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Neutrino transport and hydrodynamic stability of rotating proto-neutron stars

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ABSTRACT

Aims. We study the stability of differentially rotating, non-magnetic proto-neutron stars.

Methods. The stability is considered by making use of a linear analysis and taking neutrino transport into account.

Results. When neutrino transport is efficient, the star can be subject to a diffusive instability that can occur even in the convectively stable region. The instability arises on a time scale that is comparable to the time scale of thermal diffusion. Hydrodynamic motions driven by the instability can lead to anisotropy in the neutrino flux since the instability is suppressed near the equator and rotation axis.

Key words. stars: neutron – stars: rotation – stars: supernovae: general – hydrodynamics – instabilities

1. Introduction

Newly born neutron stars are subject to various hydrodynamic and hydromagnetic instabilities shortly after core collapse. These instabilities can play an important role in enhancing neutrino luminosities and increasing the energy deposition efficiency. Convection is probably among the best-studied instabilities in proto-neutron stars (PNSs) where it can be driven by both lepton and entropy gradients (see, e.g., Epstein 1979; Arnett 1987; Burrows & Lattimer 1988). The development of negative entropy and lepton gradients is common to many simulations of core collapse supernovae (Bruenn & Mezzacappa 1994; Bruenn et al. 1995; Thompson et al. 2005) and evolutionary models of PNSs (Burrows & Lattimer 1986; Keil & Janka 1995; Sumiyoshi et al. 1995; Pons et al. 1999; Dessart et al. 2006). Numerical simulations also indicate the presence of instabilities in PNSs, however, conclusions regarding the efficiency of turbulent transport and the duration of unstable phase depend on the code used and can differ significantly; compare, for instance, the simulations by Keil et al. (1996) and Mezzacappa et al. (1998).

The nature of instabilities arising in PNSs has been considered by a number of authors (Grossman et al. 1993; Bruenn & Dineva 1996; Miralles et al. 2000, 2002). The true stability criteria for non-magnetic and non-rotating PNSs (Miralles et al. 2000) indicate the presence of two essentially different instabilities with the convectively unstable region surrounded by the neutron-finger unstable region, the latter typically involving a larger portion of the stellar material. Due to the cooling, the temperature and lepton gradients are progressively reduced, and both instabilities disappear completely after ∼30−40 s. The growth time of convective and neutron-finger instabilities differs substantially, and therefore the efficiency of turbulent transport in the convective and neutron-finger unstable zones can be different as well. This difference is crucial for many MHD-phenomena in PNSs, such as turbulent dynamo action (see, e.g., Bonanno et al. 2003, 2005) and transport of the angular momentum. Note that fast rotation of PNSs can substantially modify convection and make convective motions anisotropic and constrained to the polar regions (Fryer & Heger 2000; Miralles et al. 2004). This mechanism is a natural way of creating anisotropic energy and momentum transport by convective motions, which only requires that the angular velocity be of the order of the Brunt-Väisälä or leptonic buoyant frequences.

The hydrodynamic stability properties of proto-neutron stars, however, can be much more complex than is believed at the moment. Recently, for example, Bruenn et al. (2004) have argued that a new doubly diffusive instability can occur in an extensive region below the neutrinosphere, which the authors refer to as "lepto-entropy fingers". This instability is driven mainly by the entropy equilibration due to a lepton fraction difference and lepton equilibration that is in turn caused by the entropy difference and that may play an important role in enhancing the neutrino emission.

The possible presence of the magnetic field and differential rotation in a core collapse supernovae favours a number of other instabilities that can also be important in enhancing anisotropic turbulent transport. For example, the magnetorotational instability in the context of core collapse was first considered by Akiyama et al. (2003). The authors argue that the instability must inevitably occur in core collapse and that it has the capacity to produce fields that are strong enough to affect, if not cause, the explosion. Thompson et al. (2005) constructed one-dimensional models, including rotation and magnetic fields, to study the mechanism of energy deposition. They explore several mechanisms for viscosity and argue that the turbulent viscosity caused by the magnetorotational instability is the most effective. The nonaxisymmetric magnetorotational instability in PNSs has been considered by Masada et al. (2006a). The authors have obtained the criterion and growth rate of instability and argue that the nonlinear evolution may lead to enhancement of the neutrino luminosity. The effect of neutrino radiation on this instability was then considered by Masada et al. (2006b). Neutrino transport is rather fast in PNSs and can essentially modify the standard magnetorotational instability.

In the present paper, we consider the effect of neutrino transport on the stability of differential rotation in PNSs, which can appear for a number of reasons. From theoretical modelling and simple analytic considerations, it is commonly accepted that the core collapse of a rotating progenitor leads to differential rotation for newly-born neutron stars (Zwerger & Müller 1997; Rampp et al. 1998; Liu 2002; Dimmelmeier et al. 2002; Müller et al. 2004), mainly due to conservation of the angular momentum during collapse. It is also possible that PNSs are differentially rotating due to turbulent transport of the angular momentum during the convective phase. The boundaries of the convective zone move inward when the PNS cools down and, in the region where convection stops, the angular velocity profile will evolve under the influence of shear viscosity and, what is more likely, due to development of various instabilities caused by differential rotation.

We show that this rotation can be unstable if it is accompanied by sufficiently rapid diffusion of heat and lepton number and if the angular velocity depends on the vertical coordinate. Contrary to the magnetorotational instability, the instability considered in this paper does not require the presence of the magnetic field. Turbulent motions caused by the instability can lead to anisotropic turbulent transport in convectively stable regions. The instability can be important when considering various dynamo models, the transport of energy, and the angular momentum in PNSs.

The paper is organised as follows. In Sect. 2, we consider the basic equations governing the instability in differentially rotating PNSs and derive the dispersion equation. The stability criterion and the growth rate of instability are treated in Sect. 3. A discussion of our results is presented in Sect. 4.

2. Dispersion equation

We now consider the PNS rotating with the angular velocity $\Omega = \Omega(s, z)$, where s, φ, and z are cylindrical coordinates. In the unperturbed state, the PNS is assumed to be in hydrostatic equilibrium,

$$\frac{\nabla p}{\rho} = \boldsymbol{G}\,, \quad \boldsymbol{G} = \boldsymbol{g} + \Omega^2 \boldsymbol{s} \tag{1}$$

where $\boldsymbol{g}$ is the gravity and $\boldsymbol{e}_s$ is the radial unit vector. We assume that the matter inside a PNS is in chemical equilibrium; thus, the pressure p is generally a function of the density ρ, temperature T, and lepton fraction $Y = (n_e + n_\nu)/n$, with n_e and n_ν the number densities of electrons and neutrinos, respectively, and $n = n_p + n_n$ is the number density of baryons. Therefore, we have $\nabla \ln p = (\partial \ln p / \partial \ln \rho)_{TY} (\nabla \ln \rho + \beta \nabla \ln T + \delta \nabla Y)$, where β and δ are the coefficients of thermal and chemical expansion; $\beta = -(\partial \ln \rho / \partial \ln T)_{pY}$, $\delta = -(\partial \ln \rho / \partial Y)_{pT}$. Then, taking the curl of Eq. (1), we obtain

$$\boldsymbol{G} \times \left(\beta \frac{\nabla T}{T} + \delta \nabla Y \right) = -\boldsymbol{e}_\varphi s \frac{\partial \Omega^2}{\partial z}.$$

If Ω depends on z, hydrostatic equilibrium can be reached only if gradients of the temperature or lepton fraction have components perpendicular to $\boldsymbol{G}$. If $g \gg s\Omega^2$, then the variations of T and Y on a spherical surface r = const. can be estimated as

$$\max\left(\frac{\delta T_\perp}{T}, \delta Y_\perp \right) \sim \frac{1}{\beta + \delta} \frac{s^2}{g} \frac{\partial \Omega^2}{\partial z}.$$

Hydrostatic equilibrium can be satisfied if variations are small, or if $s\Omega^2/g(\beta+\delta) \ll 1$ (we assume that $\partial\Omega/\partial z \sim \Omega/s$). Typically, $\beta+\delta \sim 0.1$ or larger in PNSs (see Miralles et al. 2002) and, hence, hydrostatic equilibrium can be reached for PNSs with $\partial\Omega/\partial z \neq 0$ if the rotation period is longer than ∼1−2 ms. Most likely, the majority of PNSs satisfies this requirement and, therefore, the unperturbed state with $\Omega(s, z)$ is justified for these PNSs.

We consider axisymmetric short wavelength perturbations with the space-time dependence $\exp(\gamma t - \mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})$ where $\boldsymbol{k} = (k_s, 0, k_z)$ is the wavevector, $|\boldsymbol{k} \cdot \boldsymbol{r}| \gg 1$. Small perturbations will be indicated by subscript 1, whilst unperturbed quantities will have no subscipt. Since the growth time of the instability in PNSs is typically longer than the period of a sound wave with the same wavelength, we use the Boussinesq approximation. Then, the linearized momentum and continuity equations governing the behaviour of small perturbations read as

$$\gamma \boldsymbol{v}_1 + 2\boldsymbol{\Omega} \times \boldsymbol{v}_1 + \boldsymbol{e}_\varphi s (\boldsymbol{v}_1 \cdot \nabla \Omega) = \frac{\mathrm{i}\boldsymbol{k} p_1}{\rho} + \boldsymbol{G}\frac{\rho_1}{\rho} - \nu k^2 \boldsymbol{v}_1, \tag{2}$$

$$\boldsymbol{k} \cdot \boldsymbol{v}_1 = 0, \tag{3}$$

where $\boldsymbol{v}_1$, p_1, and ρ_1 are perturbations of the velocity, pressure, and density, respectively, and ν is the kinematic viscosity and $\boldsymbol{e}_\varphi$ the unit vector in the azimuthal direction. Since in the Boussinesq approximation, the perturbations of pressure are negligible, the perturbations of density can be expressed in terms of the perturbations of temperature and lepton fraction,

$$\rho_1 \approx -\rho \left(\beta \frac{T_1}{T} + \delta Y_1 \right). \tag{4}$$

During the initial evolution, the PNS is opaque to neutrinos, and the neutrino transport can be treated in the diffusion approximation (Imshenik & Nadezhin 1972). Then, the linearized thermal balance and diffusion equations read as

$$\frac{\dot{T}_1}{T} - \boldsymbol{v}_1 \cdot \frac{\Delta \nabla T}{T} = \kappa_T \frac{\Delta T_1}{T} + \kappa_Y \Delta Y_1, \tag{5}$$

$$\dot{Y}_1 + \boldsymbol{v}_1 \cdot \nabla Y = \lambda_T \frac{\Delta T_1}{T} + \lambda_Y \Delta Y_1 \tag{6}$$

where λ_T, λ_Y, κ_T, and κ_Y are the corresponding kinetic coefficients (see Miralles et al. 2000). We denote the superadiabatic temperature gradient as

$$\Delta \nabla T = \left(\frac{\partial T}{\partial p} \right)_{s,Y} \nabla p - \nabla T. \tag{7}$$

The dispersion equation corresponding to Eqs. (2)–(6) is

$$\gamma^4 + a_3 \gamma^3 + a_2 \gamma^2 + a_1 \gamma + a_0 = 0, \tag{8}$$

where

$$a_3 = 2\omega_\nu + \omega_T + \omega_Y,$$
$$a_2 = \omega_T \omega_Y - \omega_{TY} \omega_{YT} + 2\omega_\nu (\omega_T + \omega_Y) + \omega_\nu^2 + q^2 - (\omega_g^2 + \omega_L^2),$$

$$a_1 = 2\omega_\nu(\omega_T\omega_Y - \omega_{TY}\omega_{YT}) + (\omega_T + \omega_Y)(\omega_\nu^2 + q^2)$$
$$-\omega_\nu(\omega_g^2 + \omega_L^2) - \omega_g^2\left(\omega_Y - \frac{\delta}{\beta}\omega_{YT}\right) - \omega_L^2\left(\omega_T - \frac{\beta}{\delta}\omega_{TY}\right).$$
$$a_0 = -\omega_\nu\left[\omega_g^2\left(\omega_Y - \frac{\delta}{\beta}\omega_{YT}\right) + \omega_L^2\left(\omega_T - \frac{\beta}{\delta}\omega_{TY}\right)\right]$$
$$+(\omega_\nu^2 + q^2)(\omega_T\omega_Y - \omega_{TY}\omega_{YT}).$$

In these expressions, we introduced the characteristic frequencies

$$q^2 = \frac{k_z}{s^3k^2}\left(k_z\frac{\partial}{\partial s} - k_s\frac{\partial}{\partial z}\right)(s^4\Omega^2),\ \omega_\nu = \nu k^2,$$
$$\omega_T = \kappa_T k^2,\ \omega_Y = \lambda_Y k^2,$$
$$\omega_{YT} = \lambda_T k^2,\ \omega_{TY} = \kappa_Y k^2,$$
$$\omega_g^2 = -\beta\boldsymbol{A}\cdot\frac{\Delta\nabla T}{T},\ \omega_L^2 = \delta\boldsymbol{A}\cdot\nabla Y,$$

where $\boldsymbol{A} = \boldsymbol{G} - \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{G})/k^2$. The quantities ω_ν, ω_T, and ω_Y are the inverse time scales of the dissipation of perturbations due to viscosity, thermal conductivity, and diffusivity, respectively; ω_{YT} characterises the rate of thermodiffusion, and ω_{TY} describes the rate of heat transport due to the chemical inhomogeneity; ω_g is the frequency of the buoyant wave; ω_L characterises the dynamical time scale of the processes associated with the lepton gradient, and q^2 describes the influence of differential rotation on stability.

If rotation is negligible ($q^2 \approx 0$), then Eq. (8) reduces to a cubic one,

$$\gamma^3 + b_2\gamma^2 + b_1\gamma + b_0 = 0, \qquad (9)$$

where

$$b_2 = \omega_\nu + \omega_T + \omega_Y,$$
$$b_1 = \omega_T\omega_Y - \omega_{TY}\omega_{YT} + \omega_\nu(\omega_T + \omega_Y) - (\omega_g^2 + \omega_L^2),$$
$$b_0 = -\omega_g^2\left(\omega_Y - \frac{\delta}{\beta}\omega_{YT}\right) - \omega_L^2\left(\omega_T - \frac{\beta}{\delta}\omega_{TY}\right)$$
$$+\omega_\nu(\omega_T\omega_Y - \omega_{TY}\omega_{YT}).$$

This equation is equivalent to the dispersion Eq. (17) derived by Miralles et al. (2000).

3. Criteria and the growth rate of instability

Equation (8) describes four essentially different modes that exist in a differentially rotating PNS. The condition that one of the roots has a positive real part (unstable) is equivalent to requiring that at least one of the following inequalities be satisfied:

$$a_3 < 0, \qquad (10)$$

$$a_0 < 0, \qquad (11)$$

$$a_3a_2 - a_1 < 0, \qquad (12)$$

$$a_1(a_3a_2 - a_1) - a_3^2a_0 < 0 \qquad (13)$$

(see, e.g., Aleksandrov et al. 1963). The quantity a_3 is positive, therefore condition (10) will never apply.

We have from condition (11)

$$-\nu\omega_g^2\left(\lambda_Y - \frac{\delta}{\beta}\lambda_T\right) - \nu\omega_L^2\left(\kappa_T - \frac{\beta}{\delta}\kappa_Y\right)$$
$$+(q^2 + \omega_\nu^2)(\kappa_T\lambda_Y - \kappa_Y\lambda_T) < 0. \qquad (14)$$

Note that, typically, the Prandtl number (the ratio of kinematic viscosity and thermal diffusivity) is small in PNSs (see, e.g., Thompson & Duncan 1993; Masada et al. 2006b), and the stabilizing influence of stratification is reduced.

Substituting expressions for the coefficients a_0, a_1, a_2, and a_3 into conditions (12)–(13), we obtain rather cumbersome criteria of instability. However, these criteria can be much simplified if we take into account that the frequencies have a different order of magnitude in PNSs. The dissipative frequencies ω_T, ω_Y, ω_{TY}, and ω_{YT} are approximately comparable to each other, and $\sim$1−10 s^{-1} for perturbations with the wavelength of the order of the density lengthscale. The dynamical frequencies, ω_g and ω_L, are typically much higher at $\sim 10^3$ s^{-1}. Therefore, if analysing the conditions (12)–(13), we can restrict ourselves by terms of the highest order in dynamical frequencies and q^2 since the angular velocity of PNSs is very likely much higher than 1−10 s^{-1}. The criteria of instability (12)–(13) read in the approximation as

$$-\omega_g^2\left(\nu + \kappa_T + \frac{\delta}{\beta}\lambda_T\right) - \omega_L^2\left(\nu + \lambda_Y + \frac{\beta}{\delta}\kappa_Y\right) + 2\nu q^2 < 0, \qquad (15)$$

$$-\omega_g^2\left(\nu + \lambda_Y - \frac{\delta}{\beta}\lambda_T\right) - \omega_L^2\left(\nu + \kappa_T - \frac{\beta}{\delta}\kappa_Y\right) + q^2(\kappa_T + \lambda_Y) < 0. \qquad (16)$$

Conditions (15) and (16) represent neutron-finger and convective instability modified by rotation. If $\Omega = 0$, these conditions agree with the criteria derived by Miralles et al. (2000). The effect of rotation is important for convective and neutron-finger instabilities if the angular velocity is comparable to the Brunt-Väisälä or leptonic buoyant frequences. In this paper, we concentrate on condition (14), which represents a new instability in PNSs associated to differential rotation and neutrino transport.

We next consider the case of a moderately rapid rotation when the PNS is far from the rotational distortion and $g \gg \Omega^2 s$. Then, $\Delta\nabla T$, ∇Y, and $\boldsymbol{g}$ are approximately radial, and

$$\omega_g^2 = -\beta\,\frac{k_\theta^2}{k^2}\,(\boldsymbol{G}\cdot\Delta\nabla T) = -\frac{k_\theta^2}{k^2}\omega_{BV}^2, \qquad (17)$$

$$\omega_L^2 = \delta\,\frac{k_\theta^2}{k^2}(\boldsymbol{G}\cdot\nabla Y) = -\frac{k_\theta^2}{k^2}\omega_{LO}^2, \qquad (18)$$

where $\omega_{BV}^2 = \beta(\boldsymbol{g}\cdot\Delta\nabla T)$ is the Brunt-Väisälä frequency, and $\omega_{LO}^2 = \delta(\boldsymbol{G}\cdot\nabla Y)$ is the characteristic frequency of leptonic buoyant oscillations; k_θ is the θ-component of a wavevector, $k_\theta^2 = k_s^2\cos^2\theta - 2k_sk_z\cos\theta\sin\theta + k_z^2\sin^2\theta$, θ is the polar angle. Substituting expressionns (17)–(18) into condition (14), we have

$$q^2 + \omega_\nu^2 + \frac{k_\theta^2}{k^2}\,\xi\,\omega_b^2 < 0, \qquad (19)$$

where

$$\omega_b^2 = \omega_{BV}^2 + \omega_{LO}^2\frac{\kappa_T - \beta\kappa_Y/\delta}{\lambda_Y - \delta\lambda_T/\beta},\quad \xi = \frac{\nu(\lambda_Y - \delta\lambda_T/\beta)}{\kappa_T\lambda_Y - \kappa_Y\lambda_T}.$$

Here, ω_b is the characteristic buoyant frequency that is determined by the Bruni-Väisälä frequency and the frequency of leptonic buoyant oscillations.

Since $\kappa_T\lambda_Y - \kappa_Y\lambda_T > 0$ from thermodynamics (see Pons et al. 1999), rotation-induced instability can occur in a stable stratification only if

$$q^2 = \frac{k_z}{s^3k^2}\left(k_z\frac{\partial}{\partial s} - k_s\frac{\partial}{\partial z}\right)(s^4\Omega^2) < 0. \qquad (20)$$

This condition is satisfactory if either the specific angular momentum, $s^2\Omega$, decreases as the cylindrical radius increases or Ω depends on z such that $\partial\Omega/\partial z \neq 0$.

Condition (19) depends on the direction of the wavevector and can be written as

$$F \equiv \omega_\nu^2 + A\frac{k_z^2}{k^2} - B\frac{k_s k_z}{k^2} + C\frac{k_s^2}{k^2} < 0, \quad (21)$$

where

$$A = 4\Omega^2 + s\Omega_s^2 + \xi\omega_b^2 \sin^2\theta,$$

$$B = s\Omega_z^2 + \xi\omega_b^2 \sin 2\theta,$$

$$C = \xi\omega_b^2 \cos^2\theta.$$

In these expressions, we denote

$$\Omega_s^2 = \frac{\partial\Omega^2}{\partial s}, \quad \Omega_z^2 = \frac{\partial\Omega^2}{\partial z}\cdot \quad (22)$$

Since the dependence of F on the direction of $\boldsymbol{k}$ is quadratic, we can obtain that the relative maximum of F corresponds to

$$\frac{k_z^2}{k^2} = \frac{1}{2}\left[1 \pm \sqrt{\frac{(A-C)^2}{(A-C)^2 + B^2}}\right]. \quad (23)$$

The maximum value of F corresponding to these k_z^2/k^2 yields the following condition of instability:

$$2\omega_\nu^2 + A + C \pm \sqrt{B^2 + (A-C)^2} < 0. \quad (24)$$

The two conditions for instability then follow from the above expression:

$$A + C + 2\omega_\nu^2 = \kappa^2 + \xi\omega_b^2 + 2\omega_\nu^2 < 0, \quad (25)$$

or

$$B^2 + (A-C)^2 > (A + C + 2\omega_\nu^2)^2, \quad (26)$$

where $\kappa^2 = 4\Omega^2 + s\Omega_s^2$ is the epicyclic frequency.

Inequality (25) can be fulfilled only if differential rotation satisfies the Rayleigh criterion, $\partial(s^2\Omega)/\partial s < 0$. This criterion requires differential rotation that decreases very rapidly with the cylindric radius and usually is not fulfilled in stars. Condition (26) can be rewritten as

$$B^2 - 4(A + \omega_\nu)(C + \omega_\nu) > 0, \quad (27)$$

or

$$s^2(\Omega_z^2)^2 + 2s\Omega_z^2\xi\omega_b^2 \sin 2\theta - 4\kappa^2\xi\omega_b^2 - 4\omega_\nu^2(\kappa^2 + \xi\omega_b^2 + \omega_\nu^2) > 0. \quad (28)$$

If stratification is stable ($\xi\omega_b^2 > 0$) and rotation satisfies the Rayleigh stability criterion ($\kappa^2 > 0$), then the rotation-driven instability can occur only if $\partial\Omega/\partial z \neq 0$. Generally, such rotation can often be achieved in PNSs. Note, however, that even if the angular velocity depends on z, the instability does not arise near the rotation axis where s is small and near the equator where $\partial\Omega/\partial z = 0$.

Since differential rotation is likely to be very strong in PNSs, we can estimate $s\Omega_z^2 \sim \Omega^2$. The "viscous" frequency is negligible in Eq. (28) if the wavelength of perturbations $\lambda = 2\pi/k$ satisfies the condition

$$\lambda > \lambda_{\rm cr} = 2\pi\sqrt{\nu/\Omega}. \quad (29)$$

Since $\nu \sim 10^8$–10^9 cm^2/s in PNSs (see, e.g., Thompson & Duncan 1993; Masada et al. 2006), we can estimate $\lambda_{\rm cr}$ as 10^4 cm, which allows a comfortable range of unstable wavelengths. For $\lambda > \lambda_{\rm cr}$, the necessary condition of instability is

$$(s\Omega_z^2)^2 + 2s\Omega_z^2\xi\omega_b^2 \sin 2\theta - 4\kappa^2\xi\omega_b^2 > 0. \quad (30)$$

If stratification is stable ($\xi\omega_b^2 > 0$), this inequality is equivalent to the conditions

$$s\Omega_z^2 > \xi\omega_b^2 \sin 2\theta + \sqrt{\xi^2\omega_b^4 \sin^2 2\theta + 4\kappa^2\xi\omega_b^2}, \quad (31)$$

$$s\Omega_z^2 < \xi\omega_b^2 \sin 2\theta - \sqrt{\xi^2\omega_b^4 \sin^2 2\theta + 4\kappa^2\xi\omega_b^2}. \quad (32)$$

The first condition requires a large positive radial gradient of Ω^2 that seems to be unlikely in PNS. On the contrary, the second condition can be satisfied if the vertical gradient of the angular velocity is negative. If rotation is sufficiently fast and $\kappa^2 \sim \Omega^2 \gg \xi\omega_b^2$, condition (32) yields

$$\frac{|s\Omega_z^2|}{\sqrt{\kappa^2}} \sim \Omega \gg 2\sqrt{\xi}\omega_b. \quad (33)$$

Since $\xi \sim \nu/\kappa_T \ll 1$, condition (33) can be fulfilled in many PNSs even if $\Omega < \omega_b$.

Many studies model rotation of the collapsing core by the angular velocity profile that depends on the spherical radius r alone, $\Omega = \Omega(r)$ (see, e.g., Akiyama et al. 2003; Kotake et al. 2004; Thompson et al. 2005; Sawai et al. 2005). Such a shellular rotation can be justified if the progenitor rotates with the angular velocity that depends on r alone (Mönchmeyer & Müller 1989). In this case, conservation of the angular momentum in the course of collapse leads straightforwardly to a shellular rotation of the collapsing core at the beginning of evolution, at least (see, e.g., Akiyama et al. 2003). For shellular rotation, we have $\Omega_s^2 = \Omega_r^2 \sin\theta$ and $\Omega_z = \Omega_r^2 \cos\theta$ where $\Omega_r^2 = {\rm d}\Omega^2/{\rm d}r$. Then, the criterion of instability reads

$$\frac{|r\Omega_r^2|}{\sqrt{\kappa^2}} \sin 2\theta \sim \Omega \gg 4\sqrt{\xi}\omega_b. \quad (34)$$

In this case, the instability occurs neither near the rotation axis nor near the equator but is strongly constrained to the polar angle $\theta \sim \pi/4$.

The growth rate of instability can be estimated from Eq. (8) if we assume that $\omega_b \sim \omega_{BV} > \Omega$ and condition (30) is satisfied. Then, by making use of the perturbation procedure, we obtain for the unstable root, with accuracy in the lowest order in ν and Ω^2,

$$\gamma \approx -\frac{a_0}{a_1} \approx -\frac{k^4(\kappa_T\lambda_Y - \kappa_Y\lambda_T)}{k_\theta^2\omega_b^2(\lambda_Y - \delta\lambda_Y/\beta)}\left(\frac{k_\theta^2}{k^2}\xi\omega_b^2 + q^2 + \omega_\nu^2\right). \quad (35)$$

This root is positive (unstable) if criterion (19) is satisfied. Under conditions (29) and (33), expression (35) yields the following estimate for the growth rate of perturbations with $k_\theta \sim k$:

$$\gamma \sim \omega_T \frac{\Omega^2}{\omega_b^2}\cdot \quad (36)$$

Perturbations with small k_θ (that corresponds to $k \approx k_r$) can grow even faster. In this case, we also should keep in a_1 the terms proportional to q^2 and obtain that $\gamma \sim \omega_T$. The growth rate of instability is rather high, particularly, for perturbations with a relatively short wavelength (but still satisfying condition (29)). Generally, γ can be comparable to the growth rate of other diffusive instabilities in PNSs, such as the neutron-finger instability.

4. Discussion

We have derived the criteria and growth rate of the rotation-driven instability in non-magnetic PNSs. This instability is associated with differential rotation and neutrino transport and can even arise in convectively stable regions. This instability is representative of a wide class of the so-called doubly diffusive instabilities known in stellar hydrodynamics (see, e.g., Acheson 1978). In fact, this instability is the analogue of the well-known Goldreich-Schubert-Fricke instability for the case of neutrino transport (see Goldreich & Schubert 1968; Fricke 1969). The criterion of the Goldreich-Schubert-Fricke instability in ordinary stars is $\partial\Omega/\partial z \neq 0$. However, neutrino transport is qualitatively different from the radiative one even in diffusive approximation because, apart from transporting heat, it also involves transport of the lepton number (Eqs. (5) and (6)), which is important for the pressure balance. That is the reason the criterion of instability in PNSs (30) is so different from the standard condition $\partial\Omega/\partial z \neq 0$ derived by Goldreich & Schubert (1968) and Fricke (1969). In PNSs, however, the instability also occurs only if the angular velocity depends on the vertical coordinate.

During the early evolution stage, a rotation profile can be rather complex in PNSs and, therefore, the necessary condition of instability (33) can be fulfilled. For example, 2D calculations of the early rotation evolution of PNSs by Ott et al. (2006) indicate the presence of differential rotation in PNSs at 200 ms after bounce. Differential rotation is moderately strong in the inner region of a PNS with $r < 10$ km, where Ω varies by a factor 1.5–3 depending on the model. In the outer region ($r > 10$ km), differential rotation is substantially stronger.

The growth rate of the considered instability is small compared to that of convection; nevertheless, it is rather high and, in general, can be comparable to the growth rate of other diffusive instabilities such as the neutron-finger one. For example, even if rotation is slow and $\Omega \sim 0.1\omega_b$, the growth time of perturbations with a lengthscale close to the pressure scale-height is $\sim$1 s. Perturbations with a shorter wavelength grow much faster. Most likely, then, the unstable motions reach saturation in PNSs. The turbulent velocity in saturation can be estimated by making use of the standard mixing-length approximation (see, e.g., Schwarzschild 1958). In the case of diffusive instabilities, this approximation agrees very well with numerical simulations (see Arlt & Urpin 2004). Then, the turbulent velocity with the lengthscale λ is given by

$$v(\lambda) \sim \lambda\gamma(\lambda) \sim \frac{\kappa_T}{\lambda}\,\frac{\Omega^2}{\omega_b^2}, \qquad (37)$$

where $\gamma(\lambda)$ is the growth rate of perturbations with the wavelength λ obtained from the linear theory of instability. If $\Omega \sim 0.1{-}1\ \omega_b$, then the velocity in the largest scale comparable to the pressure scale-height is $\sim 10^5{-}10^7$ cm/s. Such motions can essentially enhance, for example, the angular momentum transport that is important for the dynamo action in PNSs. The corresponding coefficient of turbulent diffusion is

$$\nu_T \sim \kappa_T\,\frac{\Omega^2}{\omega_b^2}, \qquad (38)$$

and it can be comparable to the coefficient of heat diffusion that is largest among the diffisive coefficients in PNSs. Note that turbulent transport caused by the instability should be substantially anisotropic since it cannot arise in regions close to the equator and rotation axis. Such anisotropic transport can perhaps lead to an anisotropic enhancement in the neutrino flux during the initial second following the core-bounce.

We have considered the instability of only short wavelength perturbations but, most likely, perturbations with a wavelength of the order of the pressure scale-heigh are unstable as well. Therefore, the instability can be the reason for a large-scale overturn of the core region with a high energy density. This overturn can generally enhance the neutrino flux and make it anisotropic, which is crucial for the explosion mechanism since even a slight modification in the neutrino cooling/heating efficiency changes it qualitatively (Dessart et al. 2006). This is particularly important because the instability operates in the regions where convection is suppressed and cannot influence the neutrino transport. For example, according to Dessart et al. (2006), convection at $t < 1$ s occurs in the region between 10 and 30 km, whereas the innermost region with the highest energy content is stable. The rotation-induced instability in the core could lead to motions that transport heat and lepton number to the outer region. The important feature of the instability is that it produces large-scale departures from the spherical symmetry. This instability cannot arise near the equator and rotation axis and, therefore, turbulent transport is suppressed in those regions.

Hydrodynamic motions induced by the instability can also contribute to the gravitational wave signals from the core-collapse supernovae. As recognised by Müller et al. (2004), the dominant contribution to the gravitational wave signal is not produced by the core bounce itself but instead by the neutrino-driven convection in the postshock region and the Ledoux convection inside the deleptonizing proto-neutron star; note, however, that Fryer et al. (2004) argue the opposite based on their results. Because of these contradictory results, the influence of convective motions on the gravitational wave signal seems to be an open issue. It is possible, however, that motions caused by the rotational instability, despite being slower than convective motions, can produce a significant gravitational signal, because the instability arises in the dense inner region of the PNS with the radius <10 km where convection does not occur during the first second after the bounce.

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