A&A 483, 285–295 (2008)
DOI: 10.1051/0004-6361:20079008

Astronomy & Astrophysics

Wave propagation in steady stratified one-dimensional cylindrical waveguides

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Received 6 November 2007 / Accepted 13 February 2008

ABSTRACT

Aims. This paper studies the propagation of longitudinal magnetic tube waves in a stratified isothermal flux tube with an internal equilibrium background flow.

Methods. The governing differential equation is solved by means of Laplace transforms and temporal and spatial solutions are developed, with boundary conditions given by various footpoint drivers, namely a monochromatic source, a delta function pulse, and a sinusoidal pulse. The effect of the background flow is to introduce an increase in amplitude of the wave perturbation and changes in phase shift when compared with the corresponding static case.

Results. Results are presented and applied to conditions in the solar atmosphere. When the source is driven continuously, the forced atmospheric oscillations are shown to have large percentage differences when compared to the corresponding static case. For the free atmospheric oscillations, percentage increases in amplitude merely a few percent are found and vary greatly in height but are practically unaltered in time. Phase shifts up to a radian are introduced and weakly depend on both height and time.

Conclusions. The results presented in this paper may have interesting observational consequences, especially when using the tools of magnetic seismology of solar atmospheric wave guides (i.e. flux tubes from photosphere to corona) in light of the present and near-future high spatial and temporal resolution space missions, e.g. Hinode, Solar Dynamics Observatory, or Solar Orbiter.

Key words. waves – Sun: atmosphere

1. Introduction

The solar atmosphere is extremely complex, being both highly structured and highly dynamic, supporting a vast array of wave and oscillatory phenomena. Theoretical descriptions of these waves and oscillations have been given fresh vigor in recent years through the detailed observational studies made with the high resolution SOHO (SOlar Heliospheric Observatory) and TRACE (Transition Region And Coronal Explorer) satellites, e.g. Aschwanden (2006), Banerjee et al. (2007), Nakariakov & Verwichte (2005) and references therein. A major question still to be addressed is the exact nature of the coupling between disturbances in the lower layers of the solar atmosphere and their manifestation as wave-like phenomena in higher layers (see e.g. the reviews Erdélyi (2006), De Pontieu et al. 2006). What is the connection between waves observed at coronal temperatures, and waves in the chromosphere and transition region (TR), or even lower to the photosphere? How is this connection affected by the dynamic state of the flux tubes (e.g. flows from photosphere to TR, etc.)?

The concept of an intense flux tube provides an obvious link between photospheric and chromospheric (or even higher) regions, in which the high energy densities associated with the dense photosphere can communicate directly to the considerably more tenuous chromosphere (plasma density in the order of 10^4 smaller) above. A recent study on intensity oscillations in the upper TR above active region plage suggests the potential role of photospheric drivers in so-called moss oscillations. De Pontieu (2003a,b) reported on strong (∼5−15%) intensity oscillations in the upper TR, with periods from 200 s to 600 s persisting typically for 4−7 cycles. They find a general correspondence between solar global acoustic p-modes and the upper TR oscillations. A model noting the link between photospheric drivers and TR oscillations is put forward by De Pontieu et al. (2004). The concept has been extended into the solar corona first by De Pontieu et al. (2005). Similar coupling was reported in sunspot regions, e.g. Marsh & Walsh (2006).

Despite the proposed links, the exact relationship between the TR oscillations and the photospheric p-modes remains unclear. Moreover, many of the upper TR oscillations are associated with upper chromospheric oscillations seen in Hα, i.e. periodic flows in spicular structures. Aspects such as these provide the motivation to study the wave propagation in the lower solar atmosphere (where the effect of gravity is important, since the gravitational scale height is comparable to the characteristic scales of the medium) with the presence of flows. The effect of steady state flows on MHD waves in a uniform magnetic slab-geometry was investigated by e.g. Nakariakov & Roberts (1995), Tirry et al. (1998), Joarder et al. (1997). They found the dispersion relation for such steady states and also have shown the presence of negative energy waves. Joarder et al. (2000), Somasundaram et al. (1999) and Narayanan (1991) generalised the slab studies to flux tubes but their derivation is valid only for limited applications and parameters. A detailed and comprehensive derivation of steady flow effects on uniform MHD waveguides in cylindrical geometry (with stratification due to gravity ignored) can be found in e.g. Terra-Homem et al. (2003). They derived the dispersion relation for uniform photospheric and coronal flux tubes, and determined the propagation windows that are Doppler shifted when compared to their static counterparts. Resonance behaviour of MHD modes in the presence of an equilibrium flow in radially inhomogeneous flux tubes was first examined by e.g. Goossens et al. (1992).

In this paper the focus is on the linear propagation of MHD waves in *stratified*, isothermal flux tubes with a *steady* background flow, and the derivation of the governing equations for linear waves. The governing equation for propagating waves is then solved for various photospheric drivers, namely a monochromatic source (mimicking solar global oscillations), a delta-function type impulse and a wider range impulse. The theoretical description is provided within the framework of the slender flux tube approximation (e.g. Roberts & Webb 1978). The classical problem of acoustic wave propagation in a general stratified atmosphere is well known, motions being governed by a Klein-Gordon type equation e.g. Lamb (1908), Rae & Roberts (1982), Ballai et al. (2006), Taroyan & Erdélyi (2008) leading to the *acoustic cut-off frequency* (ω_a, defined as $\omega_a = c_0/2H$, where c_0 is the sound speed and H is the scale height) and the well known dispersion relation $\omega^2 = k^2c_0^2 + \omega_a^2$. Thus wave propagation only occurs if $\omega > \omega_a$. Frequencies below the acoustic cut-off are therefore evanescent and are unable to transfer significant energy to higher layers in hydrostatically stratified atmospheres. Another important feature of stratification is the presence of an oscillating wake. The atmosphere behind a propagating disturbance is disturbed and oscillates at the cut-off frequency, ω_a. Rae & Roberts (1982) point out the applicability of the Klein-Gordon equation to the case of a magnetic flux tube. The speed of propagation and characteristic frequency are now modified in terms of the tube's elasticity and geometry. Generally the departure from the acoustic case is not large, save for non-adiabatic cases in which cut-off frequencies may be significantly different from the corresponding acoustic values (Roberts 2004; De Pontieu et al. 2004).

Further to the points discussed above, many recent high-resolution imaging observational studies have indicated the existence of flows in the solar atmosphere, in particular in magnetic flux concentrations. The presence of a fast wind originating from the polar coronal holes has been long established (e.g., Watanabe 1975; Gloeckner & Geiss 1998). Recent SOHO and TRACE observations have shown the presence of steady flows in the south polar coronal hole and the equatorial quiet Sun-region (Buchlin & Hassler 2000). The chromosphere also exhibits flows (often termed chromospheric down-flow) associated with thousands of small scale explosions (e.g., Innes et al. 1997; Perez et al. 1999; Teriaca et al. 1999, etc.). Background flows have been noted in arched isolated magnetic flux tubes, steady flows have been observed in slender magnetic flux tubes and in return flows from spicules. Flows cause not just considerable observable Doppler shifts, but can also be responsible for complex dynamical interactions (e.g. flow instabilities, development of boundary layers etc.). These examples and the most recent high cadence and resolution movies released on the Hinode website show the clear need to investigate the effect of such flows on the wave characteristics in the solar atmosphere. In fact one of major discoveries from Hinode is the presence ubiquitous background flow in the solar atmosphere. Since resistivity is relatively diminishing, it may follow from induction equation that the flows follow the magnetic field lines, i.e. the field-aligned equilibrium flow concept is a reasonable initial approximation in modelling efforts.

The paper is organized as follows: Sect. 2 considers a background flow in a stratified elastic tube, in which the governing equation is derived. Section 3 is concerned with the basic formulation of the solution in terms of Laplace transforms. In Sect. 4 solutions for three typical photospheric divers, namely a monochromatic source, a delta function pulse and a sinusoidal (i.e. an approximation of a spatial broadband) pulse are provided. These examples model the building blocks of the complex dynamic motion of the lower boundary of the solar atmosphere (e.g. global acoustic oscillations, photospheric buffeting, etc.). Section 5 applies the results to the Sun, and is finally followed by the Conclusions.

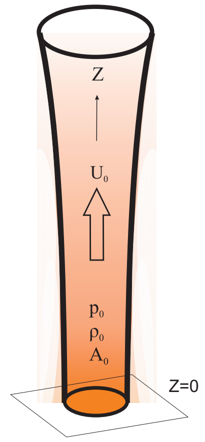

Fig. 1. Equilibrium configuration for the thin tube.

2. Wave propagation in an elastic tube with background flow

An isolated elastic tube with a background flow, embedded in a field free, isothermal and hydrostatic environment is considered. The tube is assumed to be thin, free from twists and with circular cross-section. The tube is aligned vertically, thus the z-axis is the tube axis and gravity acts such that $\boldsymbol{g} = -g\hat{z}$, with $\hat{z}$ being the unit vector in the z direction. The assumption that the atmosphere can be taken as isothermal is clearly a strong simplification, however it allows insight into a complicated physical problem. Thus the application of any results to the lower solar atmosphere needs to be carefully considered, although direct comparison to earlier studies under isothermal conditions, such as Musielak & Ulmschneider (2003), can still be made. The starting point for our theoretical discussion is the non-linear thin flux tube equations for the sausage (longitudinal) mode (e.g. Roberts & Webb 1978)

$$\rho\left(\frac{\partial v}{\partial t} + vv'\right) + p' + \rho g = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right) + \left(\frac{\rho v}{B}\right)' = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + vp' - c^2\left(\frac{\partial \rho}{\partial t} + v\rho'\right) = 0, \tag{3}$$

being equations for momentum, flux conservation and energy, where the primes indicate derivatives in the z-direction.

2.1. Equilibrium of the steady thin flux tube

Consider the equilibrium configuration of the model by letting $U_0(z)$, $B_0(z)$, $\rho_0(z)$ and $p_0(z)$ be the undisturbed quantities of bulk flow, magnetic field, density and pressure respectively, inside the flux tube and with $\partial/\partial t = 0$. The equilibrium configuration of the tube may be seen in Fig. 1 and has previously been considered for isolated arched magnetic flux tubes by Thomas (1988). For a vertical flux tube the flux conservation equation, Eq. (2), may be written

$$\frac{\rho_0'}{\rho_0} + \frac{U_0'}{U_0} = \frac{B_0'}{B_0}, \tag{4}$$

and from Eq. (1)

$$U_0 U_0' = \frac{-p_0'}{\rho_0} - g, \tag{5}$$

which may be integrated directly to give Bernoulli's equation. Assuming the external atmosphere (denoted by subscript e) is hydrostatically stratified such that one has

$$p_e = \rho_e R T_e, \tag{6}$$

$$p_e' = -\rho_e g, \tag{7}$$

and one assumes T_e is constant, thus $p_e = p_e(0)e^{(-z/H)}$ and $\rho_e = \rho_e(0)e^{(-z/H)}$ with $H = RT_e/g = p_e/\rho_e g$ being the scale height. Suppose a very rapid radiative exchange between the tube and the external atmosphere such that $T_0 = T_e$. Thus it is assumed that the bulk flow (but not it's perturbation) in the tube is isothermal such that the sound speed is given by $c_i^2 = RT_e = \text{const.}$, where c_i is a constant isothermal sound speed. Using Eqs. (4) and (5) to eliminate density one obtains

$$\frac{B_0'}{B_0} + \frac{U_0'}{U_0}\left(\frac{U_0^2}{c_i^2} - 1\right) + \frac{g}{c_i^2} = 0. \tag{8}$$

From the required condition of horizontal pressure balance one obtains

$$p_0 + \frac{B_0^2}{2\mu} = p_e, \tag{9}$$

and using Eqs. (5) and (8) to eliminate the magnetic field, one obtains

$$\left(1 - \frac{U_0^2}{c_{Ti}^2}\right)\frac{U_0'}{U_0} = \left(1 - \frac{\rho_e - \rho_0}{\rho_0}\frac{c_i^2}{v_A^2}\right)\frac{g}{c_i^2}, \tag{10}$$

where $c_{Ti}^2 = c_i^2 v_A^2/(c_i^2 + v_A^2)$. A relation describing the magnetic field expansion as a function of height may be formed by eliminating the derivatives of flow from Eqs. (8) and (10), giving

$$\left(1 - \frac{U_0^2}{c_{Ti}^2}\right)\frac{B_0'}{B_0} = \frac{-g}{v_A^2}\left(\frac{\rho_e - \rho_0}{\rho_0}\right)\left(1 - \frac{U_0^2}{c_1^2}\right), \tag{11}$$

where $c_1^2 = c_i^2(\rho_e - \rho_0)/\rho_e$. Using the horizontal pressure balance one can see that

$$\frac{\rho_e - \rho_0}{\rho_0} = \frac{v_A^2}{2c_i^2}, \quad c_1^2 = \frac{c_i^2 v_A^2}{v_A^2 + 2c_i^2}, \tag{12}$$

with $c_1^2 < c_{Ti}^2 < \min(c_i^2, v_A^2)$. Using Eqs. (12) and (10) one obtains

$$\left(1 - \frac{U_0^2}{c_{Ti}^2}\right)\frac{U_0'}{U_0} = \frac{1}{2}\frac{g}{c_i^2}, \tag{13}$$

and with Eq. (11) one arrives at

$$\left(1 - \frac{U_0^2}{c_{Ti}^2}\right)\frac{B_0'}{B_0} = -\left(1 - \frac{U_0^2}{c_1^2}\right)\frac{g}{2c_i^2}. \tag{14}$$

In order to simplify the analysis and facilitate an analytical solution it is assumed that the flow to be small, i.e. $U_0^2 \ll c_1^2 < c_{Ti}^2 < c_T^2$ at all heights, meaning the flow is much smaller than the other characteristic speeds, where $c_T^2 = c_0^2 v_A^2/(c_0^2 + v_A^2)$ with $c_0^2 = \gamma p_0/\rho_0$ Noting that $g/c_i^2 = 1/H$, we find

$$\frac{U_0'}{U_0} = \frac{1}{2H}, \quad \frac{B_0'}{B_0} = \frac{-1}{2H}, \quad \frac{\rho_0'}{\rho_0} = \frac{-1}{H}. \tag{15}$$

Thus there is a flow profile that is exponentially increasing, such that $U_0(z) = U_0(0)e^{z/2H}$. This clearly indicates that caution needs to be exercised when applications are considered. The analysis is carried out under the thin flux tube approximation, such that the magnetic field, described by $B_0(z)$, is axial. It should be noted that $B_0(z)$ is not in any way restricted by the solenoidal condition, but it allows the calculation of horizontal components of magnetic field once the vertical component is known (for more details on this issue see Roberts & Webb 1978).

2.2. Small perturbations about the steady equilibrium

Let us consider small amplitude disturbances about the equilibrium described above and find the linearised forms Eqs. (1)–(3) about the steady equilibrium state to be

$$\rho_0\frac{Du}{Dt} + \rho_0 U_0' u + p' + (U_0 U_0' + g)\rho = 0, \tag{16}$$

$$B_0\frac{D\rho}{Dt} - \rho_0\frac{DB}{Dt} - U_0 B_0\frac{\rho_0'}{\rho_0}\rho + U_0\rho_0\frac{B_0'}{B_0}B$$
$$+\rho_0 B_0\left(\frac{\rho_0'}{\rho_0} - \frac{B_0'}{B_0} + \frac{\partial}{\partial z}\right)u = 0, \tag{17}$$

$$\frac{Dp}{Dt} + p_0' u - c_0^2\left(\frac{D\rho}{Dt} + \rho_0' u\right) + U_0\frac{\rho_0'}{\rho_0}(c_0^2\rho - \gamma p) = 0, \tag{18}$$

$$p + \frac{B_0}{\mu}B = 0, \tag{19}$$

where $D/Dt = \partial/\partial t + U_0\partial/\partial z$ is the material derivative, $c_0^2 = \gamma p_0/\rho_0$ is the sound speed and where the un-subscripted quantities are the perturbed quantities and subscript 0 denote the equilibrium quantities. The magnetic tube is linked to the environment by assuming the pressure perturbation in the environment is negligible, thus it is assumed that the tube is embedded in a quiescent environment, see Roberts & Webb (1978). After Musielak et al. (1989) the following variable changes are introduced

$$p_1 = \frac{p}{\rho_0}, \quad \rho_1 = \frac{\rho}{\rho_0}, \quad B_1 = \frac{B}{B_0}, \tag{20}$$

and by taking the limit of the short wavelength assumption, where $L \ll H$, with L being a characteristic length scale, and the fact that the flow is considered much smaller than the other characteristic speeds at all heights (see Appendix A for details) one obtains

$$\frac{\partial^2 u}{\partial t^2} + 2U_0\frac{\partial^2 u}{\partial t\partial z} - c_T^2\frac{\partial^2 u}{\partial z^2}$$
$$+\frac{c_T^2}{2H}\frac{\partial u}{\partial z} + \frac{c_T^2}{H^2}\left[\frac{c_0^2}{v_A^2}\frac{\gamma - 1}{\gamma^2} + \frac{1}{2} - \frac{1}{2\gamma}\right]u = 0. \tag{21}$$

Taking the substitution

$$u = Q\left(\frac{B_0}{\rho_0}\right)^{1/2}, \tag{22}$$

Eq. (21) reduces to

$$\frac{\partial^2 Q}{\partial t^2} + 2U_0 \frac{\partial^2 Q}{\partial t \partial z} + \frac{U_0}{2H}\frac{\partial Q}{\partial t} - c_{\mathrm{T}}^2 \frac{\partial^2 Q}{\partial z^2} + \Omega_{\mathrm{T}}^2 Q = 0, \tag{23}$$

where

$$\Omega_{\mathrm{T}}^2 = \frac{c_{\mathrm{T}}^2}{H^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{c_0^2}{v_{\mathrm{A}}^2}\frac{\gamma - 1}{\gamma^2}\right), \tag{24}$$

is the longitudinal cutoff frequency. Equation (23) is the governing equation for longitudinal waves in a magnetic flux tube with a background flow subject to the applied conditions.

3. Formulation of solution by means of Laplace transforms

To solve Eq. (23) analytically Laplace transformations in time are considered (adopting the method outlined by e.g. Sutmann et al. 1998). Since the flux tube is an isothermal medium, c_{T} and Ω_{T} are constant. A wave source is introduced into the atmosphere by applying a velocity term at the footpoint of the tube. Thus the relevant boundary condition is given by $V(t)$ at $z = 0$. The atmosphere is therefore semi-bounded and it is assumed there are no initial velocity perturbations other than at the footpoint. Further, it is assumed there are no wave motions at $z = \infty$. Thus one has

$$\lim_{t\to 0,\, z\neq 0} Q(t,z) = 0, \qquad \lim_{t\to 0,\, z\neq 0} \frac{\partial Q}{\partial t} = 0, \tag{25}$$

$$\lim_{z\to 0} Q(t,z) = V(t), \qquad \lim_{z\to\infty} Q(t,z) = 0. \tag{26}$$

Thus the homogeneous governing equation, Eq. (23), is solved, with inhomogeneous boundary conditions, Eq. (26), by means of Laplace transforms. Let $q(s,z)$ be the Laplace transform in time of $Q(t,z)$ and applying the Laplace transform to Eq. (23) one obtains

$$\frac{\partial^2 q(s,z)}{\partial z^2} - \frac{2U_0 s}{c_{\mathrm{T}}^2}\frac{\partial q(s,z)}{\partial z} - \frac{1}{c_{\mathrm{T}}^2}\left(\Omega_{\mathrm{T}}^2 + s^2 + \frac{U_0 s}{2H}\right) q(s,z) = 0. \tag{27}$$

Given the exponential nature of the background flow one finds that the solution to Eq. (27) is non-trivial. In order to find an analytical solution to Eq. (23) let us solve Eq. (27) by means of perturbation methods, e.g. Holmes (1995). The problem is considered to be a regular perturbation problem, such that one can express $q(s,z)$ as

$$q(s,z) = q_0(s,z) + \varepsilon q_1(s,z) + \cdots, \tag{28}$$

where ε is a dimensionless, small parameter. Let $U_0(0)/c_{\mathrm{T}} = \varepsilon \ll 1$. This is a suitable choice as it is assumed that the background flow is strongly sub-sonic and sub-Alfvénic. Let $(\Omega_{\mathrm{T}}^2 + s^2)/c_{\mathrm{T}}^2 \equiv m^2$, $a = 2s/c_{\mathrm{T}}$ and $b = s/2Hc_{\mathrm{T}}$. Thus

$$\frac{\partial^2 q}{\partial z^2} - a\varepsilon \mathrm{e}^{z/2H}\frac{\partial q}{\partial z} - (m^2 + b\varepsilon \mathrm{e}^{z/2H})q = 0. \tag{29}$$

The solution to Eq. (29) may be approximated by (for details see Appendix B)

$$q(s,z) = V_0(s)\mathrm{e}^{-\sqrt{\Omega_{\mathrm{T}}^2+s^2}z/c_{\mathrm{T}}}\left[1 + \lambda s\right], \tag{30}$$

with $\lambda = 2U_0(0)H(\mathrm{e}^{z/2H} - 1)/c_{\mathrm{T}}^2$. After Sutmann et al. (1998) it is known that

$$\mathrm{e}^{-\sqrt{\Omega_{\mathrm{T}}^2+s^2}z/c_{\mathrm{T}}} = \mathrm{e}^{-sz/c_{\mathrm{T}}} - \frac{\Omega_{\mathrm{T}} z}{c_{\mathrm{T}}}\mathcal{L}[W(t,z)], \tag{31}$$

where

$$W(t,z) = \frac{J_1(\Omega_{\mathrm{T}}\sqrt{t^2 - (z/c_{\mathrm{T}})^2})}{\sqrt{t^2 - (z/c_{\mathrm{T}})^2)}}\mathcal{H}(t - z/c_{\mathrm{T}}). \tag{32}$$

Thus using Eq. (30) one has

$$q(s,z) = V_0(s)\left[\mathrm{e}^{-sz/c_{\mathrm{T}}} - \frac{\Omega_{\mathrm{T}} z}{c_{\mathrm{T}}}\mathcal{L}[W(t,z)]\right][1 + \lambda(z)s]\,. \tag{33}$$

Since $\mathcal{L}[\delta'(t)] = s$, using the *second shift theorem* and the *convolution theorem* in turn yields

$$\begin{aligned} q(s,z) \;=\; & \mathcal{L}[V(t - z/c_{\mathrm{T}})\mathcal{H}(t - z/c_{\mathrm{T}})] \\ & - \frac{\Omega_{\mathrm{T}} z}{c_{\mathrm{T}}}\mathcal{L}[V(t) * W(t,z)] \\ & + \lambda\mathcal{L}[(V(t - z/c_{\mathrm{T}})\mathcal{H}(t - z/c_{\mathrm{T}})) * \delta'(t)] \\ & - \frac{\Omega_{\mathrm{T}} z\lambda}{c_{\mathrm{T}}}\mathcal{L}[(V(t) * W(t,z)) * \delta'(t)], \end{aligned} \tag{34}$$

where the prime represents the first derivative of the delta function with respect to z. Equation (34) can be readily inverted to give

$$\begin{aligned} Q(t,z) \;=\; & V(t - z/c_{\mathrm{T}})\mathcal{H}(t - z/c_{\mathrm{T}}) \\ & - \frac{\Omega_{\mathrm{T}} z}{c_{\mathrm{T}}}[V(t) * W(t,z)] \\ & + \lambda[(V(t - z/c_{\mathrm{T}})\mathcal{H}(t - z/c_{\mathrm{T}})) * \delta'(t)] \\ & - \frac{\Omega_{\mathrm{T}} z\lambda}{c_{\mathrm{T}}}[(V(t) * W(t,z)) * \delta'(t)], \end{aligned} \tag{35}$$

being an equation describing the evolution of waves in a magnetic flux tube with the presence of an equilibrium background flow. It should be noted that Eq. (35) has been derived for the scaled velocity $Q(t,z)$, and so the original variable u will grow like $\mathrm{e}^{z/4H}$ in an isothermal atmosphere, corresponding to an e-folding distance of four scale heights. Thus relatively small velocities, e.g. at the photosphere will be greatly amplified in several scale heights, highlighting the likelihood of shocks and other complicating factors coming into play.

4. Atmospheric response to various photospheric drivers

In the previous section a general solution to the steady flow problem in terms of a general imposed velocity function, $V(t)$, was derived. In this section the solution for three specific imposed drivers, namely a monochromatic source, a delta-function pulse and a sinusoidal pulse, is considered. The monochromatic source and the sinusoidal pulse (essentially excitation by a piston like motion) provide a plausible model for the representation of the global solar p-modes or the turbulent motions of the convection zone. A further explanation for the observed wave excitation may be turbulent buffeting of the tube by granular motions or by the impact of traveling shocks, which may be represented by the delta-function pulse.

4.1. Monochromatic source

Firstly the boundary conditions are to be satisfied by a source of monochromatic acoustic waves, generated continuously with the frequency ω and amplitude V_0. Hence wave propagation in a thin flux tube by a periodic driver is modeled, mimicking the effect, say, of solar global oscillations on vertical steady flux tubes. Thus the required boundary condition is given by

$$V(t) = V_0 e^{-i\omega t}. \tag{36}$$

In Eq. (35) the first two terms of the general solution are the same as the corresponding static case while the last two terms are new, and are introduced by the background flow. The static case for the magnetic tube has been solved previously by e.g. Musielak & Ulmschneider (2003), being an extension of the case of a stratified atmosphere solved by Sutmann et al. (1998). Denoting the first two terms of the solution as A_1 and A_2 respectively one may quote them here directly, such that

$$A_1 = V(t - z/c_T)\mathcal{H}(t - z/c_T) = V_0 e^{-i\omega(t-z/c_T)}\mathcal{H}(t - z/c_T), \tag{37}$$

and

$$\begin{aligned} A_2 &= -\frac{\Omega_T z}{c_T}[V(t) * W(t,z)], \\ &= -V_0 e^{-i\omega t}(e^{i\omega z/c_T} - e^{-i\sqrt{\omega^2-\Omega^2}\, z/c_T}) \\ &\quad + \frac{\Omega_T z V_0}{c_T}\sqrt{\frac{2}{\pi\Omega_T}}\frac{1}{t^{3/2}}\frac{1}{\omega^2-\Omega_T^2}(\Omega_T\sin\mu + i\omega\cos\mu), \end{aligned} \tag{38}$$

valid for $t \gg z/c_T$, where $\mu = \Omega_T t - 3\pi/4$. Adding A_1 and A_2 gives the static solution. To calculate the third and fourth terms, A_3 and A_4, one may employ a fundamental property of the delta function, which shows that for some arbitrary function, $X(t)$, convoluted with the derivative of the delta function one obtains the derivative of the function, i.e. $X(t) * \delta'(t) = X'(t)$. Thus

$$\begin{aligned} A_3 &= \lambda[\{V(t - z/c_T)\mathcal{H}(t - z/c_T)\} * \delta'(t)] \\ &= -i\lambda V_0 \omega e^{-i\omega(t-z/c_T)}, \end{aligned} \tag{39}$$

and

$$\begin{aligned} A_4 &= V_0\lambda i\omega e^{-i\omega t}\left[e^{i\omega z/c_T} - e^{-i\sqrt{\omega^2-\Omega_T^2}\, z/c_T}\right] \\ &\quad + \frac{V_0 z\Omega_T\lambda}{c_T}\sqrt{\frac{2}{\pi\Omega_T}}\frac{1}{t^{3/2}}\frac{1}{\omega^2-\Omega_T^2} \\ &\quad \times\left[\Omega_T^2\cos\mu - i\omega\Omega_T\sin\mu - \frac{3}{2}\frac{1}{t}(\Omega_T\sin\mu + i\omega\cos\mu)\right]. \end{aligned} \tag{40}$$

The total solution for $Q(t,z)$ may be given by adding the four terms, giving

$$\begin{aligned} Q(t,z) &= V_0(1 - i\lambda\omega)e^{-i(\omega t + \sqrt{\omega^2-\Omega_T^2}\, z/c_T)} \\ &\quad + \frac{V_0 z\Omega_T}{c_T}\sqrt{\frac{2}{\pi\Omega_T}}\frac{1}{t^{3/2}}\frac{1}{\omega^2-\Omega_T^2} \\ &\quad \times\left[\Omega_T\left(1 - \lambda i\omega - \frac{3}{2}\frac{\lambda}{t}\right)\sin\mu \right. \\ &\quad \left. + i\omega\left(1 - \lambda\frac{i\Omega_T^2}{\omega} - \frac{3}{2}\frac{\lambda}{t}\right)\cos\mu\right]. \end{aligned} \tag{41}$$

The effect of the flow is apparent through the λ term, which has temporal dimensions and thus can be thought of as characteristic time introduced by the flow. Setting $\lambda = 0$ reduces the equation to that of the equivalent static solution given by Musielak & Ulmschneider (2003). As in the static case the solution is comprised of two parts: the so-called forced oscillations with wave frequency ω, representing the continuous nature of the driver at the footpoint and the free atmospheric oscillations at the cutoff frequency Ω_T. The effect of the flow on the forced term may be seen by examining it's real part, which can be shown to be

$$\Re[Q_{\text{Forced}}(t,z)] = V_0\sqrt{1 + (\omega\lambda)^2}\cos(\xi + \psi), \tag{42}$$

where $\xi = \omega t + (\omega^2 - \Omega_T^2)^{1/2} z/c_T$) and $\psi = \arctan(\lambda\omega)$. Thus the background flow has increased the amplitude of the forced term and introduced a phase shift. Both these effects are functions of λ, and hence the applied background flow, but interestingly they are dependant also on the wave frequency, ω. The free atmospheric oscillations, represented by the second term in Eq. (41), are dominated temporally by the $t^{-3/2}$ term, which results in a decay in time at any given height. The presence of the flow introduces a secondary temporal dependance, given by the $3\lambda/2t$ term. Despite the fact that t is considered to be large, the effect of this term may become significant when considering solar parameters. Furthermore note that in the static case the amplitude of the free oscillations is linear with height, however with the addition of the background flow this is not the case anymore. One may demonstrate these two points by considering the real part of the free oscillations in Eq. (41), and rewriting the equation as

$$\begin{aligned} \Re[Q_{\text{Free}}(t,z)] &= \frac{V_0 z\Omega_T}{c_T}\sqrt{\frac{2}{\pi\Omega_T}}\frac{1}{t^{3/2}}\frac{1}{\omega^2-\Omega_T^2} \\ &\times\left[\Omega_T\sqrt{\left(1 - \frac{3\lambda}{2t}\right)^2 + (\lambda\Omega_T)^2}\sin(\mu + \alpha)\right], \end{aligned} \tag{43}$$

where

$$\alpha = \arctan\frac{\Omega_T t}{t/\lambda - 3/2}. \tag{44}$$

The effect of the flow, therefore, is to alter the amplitude of the free oscillations (relative to the static case) in a non-trivial fashion. The change in amplitude is both a function of time, height and the value of the background flow. Furthermore, the flow introduces a phase shift which equally is dependant on time, height and the magnitude of the flow. It can also be noted that, in contrast to the forced terms, the increase in amplitude and phase shift are not dependant on the wave frequency, ω.

Consider now the special case of a monochromatic driver with a driving frequency matching the tube cut-off, i.e. $\omega = \Omega_T$. In this case one may follow a similar procedure and find

$$\begin{aligned} Q(t,z) &= V_0(1 - i\lambda\Omega_T)e^{-i\Omega_T t} \\ &\quad + \frac{z\Omega_T V_0}{c_T}\sqrt{\frac{2}{\pi\Omega_T}}\left[\frac{e^{i\mu}}{t^{3/2}}\left(\frac{1}{i\Omega_T} - \frac{\lambda}{4} - i\frac{3\lambda}{8\Omega_T t}\right)\right. \\ &\quad \left. + \frac{e^{-i\mu}}{t^{1/2}}\left(1 - \frac{\lambda}{2t} - i\Omega_T\lambda\right)\right]. \end{aligned} \tag{45}$$

As discussed in some detail by Sutmann et al. (1998) it is necessary to take $t \to \infty$ when $\omega \to \Omega_T$, thus Eq. (45) reduces to

$$Q(t,z) = V_0(1 - i\lambda\Omega_T)e^{-i\Omega_T t}, \tag{46}$$

showing only the forced term remains. In much the same way as in the previous case an amplitude increase and phase shift have been introduced, which may be seen by considering the real part of Eq. (46), such that

$$\Re[Q_{\mathrm{Forced}}(t,z)] = V_0\sqrt{1+(\Omega_{\mathrm{T}}\lambda)^2}\cos(\Omega_{\mathrm{T}}+\phi), \tag{47}$$

where $\phi = \arctan(\lambda\Omega_{\mathrm{T}})$. The amplitude increase and phase shift are functions of Ω_{T} rather than ω since the system is being driven at the cutoff frequency.

4.2. Delta function pulse

Here the source is considered to be a single pulse of δ-function form in time. Thus the lower boundary condition at $z = 0$ is

$$V(t) = \bar{V}_0\delta(t). \tag{48}$$

The delta function has dimensions $1/T$, consequently the dimensions of $\bar{V}_0$ are not velocity. This may be remedied by using a dimensionless time $\bar{t} = t/P_{\mathrm{T}}$, where $P_{\mathrm{T}} = 2\pi/\Omega_{\mathrm{T}}$. Thus

$$V(t) = \bar{V}_0\delta(P_{\mathrm{T}}\bar{t}) = \frac{\bar{V}_0}{|P_{\mathrm{T}}|}\delta(\bar{t}) = V_0\delta(\bar{t}). \tag{49}$$

Considering the general solution, Eq. (35), being constructed of four summed terms, each denoted by B_{i}, where $i = 1,2,3,4$ representing the four terms in the general solution. The first two terms are given by the static solution of Musielak & Ulmschneider (2003). Consider t to be large such that the first term, B_1, is zero. The second term, B_2, is given by expanding the Bessel function for large arguments resulting in

$$B_2 = \frac{-2\pi z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{t^{3/2}}\cos\mu. \tag{50}$$

It is easy to show that the third term, B_3, is also zero, as a direct consequence of B_1. The fourth term, B_4, is given by

$$B_4 = \frac{2\pi z\lambda V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{t^{3/2}}\left[\frac{3}{2}\frac{1}{t}\cos\mu + \Omega_{\mathrm{T}}\sin\mu\right]. \tag{51}$$

Adding the four terms gives the solution for $Q(t,z)$

$$Q(t,z) = \frac{-2\pi z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{t^{3/2}} \times\left[\cos\mu - \frac{3}{2}\frac{\lambda}{t}\cos\mu - \lambda\Omega_{\mathrm{T}}\sin\mu\right]. \tag{52}$$

Note that the delta function source results in only the free atmospheric oscillations, due to the fact that the pulse is not continuously driven and, in this analysis, has long passed. The oscillations are dominated by the decay in time, given by the $t^{-3/2}$ term and oscillate at the cutoff frequency, Ω_{T}. The effect of the flow can be emphasised by rewriting Eq. (52) in the following form

$$Q(t,z) = \frac{-2\pi z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{t^{3/2}} \times\left[\sqrt{\left(1-\frac{3\lambda}{2t}\right)^2 + (\lambda\Omega_{\mathrm{T}})^2}\cos(\mu-\alpha)\right]. \tag{53}$$

Thus the effect of the flow is much the same as for the monochromatic driver. The amplitude is modified in a non-trivial fashion, being a function of time, height and the magnitude of the background flow. A phase shift is also introduced that is also non-trivial in nature.

4.3. Sinusoidal pulse

The final case to be considered here is a sinusoidal pulse generated at the lower boundary and lasting for one wave period P, where $P = 2\pi/\omega$. Thus the boundary driver is given by

$$V(t) = Q_0\,[H(t) - H(t-P)]\,\mathrm{e}^{-\mathrm{i}\omega t}. \tag{54}$$

As previously the general solution is to be constructed of four summed terms, each denoted by C_{i}. Since large times are assumed, i.e. $t \gg z/c_{\mathrm{T}}$, one arrives at

$$C_1 = V(t - z/c_{\mathrm{T}})\mathcal{H}(t - z/c_{\mathrm{T}}) = 0, \tag{55}$$

because the two Heaviside functions cancel. The second term can be shown to be (see Sutmann et al. 1998 for details, since the steady state has no influence)

$$C_2 = \frac{-\Omega_{\mathrm{T}} z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{\omega^2-\Omega_{\mathrm{T}}^2}\frac{1}{t^{3/2}} \times\left[\Omega_{\mathrm{T}}\sin\theta - \Omega_{\mathrm{T}}\sin\mu + \mathrm{i}\omega\cos\theta - \mathrm{i}\omega\cos\mu\right]. \tag{56}$$

where $\theta = \Omega_{\mathrm{T}}(t-P) - 3\pi/4$. The third term, C_3, is zero, which can be deduced by inspecting C_1. For determining C_4 use the same technique is applied as for the other drivers, yielding

$$C_4 = -\frac{\Omega_{\mathrm{T}} z V_0\lambda}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{\omega^2-\Omega_{\mathrm{T}}^2}\frac{1}{t^{3/2}} \tag{57}$$
$$\times\left[\frac{-3}{2}\frac{1}{t}(\Omega_{\mathrm{T}}\sin\theta - \Omega_{\mathrm{T}}\sin\mu + \mathrm{i}\omega\cos\theta - \mathrm{i}\omega\cos\mu) + \Omega_{\mathrm{T}}^2\cos\theta - \Omega_{\mathrm{T}}^2\cos\mu - \mathrm{i}\omega\Omega_{\mathrm{T}}\sin\theta + \mathrm{i}\omega\Omega_{\mathrm{T}}\sin\mu\right].$$

Combining all the four terms gives

$$\begin{aligned} Q(t,z) = {} & \frac{-\Omega_{\mathrm{T}} z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{\omega^2-\Omega_{\mathrm{T}}^2}\frac{1}{t^{3/2}} \\ & \times\left[\left(\Omega_{\mathrm{T}} - \frac{3}{2}\frac{\lambda\Omega_{\mathrm{T}}}{t} - \mathrm{i}\lambda\omega\Omega_{\mathrm{T}}\right)\sin\theta \right. \\ & - \left(\Omega_{\mathrm{T}} - \frac{3}{2}\frac{\lambda\Omega_{\mathrm{T}}}{t} - \mathrm{i}\lambda\omega\Omega_{\mathrm{T}}\right)\sin\mu \\ & + \left(\mathrm{i}\omega - \frac{3}{2}\frac{\mathrm{i}\omega\lambda}{t} + \lambda\Omega_{\mathrm{T}}^2\right)\cos\theta \\ & \left. - \left(\mathrm{i}\omega - \frac{3}{2}\frac{i\lambda\omega}{t} + \lambda\Omega_{\mathrm{T}}^2\right)\cos\mu\right]. \end{aligned} \tag{58}$$

As in the case of the delta function source the solution is given by the free oscillations only, since the pulse only was driven for one wave period. The free oscillations decay as $t^{-3/2}$ and are at the cutoff frequency. However, also note that the solution is a function of the pulse frequency. It is interesting to observe the case when $\omega \to \Omega_{\mathrm{T}}$ because, as discussed in Sutmann et al. (1998), under these conditions it is necessary to impose the limit $t \to \infty$ and thus we see that there are no free oscillations present. The effect of the flow can be highlighted again by considering the real part of Eq. (58), e.g. in the following form

$$\Re[Q_{\mathrm{Free}}(t,z)] = \frac{-\Omega_{\mathrm{T}} z V_0}{c_{\mathrm{T}}}\sqrt{\frac{2}{\pi\Omega_{\mathrm{T}}}}\frac{1}{\omega^2-\Omega_{\mathrm{T}}^2}\frac{1}{t^{3/2}} \tag{59}$$
$$\times\Omega_{\mathrm{T}}\sqrt{\left(1-\frac{3\lambda}{2t}\right)^2 + (\lambda\Omega_{\mathrm{T}})^2}\,[\sin(\theta+\alpha) - \sin(\mu+\alpha)]\,.$$

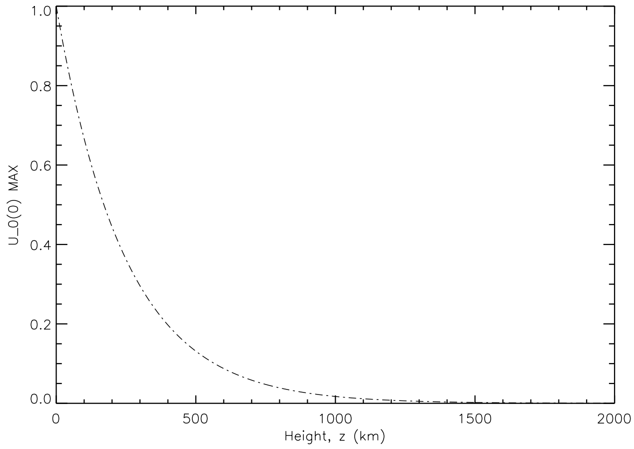

Fig. 2. The maximum value of footpoint flow, $U_0(0)$, against the maximum height considered.

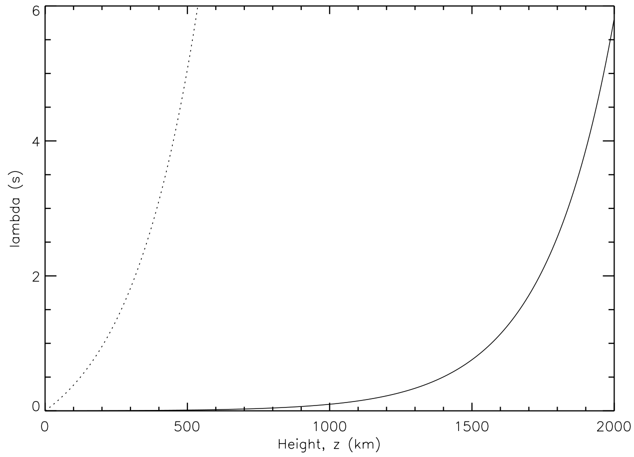

Fig. 3. The variation of the flow timescale, λ, with height plotted for two magnitudes of footpoint flow, $U_0(0)$. The dotted line corresponds to $z_{\max} = 500$, giving $U_0(0) = \exp(-500/2H)$. The solid line corresponds to $z_{\max} = 2000$.

Thus it is found that the effect of the flow, when considering the sinusoidal driver, is to introduce a modification to the amplitude and a phase shift which are both functions of time, height and the magnitude of the background flow.

5. Application to the Sun

Taking typical solar parameters, corresponding to the photospheric region, we may apply the equations derived previously to the Sun. Gravity, $g = 0.275$ km s^{-2} and $\gamma = 5/3$. A sound speed $c_0 = 7.5$ km s^{-1} and a scale height of $H = 122.7$ km yields an acoustic cutoff frequency of $\Omega_{\rm AC} = 0.0306$ s^{-1} (where $\Omega_{\rm AC} = c_0/2H$) and a corresponding acoustic cutoff period of $P_{\rm AC} \approx 200$ s. We can express the tube speed, $c_{\rm T}$, and the tube cutoff frequency, $\Omega_{\rm T}$, in terms of the sound speed and plasma beta by means of the following relations

$$c_{\rm T} = \frac{c_0}{\sqrt{1+\beta\gamma/2}}, \quad \Omega_{\rm T} = \frac{\Omega_{\rm AC}}{\sqrt{1+\beta\gamma/2}}\sqrt{\frac{9}{4} - \frac{2}{\gamma} + 2\beta\frac{\gamma-1}{\gamma}} \cdot \tag{60}$$

Thus for $\beta \to 0$ we see that $\Omega_{\rm T} \approx 1.03\Omega_{\rm AC}$ and for $\beta \gg 0$ we find $\Omega \approx 0.98\Omega_{\rm AC}$. Thus the extremes of magnetic configuration yield a very similar frequency range over which longitudinal tube waves and acoustic waves propagate. For the purpose of plotting a plasma beta of $\beta = 0.4$ shall be taken, yielding a cut-off frequency of $\Omega_{\rm T} = 0.0310$ s^{-1}, which is approximately 1% larger than an acoustic case considered by, say, Sutmann et al. (1998). Any application to the solar atmosphere will require consideration of the magnitude of the footpoint flow, $U_0(0)$. In the preceding analysis the background flow was required to remain bounded for all z. This condition is plotted in Fig. 2, which shows that for a tube examined at say, $z = 2000$ km, one is severely restricted in the maximum value of the footpoint flow one may adopt. It should be noted that in this example the magnitude of the background flow will increase with height, reaching a value of $U_0 = 1$ km s^{-1} at $z = 2000$ km. In Fig. 3 the flow timescale λ against height z is plotted. The variation in λ is shown for two different values of the footpoint flow. The dotted line in Fig. 3 corresponds to a footpoint flow given by a maximum height of $z = 500$ km (such that $U_0(0) = \exp(-500/2H)$), and consequently could only be used to maximum height of $z = 500$ km. The solid line corresponds to a footpoint flow applicable up to $z = 2000$ km.

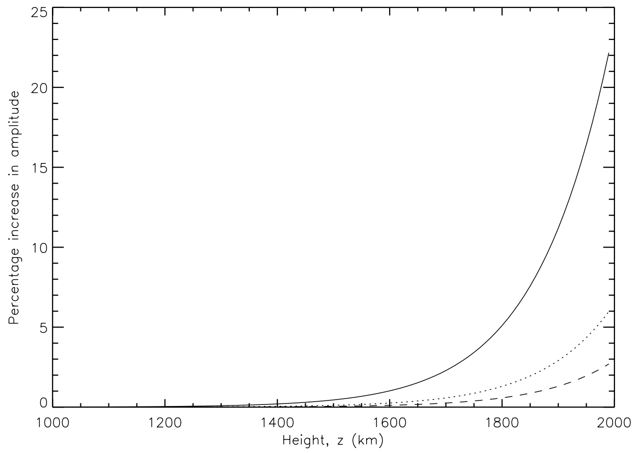

Fig. 4. The percentage increase in amplitude of the forced atmospheric oscillations between the steady and static flux tubes. The solid, dotted and dashed curves correspond to $\omega = 0.13$, 0.06 and 0.04 s^{-1} respectively. All curves are plotted to $z_{\max} = 2000$, giving $U_0(0) = \exp(-2000/2H)$.

5.1. Forced atmospheric oscillations

It has been shown that for the case of the continuously driven monochromatic source one obtains the so called forced atmospheric oscillations. Equation (42) demonstrated the real part of the forced oscillations has an amplitude increase and a phase shift when compared with its static counterpart. The percentage difference in amplitude between the static and steady cases is shown in Fig. 4. The three curves correspond to propagating wave frequencies of $\omega = 0.13$, 0.06, 0.04 s^{-1} (i.e. wave periods of 50, 100, 150 s). For a wave period of 50 s it is seen that the percentage difference in amplitude over the static case rises to the order of 25% at the limit of validity. Since the choice of imposed wave frequency relatively free, one may achieve extremely large percentage differences.

5.2. Free atmospheric oscillations

From the equations presented in Sect. 4 it can be seen that the effect of the background flow is to increase the amplitude of the resulting free atmospheric oscillations and to introduce a phase

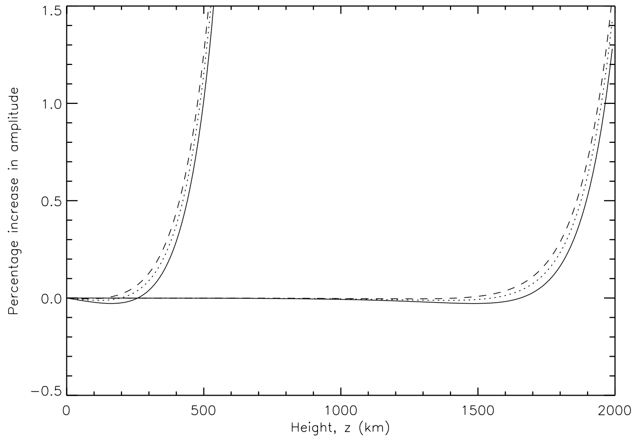

Fig. 5. The percentage increase in amplitude between the steady and static cases against height. Two sets of curves are plotted; the leftmost corresponds to a footpoint flow valid up to z = 500 km, the rightmost to a footpoint flow valid up to z = 2000 km. The solid, dotted and dashed lines correspond to times of t = 1000, 2000 and 3000 s respectively.

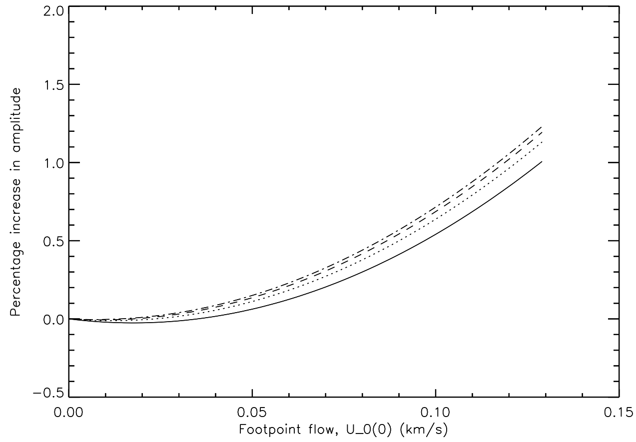

Fig. 6. The percentage increase in amplitude between the steady and static cases against footpoint flow. The curve is plotted for z = 500 km and the solid, dashed, dotted and dot-dashed lines correspond to t = 1000, 2000, 3000 and 4000 s respectively.

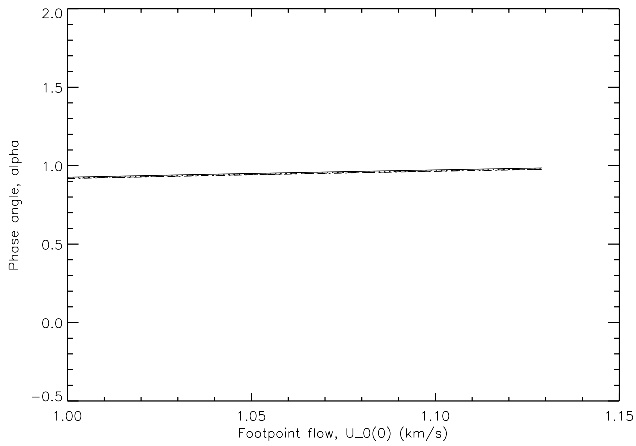

Fig. 7. The phase angle between against footpoint flow. The curve is plotted for z = 500 km and the solid, dashed, dotted and dot-dashed lines correspond to t = 1000, 2000, 3000 and 4000 s respectively.

shift. For each of the three drivers considered, the monochromatic source, the delta function source and the sinusoidal pulse, the percentage difference between the amplitude of the free atmospheric oscillations in dynamic case and the static case is the same and is given by

$$\left[\sqrt{1-\frac{3\lambda}{t}+\lambda^2\left(\frac{9}{4t^2}+\Omega_{\mathrm{T}}^2\right)}-1\right]\times 100. \qquad (61)$$

The percentage increase in amplitude between the dynamic and static case is plotted in Fig. 5. Two sets of curves are shown; the leftmost set correspond to a footpoint flow that is applicable up to z = 500 km, the rightmost to a footpoint flow valid up to z = 2000 km. The solid, dotted and dashed lines correspond to times of t = 1000, 2000 and 3000 s respectively. By inspection one may conclude that the amplitude of the free atmospheric oscillations is increased by a maximum of approximately 1.5% for both magnitudes of footpoint flow. Note that if the curves for the larger footpoint flow (i.e. for $z_{\max}$ = 500 km) were to be extrapolated to higher heights then, the percentage increase in amplitude would be considerably larger and could be readily seen in Doppler-shifted spectral lines. However this value of larger footpoint flow may only be used up to the maximum height allowable, i.e. z = 500 km.

Further one may examine the percentage increase in amplitude against the magnitude of the footpoint flow, i.e shown in Fig. 6. The curve has been plotted for a height of z = 500 km and the solid, dashed, dotted and dot-dashed lines correspond to t = 1000, 2000, 3000 and 4000 s respectively. The magnitude of $U_0(0)$ ranges from zero to approximately 0.13 km s^{-1}, being the maximum permitted for this height. There is approximately 1% increase in amplitude in the dynamic case and that there is a further small increase for larger times. Turning our attention to the introduction of the phase angle, α, by the flow, Fig. 7 shows that the phase angle is relatively constant at approximately 1 radian and is practically unaffected by time. The curve has been plotted under the same conditions as the previous figure.

6. Conclusions

The propagation of magnetic tube waves in a stratified thin, isothermal flux tube with an internal equilibrium background flow has been studied. The governing equations have been solved by Laplace transforms and perturbation analysis, and been applied for conditions appropriate to the solar atmosphere. It was found that for each of the three footpoint drivers considered, namely a monochromatic source, a delta function pulse and a sinusoidal pulse, that the effect of the flow was to introduce a change in amplitude and phase, with both changes being functions of height, in both the forced and free atmospheric oscillations. The percentage increase in amplitude for the forced terms has been shown to be strongly dependant on the applied wave frequency. For a driving period of 50 s it is shown that the difference over the static case may be up to 25% at the largest heights. Smaller periods will result in even larger differences. For the free oscillations, the dominant characteristic of the temporal decay at any given height (as $t^{-3/2}$) is relatively unaffected by the bulk flow. For a given footpoint flow it was found that the amplitude will increase with height up to the validity of the analysis. The maximum percentage increase in amplitude is given at the highest z that is analytically still valid and is around 1.5% when compared to the static case. Although this seems relatively small it must be remembered that we are severely restricted in the magnitude of the footpoint flow we may choose. It can also be noted that the frequency of the free atmospheric oscillations is the same in the steady and static cases. This is perhaps surprising, but the authors note that background flows of second order and higher have been considered small compared to the characteristic speeds, which is a possible factor. The results presented in this paper may have important observational consequences, especially when flux tube diagnostics are carried out using the tools of magnetic seismology.

Acknowledgements. R.E. acknowledges M. Kéray for patient encouragement and is grateful to NSF of Hungary (OTKA, K67746), for their financial support. J.H. acknowledges support from the Science and Technology Facilities Council (STFC), UK.

Appendix A: Short wavelength assumption

Linearising the non-linear governing equations gives

$$\rho_0\frac{Du}{Dt}+\rho_0U_0'u+p'+(U_0U_0'+g)\rho=0, \tag{A.1}$$

$$B_0\frac{D\rho}{Dt}-\rho_0\frac{DB}{Dt}-U_0B_0\frac{\rho_0'}{\rho_0}\rho+U_0\rho_0\frac{B_0'}{B_0}B$$
$$+\rho_0B_0\left(\frac{\rho_0'}{\rho_0}-\frac{B_0'}{B_0}+\frac{\partial}{\partial z}\right)u=0, \tag{A.2}$$

$$\frac{Dp}{Dt}+p_0'u-c_0^2\left(\frac{D\rho}{Dt}+\rho_0'u\right)+U_0\frac{\rho_0'}{\rho_0}(c_0^2\rho-\gamma p)=0, \tag{A.3}$$

$$p+\frac{B_0}{\mu}B=0, \tag{A.4}$$

where the un-subscripted quantities are the perturbed quantities and subscript 0 are the equilibrium quantities. One may link the tube to the environment by assuming the pressure perturbation in the environment is negligible, thus it is assumed that the tube is embedded in a quiescent environment. Considering the following variable changes

$$p_1=\frac{p}{\rho_0},\quad \rho_1=\frac{\rho}{\rho_0},\quad B_1=\frac{B}{B_0}, \tag{A.5}$$

one may recast (A.1) to (A.4) into

$$\frac{Du}{Dt}+U_0'u+\left(\frac{\partial}{\partial z}+\frac{\rho_0'}{\rho_0}\right)p_1+(U_0U_0'+g)\rho_1=0, \tag{A.6}$$

$$\frac{D\rho_1}{Dt}-\frac{DB_1}{Dt}+\left(\frac{\rho_0'}{\rho_0}-\frac{B_0'}{B_0}+\frac{\partial}{\partial z}\right)u=0, \tag{A.7}$$

$$\frac{Dp_1}{Dt}-c_0^2\frac{D\rho_1}{Dt}+\left(\frac{p_0'}{\rho_0}-c_0^2\frac{\rho_0'}{\rho_0}\right)u+U_0\frac{\rho_0'}{\rho_0}(1-\gamma)p_1=0, \tag{A.8}$$

$$p_1+v_A^2B_1=0. \tag{A.9}$$

A.1. Assumptions for reduction

In Eq. (A.6) it can be noted that $U_0U_0'=U_0^2/2H$ and it is assumed that

$$\frac{U_0^2}{2H}\ll g, \tag{A.10}$$

for all z. This could be rewritten as

$$U_0^2\ll\frac{2c_0^2}{\gamma}, \tag{A.11}$$

for all z. This is justifiable given the earlier assumption that $U_0\ll c_T^2$ for all z. In Eq. (A.8) one can rewrite the D/Dt term as $\partial/\partial t+U_0\partial/\partial z$. Comparing the z derivative term with the other pressure term in Eq. (A.8) and making the assumption that

$$U_0\frac{\partial p_1}{\partial z}\gg U_0\frac{\gamma-1}{H}p_1, \tag{A.12}$$

and if p_1 varies over some length scale L one may take

$$\frac{p_1}{L}\gg\frac{\gamma-1}{H}p_1, \tag{A.13}$$

i.e. $L\ll H$, corresponding to a short wavelength approximation in perturbed quantities. Using the same reasoning the second term in Eq. (A.6) may be neglected. Under these assumptions, substituting in our equilibrium relations and noting that $p_0'/\rho_0-c_0^2\rho_0'/\rho_0=g(\gamma-1)$, one may rewrite the governing equations as

$$\frac{Du}{Dt}+\left(\frac{\partial}{\partial z}-\frac{1}{H}\right)p_1+g\rho_1=0, \tag{A.14}$$

$$\frac{D\rho_1}{Dt}-\frac{DB_1}{Dt}+\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)u=0, \tag{A.15}$$

$$\frac{Dp_1}{Dt}-c_0^2\frac{D\rho_1}{Dt}+g(\gamma-1)u+\frac{U_0}{H}(\gamma-1)p_1=0. \tag{A.16}$$

$$\frac{Dp_1}{Dt}+v_A^2\frac{DB_1}{Dt}=0. \tag{A.17}$$

A.2. Eliminating B_1 and ρ_1

Performing the following procedure $[c_0^2/v_A^2\times$ (A.17) + $c_0^2\times$ (A.15) + (A.16)$]$ gives

$$\left(1+\frac{c_0^2}{v_A^2}\right)\frac{Dp_1}{Dt}+c_0^2\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)u+g(\gamma-1)u=0, \tag{A.18}$$

so

$$\frac{Dp_1}{Dt}=-\frac{c_T^2}{c_0^2}\left[c_0^2\left(\frac{\partial}{\partial z}-\frac{1}{2H}\right)u+g(\gamma-1)u\right]. \tag{A.19}$$

Now examining the full time derivative of Eq. (A.14) (D(A.14)$/Dt$) one finds

$$\frac{D^2u}{Dt^2}+\frac{D}{Dt}\left(\frac{\partial}{\partial z}-\frac{1}{H}\right)p_1+g\frac{D\rho_1}{Dt}=0, \tag{A.20}$$

which is

$$\left(\frac{\partial^2u}{\partial t^2}+2U_0\frac{\partial^2u}{\partial t\partial z}+U_0^2\frac{\partial^2u}{\partial z^2}+\frac{U_0^2}{2H}\frac{\partial u}{\partial z}\right)+\frac{D}{Dt}\frac{\partial p_1}{\partial z}$$
$$-\frac{1}{H}\frac{Dp_1}{Dt}+g\frac{D\rho_1}{Dt}=0. \tag{A.21}$$

It can be seen that

$$\frac{D}{Dt}\frac{\partial p_1}{\partial z}=\frac{\partial^2p_1}{\partial t\partial z}+U_0\frac{\partial^2p_1}{\partial z^2}\cdot \tag{A.22}$$

Also

$$\frac{\partial}{\partial z}\frac{Dp_1}{Dt}=\frac{\partial^2p_1}{\partial t\partial z}+\frac{U_0}{2H}\frac{\partial p_1}{\partial z}+U_0\frac{\partial^2p_1}{\partial z^2}\cdot \tag{A.23}$$

Thus since the assumption $L/H \ll 1$ has been made, it can be seen that

$$\frac{U_0}{2H}\frac{\partial p_1}{\partial z} \ll U_0 \frac{\partial^2 p_1}{\partial z^2}, \tag{A.24}$$

then

$$\frac{D}{Dt}\frac{\partial p_1}{\partial z} \approx \frac{\partial}{\partial z}\frac{Dp_1}{Dt} \,. \tag{A.25}$$

If one makes these assumptions and add the following $[-g/v_\mathrm{A}^2 \times$ (A.17) $- g \times$ (A.15)] one obtains

$$\left(\frac{\partial^2 u}{\partial t^2} + 2U_0\frac{\partial^2 u}{\partial t \partial z} + U_0^2\frac{\partial^2 u}{\partial z^2} + \frac{U_0^2}{2H}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial}{\partial z} - \frac{1}{H} - \frac{g}{v_\mathrm{A}^2}\right)\frac{Dp_1}{Dt} - g\left(\frac{\partial}{\partial z} - \frac{1}{2H}\right) u = 0. \tag{A.26}$$

Substituting Eq. (A.19) into Eq. (A.26) it is found that

$$\frac{\partial^2 u}{\partial t^2} + 2U_0\frac{\partial^2 u}{\partial t \partial z} + (U_0^2 - c_\mathrm{T}^2)\frac{\partial^2 u}{\partial z^2} + \frac{U_0^2 + c_\mathrm{T}^2}{2H}\frac{\partial u}{\partial z} + \frac{c_\mathrm{T}^2}{H^2}\left[\frac{c_0^2}{v_\mathrm{A}^2}\frac{\gamma - 1}{\gamma^2} + \frac{1}{2} - \frac{1}{2\gamma}\right] u = 0. \tag{A.27}$$

Since $U_0^2 \ll c_\mathrm{T}^2$ for all z one may neglect the terms in U_0^2 and rewrite this as

$$\frac{\partial^2 u}{\partial t^2} + 2U_0\frac{\partial^2 u}{\partial t \partial z} - c_\mathrm{T}^2\frac{\partial^2 u}{\partial z^2} + \frac{c_\mathrm{T}^2}{2H}\frac{\partial u}{\partial z} + \frac{c_\mathrm{T}^2}{H^2}\left[\frac{c_0^2}{v_\mathrm{A}^2}\frac{\gamma - 1}{\gamma^2} + \frac{1}{2} - \frac{1}{2\gamma}\right] u = 0. \tag{A.28}$$

Appendix B: Regular perturbation method

The governing equation is

$$\frac{\partial^2 Q}{\partial t^2} + 2U_0\frac{\partial^2 Q}{\partial t \partial z} + \frac{U_0}{2H}\frac{\partial Q}{\partial t} - c_\mathrm{T}^2\frac{\partial^2 Q}{\partial z^2} + \Omega_\mathrm{T}^2 Q = 0, \tag{B.1}$$

where

$$\Omega_\mathrm{T}^2 = \frac{c_\mathrm{T}^2}{H^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{c_0^2}{v_\mathrm{A}^2}\frac{\gamma - 1}{\gamma^2}\right), \tag{B.2}$$

and $U_0 = U_0(0)\exp(z/2H)$. After the Laplace transform one obtains

$$\frac{\partial^2 q}{\partial z^2} - \frac{2U_0(0)\mathrm{e}^{z/2H} s}{c_\mathrm{T}^2}\frac{\partial q}{\partial z} - \frac{1}{c_\mathrm{T}^2}\left(\Omega_\mathrm{T}^2 + s^2 + \frac{U_0(0)\mathrm{e}^{z/2H} s}{2H}\right) q = 0. \tag{B.3}$$

Let $U_0(0)/c_\mathrm{T} = \varepsilon \ll 1$. Let $(\Omega_\mathrm{T}^2 + s^2)/c_\mathrm{T}^2 = m^2$, $2s/c_\mathrm{T} = a$ and $b = s/2Hc_\mathrm{T}$. Thus

$$\frac{\partial^2 q}{\partial z^2} - a\varepsilon \mathrm{e}^{z/2H}\frac{\partial q}{\partial z} - (m^2 + b\varepsilon \mathrm{e}^{z/2H}) q = 0. \tag{B.4}$$

B.1. Boundary conditions

The equations are subject the following (time domain) boundary conditions

$$\lim_{t \to 0,\, z \neq 0} Q(t,z) = 0, \qquad \lim_{t \to 0,\, z \neq 0} \frac{\partial Q}{\partial t} = 0,$$
$$\lim_{z \to 0} Q(t,z) = A_0(t), \qquad \lim_{z \to \infty} Q(t,z) = 0. \tag{B.5}$$

Taking the Laplace transform of the last two of these four one obtains

$$\lim_{z \to 0} q(s,z) = A_0(s), \qquad \lim_{z \to \infty} q(s,z) = 0. \tag{B.6}$$

It can be noted that when perturbation analysis is used it is assumed that $q = q_0 + \varepsilon q_1 + \cdots$. By substituting and equating terms one sees that

$$\lim_{z \to 0} q_0(s,z) = A_0(s), \qquad \lim_{z \to \infty} q_0(s,z) = 0. \tag{B.7}$$

$$\lim_{z \to 0} q_1(s,z) = 0, \qquad \lim_{z \to \infty} q_1(s,z) = 0. \tag{B.8}$$

B.2. Regular perturbation

Consider this to be a regular perturbation problem, such that one can express q as

$$q = q_0 + \varepsilon q_1 + \cdots, \tag{B.9}$$

thus

$$(q_0'' + \varepsilon q_1'' + \cdots) - a\varepsilon \mathrm{e}^{z/2H}(q_0' + \cdots) - (m^2 + b\varepsilon \mathrm{e}^{z/2H})(q_0 + \varepsilon q_1 + \cdots) = 0. \tag{B.10}$$

Considering terms of order ε^0 is can be seen that

$$q_0'' - m^2 q_0 = 0, \tag{B.11}$$

thus

$$q_0 = A\mathrm{e}^{-mz} + B\mathrm{e}^{mz}, \tag{B.12}$$

to which BC's shall be applied later. Considering terms of order ε^1 one obtains

$$q_1'' - m^2 q_1 = \mathrm{e}^{z/2H}(aq_0' + bq_0), \tag{B.13}$$

thus

$$q_1'' - m^2 q_1 = (bA - amA)\mathrm{e}^{z(1/2H - m)} + (bB + amB)\mathrm{e}^{z(1/2H + m)}. \tag{B.14}$$

Solving for q_1 and first considering the complimentary function

$$q_1'' - m^2 q_1 = 0, \tag{B.15}$$

thus

$$q_1 = C\mathrm{e}^{-mz} + D\mathrm{e}^{mz}. \tag{B.16}$$

For the particular integral let

$$q_1 = E\mathrm{e}^{z(1/2H - m)} + F\mathrm{e}^{z(1/2H + m)}, \tag{B.17}$$

and substituting into the original one obtains

$$E = \frac{4H^2(bA - amA)}{1 - 4Hm}, \qquad F = \frac{4H^2(bB - amB)}{1 + 4Hm} \,. \tag{B.18}$$

Thus it can be seen that

$$\begin{aligned} q_1 &= CF + PI \\ &= Ce^{-mz} + De^{mz} + \frac{4H^2(bA - amA)}{1 - 4Hm} e^{z(1/2H-m)} \\ &\quad + \frac{4H^2(bB - amB)}{1 + 4Hm} e^{z(1/2H+m)}. \end{aligned} \tag{B.19}$$

Applying boundary conditions yields

$$\lim_{z\to\infty} q_0(s,z) = 0, \quad \Rightarrow B = 0 \tag{B.20}$$

$$\lim_{z\to 0} q_0(s,z) = V_0(s), \quad \Rightarrow A = V_0(s) \tag{B.21}$$

$$\lim_{z\to\infty} q_1(s,z) = 0, \quad \Rightarrow D = 0 \tag{B.22}$$

$$\lim_{z\to 0} q_1(s,z) = 0. \quad \Rightarrow C = \frac{4H^2 V_0(s)(am - b)}{1 - 4mH} \cdot \tag{B.23}$$

Finally recall that

$$q = q_0 + \varepsilon q_1 + \cdot, \tag{B.24}$$

and so

$$q = V_0(s)e^{-mz}\left(1 + \frac{4\varepsilon H^2(am - b)}{1 - 4Hm}(1 - e^{z/2H})\right), \tag{B.25}$$

where $\varepsilon = U_0(0)/c_{\mathrm{T}}$. Recall $m^2 = (\Omega_{\mathrm{T}}^2 + s^2)/c_{\mathrm{T}}^2$, $a = 2s/c_{\mathrm{T}}$ and $b = s/2Hc_{\mathrm{T}}$. Thus one may rewrite q as

$$q(s,z) = V_0(s)e^{-\sqrt{\Omega_{\mathrm{T}}^2+s^2}z/c_{\mathrm{T}}}\left[1 - \lambda s\right], \tag{B.26}$$

with $\lambda = 2U_0(0)H(1 - e^{z/2H})/c_{\mathrm{T}}^2$.

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