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# The Hanle effect

## Decomposition of the Stokes parameters into irreducible components

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**ABSTRACT**

*Context.* It has been shown for the weak-field Hanle effect that the Stokes parameters $I$, $Q$, and $U$ can be represented by a set of six cylindrically symmetrical functions. The proof relies on azimuthal Fourier expansions of the radiation field and of the Hanle phase matrix. It holds for a plane-parallel atmosphere and scattering processes that can be described by a redistribution matrix where redistribution in frequency is decoupled from angle redistribution and polarization.

*Aims.* We give a simpler and more general proof of the Stokes parameter decomposition using powerful new tools introduced for polarimetry, in particular the Landi Degl'Innocenti spherical tensors $\mathcal{T}^K_Q(i,\mathbf{\Omega})$.

*Methods.* The elements of the Hanle phase matrix are written as a sum of terms that depend separately on the magnetic field vector and the directions $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ of the incoming and scattered beams. The dependence on $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ is expressed in terms of the spherical tensors $\mathcal{T}^K_Q(i,\mathbf{\Omega})$ where $i$ refers to the Stokes parameters ($i = 0,\ldots,3$). A multipolar expansion in terms of the $\mathcal{T}^K_Q(i,\mathbf{\Omega})$ is then established for the source term in the transfer equation for the Stokes parameters.

*Results.* We show that the Stokes parameters have a multipolar expansion that can be written as $I_i(\nu,\mathbf{\Omega}) = \sum_{KQ}\mathcal{T}^K_Q(i,\mathbf{\Omega})I^K_Q(\nu,\theta)$ ($K = 0,1,2, -K \le Q \le +K$) where the $I^K_Q$ are nine cylindrically symmetrical, irreducible tensors, $\theta$ being the inclination of $\mathbf{\Omega}$ with respect to the vertical in the atmosphere. The proof is generalized to frequency-dependent phase matrices. It is applied both to partial frequency redistribution with angle-averaged scalar frequency redistribution functions and to complete frequency redistribution with the Hanle effect in the line core and Rayleigh scattering in the wings. Non-LTE transfer equations for the $I^K_Q$ and integral equations for the associated source functions $S^K_Q$ are established. Formal vectors and matrices constructed with $I^K_Q$, $S^K_Q$, and $\mathcal{T}^K_Q$ are introduced in order to present the results in a compact matrix notation. In particular, a simple factorized form is proposed for the Hanle phase matrix.

**Key words.** line: formation – polarization – magnetic fields – radiative transfer

## 1. Introduction

In a magnetized medium, the polarized radiation field can be described by the four Stokes parameters $I$, $Q$, $U$, and $V$, which are functions of position, frequency, and direction. For the weak-field Hanle effect[1] in a one-dimensional medium, it was shown by Faurobert-Scholl (1991) that Stokes $I$, $Q$ and $U$ can be represented by a set of six cylindrically symmetrical functions. The new functions, denoted $I^K_Q$ in the present paper, satisfy a set of non-LTE transfer equations. They are significantly simpler than the equations for the Stokes parameters because the source terms are independent of direction. The proof in Faurobert-Scholl (1991) relies on an azimuthal Fourier expansion of the linearly polarized radiation field and of the Hanle phase matrix. A simpler version of the original proof can be found in Nagendra et al. (1998). It was established for a two-level atom with no-lower level atomic polarization and a redistribution matrix of the form

$$R(\nu,\mathbf{\Omega},\nu',\mathbf{\Omega}';\boldsymbol{B}) = (1-\epsilon)[(1-\gamma)\bar{r}_{\rm II}(\nu,\nu') + \gamma\bar{r}_{\rm III}(\nu,\nu')]P_{\rm H}((\mathbf{\Omega},\mathbf{\Omega}';\boldsymbol{B}), \quad (1)$$

where $\bar{r}_{\rm II}(\nu,\nu')$ and $\bar{r}_{\rm III}(\nu,\nu')$ are angle-averaged partial frequency redistribution functions introduced in Hummer (1962), which correspond to coherent and incoherent frequency redistribution (in the atomic rest frame), $\gamma$ a branching ratio that takes elastic collisions into account, $\epsilon$ a rate of inelastic collisions, and $P_{\rm H}(\mathbf{\Omega},\mathbf{\Omega}';\boldsymbol{B})$ the Hanle phase matrix as given in Landi Degl'Innocenti & Landi Degl'Innocenti (1988). Here the prime letters $\nu'$, $\mathbf{\Omega}'$ refer to incoming photons (frequency and direction) and the unprimed ones to scattered photons. The direction $\mathbf{\Omega}$ is defined by its polar angles $\theta$, $\chi$ (see Fig. 1) and similarly $\mathbf{\Omega}'$ by polar angles $\theta'$, $\chi'$. The redistribution matrix in Eq. (1) was introduced in a heuristic way. It does not incorporate the property that the Hanle effect acts in the line core alone (Omont et al. 1973; Stenflo 1978, 1994, p. 83) but for the first time, the Hanle effect was treated with partial frequency redistribution and elastic collisions. The more systematic approach to the Hanle effect based on the density matrix formulation was limited at that time to complete frequency redistribution (e.g. Landi Degl'Innocenti et al. 1990). The decomposition method introduced in Faurobert-Scholl (1991) was successfully applied to the analysis of the solar Ca I 4227 Å line (Faurobert-Scholl 1992) with a slightly more sophisticated redistribution matrix of the form

$$R(\nu,\mathbf{\Omega},\nu',\mathbf{\Omega}';\boldsymbol{B}) = \sum_\alpha \gamma_\alpha\bar{r}_\alpha(\nu,\nu')P_\alpha(\mathbf{\Omega},\mathbf{\Omega}';\boldsymbol{B}). \quad (2)$$

[1] The weak-field Hanle effect regime implies that the magnetic field is strong enough to reduce phase coherences between Zeeman sublevels, but that the Zeeman splitting can be neglected. The absorption of radiation can be described by a scalar coefficient, independent of the magnetic field.

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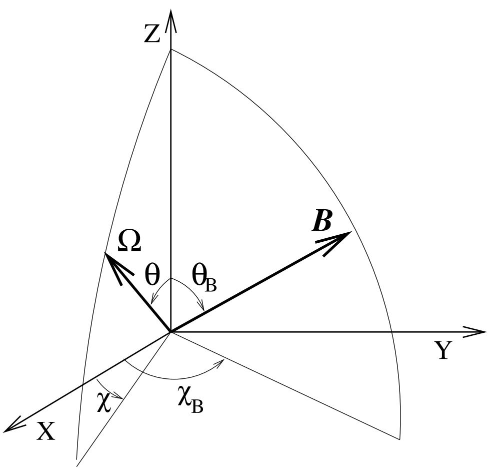


**Fig. 1.** Atmospheric reference system with the definition of $(\theta, \chi)$ and $(\theta_{\rm B}, \chi_{\rm B})$, the polar angles of direction $\boldsymbol{\Omega}$, and magnetic field vector $\boldsymbol{B}$. In the text we also introduce $(\theta', \chi')$, the polar angles of direction $\boldsymbol{\Omega}'$.

Also the Hanle effect was properly restricted to the line core. Here, $\gamma_\alpha$ are branching ratios, $\bar{r}_\alpha(\nu, \nu')$ stand for $\bar{r}_{\rm II}(\nu, \nu')$ or $\bar{r}_{\rm III}(\nu, \nu')$, and $P_\alpha$ for the Hanle, Rayleigh, or isotropic phase matrices.

A significant progress in the theoretical formulation of the Hanle effect with partial frequency redistribution was accomplished by Bommier (1997). The work is based on a quantum electro-dynamical approach with re-summation to all orders of the perturbation expansion of the atom-radiation interaction. Partial redistribution appears at order 4 in the expansion. This work clearly shows that redistribution in frequency, direction, and polarization are coupled by the presence of the magnetic field (see also Stenflo 1998), even in the weak-field limit. This coupling disappears if the expansion is limited to order 2 terms, in which case one has complete frequency redistribution, or if both the lower and upper levels of the transition are infinitely sharp (Bommier 2003). The latter model known as $R_{\rm I}$ redistribution does not apply to real lines but is sometimes used to study scattering problems.

For the weak-field Hanle effect and infinitely sharp lower level, Bommier (1997) has proposed a decomposition of the $(\nu, \nu')$ frequency space into several domains based on asymptotic expansions of the Doppler-broadened generalized profiles at line center and in the wings. In each domain the redistribution matrix can be represented by a sum of terms, each one the product of a phase matrix $P(\boldsymbol{\Omega}, \boldsymbol{\Omega}'; \boldsymbol{B})$ by a scalar frequency redistribution function $r(x, x', \Theta)$ with $\Theta$ the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$. This model is referred to as a "level-II" approximation. In the "level-III" approximation, the angle-dependent redistribution functions are replaced by their angle-averaged version. The Stokes parameters for the "level-II" and "level-III" approximations have been calculated numerically in Nagendra et al. (2002). It was shown in particular that angle-averaging has essentially no effect on Stokes $I$, some effect on Stokes $Q$, but a strong effect on Stokes $U$. These numerical calculations clearly show that averaging partial frequency redistribution functions is not justified for the accurate calculation of fine polarization effects.

In this paper we propose a new proof for the Stokes vector decomposition. It is based on a factorization of the Hanle (and Rayleigh) phase matrix in terms of the irreducible tensors for polarimetry $\mathcal{T}^K_Q(i, \boldsymbol{\Omega})$ introduced by Landi Degl'Innocenti (1984). The index $i$ (0 to 3) refers to the four Stokes parameters. A comprehensive description of the properties of these tensors can be found in Landi Degl'Innocenti & Landolfi (2004, henceforth cited as LL04). We first describe our method for a generic redistribution matrix of the form

$$R(\nu, \boldsymbol{\Omega}, \nu', \boldsymbol{\Omega}'; \boldsymbol{B}) = g(\nu, \nu') P(\boldsymbol{\Omega}, \boldsymbol{\Omega}'; \boldsymbol{B}). \tag{3}$$

We then show that it holds also for frequency-dependent phase matrices $P(\nu, \nu', \boldsymbol{\Omega}, \boldsymbol{\Omega}'; \boldsymbol{B})$, which take into account that the Hanle effect is only present in the line core. Typically, $P(\nu, \nu', \boldsymbol{\Omega}, \boldsymbol{\Omega}'; \boldsymbol{B})$ is a linear combination of Hanle and Rayleigh phase matrices with weighting functions depending on frequencies $\nu$ and $\nu'$. As mentioned above, the factorization between frequency redistribution and polarization only holds for complete redistribution and $R_{\rm I}$ redistribution. For partial redistributions $R_{\rm II}$ and $R_{\rm III}$, it is only an approximation.

The organization of the paper is as follows. In Sect. 2 we discuss the factorization of the phase matrix in terms of the tensors $\mathcal{T}^K_Q(i, \boldsymbol{\Omega})$. In Sects. 3 and 4 we give a proof of the decomposition of the Stokes vector into a set of irreducible tensors $I^K_Q$ and establish a non-LTE transfer equation for the $I^K_Q$. In Sect. 5 the main results are rewritten in symbolic matrix notation, thus rendering their structure easier to grasp. In particular we propose a factorization of the Hanle phase matrix as a product of three matrices depending separately on $\boldsymbol{\Omega}$, $\boldsymbol{\Omega}'$ and magnetic field vector $\boldsymbol{B}$. It is similar to the factorization proposed by Ivanov (2004), but our proof is simpler. In Sect. 6 we consider frequency-dependent phase matrices and apply our decomposition method to the "level-III" approximation of Bommier (1997). A few additional generalizations are mentioned in Sect. 7.

## 2. Factorization of the scattering phase matrix

Phase matrices are deduced from statistical equilibrium equations for the density matrix elements (see e.g. LL04; or Bommier 1997). These equations are usually written in a reference frame where the quantization axis is parallel to the magnetic field. We give two examples in Sects. 2.1 and 2.2 of phase matrices taken from LL04. We explain in Sect. 2.3 how to write phase matrices in the reference frame of the atmosphere, the one where the Stokes vector is calculated.

For a two-level atom, unpolarized lower level, weak-field Hanle effect, the phase matrix elements, in the reference of the magnetic field, can be written in the form (LL04),

$$P_{ij}(\boldsymbol{\Omega}, \boldsymbol{\Omega}'; B) = \sum_{KQ} X_{KQ}(B)(-1)^Q \mathcal{T}^K_Q(i, \boldsymbol{\Omega}) \mathcal{T}^K_{-Q}(j, \boldsymbol{\Omega}'). \tag{4}$$

The indices $i$ and $j$ refer to the Stokes parameters, and $B$ is the strength of the magnetic field vector. The indices $K$ and $Q$ are integers coming from the multipolar decomposition of the density matrix elements into irreducible spherical tensors. The index $K$ takes the values $K = 0, 1, 2$ and, for each value of $K$, $-K \le Q \le K$. The magnetic kernel $X_{KQ}$ depends on the magnetic field strength and on additional atomic variables omitted here to simplify notation (e.g. $J_{\rm l}$ and $J_{\rm u}$ total angular momentum of lower and upper levels, collision rates). As can be observed in Eq. (4), the elements $P_{ij}$ are sums of terms that depend separately on $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$. This property is a basic ingredient of the decomposition of the Stokes parameters into cylindrically symmetrical components $I^K_Q$.

Explicit expressions of the $\mathcal{T}^K_Q(i, \boldsymbol{\Omega})$ can be found in LL04 (Chap. 5, Table 5.6, p. 211; see also our Table 2). They are combinations of Wigner rotation matrices, hence purely geometrical factors. In Eq. (4), the directions $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ are reckoned with respect to the magnetic field direction. In this paper, the reference angles $\gamma$ and $\gamma'$ that define the direction of positive $Q$ are set to zero (see e.g. Fig. 5.9 in LL04). An important property, frequently used in the following, is the conjugation relation

$$(\mathcal{T}_Q^K(i,\mathbf{\Omega}))^* = (-1)^Q \mathcal{T}_{-Q}^K(i,\mathbf{\Omega}), \quad (5)$$

where $^*$ stands for complex conjugate.

### 2.1. Two-level atom with an unpolarized ground level

When collisions, both elastic and inelastic, are taken into account, the solution of the statistical equilibrium equations for the density matrix elements $\rho_Q^K$ of the upper level yield (see LL04, p. 534)

$$X_{KQ}(B) = \frac{W_K(J_\mathrm{l},J_\mathrm{u})}{1+\epsilon+\delta_\mathrm{u}^{(K)}} \frac{1}{1+\mathrm{i}\,QH'_\mathrm{u}}. \quad (6)$$

In terms of the rotation angle $\alpha_Q$ defined by $\tan\alpha_Q = QH'_\mathrm{u}$, one can also write

$$X_{KQ}(B) = \frac{W_K(J_\mathrm{l},J_\mathrm{u})}{1+\epsilon+\delta_\mathrm{u}^{(K)}} \cos\alpha_Q\,\mathrm{e}^{-\mathrm{i}\alpha_Q}. \quad (7)$$

Here $H'_\mathrm{u} = H_\mathrm{u}/(1+\epsilon+\delta_\mathrm{u}^{(K)})$ with $\epsilon = C_\mathrm{u,l}/A_\mathrm{u,l}$ and $\delta_\mathrm{u}^{(K)} = D^{(K)}/A_\mathrm{u,l}$ (Bommier 1997; LL04 p. 520 and 532). We recall that $C_\mathrm{u,l}$ and $A_\mathrm{u,l}$ are the inelastic collisional and radiative de-excitation rates. The parameter $\delta_\mathrm{u}^{(K)}$, with $\delta_\mathrm{u}^{(0)} = 0$, is the effective number of depolarizing collisions for the statistical tensor of rank $K$ taking place during the lifetime of the excited level. The magnetic field strength $B$ enters through the efficiency factor $H_\mathrm{u} = 2\pi\nu_\mathrm{L} g_\mathrm{u}/A_\mathrm{u,l}$ where $\nu_\mathrm{L}$ is the Larmor frequency and $g_\mathrm{u}$ the Landé factor of the upper level. We recall that $H_\mathrm{u} = \alpha_\mathrm{Q} = 0$ for resonance scattering; hence, $X_{KQ}$ is independent of $Q$. Finally, $W_K(J_\mathrm{l},J_\mathrm{u})$ are atomic depolarization parameters that can be found in LL04 (Table 10.1, p. 515). For a normal Zeeman triplet ($J_\mathrm{l} = 0$, $J_\mathrm{u} = 1$), $W_K = 1$ for all values of $K$.

### 2.2. Two-level atom in the weak anisotropy approximation

In a stellar atmosphere, the anisotropy of the radiation field is usually weak. The statistical equilibrium equations for the density matrix elements $\rho_Q^K$ can then be written as a series expansion in terms of a small parameter measuring the anisotropy of the radiation field. Each term in the expansion can then be calculated iteratively (LL04, p. 570). In this approximation, it is possible to take lower-level polarization into account and still express the scattering in terms of a phase matrix.

For example, with magnetic field and depolarizing collisions taken into account, but not inelastic collisions, the $X_{KQ}$ are of the form

$$X_{KQ}(B) = \frac{a+\mathrm{i}\,bQ}{c+\mathrm{i}\,dQ+eQ^2}, \quad (8)$$

where the coefficients $a, b, c, d, e$ depend on $K$, $J_\mathrm{l}$, $J_\mathrm{u}$, $\delta_\mathrm{l}^{(K)}$, $\delta_\mathrm{u}^{(K)}$, and the Hanle efficiency factors $H_\mathrm{l}$ and $H_\mathrm{u}$. Their expressions can be found in LL04 on p. 578.

### 2.3. Phases matrices in the atmospheric reference frame

Applying the method described in LL04 for the rotation of phase matrices between two reference frames, we find that the phase matrix elements in the atmospheric reference frame can be written as

$$P_{ij}(\mathbf{\Omega},\mathbf{\Omega}';\boldsymbol{B}) = \sum_{KQ} \mathcal{T}_Q^K(i,\mathbf{\Omega}) \sum_{Q'} \mathcal{N}_{QQ'}^K(\boldsymbol{B})(-1)^{Q'} \mathcal{T}_{-Q'}^K(j,\mathbf{\Omega}'). \quad (9)$$

Here, $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ are defined by their polar angles $\theta$, $\chi$ and $\theta'$, $\chi'$ in the reference frame $(X,Y,Z)$ of the atmosphere (see Fig. 1). The magnetic kernel $\mathcal{N}_{QQ'}^K(\boldsymbol{B})$ may be written as

$$\mathcal{N}_{QQ'}^K(\boldsymbol{B}) = \mathrm{e}^{\mathrm{i}\,(Q'-Q)\chi_\mathrm{B}} \sum_{Q''} d_{QQ''}^K(\theta_\mathrm{B}) d_{Q''Q'}^K(-\theta_\mathrm{B}) X_{KQ''}(B), \quad (10)$$

where $\theta_\mathrm{B}$ and $\chi_\mathrm{B}$ are the polar angles of the magnetic field (inclination and azimuthal) in the atmospheric reference frame (see Fig. 1). Explicit expressions for the reduced rotations matrices $d_{MM'}^J$ can be found in LL04 (Table 2.1, p. 57) or any book on angular momentum theory.

For Rayleigh scattering (zero magnetic field), $\mathcal{N}_{QQ'}^K$ is proportional to $\delta_{QQ'}$, with a proportionality factor depending on $K$ (see e.g. Eq. (6)). Equation (9) also holds for the isotropic matrix, defined by $P_{00} = 1$ and $P_{ij} = 0$ for all the other $ij$, if we set $\mathcal{N}_{QQ'}^K = \delta_{00}\delta_{K0}$.

## 3. Decomposition of the Stokes vector

We consider a redistribution matrix of the type written in Eq. (3). The transfer equation for the Stokes vector $\boldsymbol{I} = \{I_i\} = \{I,Q,U,V\}$, $(i = 0,1,2,3)$ may be written as

$$\mu\frac{\partial \boldsymbol{I}}{\partial\tau} = \varphi(x)\boldsymbol{I} - \varphi(x)\boldsymbol{G}(\tau) - \oint\int g(x,x')P(\mathbf{\Omega},\mathbf{\Omega}';\boldsymbol{B})\boldsymbol{I}(\tau,x',\mathbf{\Omega}')\,\mathrm{d}x'\,\frac{\mathrm{d}\Omega'}{4\pi}. \quad (11)$$

Henceforth frequencies are measured in Doppler width units with 0 at line center and denoted by $x$ and $x'$ for the incoming and scattered beams. The space variable is the line optical depth $\tau$ defined by $\mathrm{d}\tau = -k_\mathrm{L}\mathrm{d}z$, with $k_\mathrm{L}$ the mean frequency-integrated absorption coefficient; $\tau$ is zero at the surface and goes to plus infinity in the interior. The absorption profile $\varphi(x)$ is normalized to unity. Following standard notation, $\mu = \cos\theta$. The integration element over $\mathbf{\Omega}'$ is $\mathrm{d}\Omega' = \sin\theta'\mathrm{d}\theta'\mathrm{d}\chi'$. The vector $\boldsymbol{G}$ is a given primary source term. For simplicity we also assume that there is no incident radiation and that the medium is semi-infinite.

Each Stokes parameter satisfies a transfer equation of the form

$$\mu\frac{\partial I_i}{\partial\tau} = \varphi(x)\Big[I_i(\tau,x,\mathbf{\Omega}) - S_i(\tau,x,\mathbf{\Omega})\Big], \quad (12)$$

where $S_i(\tau,x,\mathbf{\Omega})$ are the components of the source term. It follows from Eqs. (9) and (11) that they may be written as

$$S_i(\tau,x,\mathbf{\Omega}) = G_i(\tau) + \sum_{KQ}\mathcal{T}_Q^K(i,\mathbf{\Omega})\sum_{Q'}\mathcal{N}_{QQ'}^K(\boldsymbol{B}) \times \int\frac{g(x,x')}{\varphi(x)}(J_{Q'}^K)^*(\tau,x')\,\mathrm{d}x'. \quad (13)$$

In this equation, $G_i$ is the $i$th component of the primary source $\boldsymbol{G}(\tau)$ and $J_Q^K$ the mean irreducible tensor for polarimetry (LL04, p. 208) defined by

$$J_Q^K(\tau,x) = \sum_{j=0}^{j=3}\oint\mathcal{T}_Q^K(j,\mathbf{\Omega})I_j(\tau,x,\mathbf{\Omega})\,\frac{\mathrm{d}\Omega}{4\pi}. \quad (14)$$

Since the Stokes parameters $I_i$ are real quantities, $J_Q^K$ satisfies the conjugation property in Eq. (5).

### 3.1. Expansion of the source term in the transfer equation

We observe that the scattering term on the righthand side of Eq. (13) is written as a multipolar expansion on the basis of the $\mathcal{T}^K_Q(i,\boldsymbol{\Omega})$. We now show that the primary source $\boldsymbol{G}(\tau)$ possesses a similar expansion.

If $\boldsymbol{G}(\tau)$ is unpolarized, only the component $G_0(\tau)$ is non zero, and we can write

$$G_i(\tau) = \sum_{KQ} \mathcal{T}^K_Q(i,\boldsymbol{\Omega}) G^K_Q(\tau), \tag{15}$$

with $G^0_0(\tau) = G_0(\tau)$ and all other $G^K_Q = 0$. In this paper, we use the same notation for the components with indices $i$ (0 to 3) and the components with indices $K$, $Q$. We think that it should not be a source of confusion.

If the primary source term is polarized, we follow Ivanov (1997) and introduce a diffuse field $\boldsymbol{I}^{\rm d}$ defined by $\boldsymbol{I}^{\rm d} = \boldsymbol{I} - \boldsymbol{I}^{\rm p}$ where $\boldsymbol{I}^{\rm p}$, the field created by the primary source $\boldsymbol{G}(\tau)$, is the solution of the transfer equation

$$\mu \frac{\partial \boldsymbol{I}^{\rm p}}{\partial \tau} = \varphi(x)\boldsymbol{I}^{\rm p} - \varphi(x)\boldsymbol{G}(\tau). \tag{16}$$

Simple algebra shows that the diffuse field $\boldsymbol{I}^{\rm d}$ satisfies Eq. (11) with $\boldsymbol{G}(\tau)$ replaced by an angle and frequency-dependent source term

$$\boldsymbol{G}^{\rm d}(\tau,x,\boldsymbol{\Omega}) = \oint\int \frac{g(x,x')}{\varphi(x)} P(\boldsymbol{\Omega},\boldsymbol{\Omega}';\boldsymbol{B})\boldsymbol{I}^{\rm p}(\tau,x',\boldsymbol{\Omega}')\,{\rm d}x' \frac{{\rm d}\Omega'}{4\pi}\cdot \tag{17}$$

It follows then from Eq. (9) that the components $G^{\rm d}_i(\tau,x,\boldsymbol{\Omega})$ can be expanded as in Eq. (15) with

$$G^K_Q(\tau,x) = \sum_{Q'} \mathcal{N}^K_{QQ'}(\boldsymbol{B}) \int \frac{g(x,x')}{\varphi(x)} (J^{{\rm p}K}_{Q'})^*(\tau,x')\,{\rm d}x', \tag{18}$$

where $J^{{\rm p}K}_Q(\tau,x)$ is defined as in Eq. (14) with $I^{\rm p}_j$ in place of $I_j$.

Regrouping Eqs. (13) and (15), we obtain the multipolar expansion

$$S_i(\tau,x,\boldsymbol{\Omega}) = \sum_{KQ} \mathcal{T}^K_Q(i,\boldsymbol{\Omega}) S^K_Q(\tau,x), \tag{19}$$

where

$$S^K_Q(\tau,x) = G^K_Q(\tau,x) + \sum_{Q'} \mathcal{N}^K_{QQ'}(\boldsymbol{B}) \int \frac{g(x,x')}{\varphi(x)} (J^K_{Q'})^*(\tau,x')\,{\rm d}x'. \tag{20}$$

For complete frequency redistribution, the elements $S^K_Q$ are independent of frequency since $g(x,x') = \varphi(x)\varphi(x')$.

### 3.2. Irreducible components of the Stokes vector

We now solve the transfer equation for the Stokes parameters. Combining Eq. (19) with Eq. (12), we see that the solution of Eq. (12) for a semi-infinite atmosphere with zero incident radiation can be written as

$$I_i(\tau,x,\boldsymbol{\Omega}) = \sum_{KQ} \mathcal{T}^K_Q(i,\boldsymbol{\Omega}) I^K_Q(\tau,x,\mu), \tag{21}$$

with

$$I^K_Q(\tau,x,\mu) = \int_\tau^\infty {\rm e}^{-(\tau'-\tau)\varphi(x)/\mu} S^K_Q(\tau',x)\frac{\varphi(x)}{\mu}\,{\rm d}\tau', \quad \mu > 0, \tag{22}$$

$$I^K_Q(\tau,x,\mu) = -\int_0^\tau {\rm e}^{-(\tau'-\tau)\varphi(x)/\mu} S^K_Q(\tau',x)\frac{\varphi(x)}{\mu}\,{\rm d}\tau', \quad \mu < 0. \tag{23}$$

Since the $S^K_Q$ are irreducible tensors, Eqs. (22) and (23) imply that $I^K_Q$ are also irreducible tensors. Equation (21) provides a decomposition of the four Stokes parameters in terms of nine spherical tensors $I^K_Q(\tau,x,\mu)$ that depend on $\theta$, the inclination of $\boldsymbol{\Omega}$, but not on its azimuth $\chi$. The azimuthal dependence is incorporated in the $\mathcal{T}^K_Q(i,\boldsymbol{\Omega})$. When linear polarization is decoupled from circular polarization, linear polarization can be described by six components corresponding to $K=0$, $Q=0$, and $K=2$, $Q=\pm2,\pm1,0$.

Because the Stokes parameters $I_i$ are real quantities, the $I^K_Q$ satisfy the same conjugation property as the $\mathcal{T}^K_Q(i,\boldsymbol{\Omega})$, namely

$$(I^K_Q(\tau,x,\mu))^* = (-1)^Q I^K_{-Q}(\tau,x,\mu). \tag{24}$$

The new tensors $I^K_Q(\tau,x,\mu)$ should not be mistaken for the irreducible tensor for polarimetry $\mathcal{I}^K_Q(\tau,x,\boldsymbol{\Omega}) = \sum_i \mathcal{T}^K_Q(i,\boldsymbol{\Omega}) I_i(\tau,x,\boldsymbol{\Omega})$ constructed with the Stokes parameters (see LL04, p. 208). The latter are written with calligraphic $\mathcal{I}$ and depends on the two polar angles $\theta$ and $\chi$ defining the direction $\boldsymbol{\Omega}$, in contrast with the $I^K_Q$ that only depend on $\theta$.

The components $I^K_Q$ for $K,Q \neq 0$ are complex numbers. It is possible to rewrite the expansion of the Stokes parameters in terms of real components. Using the conjugation property in Eq. (24), we can define for $K=1,2$ the real components

$$I^{xK}_Q = \frac{1}{2}[I^K_Q + (-1)^Q I^K_{-Q}] = \Re(I^K_Q), \quad Q > 0, \tag{25}$$

$$I^{yK}_Q = -\frac{\rm i}{2}[I^K_Q - (-1)^Q I^K_{-Q}] = \Im(I^K_Q), \quad Q > 0. \tag{26}$$

Combining Eqs. (21), (25), and (26) with explicit expressions of $\mathcal{T}^K_Q(i,\boldsymbol{\Omega})$, we obtain the expansions given in Eqs. (B.1) to (B.4) for the Stokes parameters.

The expansion of the Stokes parameters, written in compact form in Eq. (21) and in expanded form in Eqs. (B.1) to (B.4), is the main result of this paper. These equations almost explicitly yield the $\theta$ dependence of the Stokes parameters since $I^{xK}_Q$ and $I^{yK}_Q$ have a slow variation with $\theta$ of the limb-darkening type. They also explicitly yield the azimuthal Fourier components of the Stokes parameters. The latter are easily related to the $I^K_Q$. We introduce azimuthal Fourier expansions:

$$I_i(\tau,x,\boldsymbol{\Omega}) = \sum_l \tilde{I}^{(l)}_i(\tau,x,\theta){\rm e}^{{\rm i}\,l\chi}, \quad -\infty < l < +\infty, \tag{27}$$

with $\tilde{I}^{(l)}_i$ the Fourier components. Now we point out that the spherical tensors (see Table 5.6 in LL04) can be written as

$$\mathcal{T}^K_Q(i,\boldsymbol{\Omega}) = \tilde{\mathcal{T}}^K_Q(i,\theta){\rm e}^{{\rm i}\,Q\chi}. \tag{28}$$

Identifying the expansions in Eqs. (21) and (27), we obtain

$$\tilde{I}^{(l)}_{\rm I}(\tau,x,\theta) = \sum_K \tilde{\mathcal{T}}^K_l(i,\theta) I^K_l(\tau,x,\mu), \quad -2 \le l \le 2. \tag{29}$$

As a final remark, we want to recall that $I_0^0$ is the solution of the unpolarized radiative transfer equation for the problem under consideration (see Eq. (11)). The component $I_0^2$ is of the order of magnitude of $S_0^2$, and the latter is controlled by the anisotropy of the radiation field. The others components $I_Q^K$ are about one order of magnitude smaller than $I_0^2$ (see e.g. Faurobert-Scholl 1991; Nagendra et al. 1998).

## 4. Non-LTE transfer equation for the tensors $I_Q^K$

It follows from Eqs. (22) and (23) that we have

$$\mu \frac{\partial I_Q^K}{\partial \tau} = \varphi(x) \left[ I_Q^K(\tau, x, \mu) - S_Q^K(\tau, x) \right], \tag{30}$$

where $S_Q^K$ is given in Eq. (20). To obtain a non-LTE transfer equation for $I_Q^K$, we must express $J_Q^K$ in terms of $I_Q^K$. Inserting Eq. (21) into (14), we obtain

$$(J_{Q'}^K)^*(\tau, x) = \sum_j \sum_{K'Q''} \oint (\mathcal{T}_{Q'}^K)^*(j, \mathbf{\Omega}') \mathcal{T}_{Q''}^{K'}(j, \mathbf{\Omega}') I_{Q''}^{K'}(\tau, x, \mu') \frac{\mathrm{d}\Omega'}{4\pi}. \tag{31}$$

The integration with respect to the azimuthal angle $\chi'$ only acts on the product $(\mathcal{T}_{Q'}^K)^* \mathcal{T}_{Q''}^{K'}$, since $I_Q^K$ depends on $\theta$ but not on $\chi$. The dependence of this product on $\chi'$ is of the form $\mathrm{e}^{-\mathrm{i}(Q'-Q'')\chi'}$ (see Eq. (28)). Hence, $Q' = Q''$ is necessary for the integration over $\chi'$ to be non zero. We thus obtain

$$(J_{Q'}^K)^*(\tau, x) = \sum_{K'} \frac{1}{2} \int_{-1}^{+1} \Psi_{Q'}^{K,K'}(\mu') I_{Q'}^{K'}(\tau, x, \mu') \, \mathrm{d}\mu', \tag{32}$$

where

$$\Psi_{Q'}^{K,K'}(\mu') = \sum_{j=0}^{j=3} (-1)^{Q'} \mathcal{T}_{-Q'}^K(j, \mathbf{\Omega}') \mathcal{T}_{Q'}^{K'}(j, \mathbf{\Omega}'). \tag{33}$$

It follows from the definition and conjugation property of $\mathcal{T}_Q^K(i, \mathbf{\Omega})$ that $\Psi_{Q'}^{K,K'}(\mu)$ are real quantities, even functions of $\mu$ and that they vanish unless $K$ and $K'$ are both even or both odd. Only 8 elements are non zero. They can be found in this paper (Appendix A) or LL04 (Appendix A.20) (see also Landi Degl'Innocenti et al. 1990; Frisch 1999). The $\Psi_{Q'}^{K,K'}(\mu)$ are a subset corresponding to $Q = Q'$ of the $\Gamma_{KQ,K'Q'}(\mathbf{\Omega})$ introduced in LL04 (Appendix A.20).

The set of Eqs. (20), (30), and (32) provides a non-LTE transfer equation for the $I_Q^K$. It has been used for numerical work (Faurobert-Scholl 1991; Nagendra et al. 1998, 1999, 2003; Fluri et al. 2003) and recently to investigate the Hanle effect due to a random magnetic field (Frisch 2006).

## 5. Formulation with vector and matrix notation

We now rewrite the results of the preceding sections in more compact expressions with the help of formal vectors and matrices. We focus on the results concerning the atmospheric reference frame and on linear polarization, i.e. $K = 0$ and $K = 2$. Circular polarization corresponding to $K = 1$ and linear polarization are decoupled unless the primary irradiation source $\boldsymbol{G}$ is circularly polarized. We briefly indicate how to include circular polarization.

**Table 1.** Definition of the index $l$ of the matrices $T^{\text{out}}$, also yielding the index $p$ of $T^{\text{in}}$.

| $l$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $KQ$ | 0, 0 | 2, 0 | 2, 1 | 2, −1 | 2, 2 | 2, −2 |

### 5.1. Factorization of the phase matrix

It has been known since Sekara (1963) that the Rayleigh phase matrix describing resonance polarization in a cylindrically symmetrical atmosphere can be factorized as the product of two 2 × 2 matrices that depend separately on the direction of the incoming and scattered photons. This factorization, which is not unique (see e.g. Van de Hulst 1980), has been used quite extensively in the literature (e.g. Stenflo & Stenholm 1976; Rees 1978; Ivanov 1995).

When cylindrical symmetry is broken, the three Stokes parameters $I$, $Q$, and $U$ are needed to describe linear polarization created by resonance scattering. Recently Loskutov (2004) has shown that the 3 × 3 scattering phase matrix can also be factorized as a product of two matrices depending separately on $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. This factorization is the starting point for the factorization of the Hanle phase matrix proposed by Ivanov (2004). These factorizations are easy to obtain with the help of the irreducible tensors $\mathcal{T}_Q^K$.

For resonance scattering, the elements of the Rayleigh phase matrix in the atmospheric reference frame can written according to Eq. (9) as

$$[P_{\mathrm{R}}]_{ij}(\mathbf{\Omega}, \mathbf{\Omega}') = \sum_{KQ} \mathcal{T}_Q^K(i, \mathbf{\Omega})(-1)^Q \mathcal{T}_{-Q}^K(j, \mathbf{\Omega}'). \tag{34}$$

This expression suggests the introduction of the 3 rows × 6 columns matrix $T^{\text{out}}(\mathbf{\Omega})$, the elements of which are defined as

$$T_{ml}^{\text{out}}(\mathbf{\Omega}) = \mathcal{T}_Q^K(i, \mathbf{\Omega}), \tag{35}$$

with $m = 0, 1, 2$ and $l = 0, \ldots, 5$. The correspondence between the index $l$ and the set of indices $K$ and $Q$ is shown in Table 1. The index $m$ corresponds to index $i$. We now introduce $T^{\text{in}}(\mathbf{\Omega})$, a 6 rows × 3 columns matrix, with elements defined by

$$T_{pq}^{\text{in}}(\mathbf{\Omega}) = (-1)^Q \mathcal{T}_{-Q}^K(j, \mathbf{\Omega}), \tag{36}$$

with $p = 0, \ldots, 5$ and $q = 0, 1, 2$ corresponding to $j$. The index $p$ is defined as the index $l$ (see Table 1). The factorization of the Rayleigh phase matrix is thus simply

$$P_{\mathrm{R}}(\mathbf{\Omega}, \mathbf{\Omega}') = T^{\text{out}}(\mathbf{\Omega}) T^{\text{in}}(\mathbf{\Omega}'). \tag{37}$$

Using the conjugation relation in Eq. (5), it is easy to verify that $T^{\text{in}}$ is the transpose conjugate of $T^{\text{out}}$. Equation (36) gives the relation between the elements $T_{p,q}^{\text{in}}(\mathbf{\Omega})$ and $\mathcal{T}_Q^K(i, \mathbf{\Omega})$. The elements of $T^{\text{in}}$ are given in Table 2 for a reference angle $\gamma = 0$ (see LL04, Table 5.6). This angle defines the direction of positive $Q$. Its value is irrelevant for the factorization of the phase matrix. For resonance polarization with cylindrical symmetry, $T^{\text{in}}$ and $T^{\text{out}}$ reduce to 2 × 2 matrices. The indices $p$ and $q$ take the values 0 and 1 only.

For the Hanle effect (see Eq. (9)), if we only consider linear polarization, we can write the 3 × 3 phase matrix as

$$P(\mathbf{\Omega}, \mathbf{\Omega}'; \boldsymbol{B}) = T^{\text{out}}(\mathbf{\Omega}) \hat{N}(\boldsymbol{B}) T^{\text{in}}(\mathbf{\Omega}'). \tag{38}$$

The magnetic kernel $\hat{N}(\boldsymbol{B})$ is a 6 × 6 matrix. Its elements $\hat{N}_{l,p}$ are equal to $\mathcal{N}_{QQ'}^K$ with the indices $l$ and $p$ defined as in Table 1 ($l$ corresponds to $KQ$ and $p$ to $KQ'$). Henceforth all 6 × 6 matrices

**Table 2.** Definition of the elements $T^{\rm in}_{p,q}(\mathbf{\Omega})$, $p = 0, 1, 2$; $q = 0, \ldots, 5$.

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $1$ | $0$ | $0$ |
| 1 | $\frac{1}{2\sqrt{2}}(3\cos^2\theta - 1)$ | $-\frac{3}{2\sqrt{2}}\sin^2\theta$ | $0$ |
| 2 | $-\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\chi}$ | $-\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{\mathrm{i}\chi}$ | $\mathrm{i}\,\frac{\sqrt{3}}{2}\sin\theta\,\mathrm{e}^{\mathrm{i}\chi}$ |
| 3 | $\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{-\mathrm{i}\chi}$ | $\frac{\sqrt{3}}{2}\sin\theta\cos\theta\,\mathrm{e}^{-\mathrm{i}\chi}$ | $\mathrm{i}\,\frac{\sqrt{3}}{2}\sin\theta\,\mathrm{e}^{-\mathrm{i}\chi}$ |
| 4 | $\frac{\sqrt{3}}{4}\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\chi}$ | $-\frac{\sqrt{3}}{4}(1+\cos^2\theta)\,\mathrm{e}^{2\mathrm{i}\chi}$ | $2\mathrm{i}\,\frac{\sqrt{3}}{4}\cos\theta\,\mathrm{e}^{2\mathrm{i}\chi}$ |
| 5 | $\frac{\sqrt{3}}{4}\sin^2\theta\,\mathrm{e}^{-2\mathrm{i}\chi}$ | $-\frac{\sqrt{3}}{4}(1+\cos^2\theta)\,\mathrm{e}^{-2\mathrm{i}\chi}$ | $-2\mathrm{i}\,\frac{\sqrt{3}}{4}\cos\theta\,\mathrm{e}^{-2\mathrm{i}\chi}$ |

will carry a hat. Actually $\hat{N}(\boldsymbol{B})$ is a block diagonal matrix. The first block contains $\mathcal{N}^0_{00} = \hat{N}_{00}$. Thus all the elements of the first line and first column are zero, except $\hat{N}_{00}$. The second block, containing the elements $\mathcal{N}^2_{QQ'}$, is a $5 \times 5$ matrix.

To take circular polarization into account, it suffices to include the elements corresponding to $K = 1$ in $T^{\rm in}$, $T^{\rm out}$, and $\hat{N}(\boldsymbol{B})$. The matrix $T^{\rm out}$ becomes a $4 \times 9$ matrix, $T^{\rm in} = (T^{\rm out})^\dagger$ a $9 \times 4$ matrix, and $\hat{N}(\boldsymbol{B})$ a $9 \times 9$ matrix, containing a $3 \times 3$ block with the elements $\mathcal{N}^1_{QQ'}$ $(Q, Q' = 0, \pm 1)$.

### 5.2. Matrix decomposition of the magnetic kernel $\hat{N}(\boldsymbol{B})$

We now consider the atomic model described in Sect. 2.1. We introduce the identity matrix, denoted $\hat{\mathcal{I}}$, a diagonal matrix $\hat{\mathcal{E}}$ with $\hat{\mathcal{E}}_{00} = \epsilon/(1+\epsilon)$, and $\hat{\mathcal{E}}_{kk} = (\epsilon + \delta^{(2)}_{\rm u})/(1 + \epsilon + \delta^{(2)}_{\rm u})$ for $k = 1, \ldots, 5$, and another diagonal matrix $\hat{W}$ with $\hat{W}_{00} = W_0$ and $\hat{W}_{kk} = W_2$ for $k = 1, \ldots, 5$. The dependence on the magnetic field can be represented by a matrix $\hat{M}(\theta_{\rm B}, \chi_{\rm B}, H)$ where $\theta_{\rm B}$ and $\chi_{\rm B}$ are the polar angles of the magnetic field and $H$ the efficiency factor. Its elements $\mathcal{M}^K_{QQ'}$ are given in Eq. (10) with $X_{KQ''} = 1/(1 + \mathrm{i}\,Q''H)$. For the two-level atom considered in Sect. 2.1,

$$\hat{N}(\boldsymbol{B}) = (\hat{\mathcal{I}} - \hat{\mathcal{E}})\hat{W}\hat{M}(\theta_{\rm B}, \chi_{\rm B}, H'_{\rm u}). \tag{39}$$

Equation (10) shows that $\hat{M}$ can be factorized as a product of $6 \times 6$ matrices, namely

$$\hat{M}(\theta_{\rm B}, \chi_{\rm B}, H) = \hat{U}(\chi_{\rm B})\hat{m}(\theta_{\rm B}, H)\hat{U}(-\chi_{\rm B}), \tag{40}$$

where

$$\hat{m}(\theta_{\rm B}, H) = \hat{D}(\theta_{\rm B})[\hat{\mathcal{I}} + \mathrm{i}\,H\hat{\boldsymbol{Q}}]^{-1}\hat{D}(-\theta_{\rm B}), \tag{41}$$

$$\hat{U}(\chi_{\rm B}) = \mathrm{diag}[1, 1, \mathrm{e}^{-\mathrm{i}\chi_{\rm B}}, \mathrm{e}^{\mathrm{i}\chi_{\rm B}}, \mathrm{e}^{-2\mathrm{i}\chi_{\rm B}}, \mathrm{e}^{2\mathrm{i}\chi_{\rm B}}], \tag{42}$$

$$\hat{\boldsymbol{Q}} = \mathrm{diag}[0, 0, 1, -1, 2, -2]. \tag{43}$$

The matrix $\hat{D}(\theta_{\rm B})$ is constructed with the elements $d^K_{QQ'}(\theta_{\rm B})$. The Hanle phase matrix $P_{\rm H}(\mathbf{\Omega}, \mathbf{\Omega}'; \boldsymbol{B})$ is given by Eqs. (38) and (39) with $\hat{\mathcal{E}} = 0$. In the magnetic reference frame, one simply has $\hat{M} = [\hat{\mathcal{I}} + \mathrm{i}\,H\hat{\boldsymbol{Q}}]^{-1}$. When the magnetic field is zero, $\hat{M}(H)$ reduces to the identity matrix $\hat{\mathcal{I}}$.

### 5.3. Transfer equation in vector notation

We introduce the formal 6-component vector

$$\mathcal{I} = \{I_k\} = \{I^0_0, I^2_0, I^2_{+1}, I^2_{-1}, I^2_{+2}, I^2_{-2}\}, \tag{44}$$

with $k = 0, \ldots, 5$. We also introduce a vector $\mathcal{S}$ constructed with $S^K_Q$, a vector $\mathcal{G}$ constructed with $G^K_Q$, and a vector $\mathcal{J}$ constructed with $J^K_Q$. The decomposition of the Stokes vector $\boldsymbol{I}$ into its irreducible components (see Eq. (21)) may be written as

$$\boldsymbol{I}(\tau, x, \mathbf{\Omega}) = T^{\rm out}(\mathbf{\Omega})\mathcal{I}(\tau, x, \mu). \tag{45}$$

In vector notation, Eq. (30) becomes

$$\mu\frac{\partial \mathcal{I}}{\partial \tau} = \varphi(x)[\mathcal{I}(\tau, x, \mu) - \mathcal{S}(\tau, x)], \tag{46}$$

where

$$\mathcal{S}(\tau, x) = \mathcal{G}(\tau, x) + \hat{N}(\boldsymbol{B})\int_{-\infty}^{+\infty}\frac{g(x, x')}{\varphi(x)}(\mathcal{J})^*(\tau, x')\,\mathrm{d}x', \tag{47}$$

$$\mathcal{J}(\tau, x) = \frac{1}{2}\int_{-1}^{+1}\hat{\Psi}(\mu')\mathcal{I}(\tau, x, \mu')\,\mathrm{d}\mu', \tag{48}$$

and

$$\hat{\Psi}(\mu) = T^{\rm in}(\mathbf{\Omega})T^{\rm out}(\mathbf{\Omega}). \tag{49}$$

The elements of the $6 \times 6$ matrix $\hat{\Psi}(\mu)$ are the $\Psi^{KK'}_{Q'}(\mu)$ introduced in Eq. (33) (see also Appendix A).

In the same way as we have associated a 6-component vector $\mathcal{I}$ to the complex irreducible components $I^K_Q$, we can build a 6-component real vector $\mathcal{R}$ with the real components $I^K_0$, $I^{xK}_Q$, and $I^{yK}_Q$ $(Q > 0)$. The relation between the complex vector $\mathcal{I}$ and the real vector $\mathcal{R}$ can be written as $\mathcal{I} = \hat{T}\mathcal{R}$, with

$$\hat{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathrm{i} \\ 0 & 0 & 0 & 0 & 1 & -\mathrm{i} \end{bmatrix}. \tag{50}$$

The vector $\mathcal{R}$ satisfies the same transfer equation as $\mathcal{I}$ (see Eq. (46)) with $\hat{N}(\boldsymbol{B})$ changed to

$$\hat{N}^{\rm r}(\boldsymbol{B}) = \hat{T}^{-1}\hat{N}(\boldsymbol{B})\hat{T}. \tag{51}$$

The elements of $\hat{N}^{\rm r}(\boldsymbol{B})$ are real. All the matrices with superscript r have real elements.

For the example of Sect. 5.2, the matrix $\hat{N}^{\rm r}(\boldsymbol{B})$ is given by Eqs. (39) to (42) with $\hat{m}$ and $\hat{U}$ replaced by $\hat{m}^{\rm r}(\theta_{\rm B}, H) = \hat{T}^{-1}\hat{m}(\theta_{\rm B}, H)\hat{T}$, and $\hat{U}^{\rm r}(\chi_{\rm B}) = \hat{T}^{-1}\hat{U}(\chi_{\rm B})\hat{T}$. The elements of $\hat{m}^{\rm r}(\theta_{\rm B}, H)$ can be found in Appendix C. They were first given in Faurobert-Scholl (1991) (see also Nagendra et al. 1998).

### 5.4. Integral equation for the source vector

For a two-level atom, unpolarized ground level, and complete frequency redistribution, $S^K_Q$ is proportional to the density matrix element $\rho^K_Q$ (see e.g. Landi Degl'Innocenti & Bommier 1994; Manso Sainz & Trujillo Bueno 1999). Integral equations for the $S^K_Q$ or $\rho^K_Q$ have been given in e.g. Landi Degl'Innocenti et al. (1990), Landi Degl'Innocenti & Bommier (1994; see also Faurobert-Scholl 1991; Nagendra et al. 1998; Frisch 1999). Integral equations for partial frequency redistribution have been established by Faurobert-Scholl (1991) for the redistribution matrix written in Eq. (1) and by Bommier (2003) for a more general redistribution matrix, but only in the magnetic reference frame (i.e. magnetic field parallel to the normal to the atmosphere).

The integral equation for $S^K_Q$ can be deduced from the solution of the transfer equation for the Stokes parameters and the multipolar expansion of the source term $S_i(\tau, x, \mathbf{\Omega})$. However, it is easier to insert the formal solution of the transfer equation for $I^K_Q$ into the expression of the source term $S^K_Q$ given by Eqs. (20) and (32).

In vector notation the integral equation for $S_Q^K$ can be written as

$$\mathcal{S}(\tau, x) = \mathcal{G}(\tau, x) + \hat{N}(\boldsymbol{B}) \int_0^\infty \int_{-\infty}^{+\infty} \hat{K}(\tau-\tau', x, x') \mathcal{S}(\tau', x')\, \mathrm{d}x'\, \mathrm{d}\tau', \quad (52)$$

where

$$\hat{K}(\tau, x, x') = \frac{g(x, x')\varphi(x')}{\varphi(x)} \int_0^1 \mathrm{e}^{-|\tau|\varphi(x')/\mu'} \hat{\Psi}(\mu') \frac{\mathrm{d}\mu'}{2\mu'}. \quad (53)$$

For complete frequency redistribution, $\mathcal{S}$ satisfies a convolution type integral equation with a kernel depending on optical depth alone.

Real vectors can also be associated to $\mathcal{S}$ and $\mathcal{J}$ by applying the matrix $\hat{T}$. The real vector associated to $\mathcal{S}$ satisfies the integral Eq. (52) where $\hat{N}(\boldsymbol{B})$ is replaced by $\hat{N}^{\mathrm{r}}(\boldsymbol{B})$. The convolution integral equations in Faurobert-Scholl (1991) and Nagendra et al. (1998) are written for these real vectors.

Integral equations for source functions or source vectors have served as starting points for numerical methods of the operator perturbation type for complete and partial frequency redistribution. After being introduced in the late eighties for scalar non-LTE transfer problems (see e.g. Trujillo Bueno & Fabiani Bendicho 1995, and references therein), they were extended to resonance scattering (Faurobert-Scholl et al. 1997; Trujillo Bueno & Manso Sainz 1999) and later to the Hanle effect (Nagendra et al. 1998, 1999, 2003; Manso Sainz & Trujillo Bueno 1999, 2003; Fluri et al. 2003). These iterative methods require the formal solution of polarized transfer equations with a given source term. Clearly, the solution of Eq. (46) for the vector $\mathcal{I}$ is simpler than the solution of Eq. (12) for the Stokes parameters.

## 6. Frequency-dependent phase matrix

We explain in Sect. 6.1 how the results of the preceding sections can be generalized to redistribution matrices with a frequency-dependent phase matrix $P(\boldsymbol{\Omega}, \boldsymbol{\Omega}', x, x'; \boldsymbol{B})$. The method is illustrated in Sect. 6.2 with the "level-III" approximation of Bommier (1997) for partial frequency redistribution.

### 6.1. Stokes vector decomposition

We assume that the redistribution matrix elements $R_{ij}$ can be written as on the righthand side of Eq. (9) but that $\mathcal{N}_{QQ'}^K$ is frequency dependent. In matrix notation, we now have

$$R(x, \boldsymbol{\Omega}, x', \boldsymbol{\Omega}'; \boldsymbol{B}) = T^{\mathrm{out}}(\boldsymbol{\Omega})\hat{N}(x, x'; \boldsymbol{B})T^{\mathrm{in}}(\boldsymbol{\Omega}'). \quad (54)$$

For conveniency we have also included the scalar redistribution function $g(x, x')$ in $\hat{N}$. Since the redistribution matrix depends on $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ separately, we can expand the components $S_i(\tau, x, \boldsymbol{\Omega})$ of the source vector as in Eq. (19) and the Stokes parameters as in Eq. (21).

The non-LTE transfer equation for the 6-component formal vector $\mathcal{I}$ can be written as in Eq. (46) with

$$\mathcal{S}(\tau, x) = \mathcal{G}(\tau, x) + \int_{-\infty}^{+\infty} \frac{1}{\varphi(x)} \hat{N}(x, x'; \boldsymbol{B})(\mathcal{J})^*(\tau, x')\, \mathrm{d}x', \quad (55)$$

where $\mathcal{J}$ is defined as in Eq. (48).

We can still write an integral equation for $\mathcal{S}(\tau, x)$. It is now of the form

$$\mathcal{S}(\tau, x) = \mathcal{G}(\tau, x) + \int_0^\infty \int_{-\infty}^{+\infty} \hat{K}(\tau-\tau', x, x'; \boldsymbol{B})\, \mathcal{S}(\tau', x')\, \mathrm{d}x'\, \mathrm{d}\tau', \quad (56)$$

where

$$\hat{K}(\tau, x, x'; \boldsymbol{B}) = \hat{N}(x, x'; \boldsymbol{B}) \frac{\varphi(x')}{\varphi(x)} \int_0^1 \mathrm{e}^{-|\tau|\varphi(x')/\mu'} \hat{\Psi}(\mu') \frac{\mathrm{d}\mu'}{2\mu'}. \quad (57)$$

### 6.2. Partial frequency redistribution with angle-averaged frequency redistribution function

As mentioned in Sect. 1, Bommier (1997) has proposed approximate redistribution matrices for the weak-field Hanle effect. The "level-III" approximation (with angle-averaged redistribution function) is employed in Fluri et al. (2003) to analyze the effects of elastic and inelastic collisions on the polarization. In this work, the Stokes vector is expanded in cylindrically symmetrical components, and this decomposition is used to set up an ALI method (accelerated lambda iteration method). A simple rectangular frequency domain decomposition is introduced for the specific needs of the ALI method, but the final results correspond to the original domains of Bommier (1997). The simplified decomposition respects the number and overall shapes of the different regions. Original and simplified decompositions are displayed in Figs. 1 and 2 of Fluri et al. (2003).

Here we propose a simple expression of the magnetic kernel $\hat{N}(x, x'; \boldsymbol{B})$ for the "level-III" redistribution matrix with rectangular domain decomposition. Following Bommier (1997), we write the redistribution matrix as

$$R(x, \boldsymbol{\Omega}, x', \boldsymbol{\Omega}'; \boldsymbol{B}) = R_{\mathrm{III}}(x, \boldsymbol{\Omega}, x', \boldsymbol{\Omega}'; \boldsymbol{B}) + R_{\mathrm{II}}(x, \boldsymbol{\Omega}, x', \boldsymbol{\Omega}'; \boldsymbol{B}). \quad (58)$$

The term with $R_{\mathrm{III}}$ corresponds to complete frequency redistribution, and the term with $R_{\mathrm{II}}$ to frequency coherent scattering (in the atomic reference frame). The contribution from coherent scattering is essential for interpreting strong resonance lines.

For $R_{\mathrm{III}}$, the rectangular domain decomposition can be described by a function

$$f_{\mathrm{III}}(x) = \begin{cases} 0, & |x| \le x_{\mathrm{c}} \\ 1, & |x| > x_{\mathrm{c}}, \end{cases} \quad (59)$$

and for $R_{\mathrm{II}}$ by a function

$$f_{\mathrm{II}}(x, x') = \begin{cases} 0, & |x| \text{ and } |x'| \le x_{\mathrm{c}} \\ 1, & |x| \text{ or } |x'| > x_{\mathrm{c}}. \end{cases} \quad (60)$$

Here $x_{\mathrm{c}}$ is a cut-off frequency between core and wings that can be defined by $\sqrt{\pi} \exp(-x_{\mathrm{c}}^2) \simeq a/(a^2 + x_{\mathrm{c}}^2)$ with $a$ the line profile damping parameter.

In the magnetic reference frame, the elements of the redistribution matrices take the form

$$[R_{\mathrm{III,II}}]_{ij} = \bar{r}_{\mathrm{III,II}}(x, x') \times \sum_{KQ} X_{KQ}^{\mathrm{III,II}}(x, x'; B)(-1)^Q \mathcal{T}_Q^K(i, \boldsymbol{\Omega})\mathcal{T}_{-Q}^K(j, \boldsymbol{\Omega}'), \quad (61)$$

with $\bar{r}_{\mathrm{II}}$ and $\bar{r}_{\mathrm{III}}$ the Hummer (1962) angle-averaged scalar frequency redistribution functions. The magnetic kernels $X_{KQ}^{\mathrm{III,II}}$, which are now frequency-dependent, can be written as

$$X_{KQ}^{\rm III} = W_{\rm K}(J_{\rm l}, J_{\rm u})\Big[\frac{\beta^{(K)}}{1+{\rm i}\,Q\beta^{(K)}H_{\rm u}} - \frac{\alpha}{1+{\rm i}\,Q\alpha H_{\rm u}}\Big] \times\Big[1+{\rm i}\,Q\alpha H_{\rm u} f_{\rm III}(x)\Big]\Big[1+{\rm i}\,Q\alpha H_{\rm u} f_{\rm III}(x')\Big] \quad (62)$$

and

$$X_{KQ}^{\rm II} = W_{\rm K}(J_{\rm l}, J_{\rm u})\Big[\frac{\alpha}{1+{\rm i}\,Q\alpha H_{\rm u}}\Big]\times\Big[1+{\rm i}\,Q\alpha H_{\rm u} f_{\rm II}(x,x')\Big]. \quad (63)$$

The definitions of $\beta^{(K)}$ and $\alpha$ are $\beta^{(K)} = 1/(1+\epsilon+\delta_{\rm u}^{(K)})$ and $\alpha = 1/(1+\epsilon+\epsilon_{\rm E})$, where $\epsilon_{\rm E} = \Gamma_{\rm E}/A_{\rm u,l}$, with $\Gamma_{\rm E}$ the elastic collision rate (see Bommier 1997).

For $R_{\rm III}$ there are three different regions: region 1 corresponding to the line core ($|x|, |x'| \le x_{\rm c}$), region 2 corresponding to $|x|$ in the line core and $|x'|$ outside the core, and vice-versa, and finally region 3 with both $|x|$ and $|x'|$ outside the core. For $R_{\rm II}$ we have two different regions corresponding to $|x|, |x'| \le x_{\rm c}$ (line core) and $|x|, |x'| > x_{\rm c}$ (line wings). They are referred to as regions 4 and 5 in Fluri et al. (2003).

We now give the redistribution matrix in the atmospheric reference frame, using matrix notation. For this purpose we introduce a diagonal matrix $\hat{\mathcal{B}}$ with elements $\hat{\mathcal{B}}_{00} = \beta^{(0)}$ and $\hat{\mathcal{B}}_{kk} = \beta^{(2)}$ for all $k = 1, \ldots, 5$, and a diagonal matrix $\hat{\mathcal{F}}$ with elements $\hat{\mathcal{F}}_{00} = 1 - \alpha/\beta^{(0)}$ and $\hat{\mathcal{F}}_{kk} = 1 - \alpha/\beta^{(2)}$ for all $k = 1, \ldots, 5$. To simplify notation, we use $\hat{M}(H)$ as shorthand for $\hat{M}(\theta_{\rm B}, \chi_{\rm B}, H)$ (see Eq. (40)) and $H$ instead of $H_{\rm u}$.

The matrix $\hat{N}(x, x'; \boldsymbol{B})$ in Eq. (54) can be written as

$$\hat{N}(x,x';\boldsymbol{B}) = \hat{W}\Big[\bar{r}_{\rm II}(x,x')\hat{M}_{\rm II} + \bar{r}_{\rm III}(x,x')\hat{M}_{\rm III}\Big]. \quad (64)$$

For $R_{\rm III}$ scattering,

$$\hat{M}_{\rm III} = \hat{M}_{\rm III}^{(1)} + \hat{M}_{\rm III}^{(2)} + \hat{M}_{\rm III}^{(3)}, \quad (65)$$

where the superscripts refer to the three different regions introduced above. The corresponding phase matrices can be written as

$$\hat{M}_{\rm III}^{(1)} = [1 - f_{\rm II}(x,x')]\Big[\hat{\mathcal{B}}\hat{M}(\beta^{(2)}H) - \alpha\hat{M}(\alpha H)\Big], \quad (66)$$

$$\hat{M}_{\rm III}^{(2)} = [f_{\rm III}(x) + f_{\rm III}(x')][1 - f_{\rm III}(x) f_{\rm III}(x')] \times[\hat{\mathcal{B}} - \alpha\hat{\mathcal{I}}]\hat{M}(\beta^{(2)}H), \quad (67)$$

$$\hat{M}_{\rm III}^{(3)} = f_{\rm III}(x) f_{\rm III}(x')\hat{\mathcal{F}}\Big[[\hat{\mathcal{B}} - \alpha\hat{\mathcal{I}}]M(\beta^{(2)}H) + \alpha\hat{\mathcal{I}}\Big]. \quad (68)$$

Equations (66)–(68) correspond to Eqs. (107), (109), and (110) in Bommier (1997). Equation (68) shows that in the wings, i.e. when both $|x|$ and $|x'|$ are large, the Hanle effect does not disappear when elastic collisions are taken into account (see also LL04, Sect. 10.6, p. 534).

For $R_{\rm II}$ scattering,

$$\hat{M}_{\rm II} = \alpha\Big[[1 - f_{\rm II}(x,x')]\hat{M}(\alpha H) + f_{\rm II}(x,x')\hat{\mathcal{I}}\Big]. \quad (69)$$

We recover Eqs. (112) and (113) in Bommier (1997) with the Hanle effect in the line core and Rayleigh scattering outside the core. When $\epsilon_{\rm E} = \delta_{\rm u}^{(2)} = 0$, all the matrices $\hat{M}_{\rm III}^{(j)}$, $j = 1, 2, 3$, are zero, hence frequency redistribution is of the $R_{\rm II}$ type. If $\alpha = 0$, redistribution is purely of the $R_{\rm III}$ type. These limiting cases are discussed in detail in Bommier (1997).

When angle-averaging is relaxed ("level-II" approximation), the elements of the redistribution matrix are given by Eq. (61) with the angle-dependent redistribution functions $r_{\rm III,II}(x, x', \Theta)$. One can still write a multipolar expansion similar to Eq. (19) for the components $S_i(\tau, x, \boldsymbol{\Omega})$ of the source vector, but now $S_Q^K$ depends on $\boldsymbol{\Omega}$. A generalization of some of the results obtained in the preceding sections, like the construction of an integral equation for the $S_Q^K$, may still be possible. This generalization is beyond the scope of the present work, but we plan to consider it in the future.

We note that in the case of complete frequency redistribution, the variation of the Hanle effect along the line profile (see e.g. LL04, p. 504) can be taken into account heuristically by a decomposition of the frequency space. One can, for example, use the same decomposition as for $R_{\rm II}$. Then according to Eq. (39), the matrix $\hat{N}(x, x'; \boldsymbol{B})$ becomes

$$\hat{N}(x,x';\boldsymbol{B}) = (\hat{\mathcal{I}} - \hat{\mathcal{E}})\hat{W}\varphi(x)\varphi(x') \times\Big[[1 - f_{\rm II}(x,x')]\hat{M}(\theta_{\rm B},\chi_{\rm B},H'_{\rm u}) + f_{\rm II}(x,x')\hat{\mathcal{I}}\Big]. \quad (70)$$

To make contact with the notation used for partial redistribution, we note that $(\hat{\mathcal{I}} - \hat{\mathcal{E}}) = \hat{\mathcal{B}}$ and $H'_{\rm u} = \beta^{(2)}H$.

## 7. Concluding remarks

In Sects. 2 to 6 we have considered redistribution matrices in which the scalar frequency redistribution function is independent of the direction of the incoming and scattered beams. As pointed out in Sect. 6, the structure of the redistribution matrix will undergo a drastic change if the frequency redistribution function has the form $r(x, x', \Theta)$ with $\Theta$ the angle between $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$ and some of the results of the preceding sections will certainly no longer hold. However, when this function has azimuthal symmetry, i.e. of the form $r(x, \theta, x', \theta')$, with $\theta$ and $\theta'$ the inclination of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}'$, the results of the preceding sections are easy to generalize. Such azimuthally symmetrical frequency redistribution functions arise naturally as a zeroth-order term in azimuthal Fourier expansions of redistribution functions (see Domke & Hubeny 1988; also Frisch et al. 2001). The proof of the Stokes vector decomposition proceeds as described in Sect. 3, except that the integration over the full solid angle in Eq. (14) must be replaced by averaging over the azimuthal angle $\chi$. The multipolar components $S_Q^K$ of the source vector now depend on the angle $\theta$. The Stokes parameters still satisfy the expansion written in Eq. (21) where the $I_Q^K$ now depend on $\theta$ and not only $\cos\theta$.

To simplify proofs and notations, we have assumed that the atmospheric parameters (e.g. temperature) and magnetic field are independent of space. The magnetic field in the weak-field Hanle limit has a purely local action (it does not enter into the absorption coefficient); hence, uniformity can be relaxed, as long as the redistribution matrix can be factorized as in Eq. (54). Of course, if the temperature is varying in the medium, the Doppler width will not be constant, and the dimensionless frequency variable $x$ cannot be used anymore.

For a correct treatment of spectral line polarization, continuum polarization has to be taken into account. In the visible wavelength range, continuum polarization is due to Thomson scattering by free electrons and Rayleigh scattering by hydrogen molecules. For both processes, the scattering phase matrix is the Rayleigh one, and we have seen that $P_{\rm R}(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = T^{\rm out}(\boldsymbol{\Omega})T^{\rm in}(\boldsymbol{\Omega}')$. Therefore the total source function (with line and continuum contributions) can also be written as in Eq. (19), and the proof of the decomposition of the Stokes parameters in irreducible components remains unchanged.

From a numerical point of view, it is definitely more interesting to work with the cylindrically symmetrical irreducible components $I^K_Q$, whenever possible, than with the Stokes parameters themselves. For example, with operator type perturbation methods, a formal solution of the radiative transfer equation with a known source function is needed at each step in the iterative process. A 1D formal solver is sufficient when working with the $I^K_Q$ (see e.g. Nagendra et al. 1998; Fluri et al. 2003). A somewhat more elaborate solution method is needed to solve the transfer equation for the Stokes parameters (e.g. Fabiani Bendicho & Trujillo Bueno 1999; also Landi Degl'Innocenti et al. 1990).

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## Appendix A: Elements of the matrix $\hat{\mathbf{\Psi}}(\mu)$

$$\hat{\Psi}_{11} = \Psi^{00}_0 = 1, \tag{A.1}$$

$$\hat{\Psi}_{12} = \hat{\Psi}_{21} = \Psi^{20}_0 = \frac{1}{2\sqrt{2}}(3\mu^2 - 1), \tag{A.2}$$

$$\hat{\Psi}_{22} = \Psi^{22}_0 = \frac{1}{4}(5 - 12\mu^2 + 9\mu^4), \tag{A.3}$$

$$\hat{\Psi}_{33} = \hat{\Psi}_{44} = \Psi^{22}_1 = \frac{3}{4}(1 - \mu^2)(1 + 2\mu^2), \tag{A.4}$$

$$\hat{\Psi}_{55} = \hat{\Psi}_{66} = \Psi^{22}_2 = \frac{3}{8}(1 + \mu^2)^2. \tag{A.5}$$

For completeness we mention that the $\Psi^{K,K'}_{Q'}(\mu)$ corresponding to circular polarization are $\Psi^{11}_0(\mu) = 3\mu^2/2$ and $\Psi^{11}_1(\mu) = 3(1 - \mu^2)/4$.

## Appendix B: Expansion of the Stokes parameters in real irreducible components

Combining Eqs. (21), (25), and (26) of the text with explicit expressions of the $\mathcal{T}^K_Q(i, \mathbf{\Omega})$ for the reference angle $\gamma = 0$, we obtain

$$\begin{aligned} I(\tau, x, \mathbf{\Omega}) = {} & I^0_0 + \frac{1}{2\sqrt{2}}(3\cos^2\theta - 1)\, I^2_0 \\ & - \sqrt{3}\cos\theta\sin\theta\,(I^{x2}_1\cos\chi - I^{y2}_1\sin\chi) \\ & + \frac{\sqrt{3}}{2}(1 - \cos^2\theta)(I^{x2}_2\cos 2\chi - I^{y2}_2\sin 2\chi), \end{aligned} \tag{B.1}$$

$$\begin{aligned} Q(\tau, x, \mathbf{\Omega}) = {} & -\frac{3}{2\sqrt{2}}(1 - \cos^2\theta)\, I^2_0 \\ & - \sqrt{3}\cos\theta\sin\theta\,(I^{x2}_1\cos\chi - I^{y2}_1\sin\chi) \\ & - \frac{\sqrt{3}}{2}(1 + \cos^2\theta)(I^{x2}_2\cos 2\chi - I^{y2}_2\sin 2\chi), \end{aligned} \tag{B.2}$$

$$\begin{aligned} U(\tau, x, \mathbf{\Omega}) = {} & \sqrt{3}\sin\theta\,(I^{x2}_1\sin\chi + I^{y2}_1\cos\chi) \\ & + \sqrt{3}\cos\theta\,(I^{x2}_2\sin 2\chi + I^{y2}_2\cos 2\chi), \end{aligned} \tag{B.3}$$

$$V(\tau, x, \mathbf{\Omega}) = \sqrt{\frac{3}{2}}\cos\theta\, I^1_0 - \sqrt{3}\sin\theta\,(I^{x1}_1\cos\chi - I^{y1}_1\sin\chi). \tag{B.4}$$

To simplify the notation, the dependence on $\tau$, $x$, $\mu$ of the functions $I^K_0$, $I^{xK}_Q$, and $I^{yK}_Q$ has not been explicitly indicated. Similar expansions have been given in Nagendra et al. (1998) for $I$, $Q$, and $U$ and, for the corresponding source functions, in Bommier et al. (1991) and Manso Sainz & Trujillo Bueno (1999).

## Appendix C: Elements of the matrix $\hat{\mathbf{m}}^{\mathrm{r}}(\mathbf{B})$

It follows from the symmetries of $\mathcal{M}^K_{QQ'}$ that the $6 \times 6$ matrix $\hat{m}^{\mathrm{r}}(\theta_{\mathrm{B}}, H)$ introduced in Sect. 5.3 has the form

$$\hat{m}^{\mathrm{r}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & 2m_{12} & m_{22} & m_{23} & m_{24} & m_{25} \\ 0 & -2m_{13} & -m_{23} & m_{33} & m_{34} & m_{35} \\ 0 & 2m_{14} & m_{24} & -m_{34} & m_{44} & m_{45} \\ 0 & -2m_{15} & -m_{25} & m_{35} & -m_{45} & m_{55} \end{bmatrix}, \tag{C.1}$$

where

$$\begin{aligned}
m_{11} &= \Re(\mathcal{M}^2_{00}) = 1 - 3S^2_{\mathrm{B}}H^2\Big[\frac{C^2_{\mathrm{B}}}{1 + H^2} + \frac{S^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{12} &= \Re(\mathcal{M}^2_{01}) = -\sqrt{\frac{3}{2}}C_{\mathrm{B}}S_{\mathrm{B}}H^2\Big[\frac{2C^2_{\mathrm{B}} - 1}{1 + H^2} + \frac{2S^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{13} &= -\Im(\mathcal{M}^2_{01}) = \sqrt{\frac{3}{2}}S_{\mathrm{B}}H\Big[\frac{C^2_{\mathrm{B}}}{1 + H^2} + \frac{S^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{14} &= \Re(\mathcal{M}^2_{02}) = \sqrt{\frac{3}{2}}S^2_{\mathrm{B}}H^2\Big[\frac{C^2_{\mathrm{B}}}{1 + H^2} - \frac{1 + C^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{15} &= -\Im(\mathcal{M}^2_{02}) = -\sqrt{\frac{3}{2}}S^2_{\mathrm{B}}C_{\mathrm{B}}H\Big[\frac{1}{1 + H^2} - \frac{1}{1 + 4H^2}\Big], \\
m_{22} &= \Re(\mathcal{M}^2_{11} - \mathcal{M}^2_{1-1}) = 1 - H^2\Big[\frac{(1 - 2C^2_{\mathrm{B}})^2}{1 + H^2} + \frac{4S^2_{\mathrm{B}}C^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{23} &= -\Im(\mathcal{M}^2_{11}) = -C_{\mathrm{B}}H\Big[\frac{1 - 2C^2_{\mathrm{B}}}{1 + H^2} - \frac{2S^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{24} &= \Re(\mathcal{M}^2_{12} + \mathcal{M}^2_{1-2}) = -C_{\mathrm{B}}S_{\mathrm{B}}H^2\Big[\frac{1 - 2C^2_{\mathrm{B}}}{1 + H^2} + \frac{2(1 + C^2_{\mathrm{B}})}{1 + 4H^2}\Big], \\
m_{25} &= -\Im(\mathcal{M}^2_{12} - \mathcal{M}^2_{1-2}) = S_{\mathrm{B}}H\Big[\frac{1 - 2C^2_{\mathrm{B}}}{1 + H^2} + \frac{2C^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{33} &= \Re(\mathcal{M}^2_{11} + \mathcal{M}^2_{1-1}) = 1 - H^2\Big[\frac{C^2_{\mathrm{B}}}{1 + H^2} + \frac{4S^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{34} &= \Im(\mathcal{M}^2_{12} + \mathcal{M}^2_{1-2}) = S_{\mathrm{B}}H\Big[\frac{C^2_{\mathrm{B}}}{1 + H^2} - \frac{1 + C^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{35} &= \Re(\mathcal{M}^2_{12} - \mathcal{M}^2_{1-2}) = C_{\mathrm{B}}S_{\mathrm{B}}H^2\Big[\frac{1}{1 + H^2} - \frac{4}{1 + 4H^2}\Big], \\
m_{44} &= \Re(\mathcal{M}^2_{22} + \mathcal{M}^2_{2-2}) = 1 - H^2\Big[\frac{C^2_{\mathrm{B}}S^2_{\mathrm{B}}}{1 + H^2} + \frac{(1 + C^2_{\mathrm{B}})^2}{1 + 4H^2}\Big], \\
m_{45} &= -\Im(\mathcal{M}^2_{22} - \mathcal{M}^2_{2-2}) = C_{\mathrm{B}}H\Big[\frac{S^2_{\mathrm{B}}}{1 + H^2} + \frac{1 + C^2_{\mathrm{B}}}{1 + 4H^2}\Big], \\
m_{55} &= \Re(\mathcal{M}^2_{22} - \mathcal{M}^2_{2-2}) = 1 - H^2\Big[\frac{S^2_{\mathrm{B}}}{1 + H^2} + \frac{4C^2_{\mathrm{B}}}{1 + 4H^2}\Big],
\end{aligned} \tag{C.2}$$

with

$$C_{\mathrm{B}} = \cos\theta_{\mathrm{B}}, \quad S_{\mathrm{B}} = \sin\theta_{\mathrm{B}}. \tag{C.3}$$

The elements $M_{kk'}$ with $k = 1, 2$, and $k' = 3, 4, 5$ have a different sign from the elements $M_{kk'}$ in Nagendra et al. (1998) because of a different sign for some of the $I^K_Q$. The expressions in Eq. (C.2) can also be expressed in terms of the two Hanle angles $\alpha_1$ and $\alpha_2$ defined by

$$\cos\alpha_1 = \frac{1}{\sqrt{1+H^2}}; \quad \sin\alpha_1 = \frac{H}{\sqrt{1+H^2}},$$
$$\cos\alpha_2 = \frac{1}{\sqrt{1+4H^2}}; \quad \sin\alpha_2 = \frac{2H}{\sqrt{1+4H^2}}. \qquad \text{(C.4)}$$

The matrix $\hat{U}^{\rm r}(\chi_{\rm B}) = \hat{T}^{-1}\hat{U}(\chi_{\rm B})\hat{T}$ introduced in Sect. 5.3 may be written as

$$\hat{U}^{\rm r} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\chi_{\rm B} & \sin\chi_{\rm B} & 0 & 0 \\ 0 & 0 & -\sin\chi_{\rm B} & \cos\chi_{\rm B} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\chi_{\rm B} & \sin 2\chi_{\rm B} \\ 0 & 0 & 0 & 0 & -\sin 2\chi_{\rm B} & \cos 2\chi_{\rm B} \end{bmatrix}. \qquad \text{(C.5)}$$

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