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Radiation-induced torques on spheroids

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ABSTRACT

Radiation-induced torques on ellipsoids of revolution are discussed. Exact formulae for the thermal YORP torques are given in terms of elliptic integrals. It is demonstrated that in the absence of thermal inertia, the average values of dynamically significant projections of these torques are zero if a spheroid rotates around the axis of maximum inertia and if there are no resonances between rotation and orbital motion. The thermal lag leads to a systematic drift in the obliquity, but it does not affect the rotation period. The direct radiation pressure torques on spheroids are shown to be zero.

Key words. methods: analytical – celestial mechanics – minor planets, asteroids

1. Introduction

Radiation forces and torques on small bodies in the Solar System have attracted considerable attention during the past decade or so (e.g. Bottke et al. 2002, 2006). This is because, unlike gravitational perturbations, these effects can in the long-term permanently increase or decrease orbital and/or rotational energy. As a result, the orbital semimajor axis is also secularly changed and the body may migrate from one heliocentric zone to another. Similarly, rotation rate and obliquity of the spin axis could be permanently changed such that a normal rotator may be moved to the category of fast or slow, or even tumbling, rotators. Radiation forces and torques have been thus identified to drive the most important transport processes for small bodies in the Solar System. As an example, radiation torques are (i) a key element in explaining peculiar distribution of the rotation rate and the pole orientation of large-size asteroids in the Koronis family (e.g. Slivan 2002; Slivan et al. 2003; Vokrouhlický et al. 2003); (ii) assist depletion of the main asteroid belt to the planet crossing zone (e.g. Morbidelli & Vokrouhlický 2003); (iii) produce uneven distribution of small asteroids in the asteroid families (e.g. Vokrouhlický et al. 2006a,b,c); (iv) may be a viable mechanism of binary asteroid formation (e.g. Pravec & Harris 2007) or tumbling asteroids (e.g. Pravec et al. 2005; Vokrouhlický et al. 2007). The effects of radiation torques have been also directly measured (Lowry et al. 2007; Taylor et al. 2007; Kaasalainen et al. 2007) and many more detections are likely to result from large projects equipped with a stable photometry system such as Pan-STARRS (e.g. Ďurech et al. 2007).

The effects of radiation forces in thermal infrared, dubbed the Yarkovsky effect, have been thoroughly investigated by analytical methods (e.g. Rubincam 1995, 1998; Vokrouhlický 1998a, 1999), semi-analytical methods (e.g. Vokrouhlický & Farinella 1999) and using a numerical approach (e.g. Čapek & Vokrouhlický 2005). In contrast, the effects of radiation torques, in both optical and infrared wavebands, commonly called the Yarkovsky-O'Keefe-Radzievskii-Paddack (YORP) effect (Rubincam 2000), have been analysed so far mainly by numerical means (e.g. Rubincam 2000; Vokrouhlický & Čapek 2002; Čapek & Vokrouhlický 2004; Vokrouhlický et al. 2007). Scheeres (2007) developed a semi-analytical method which numerically precomputes coefficients of the YORP torque Fourier representation and then uses a linearized theory for rotation rate and pole orientation. So far no analytical analysis of the YORP effect has been developed. Several important YORP results acquired numerically to date thus are not well understood (e.g., why YORP ceases in the long term to act on bodies with ~55° and ~125° obliquity; what is the fundamental shape parameter that makes YORP operational; why YORP does not seem to affect bodies that have ellipsoidal shape, etc.).

In this paper, we attempt an analytical modeling of YORP. We consider YORP torques acting on ellipsoids of revolution (spheroids; Vokrouhlický (1998b) developed an analogous theory for the Yarkovsky force). Our goal is to analytically prove that YORP does not secularly change either rotation rate or obliquity.

Unlike the Yarkovsky effect, the radiation torques may operate even for a zero thermal inertia Γ of the surface (e.g. Rubincam 2000; Vokrouhlický & Čapek 2002). Hereafter we assume that the surface has $\Gamma = 0$, however we shall argue that our conclusions will be the same even in the $\Gamma \neq 0$ case. The radiation torques are generally due to both (i) the incoming sunlight and its surface reflection in the optical band, and (ii) the proper thermal radiation of the surface. The notion of YORP has been used vaguely in the past years, since some works assumed YORP was just the torque due to the thermal radiation of the surface, while others included also the torque due to surface scattered sunlight in the optical band.

After a brief discussion of three radiation-related effects in Sect. 2, we proceed with the major thermal component (Sect. 3;

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note the albedo of Solar System bodies is typically low so that most of the sunlight is absorbed), while in Sect. 4 we deal with the torques due to absorbed sunlight and its component scattered by the surface in the optical band.

2. Radiation-induced torques

The radiation flux reaching an infinitesimal surface element of a celestial body leads to three kinds of force d$\boldsymbol{f}$ that may influence the body rotation: direct radiation pressure, thermal radiation force, and scattered radiation force. In all cases the resulting torque $\boldsymbol{T}$ is obtained after integration over the whole body surface

$$\boldsymbol{T} = \int \boldsymbol{r} \times \mathrm{d}\boldsymbol{f}, \tag{1}$$

where the radius vector $\boldsymbol{r}$ points to the surface element responsible for d$\boldsymbol{f}$. In this paper we will consider torques $\boldsymbol{T}$ expressed in the "body frame" – a right-handed Cartesian system with the origin O at the centre of mass of the illuminated body and the axes aligned with the principal axes of inertia. The Ox axis (and its unit vector $\boldsymbol{e}_x$) will be aligned with the minimum inertia axis, whereas Oz and $\boldsymbol{e}_z$ will lie on the maximum inertia axis.

2.1. Direct radiation pressure

When photons hit an infinitesimal surface element dS, their momentum is transferred resulting in a force

$$\mathrm{d}\boldsymbol{f} = -\frac{\Phi}{v_\mathrm{c}} \max\left(0, \frac{\mathrm{d}\boldsymbol{S}}{||\mathrm{d}\boldsymbol{S}||} \cdot \boldsymbol{n}_\odot\right) \boldsymbol{n}_\odot \, \mathrm{d}S, \tag{2}$$

where Φ is the radiation flux at a given distance from the Sun, v_c is the speed of light, and $\boldsymbol{n}_\odot$ is the unit vector directed to the Sun. The vector d$\boldsymbol{S}$ has a length equal to the area of the surface element dS, and the direction of the local outward normal vector $\boldsymbol{n}$, i.e. $\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S$. Thus the maximum function in Eq. (2) cuts off this part of the body, where the Sun is below the horizon, and $\mathrm{d}\boldsymbol{S} \cdot \boldsymbol{n}_\odot = 0$, or $\boldsymbol{n} \cdot \boldsymbol{n}_\odot = 0$ is the implicit equation of terminator line. The unit vector directed to the Sun has the components in the body frame that will be either denoted as $x_\odot$, $y_\odot$, $z_\odot$, or expressed in terms of the polar variables: longitude $\lambda_\odot$ and the sine/cosine of latitude $s_\odot$, $c_\odot$

$$\boldsymbol{n}_\odot = \begin{pmatrix} x_\odot \\ y_\odot \\ z_\odot \end{pmatrix} = \begin{pmatrix} c_\odot \cos\lambda_\odot \\ c_\odot \sin\lambda_\odot \\ s_\odot \end{pmatrix}. \tag{3}$$

2.2. Radiation scattered in the optical band

Part of the incoming radiation – characterized by the "albedo coefficient" – is directly scattered by the surface in the optical waveband. In principle, this radiation component can also produce a net torque on the body. In what follows we give its brief description.

The interaction of sunlight with planetary surfaces is complicated, but well-studied. Let $\mu_\odot$ stand for a cosine of zenith angle measured from the local normal vector $\boldsymbol{n}$, i.e. $\mu_\odot = \boldsymbol{n} \cdot \boldsymbol{n}_\odot$); the radiation (specific) intensity I of the reflected sunlight at a local direction $\boldsymbol{n}_r$ (with μ_r a cosine of zenith angle measured from the local normal vector $\boldsymbol{n}$, i.e. $\mu_r = \boldsymbol{n} \cdot \boldsymbol{n}_r$) is given by

$$I(\boldsymbol{n}_r; \boldsymbol{n}_\odot) = I(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot) = \Phi\,\rho(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot). \tag{4}$$

Here Φ is the intensity (flux) of the incident solar radiation as above and $\rho(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot)$ is the reflectance (scattering) function. The classical formulations of ρ by Hapke and Lumme-Bowell are reviewed, for instance, by Bowell et al. (1989).

First, we note that the energy flux absorbed and conducted to the body through a surface element d$\boldsymbol{S}$ is given by

$$\mathcal{E} = \Phi\mu_\odot \left[1 - A_\mathrm{H}(\mu_\odot)\right], \tag{5}$$

where

$$A_\mathrm{H}(\mu_\odot) = \frac{1}{\Phi\mu_\odot}\left[\int_{\Omega_+} \mathrm{d}\Omega_r \; \boldsymbol{n}_r \, I(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot)\right] \cdot \mathrm{d}\boldsymbol{S} \tag{6}$$

is the hemispheric albedo (e.g. Irvine 1975; Bowell et al. 1989). The integration domain Ω_+ denotes a half space above the surface element, such that if we use a spherical angle parametrization of $\boldsymbol{n}_r$

$$\boldsymbol{n}_r = \begin{pmatrix} \sqrt{1-\mu_r^2}\,\cos\phi_r \\ \sqrt{1-\mu_r^2}\,\sin\phi_r \\ \mu_r \end{pmatrix}, \tag{7}$$

we have: $\int_{\Omega_+} \mathrm{d}\Omega_r = \int_0^1 \mathrm{d}\mu_r \int_0^{2\pi} \mathrm{d}\phi_r$. In general, the hemispheric albedo (6) is a function of the zenith angle $\mu_\odot$ of the incident solar radiation. Only in very special and simple cases, where diffuse reflection obeys Lambert law

$$I(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot) = A\Phi\,\frac{\mu_\odot}{\pi}, \tag{8}$$

we do have $A_\mathrm{H} = A = \mathrm{const}$.

The infinitesimal radiation recoil force due to the scattered radiation in the optical band is given by (e.g. Mihalas 1978)

$$\mathrm{d}\boldsymbol{f} = -\frac{1}{v_\mathrm{c}}\left[\int_{\Omega_+} \mathrm{d}\Omega_r \; \boldsymbol{n}_r\,\boldsymbol{n}_r\,I(\mu_r, \mu_\odot; \boldsymbol{n}_r \cdot \boldsymbol{n}_\odot)\right] \cdot \mathrm{d}\boldsymbol{S}, \tag{9}$$

that leads to the general form

$$\begin{aligned} \mathrm{d}\boldsymbol{f} \;=\; & \left[a_1\,(\boldsymbol{n}\cdot\boldsymbol{n}_\odot) + a_2\,(\boldsymbol{n}\cdot\boldsymbol{n}_\odot)^2 + \ldots\right] \mathrm{d}\boldsymbol{S} \\ & + \left[b_1\,(\boldsymbol{n}\cdot\boldsymbol{n}_\odot) + b_2\,(\boldsymbol{n}\cdot\boldsymbol{n}_\odot)^2 + \ldots\right] \boldsymbol{n}_\odot\,\mathrm{d}S. \end{aligned} \tag{10}$$

Here $(a_1, a_2, \ldots; b_1, b_2, \ldots)$ are some coefficients that depend on the scattering function ρ. In the special case of Lambert diffusion (8) we have

$$a_1 = -\frac{2}{3}\frac{\Phi A}{v_\mathrm{c}}, \tag{11}$$

while all other coefficients are zero. For the combination of diffuse and specular reflection we have (a_1, a_2) nonzero (e.g. Milani et al. 1987), while for more realistic scattering laws all a- and b-coefficients are nonzero.

2.3. Thermal radiation

Part of the incident sunlight is absorbed and later re-emitted in the infrared waveband. Assuming that thermal emission of the body has the characteristics of blackbody radiation with isotropy in all directions (see, however, discussion in Lagerros 1998), we have

$$I_\mathrm{Thermal}(\boldsymbol{n}_r) = \frac{\sigma T^4}{\pi}, \tag{12}$$

which gives a thermal recoil force

$$\mathrm{d}\boldsymbol{f} = -\frac{2}{3}\frac{\sigma T^4}{v_\mathrm{c}}\,\mathrm{d}\boldsymbol{S}. \tag{13}$$

Here σ is the Stefan-Boltzmann constant and T is the temperature of the surface element dS. To obtain T, one must solve the heat diffusion in the body, a sufficiently complicated problem such that we restrict ourselves in this work to the zero-conductivity limit where $\sigma T^4 = \mathcal{E} = \Phi\mu_\odot\,[1 - A_\mathrm{H}(\mu_\odot)]$. If moreover $A_\mathrm{H} \sim A = \mathrm{const.}$, we have

$$\mathrm{d}\boldsymbol{f} = -\frac{2}{3}\frac{(1-A)\Phi}{v_\mathrm{c}}\,(\boldsymbol{n}\cdot\boldsymbol{n}_\odot)\,\mathrm{d}\boldsymbol{S}, \tag{14}$$

(compare with Bottke et al. 2002, 2006), and this approximation will be used in the following discussion. This force, substituted into Eq. (1), leads to the YORP torque understood as the thermal radiation effect. The recoil force due to the reflected sunlight in the optical band has the same functional structure as in the thermal band – Eq. (14) – such that their composite effect is obtained by replacing $(1-A) \rightarrow 1$.

3. Zero conductivity limit of thermal torques for spheroids

Using the assumptions presented in Sect. 2.3, we consider the YORP torque

$$\boldsymbol{T} = -\frac{2}{3}\frac{(1-A)\,\Phi}{v_\mathrm{c}}\int \max\left(0,\,\frac{\mathbf{d}\boldsymbol{S}}{\|\mathrm{d}\boldsymbol{S}\|}\cdot\boldsymbol{n}_\odot\right)\boldsymbol{r}\times\mathrm{d}\boldsymbol{S}, \tag{15}$$

where the integral is taken over the surface of an ellipsoid of revolution. Later we will also assume the rotation about the maximum inertia axis, and for this reason we distinguish two cases: oblate spheroids with $a = b > c$, and prolate spheroids with $a > b = c$. In both cases the semi-axes a and b lie in the Oxy plane.

3.1. Oblate spheroid

3.1.1. Exact solution

Let the body be an ellipsoid of revolution with the semi-axes $c < b = a$. In terms of the longitude λ and parametric latitude φ, the ellipsoid surface is defined as

$$\boldsymbol{r} = a\begin{pmatrix}\nu\cos\lambda\\ \nu\sin\lambda\\ \eta\,\mu\end{pmatrix}, \tag{16}$$

where

$$\mu = \sin\varphi, \quad \nu = \sqrt{1-\mu^2} = \cos\varphi, \tag{17}$$

and

$$\eta = \sqrt{1-e^2}, \quad e = \sqrt{1-\left(\frac{c}{a}\right)^2}. \tag{18}$$

The oriented surface element vector for the ellipsoid is

$$\mathrm{d}\boldsymbol{S} = \left(\frac{\partial\boldsymbol{r}}{\partial\lambda}\times\frac{\partial\boldsymbol{r}}{\partial\mu}\right)\mathrm{d}\lambda\,\mathrm{d}\mu = a^2\begin{pmatrix}\eta\,\nu\cos\lambda\\ \eta\,\nu\sin\lambda\\ \mu\end{pmatrix}\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{19}$$

and its length is given by

$$\|\frac{\mathbf{d}\mathbf{S}}{\mathrm{d}\lambda\mathrm{d}\mu}\| = a^2\,\sqrt{\eta^2 + e^2\,\mu^2}, \tag{20}$$

so the unit normal vector is

$$\boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{S}}{\|\mathrm{d}\boldsymbol{S}\|} = \frac{1}{\sqrt{\eta^2+e^2\,\mu^2}}\begin{pmatrix}\eta\,\nu\cos\lambda\\ \eta\,\nu\sin\lambda\\ \mu\end{pmatrix}. \tag{21}$$

The cross product in Eq. (15) for the ellipsoid has the explicit form

$$\boldsymbol{r}\times\mathrm{d}\boldsymbol{S} = a^3\,e^2\,\mu\,\nu\begin{pmatrix}\sin\lambda\\ -\cos\lambda\\ 0\end{pmatrix}\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{22}$$

and we can already state an important conclusion: the z-component of the YORP torque $T_z = \boldsymbol{T}\cdot\boldsymbol{e}_z$ for the ellipsoid (16) is 0.

Substituting Eq. (22) into the general formula (15), we have

$$\boldsymbol{T}_\mathrm{o} = \alpha\,e^2\int_{-1}^{1}\int_{0}^{2\pi}\max(0,\,\boldsymbol{n}\cdot\boldsymbol{n}_\odot)\,\mu\,\nu\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{23}$$

where

$$\alpha = \frac{2}{3}\frac{(1-A)\,\Phi\,a^3}{v_\mathrm{c}} = \frac{2}{3}\frac{(1-A)\,\Phi_0\,a^3}{v_\mathrm{c}}\frac{R_0^2}{R_\odot^2}, \tag{24}$$

and Φ_0 is the energy flux at the nominal distance $R_0 = 1$ AU, known as the solar constant, whereas $R_\odot$ is the actual distance from the Sun.

Now, in order to evaluate the integral (23) over the illuminated surface, we have to establish the integration limits defined by the terminator equation $\boldsymbol{n}\cdot\boldsymbol{n}_\odot = 0$. For the ellipsoid

$$\boldsymbol{n}\cdot\boldsymbol{n}_\odot = \frac{s_\odot\,\mu + c_\odot\,\eta\,\nu\,\cos(\lambda-\lambda_\odot)}{\sqrt{\eta^2+e^2\,\mu^2}}, \tag{25}$$

and the terminator extends in parametric latitude from μ_1 to μ_2, defined as

$$\mu_1 = -\frac{c_\odot\eta}{\sqrt{1-c_\odot^2\,e^2}} \leq 0, \qquad \mu_2 = -\mu_1 \geq 0. \tag{26}$$

Hence the integral over longitude in Eq. (23) should be taken in the limits $0 \leq \lambda \leq 2\pi$ for the "polar day" zone, whereas on partially illuminated latitudes the integration limits are

$$\lambda_{1,2} = \lambda_\odot \pm \arccos\left(-\frac{s_\odot\,\mu}{\eta\,c_\odot\,\nu}\right), \tag{27}$$

with the minus sign for λ_1 and plus for λ_2. Thus the total YORP torque becomes the sum $\boldsymbol{T}_\mathrm{o} = \boldsymbol{T}_1 + \boldsymbol{T}_2$, where

$$\boldsymbol{T}_1 = \alpha\,e^2\int_{\mu_2}^{1}\int_{0}^{2\pi}(\boldsymbol{n}\cdot\boldsymbol{n}_\odot)\,\mu\,\nu\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}\mathrm{d}\lambda\,\mathrm{d}\mu, \tag{28}$$

and

$$\boldsymbol{T}_2 = \alpha\,e^2\int_{\mu_1}^{\mu_2}\int_{\lambda_1}^{\lambda_2}(\boldsymbol{n}\cdot\boldsymbol{n}_\odot)\,\mu\,\nu\begin{pmatrix}-\sin\lambda\\ \cos\lambda\\ 0\end{pmatrix}\mathrm{d}\lambda\,\mathrm{d}\mu. \tag{29}$$

Thanks to symmetry, the above expressions hold true regardless of the position of the Sun (i.e. of the sign of $s_\odot$).

The integration of $\boldsymbol{T}_1$ is elementary. First we integrate with respect to λ obtaining

$$\boldsymbol{T}_1 = \alpha\, e^2\, \eta\, \pi \begin{pmatrix} -y_\odot \\ x_\odot \\ 0 \end{pmatrix} \int_{\mu_2}^{1} \frac{\mu\,(1-\mu^2)}{\sqrt{\eta^2 + e^2\,\mu^2}}\,\mathrm{d}\mu, \tag{30}$$

and then perform the integration with respect to μ, rendering

$$\boldsymbol{T}_1 = \frac{2\,\eta\,\pi\,\alpha}{3\,e^2}\left[1 + \frac{\eta\,(\eta^2 - 3(1 - c_\odot^2\,e^2))}{2\,(1 - c_\odot^2\,e^2)^{3/2}}\right]\begin{pmatrix} -y_\odot \\ x_\odot \\ 0 \end{pmatrix}. \tag{31}$$

The second part of $\boldsymbol{T}$ is more cumbersome. The integration with respect to λ is relatively straightforward and leads to

$$\boldsymbol{T}_2 = \alpha\, W_2 \begin{pmatrix} -y_\odot \\ x_\odot \\ 0 \end{pmatrix}, \tag{32}$$

where

$$W_2 = \frac{e^2}{c_\odot}\int_{\mu_1}^{\mu_2} \frac{\mu\,\sqrt{1-\mu^2}}{\sqrt{\eta^2 - \mu^2\,e^2}}\left[s_\odot \mu \sqrt{1 - \frac{s_\odot^2 \mu^2}{c_\odot^2 \eta^2 (1-\mu^2)}} + c_\odot \eta \sqrt{1-\mu^2}\,\arccos\left(-\frac{s_\odot \mu}{c_\odot \eta\,\sqrt{1-\mu^2}}\right)\right]\mathrm{d}\mu. \tag{33}$$

Let us take the argument of the arccos function as a new integration variable:

$$\xi = \frac{\sigma\,\mu}{\eta\,\sqrt{1-\mu^2}}, \tag{34}$$

where

$$\sigma = \frac{s_\odot}{c_\odot}. \tag{35}$$

Then we have

$$\mathrm{d}\mu = \frac{\eta\,\sigma^2}{(\sigma^2 + \eta^2\xi^2)^{3/2}}\,\mathrm{d}\xi, \tag{36}$$

with $\xi(\mu_1) = -1$, $\xi(\mu_2) = 1$. The quadrature W_2 becomes

$$W_2 = -e^2\,\eta^2\,\sigma^4 \int_{-1}^{1} \frac{\xi\left[\xi\,\sqrt{1-\xi^2} + \arccos(-\xi)\right]}{\sqrt{\sigma^2+\xi^2}\,\left(\sigma^2 + \eta^2\xi^2\right)^{5/2}}\,\mathrm{d}\xi, \tag{37}$$

and we can use integration by parts, with the expression in the square bracket as one of factors. The result

$$W_2 = \frac{\eta^2\pi\,\sqrt{1+\sigma^2}\left(2\,\eta^2 - \eta^2\sigma^2 + 3\sigma^2\right)}{3e^2\,(\eta^2+\sigma^2)^{3/2}} + \frac{2\,\eta^2}{3\,e^2}\int_{-1}^{1} \frac{\sqrt{1-\xi^2}\,\sqrt{\xi^2+\sigma^2}\left(\eta^2\sigma^2 - 3\sigma^2 - 2\eta^2\xi^2\right)}{(\eta^2\xi^2+\sigma^2)^{3/2}}\,\mathrm{d}\xi, \tag{38}$$

is significantly better than (37). The remaining integrand is a purely algebraic, even function of ξ, so we can perform another change of variables, using

$$t = \xi^2, \tag{39}$$

that turns W_2 into

$$W_2 = \frac{\eta^2\pi\,\sqrt{1+\sigma^2}\left(2\,\eta^2 - \eta^2\sigma^2 + 3\sigma^2\right)}{3e^2\,(\eta^2+\sigma^2)^{3/2}} - \frac{2\,\eta^2}{3\,e^2}\,W_3, \tag{40}$$

where

$$\begin{aligned} W_3 &= \int_0^1 \frac{\sqrt{1-t}\,\sqrt{t+\sigma^2}\left((3-\eta^2)\,\sigma^2 + 2\eta^2 t\right)}{\sqrt{t}\,(\eta^2 t + \sigma^2)^{3/2}}\,\mathrm{d}t \\ &= \frac{(2+e^2)\,\sigma^2}{\eta^3}\int_0^1 \frac{\sqrt{(1-t)\,(t+\sigma^2)}}{\sqrt{t}\,(t+\sigma^2\eta^{-2})^{3/2}}\,\mathrm{d}t \\ &\quad + \frac{2}{\eta}\int_0^1 \frac{\sqrt{t\,(1-t)\,(t+\sigma^2)}}{(t+\sigma^2\eta^{-2})^{3/2}}\,\mathrm{d}t. \end{aligned} \tag{41}$$

Both quadratures can be identified in the elliptic integral tables of Byrd & Friedman (1971) and so, collecting all intermediate results, we obtain the final form of the YORP torque for an oblate spheroid

$$\boldsymbol{T}_\mathrm{o} = \alpha\, s_\odot\, W_\mathrm{o} \begin{pmatrix} -y_\odot \\ x_\odot \\ 0 \end{pmatrix} = \alpha\, W_\mathrm{o}\,(\boldsymbol{e}_z \cdot \boldsymbol{n}_\odot)\,(\boldsymbol{e}_z \times \boldsymbol{n}_\odot), \tag{42}$$

with

$$W_\mathrm{o} = \frac{4}{3}\,\frac{1-e^2}{e^2\,c_\odot^2}\left[\mathrm{E}(e\,c_\odot) - (1+c_\odot^2)\,\mathrm{K}(e\,c_\odot) + c_\odot^2\,\Pi(e^2, e\,c_\odot)\right], \tag{43}$$

expressed in terms of complete elliptic integrals of the first (K), second (E) and third kind (Π)

$$\mathrm{K}(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)\,(1-k^2\,t^2)}}, \tag{44}$$

$$\mathrm{E}(k) = \int_0^1 \frac{\sqrt{1-k^2\,t^2}}{\sqrt{1-t^2}}\,\mathrm{d}t, \tag{45}$$

$$\Pi(n,k) = \int_0^1 \frac{\mathrm{d}t}{(1-n\,t^2)\,\sqrt{(1-t^2)\,(1-k^2\,t^2)}}. \tag{46}$$

We emphasize that $T_z = 0$, and $T_x^2 + T_y^2$ does not depend on solar longitude $\lambda_\odot$.

3.1.2. Series approximation

Considering the eccentricity of spheroid as a small parameter, we can expand the elliptic integrals in $\boldsymbol{T}_\mathrm{o}$ using the standard power series expressions (Byrd & Friedman 1971)

$$\mathrm{K}(k) = \frac{\pi}{2}\sum_{j\geq 0}\left(\frac{(2j-1)!!}{(2j)!!}\right)^2 k^{2j}, \tag{47}$$

$$\mathrm{E}(k) = \frac{\pi}{2}\sum_{j\geq 0}\left(\frac{(2j-1)!!}{(2j)!!}\right)^2 \frac{k^{2j}}{2j-1}, \tag{48}$$

$$\Pi(n,k) = \frac{\pi}{2}\sum_{j\geq 0}\sum_{q=0}^{j} \frac{(2j)!\,(2q)!\,k^{2q}\,n^{j-q}}{4^j\,4^q\,(j!)^2\,(q!)^2}. \tag{49}$$

Then W_o becomes a sum

$$W_\mathrm{o} = \frac{\pi}{4} \sum_{j \geq 1} w_\mathrm{j} e^{2j}, \tag{50}$$

where the first three terms are

$$w_1 = 1, \tag{51}$$

$$w_2 = \frac{1}{12}\left(3 - 5 s_\odot^2\right), \tag{52}$$

$$w_3 = \frac{5}{384}\left(9 - 38 s_\odot^2 + 21 s_\odot^4\right), \tag{53}$$

and, generally, each w_j is a $j-1$ degree polynomial of $s_\odot^2$.

3.2. Prolate spheroid

3.2.1. Exact solution

Consider now the case of a prolate spheroid with semi-axes $c = b < a$. A closed form expression for the YORP torque in this case can also be obtained; however, instead of repeating the tedious path from the previous section, we can start with Eq. (42) and use some elementary transformations. Two basic steps consist of first stretching out the shortest axis of the oblate spheroid, and then swapping the coordinate axes to maintain the convention of Ox aligned with the minimum inertia axis.

Stretching out the spheroid, i.e. making $c > a$, can be easily achieved by using an imaginary eccentricity, such that for some positive, real ϵ

$$e = i\,\epsilon, \qquad e^2 = -\epsilon^2, \tag{54}$$

and so

$$c = a\sqrt{1-e^2} = a\sqrt{1+\epsilon^2} \geq a. \tag{55}$$

Elliptic integrals in Eq. (42) can be transformed using the classical formulae for the imaginary modulus (Byrd & Friedman 1971)

$$\mathrm{K}(ik) = \frac{1}{\sqrt{1+k^2}}\,\mathrm{K}(k_1), \tag{56}$$

$$\mathrm{E}(ik) = \sqrt{1+k^2}\,\mathrm{E}(k_1), \tag{57}$$

$$\Pi(n, ik) = \frac{1}{(n+k^2)\,\sqrt{1+k^2}}\left[k^2\,\mathrm{K}(k_1) + n\,\Pi(n_1, k_1)\right], \tag{58}$$

$$k_1 = \frac{k}{\sqrt{1+k^2}}, \tag{59}$$

$$n_1 = \frac{n+k^2}{1+k^2}. \tag{60}$$

The negative parameter of Π can be reduced using

$$\Pi(-n,k) = \frac{k^2}{n+k^2}\,\mathrm{K}(k) + \frac{n\,(1-k^2)}{(n+1)\,(n+k^2)} \times \Pi\left((n+k^2)(n+1)^{-1}, k\right). \tag{61}$$

So, if the new, stretched ellipsoid has the semi-axes a', b', c' such that

$$a' = a\sqrt{1+\epsilon^2}, \quad b' = b, \quad c' = a, \tag{62}$$

its new eccentricity e' can be determined from $c' = a'\sqrt{1-e'^2}$ and thus

$$a\sqrt{1+\epsilon^2}\,\sqrt{1-e'^2} = a,$$

leads to

$$e' = \frac{\epsilon}{\sqrt{1+\epsilon^2}}, \qquad \epsilon = \frac{e'}{\sqrt{1-e'^2}}. \tag{63}$$

Concluding the first step, we obtain the YORP torque for the spheroid with semi-axes a' (aligned with Oz), and $b' = c'$ (aligned with Oy and Ox) in the form

$$\boldsymbol{T}_\mathrm{p} = W \begin{pmatrix} -y_\odot \\ x_\odot \\ 0 \end{pmatrix}, \tag{64}$$

$$W = \frac{8}{9}\,\frac{(1-A)\Phi\,a'^3}{v_\mathrm{c}}\,\frac{1-e'^2}{e'^2}\,\frac{s_\odot}{c_\odot^2\,\sqrt{1-e'^2 s_\odot^2}} \times\Big[(1-e'^2)\,(1+c_\odot^2)\,\mathrm{K}(k_3) - (1-e'^2 s_\odot^2)\mathrm{E}(k_3) - c_\odot^2(1-e'^2)^2\Pi\left(e'^2, k_3\right)\Big]. \tag{65}$$

$$k_3 = \frac{e' c_\odot}{\sqrt{1-e'^2 s_\odot^2}}. \tag{66}$$

The second step consists of swapping the coordinate axes. A right-handed system with Ox' aligned with a' is obtained if

$$x' = z, \quad y' = -y, \quad z' = x. \tag{67}$$

This means that instead of (64) we will have

$$\boldsymbol{T}_\mathrm{p} = W \begin{pmatrix} 0 \\ -x_\odot \\ y_\odot \end{pmatrix}, \tag{68}$$

with the zero component along the symmetry axis, similarly to the oblate case. But changing the axes also affects the coordinates of the Sun:

$$x'_\odot = z_\odot, \quad y'_\odot = -y_\odot, \quad z'_\odot = x_\odot, \tag{69}$$

so that

$$\begin{aligned} c_\odot \cos\lambda_\odot &= s'_\odot, \\ c_\odot \sin\lambda_\odot &= -c'_\odot \sin\lambda'_\odot, \\ s_\odot &= c'_\odot \cos\lambda'_\odot. \end{aligned} \tag{70}$$

Hence

$$\boldsymbol{T}_\mathrm{p} = W \begin{pmatrix} 0 \\ -z'_\odot \\ y'_\odot \end{pmatrix}, \tag{71}$$

and

$$W = \frac{8}{9}\,\frac{(1-A)\Phi\,a'^3}{v_\mathrm{c}}\,\frac{1-e'^2}{e'^2} \times \frac{c'_\odot\cos\lambda'_\odot}{\left(1-(c'_\odot\cos\lambda'_\odot)^2\right)\sqrt{1-(e'c'_\odot\cos\lambda'_\odot)^2}} \times\Big[(1-e'^2)\left(2-(c'_\odot\cos\lambda'_\odot)^2\right)\mathrm{K}(k_\mathrm{p}) - \left(1-(e'\,c'_\odot\cos\lambda'_\odot)^2\right)\mathrm{E}(k_\mathrm{p}) - \left(1-(c'_\odot\cos\lambda'_\odot)^2\right)(1-e'^2)^2\,\Pi\left(e'^2, k_\mathrm{p}\right)\Big], \tag{72}$$

$$k_\mathrm{p} = e'\sqrt{\frac{1-(c'_\odot\cos\lambda'_\odot)^2}{1-(e'\,c'_\odot\cos\lambda'_\odot)^2}}. \tag{73}$$

Thus we have achieved the transformation, so we can drop the "prime" symbols and, using the basic definitions (3, 18, 24) in the Oxyz system aligned with the a, b, c semi-axes respectively, the final YORP torques formula becomes

$$\boldsymbol{T}_{\rm p} = \alpha\, x_\odot\, W_{\rm p} \begin{pmatrix} 0 \\ z_\odot \\ -y_\odot \end{pmatrix} = -\alpha\, W_{\rm p}\, (\boldsymbol{e}_x \cdot \boldsymbol{n}_\odot)\, (\boldsymbol{e}_x \times \boldsymbol{n}_\odot), \tag{74}$$

$$W_{\rm p} = \frac{4}{3} \frac{\eta^2}{e^2\,(1 - x_\odot^2)\,\sqrt{1 - e^2\, x_\odot^2}} \Big[\eta^2 (x_\odot^2 - 2) \mathrm{K}(k_{\rm p}) + (1 - e^2\, x_\odot^2) \mathrm{E}(k_{\rm p}) + (1 - x_\odot^2)\, \eta^4\, \Pi(e^2, k_{\rm p}) \Big], \tag{75}$$

$$k_{\rm p} = e\, \sqrt{\frac{1 - x_\odot^2}{1 - e^2\, x_\odot^2}}. \tag{76}$$

We recall that $e = \sqrt{1 - (c/a)^2}$ as usually.

3.2.2. Series approximation

Using the series expansion of elliptic integrals together with binomial expansion formulae, we can approximate $\boldsymbol{T}_{\rm p}$ using

$$W_{\rm p} = \frac{\pi}{4} \sum_{j \geq 1} v_{\rm j} e^{2j}, \tag{77}$$

with the leading terms

$$v_1 = 1, \tag{78}$$

$$v_2 = \frac{1}{12} \left(9 - 5 x_\odot^2 \right), \tag{79}$$

$$v_3 = \frac{1}{384} \left(-51 - 110 x_\odot^2 + 105 x_\odot^4 \right). \tag{80}$$

Each v_j term is a $j - 1$ degree polynomial of $x_\odot^2$.

3.3. Significant components

Suppose that the spheroidal body moves on a heliocentric orbit and it rotates around the axis of maximum inertia. Two dynamically significant components of the YORP torque are $T_{\rm s}$, responsible for the variations in the rotation rate, and $T_\perp$ affecting the obliquity of the spin axis with respect to the orbit. The two components are defined in terms of the scalar products (e.g. Rubincam 2000; Vokrouhlický & Čapek 2002; Čapek & Vokrouhlický 2004)

$$T_{\rm s} = \boldsymbol{T} \cdot \boldsymbol{s}, \quad T_\perp = \frac{\cos\varepsilon\, T_{\rm s} - \boldsymbol{T} \cdot \boldsymbol{n}_{\rm o}}{\sin\varepsilon}, \tag{81}$$

where $\boldsymbol{s}$ is the unit vector directed along the spin vector, $\boldsymbol{n}_{\rm o}$ is the unit vector normal to the orbital plane, and ε is the obliquity angle between $\boldsymbol{n}_{\rm o}$ and $\boldsymbol{s}$.

Vector $\boldsymbol{s}$ expressed in the body frame is simply $\boldsymbol{s} = (0, 0, 1)^{\rm T}$. The same simplicity is attained by $\boldsymbol{n}_{\rm o}$ expressed in the orbital frame, but we have to transform it to the body frame in order to evaluate the scalar product with $\boldsymbol{T}$. This transformation is achieved by two rotations (Fig. 1)

$$\boldsymbol{n}_{\rm o} = \mathbf{R}_3(-\Omega)\, \mathbf{R}_1(-\varepsilon) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\varepsilon \sin\Omega \\ -\sin\varepsilon \cos\Omega \\ \cos\varepsilon \end{pmatrix}, \tag{82}$$

where Ω is the longitude of the ascending node of the orbit on the plane normal to the spin axis.

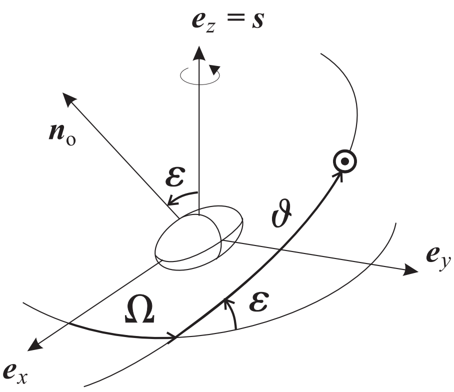

Fig. 1. A spheroid (prolate), the Sun, and the 3-1-3 Euler angles.

We will also need the expressions of the solar position in terms of Ω, ε, and the argument of latitude ϑ. This is achieved by the sequence of 3-1-3 rotations (Fig. 1)

$$\begin{pmatrix} x_\odot \\ y_\odot \\ z_\odot \end{pmatrix} = \mathbf{R}_3(-\Omega)\, \mathbf{R}_1(-\varepsilon)\, \mathbf{R}_3(-\vartheta) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{83}$$

leading to the standard formulae

$$\begin{pmatrix} x_\odot \\ y_\odot \\ z_\odot \end{pmatrix} = \begin{pmatrix} \cos\vartheta \cos\Omega - \cos\varepsilon \sin\vartheta \sin\Omega \\ \cos\vartheta \sin\Omega + \cos\varepsilon \sin\vartheta \cos\Omega \\ \sin\varepsilon \sin\vartheta \end{pmatrix}. \tag{84}$$

If we assume Keplerian orbit of the Sun and the rotation around the axis of maximum inertia (spin vector aligned with $\boldsymbol{e}_z$), then the obliquity angle ε is constant, the longitude of the ascending node Ω reflects the "daily" rotation of the body and the argument of latitude ϑ varies on a "yearly" scale due to the orbital motion.

3.3.1. Oblate spheroid

For the oblate spheroid, where the torque is given by Eq. (42), we conclude that

$$T_{\rm s} = 0, \tag{85}$$

hence there is no YORP effect on the rotation rate. The evaluation of $T_\perp$ requires some simple manipulations with trigonometric functions, rendering

$$T_\perp = \frac{\alpha}{2}\, W_{\rm o}\, \sin\varepsilon\, \sin 2\vartheta, \tag{86}$$

where we substitute

$$c_\odot = \sqrt{1 - \sin^2\varepsilon \sin^2\vartheta} \tag{87}$$

inside $W_{\rm o}$.

As one could expect from the rotational symmetry, there is no dependence on Ω. This means that Eq. (86) is also the average of $T_\perp$ over the revolution period, provided the body rotates fast enough to assume a constant ϑ during one revolution. But the values of $T_\perp$ will vary with time because of the orbital motion. The true anomaly of the Sun is present not only in $\vartheta = f + \omega$, the sum of the true anomaly and of the argument of perihelion, but also in the coefficient α that depends on the distance $R_\odot(f)$. Nevertheless, the average of $T_\perp$ over one orbital period becomes zero. This can be easily demonstrated if we use the change of variables from mean anomaly to the true anomaly in the quadrature defining the mean value

$$\langle T_\perp \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_\perp \frac{R_\odot^2(f)}{a_\odot^2 \sqrt{1-e_\odot^2}} \, \mathrm{d}f, \tag{88}$$

involving the semi-major axis $a_\odot$ and the eccentricity $e_\odot$ of the orbit ($e_\odot < 1$). Recalling the definition (24), we observe that $R_\odot(f)$ in α and in the variable change factor cancel out, so the integrand depends on f only through ϑ; thus we can directly replace $\mathrm{d}f$ in Eq. (88) by $\mathrm{d}\vartheta$. But then we note that according to Eq. (86), the integrand in Eq. (88) is an odd function of ϑ, which is a sufficient condition to claim

$$\langle T_\perp \rangle = 0. \tag{89}$$

3.3.2. Prolate spheroid

Considering the torque (74) on a prolate spheroid, we assume that it also rotates around the axis of maximum inertia (the Oz axis, which is no longer the axis of rotational symmetry). This time, using the torque (74), we find that T_s is not zero in general, because

$$T_\mathrm{s} = -\alpha \, W_\mathrm{p} \, x_\odot \, y_\odot \tag{90}$$

is an asymmetric, periodic function of Ω, f, and ϑ. In these circumstances, we resort to the series approximation of W_p presented in Sect. 3.2.2. Recalling that W_p is a sum of polynomials in $x_\odot^2$, we note that $x_\odot \, y_\odot$ will be multiplied either by a constant term or by some power of

$$x_\odot^2 = \frac{1}{4} \left[1 + \cos^2\varepsilon + \sin^2\varepsilon \, (\cos 2\vartheta + \cos 2\Omega) + \frac{(1-\cos\varepsilon)^2}{2} \cos 2(\vartheta - \Omega) + \frac{(1+\cos\varepsilon)^2}{2} \cos 2(\vartheta + \Omega) \right], \tag{91}$$

involving only the cosines of ϑ and Ω. But the term $x_\odot y_\odot$ consists only of the sines of these angles

$$x_\odot \, y_\odot = \frac{1}{4} \sin^2\varepsilon \, \sin 2\Omega - \frac{(1-\cos\varepsilon)^2}{8} \sin 2(\vartheta - \Omega) + \frac{(1+\cos\varepsilon)^2}{8} \sin 2(\vartheta + \Omega), \tag{92}$$

and this means that no term independent of Ω can occur if we reduce $x_\odot^{2j+1} y_\odot$ to the form of a trigonometric polynomial. Such a polynomial will consist of sine terms with arguments $2 j_1 \Omega + 2 j_2 \vartheta$, where j_2 is any integer, but j_1 is a nonzero integer. Thus, the average of T_s over one revolution period becomes zero. Of course, this means that the second averaging, over the orbital period, also results in

$$\langle T_\mathrm{s} \rangle = 0, \tag{93}$$

and there is no secular variation in the rotation period of a prolate ellipsoid.

The second component for the prolate ellipsoid is equal to

$$T_\perp = \alpha \, W_\mathrm{p} \, x_\odot \, z_\odot \, \cos\Omega. \tag{94}$$

Repeating the arguments from the case of T_s,

$$x_\odot \, z_\odot \, \cos\Omega = \frac{\sin\varepsilon}{8} \left[2 \sin 2\vartheta - 2 \cos\varepsilon \sin 2\Omega + (1-\cos\varepsilon) \sin 2(\vartheta - \Omega) + (1+\cos\varepsilon) \sin 2(\vartheta + \Omega) \right]. \tag{95}$$

The presence of $\sin 2\vartheta$ alone is enough to conclude that after a product by a constant or by any power of $x_\odot^2$, periodic terms $\sin 2 j \vartheta$ will remain after the averaging with respect to Ω. Thus the average of $T_\perp$ over the rotation period will be nonzero, but then the second averaging with respect to ϑ will finally bring us to

$$\langle T_\perp \rangle = 0. \tag{96}$$

3.4. Stability problem

The double averaging presented in this paper makes sense only if the principal axis rotation state is stable. In other words, small deviations of the rotation axis from the direction of the maximum inertia should not result in a systematic drift away from the initial position. The proof that no such drift arises for a precessing spheroid and the YORP torque has been worked out in a more general context and we will present it in a separate paper. Nevertheless, restricting the discussion to the vicinity of the principal axis rotation, one may draw the same conclusion from the results of Scheeres (2007). For this purpose, we take the Fourier series approximation of the YORP torques in terms of the solar longitude $\lambda_\odot$. For oblate spheroids we have a single harmonic

$$\boldsymbol{T}_\mathrm{o} = \boldsymbol{C}_1 \cos\lambda_\odot + \boldsymbol{D}_1 \sin\lambda_\odot, \tag{97}$$

with $C_{1,x} = D_{1,y} = C_{1,z} = D_{1,z} = 0$, and $C_{1,y} = -D_{1,x}$, using the notation of Scheeres (2007). Prolate spheroid torques are less trivial:

$$\boldsymbol{T}_\mathrm{p} = \sum_{n \geq 1} \boldsymbol{C}_n \cos n\lambda_\odot + \boldsymbol{D}_n \sin n\lambda_\odot, \tag{98}$$

where even coefficients are

$$\boldsymbol{C}_{2n} = \boldsymbol{0}, \quad \boldsymbol{D}_{2n} = \begin{pmatrix} 0 \\ 0 \\ D_{2n,z} \end{pmatrix}, \tag{99}$$

and odd ones are

$$\boldsymbol{C}_{2n+1} = \begin{pmatrix} 0 \\ C_{2n+1,y} \\ 0 \end{pmatrix}, \quad \boldsymbol{D}_{2n+1} = \boldsymbol{0}. \tag{100}$$

In the absence of thermal lag, such a set of coefficients will not induce a systematic trend in the inclination of the spin axis according to the linear theory of Scheeres (2007).

3.5. Thermal inertia

In all the previous considerations, we have used Rubincam's approximation, assuming that incoming energy is re-emitted instantaneously. In other words, we have neglected the thermal inertia of the body. But the extension of the YORP-related results to the case of nonzero thermal inertia is quite simple if we assume rotation around the principal axis. It is enough to replace the solar position vector $\boldsymbol{n}_\odot = (x_\odot, y_\odot, z_\odot)^\mathrm{T}$ with some "delayed Sun" vector $\boldsymbol{N}_\odot = (X_\odot, Y_\odot, Z_\odot)^\mathrm{T}$ defined as

$$\boldsymbol{N}_\odot = \boldsymbol{R}_3(-\delta) \, \boldsymbol{n}_\odot, \tag{101}$$

where the angle δ represents the constant lag. Note that $z_\odot = Z_\odot = s_\odot$, and $x_\odot^2 + y_\odot^2 = X_\odot^2 + Y_\odot^2 = c_\odot^2$. Hence, for an oblate ellipsoid, only the azimuth of the YORP torque vector will change but not its magnitude. Thus, using the 'delayed Sun' has no influence on the conclusions concerning the average values of T_s, because the scalar product $\boldsymbol{T} \cdot \boldsymbol{s}$ is invariant with respect to rotation around $\boldsymbol{s}$.

But the situation is different in the case of $T_\perp$. Note that the definition of $\boldsymbol{n}_\mathrm{o}$ given by Eq. (82) is not modified by the occurrence of lag, hence the value of product $\boldsymbol{T} \cdot \boldsymbol{n}_\mathrm{o}$ must depend on the lag angle δ. Resorting to the series expansions, we can show that the rotation/orbit average of $T_\perp$ for oblate spheroids is

$$\langle T_\perp \rangle = -\,\alpha' \, \sin\varepsilon \, \cos\varepsilon \, \sin\delta \sum_{j\geq 1} e^{2j}\, b_j, \tag{102}$$

where

$$\alpha' = \frac{\pi}{12} \, \frac{(1-A)\Phi_0 a^3}{v_\mathrm{c}\sqrt{1-e_\odot^2}} \, \frac{R_0^2}{a_\odot^2}, \tag{103}$$

and the first few terms are

$$\begin{aligned} b_1 &= 1, \\ b_2 &= \tfrac{1}{16}\,(4 - 5\sin^2\varepsilon), \\ b_3 &= \tfrac{5}{1024}\,(24 - 76\sin^2\varepsilon + 35\sin^4\varepsilon). \end{aligned} \tag{104}$$

Similarly, for prolate spheroids

$$\langle T_\perp \rangle = -2\,\alpha' \, \sin\varepsilon \, \cos\varepsilon \, \sin\delta \sum_{j\geq 1} e^{2j}\, c_j, \tag{105}$$

with

$$\begin{aligned} c_1 &= 1, \\ c_2 &= \tfrac{1}{64}\,(28 + 15\sin^2\varepsilon), \\ c_3 &= -\tfrac{1}{8192}\,(1448 + 780\sin^2\varepsilon - 875\sin^4\varepsilon). \end{aligned} \tag{106}$$

Notably, these expressions agree with the linear theory of Scheeres (2007), if we use the Fourier coefficients defined in Sect. 3.4.

4. Direct radiation pressure for spheroids

The direct radiation pressure (DRP) torque exerted on an illuminated body is given by Eqs. (1) and (2), i.e.

$$\boldsymbol{T} = \int \boldsymbol{r} \times \mathrm{d}\boldsymbol{f}, \tag{107}$$

where

$$\mathrm{d}\boldsymbol{f} = -\frac{\Phi}{v_\mathrm{c}} \, \max\left(0, \, \boldsymbol{N}\cdot\boldsymbol{n}_\odot\right) \, \boldsymbol{n}_\odot \, \mathrm{d}\lambda \, \mathrm{d}\mu, \tag{108}$$

and $\boldsymbol{N}$ is the normal vector

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial\lambda} \times \frac{\partial \boldsymbol{r}}{\partial\mu} \cdot \tag{109}$$

Considering the oblate ellipsoid of revolution with $a = b > c$, we have

$$\boldsymbol{T}_\mathrm{d} = -\gamma \int_{-\pi}^{\pi}\int_{-1}^{1} \boldsymbol{U}\, \mathrm{d}\lambda \, \mathrm{d}\mu, \tag{110}$$

where $\gamma = \Phi\, a^3 / v_\mathrm{c}$ and

$$\begin{aligned} \boldsymbol{U} = {} & (s_\odot \mu + c_\odot \eta \nu \cos(\lambda_\odot - \lambda)) \\ & \times \begin{pmatrix} -c_\odot \eta \mu \, \sin\lambda_\odot + s_\odot \nu \, \sin\lambda \\ c_\odot \eta \mu \, \cos\lambda_\odot - s_\odot \nu \, \cos\lambda \\ c_\odot \eta \, \sin(\lambda_\odot - \lambda) \end{pmatrix}. \end{aligned} \tag{111}$$

For the sake of brevity, we use $\nu = \sqrt{1-\mu^2}$ as above.

The terminator equation and the partition into the polar day, day-and-night, and polar night zones are made exactly as in Sect. 3.1.1. But this time the resulting quadratures are elementary and we do not discuss the details, providing only the final formulae. Thus, in the polar day zone

$$\boldsymbol{T}_{\mathrm{d},1} = \gamma\,\pi \, \frac{c_\odot \eta^2 s_\odot^3}{(1 - c_\odot^2 e^2)^{\frac{3}{2}}} \begin{pmatrix} y_\odot \\ -x_\odot \\ 0 \end{pmatrix}. \tag{112}$$

Remarkably, in the partially illuminated area we obtain

$$\boldsymbol{T}_{\mathrm{d},2} = -\boldsymbol{T}_{\mathrm{d},1}. \tag{113}$$

The final conclusion that

$$\boldsymbol{T}_\mathrm{d} = \boldsymbol{T}_{\mathrm{d},1} + \boldsymbol{T}_{\mathrm{d},2} = \boldsymbol{0}, \tag{114}$$

proves that no DRP torque is exerted on an ellipsoid of revolution – neither oblate nor prolate.

5. Conclusions

The three principal results are:

1. We have obtained the exact formulae for the YORP torque on an ellipsoid of revolution as a function of the solar position in the body frame. To our knowledge it is the first and only exact formula for a nontrivial body shape[1]. As such, it can be a useful benchmark for the common algorithms using triangulated surface, Fourier series or other approximate tools that model the YORP torques.
2. We have demonstrated that in the absence of thermal inertia, if the spheroid rotates around its principal axis of inertia, the two dynamically significant projections of the YORP torque vanish during the double averaging (with respect to the rotation and with respect to the orbital motion), provided the two periods are not commensurable. Thus we confirm the common opinion that the YORP torques do not contribute to the long-term rotational evolution of a spheroid. But if the thermal inertia is taken into account, the spin axis will suffer a systematic deviation from the initial obliquity, although there is still no secular trend in the rotation rate.
3. We have demonstrated that the direct radiation pressure torques on ellipsoids of revolution vanish identically regardless of the solar position.

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[1] Scheeres (2007) approached this goal quite closely, providing the quadrature expressions for the YORP torques Fourier series of a triangulated body shape.

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