and use, for instance, an open FLRW model the redshift ranges where this distinction becomes detectable could also change. Indeed, by means of differential densities calculated from the CNOC2 galaxy redshift survey data in the range $0.05 \le z \le 1$ in a FLRW model with $\Omega_{m_0} = 0.3$ and $\Omega_{\Lambda_0} = 0.7$, we are able to detect such a distinction for $z \approx 0.1$ (see A07) but, to do so we need to obtain indirectly the number counts by employing a method capable of calculating them from the galaxy luminosity function (LF). Presently the LF is evaluated from detailed observations of the apparent magnitude of galaxies and such magnitude measurements can go to redshifts much higher than the unity. To extract number counts from the LF, we require a relativistic theory connecting the theoretical aspects of a cosmological model with the usual procedures carried out by astronomers when quantifying galaxy catalogues. Ribeiro & Stoeger (2003) developed such a method, A07 applied to the CNOC2 galaxy survey, and Iribarrem et al. (2008, in preparation) applied to the FORS Deep Field redshift survey. These articles dealt with issues such as source evolution and K-correction (see also Ribeiro 2002) and the reader interested in those topics is referred to these articles since it is beyond the scope of this work to present a detailed discussion of these issues.

Despite this, a simpler discussion about the relationship between number counts and magnitudes can be presented in the context of this paper. We define the bolometric *apparent magnitude* m to be given by the following equation,

$$m = -2.5 \log \left(\frac{L}{4\pi {d_L}^2} \right) + \text{const.}, \tag{82}$$

where L is the intrinsic bolometric luminosity of a cosmological source, assumed point like (Ribeiro 2002), and the constant is due to the calibration of the magnitude system. Since by definition the bolometric *absolute magnitude* M is defined to be the apparent magnitude of a source located at a distance of 10 pc, the *distance modulus* is defined as follows,

$$\mu \equiv m - M = 5 \log d_L + 25. \tag{83}$$

In this equation, the luminosity distance is measured in Mpc. Thus, if astronomical observations provide measurements of the pair $(m; z)$ for cosmological sources, which is common for sources of redshifts higher than unity, then, by assuming a cosmological model, we have the function $d_L = d_L(z)$ from which we can calculate L using Eq. (82) and then derive M and μ for these sources. Therefore, the distance modulus is another way of representing observed magnitudes and redshifts of cosmological sources.

Our aim is to relate the number counts given for each distance adopted in this paper to the distance modulus. This can be done if we use the reciprocity theorem provided by Eq. (1) to write ${}^{D}\!N_A$, ${}^{D}\!N_G$, and ${}^{D}\!N_L$, as given by Eq. (6), in terms of the luminosity distance and then use Eq. (83) to write the final expressions in terms of the distance modulus. The results may be written as follows,

$${}^{D}\!N_A = \left[B \frac{10^{0.2(\mu-25)}}{(1+z)^2} \right]^D, \tag{84}$$

$${}^{D}\!N_G = \left[B \frac{10^{0.2(\mu-25)}}{(1+z)} \right]^D, \tag{85}$$

$${}^{D}\!N_L = \left[B\, 10^{0.2(\mu-25)} \right]^D, \tag{86}$$

$${}^{D}\!N_Z = \left[\frac{cB}{H_0} z \right]^D, \tag{87}$$

where the last equation is the result of simply substituting the definition given by Eq. (2) for the redshift distance, directly into Eq. (6) when taking $i = z$.

To express the number counts above only in terms of the distance modulus, we require the function $z = z(\mu)$ and to derive it we need to adopt a cosmological model as we did in the previous sections. In terms of observations, a cosmological model is required from the beginning of this approach, otherwise it would

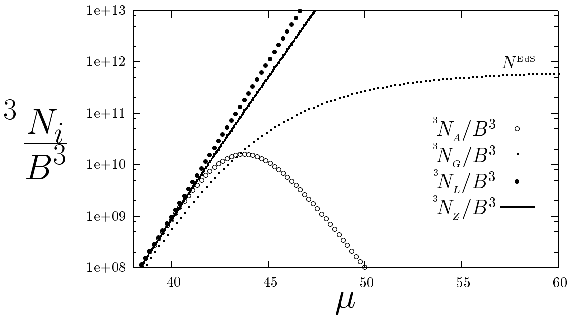

Fig. 11. This graph shows number counts versus distance modulus for the case $D = 3$. The constant B is given by Eq. (6) which, for the EdS cosmology, turns out to be equal to Eq. (52). The results are very similar to the ones shown in Fig. 3, meaning that the same conclusions reached there apply here.

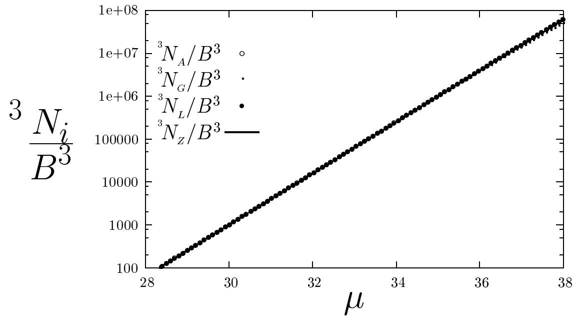

Fig. 12. This is the same plot shown in Fig. 11, but with a distance modulus range equivalent to small redshifts interval.

be impossible to find absolute magnitudes and, therefore, the distance moduli. As in previous sections, we adopt the EdS cosmology, which corresponds to using the luminosity distance as given in Eq. (14). Recalling that $d_{\rm L} = 0$ when $z = 0$ Eq. (14) can be inverted to produce $z = z(d_{\rm L})$ and, after considering Eq. (83), we finally obtain the function $z(\mu)$ given by,

$$1 + z = \frac{1}{2} + \frac{H_0}{2c} 10^{0.2(\mu-25)} + \sqrt{\frac{H_0}{2c} 10^{0.2(\mu-25)} + \frac{1}{4}}\,. \qquad (88)$$

We note that Eqs. (14) and (83) imply that the small redshifts interval $0.001 \le z \le 0.1$ corresponds to $4.3\,\text{Mpc} \le d_{\rm L} \le 439\,\text{Mpc}$ and $28.2 \le \mu \le 38.2$, if we assume that $H_0 = 70$ km s^{-1} Mpc^{-1}.

For number counts to correspond to OH, we must choose $D = 3$, which implies that the EdS cosmology number counts distribution is derived as in Sect. 4.1, that is, is given by $N^{\rm EdS}(\mu) = {}^3N_{\rm G}(\mu)$. Thus, only the expression ${}^3N_{\rm G}(\mu)$ produces a number count distribution that is both OH and SH whereas the remaining functions ${}^3N_{\rm A}(\mu)$, ${}^3N_{\rm L}(\mu)$, and ${}^3N_Z(\mu)$ are OH, but are not SH. Figures 11 and 12 show graphs of these functions where it is clear that they are similar to those plotted in Figs. 3 and 4. Therefore, although these four expressions are built with a geometrical part consistent with EdS cosmology, this does not imply that the counts will be SH and OH, showing again the ambiguous nature of these expressions as far as "homogeneity" is concerned.

7. The relativity of observational homogeneity

We have seen how the concept of homogeneity applied to cosmological models is prone to ambiguities. Attempting to distinguish between spatial and observational homogeneities is a means of diminishing the ambiguities that, as seen above, have not been eliminated; this is because using various distance measures to calculate cosmological densities is still a source of ambiguity. At this point we recall some well known concepts that may help to clarify the physical interpretation of the effects discussed above.

According to the reciprocity theorem given by Eq. (1), all distance definitions discussed above become equal at $z = 0$. This means that if a signal such as pulses emitted at unit time intervals are emitted at the rest frame of the source and an observer measures the rate of change of the same signal, these rates of change are, by definition, the redshift z. In particular, the observed frequencies ν of light or radio waves are related to z as $1 + z = \nu_{\rm emi}/\nu_{\rm obs}$. We can think of this as a *time dilation* effect (Ellis 1971). So, if a proper time interval dt is observed to elapse between particular signals, then

$$\frac{{\rm d}t_{\rm emitted}}{{\rm d}t_{\rm observed}} = \frac{\nu_{\rm emitted}}{\nu_{\rm observed}} = 1 + z. \qquad (89)$$

This relationship is true regardless of the separation of emitter and observer and implies that the difference between two distance measures in cosmology can be thought to be, in effect, a result of the time dilation between emitter and observer located in different reference frames in relative motion with one another. So, we can consider Eq. (1) to be produced by the relativity of time intervals. Inasmuch as the observable distances discussed in the previous sections are used to build observational densities, then we conclude that the concept of OH as defined in Eq. (36) must also be relative. This means that similarly to the concept of a cosmological distance, we can talk about the relativity of OH and, therefore, we must abandon the notion of a "true", or unique, homogeneity of the observable Universe. In a similar way to the statement of McVittie (1974) about cosmological distances, we conclude that measuring the possible homogeneity of the large-scale structure in the Universe depends on circumstances, that is, on the method of measurement.

The reasoning presented above can then help us to understand the limitations of the generic concept of homogeneity widely used in cosmology. It has its origins in the assumption of the maximal spatial isotropy in the Riemannian spacetime manifold, which then follows, as a mathematical result, that a perfect fluid cosmology metric ends up with its fluid variables (density and pressure) being time dependent only (Stephani et al. 2003, pp. 173, 210–212; Weinberg 1972, pp. 403, 412–415). This means that the local density ρ that appears in the right hand side of Einstein's field equations becomes a function of the time coordinate only. Hence, a spatially-isotropic spacetime is, by mathematical requirement, spatially homogeneous as well. Due to this widely known result, it is usual to refer to the standard FLRW family of cosmological models as being characterized by isotropy and homogeneity. The adjective "spatial" is often dropped from appearing in front of the term homogeneity when the most basic features of the standard cosmology are described (e.g., see Peacock 1999, p. 65).

Such an economy of language could, perhaps, have been thought harmless, but as a side effect it has in practice created a simplistic, but wrong, impression that all types of densities that can be derived in these cosmologies must also be homogeneous, that is, eventually become a constant value. As discussed in the previous sections, in contrast to this simplistic view it is possible to define densities in standard cosmologies that have different types of physical homogeneity[8]. The concept of OH is therefore fundamentally different from the concept of SH and, therefore, it is a misleading use of language to call the standard FLRW family of cosmological models simply isotropic and homogeneous. They are in fact isotropic and spatially homogeneous and either can or cannot be observationally homogeneous as well.

8. Conclusion

In this paper we have presented an analysis of the physical consequences of the distinction between the usual concept of spatial homogeneity, as defined by the Cosmological Principle, and the concept of observational homogeneity. This distinction is based on calculating observational areas, volumes and densities with four cosmological distance measures d_i (i = A, G, L, Z), namely the area distance d_{A}, the galaxy area distance d_{G}, the luminosity distance d_{L}, and the redshift distance d_{Z}. Our aim was to simulate number counts as if they were actual observations. To do so in a general way, we have adopted the generalized number-distance relation $\overset{D}{N}_i = (B\,d_i)^D$ to obtain the differential density ${}^D\gamma_i$ and the integral differential density ${}^D\gamma_i^*$, where the latter is the observed radial number density $\overset{D}{[n]}_i$. In this way, these densities become a function of the fractal dimension D. We then reviewed the results of Ribeiro (2001b, 2005) and Albani et al. (2007), where those equations were applied to the Einstein-de Sitter cosmological model and the open FLRW cosmology with $\Omega_{m_0} = 0.3$, $\Omega_{\Lambda_0} = 0.7$, concluding that a spatially-homogeneous cosmology does not necessarily possess observational homogeneity. These features are only present if the galaxy area distance d_{G}, which in EdS cosmology is equivalent to the comoving distance apart from a constant, is used to calculate both differential densities.

Models with and without observational homogeneity by construction were studied by means of setting $D = 3$ and $D = 2$ respectively in the generalized number-distance relation. It was found that models with $D = 3$ do not seem to remain spatially homogeneous as well. The only exception appears to be when one adopts the galaxy area distance d_{G}. Models with $D = 2$ were developed to be observationally inhomogeneous, although the integral differential density was constructed with the galaxy area distance d_{G}, which when plotted versus redshift shows a power-law decay for $z < 2.5$ that, after a transition, turns into a constant value for $z > 10$. We have also studied the behavior of the dimension D for the spatially-homogeneous EdS cosmology, showing that it tends to $D = 3$ as $z \to 0$ for all distance measures, tends to $D = 0$ as $z \to \infty$ for d_{A}, d_{G}, and d_{L}, but remains $D = 3$ for d_{G} at the Big Bang singularity hypersurface. Finally, we have also studied functions of number counts versus distance modulus with the various distance definitions and reached conclusions similar to models with $D = 3$ and functions of number counts versus redshift. The paper finishes with a conceptual discussion arguing that due to the relativity of time intervals for pulses emitted and observed at different reference frames, and in view of the reciprocity theorem linking various cosmological distances by means of $(1 + z)$ factors, we can conclude that the concept of observational homogeneity should also be relative.

To end this paper, it is important to emphasize that the conceptual distinction discussed above between different types of homogeneity in the standard cosmological model is fundamental and has important consequences for observational cosmology. In view of the fact that such a distinction is not generally recognized in the literature of observational cosmology, it is our opinion that it should be considered by all those who empirically probe the possible observational homogeneity of the large-scale distribution of galaxies in the Universe.

[8] Note that in this paper the term homogeneity was used with a physical meaning related to the average density, which can, in principle, be empirically determined, directly, or indirectly, by means of astronomical observations. Therefore, homogeneity has a wider meaning than the strict mathematical sense of spacetimes admitting isometries due to groups of motions (Stephani et al. 2003, pp. 157, 171).

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