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A generalized $\sqrt{\epsilon}$-law (Research Note)

The role of unphysical source terms in resonance line polarization transfer and its importance as an additional test of NLTE radiative transfer codes

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ABSTRACT

Context. A derivation of a generalized $\sqrt{\epsilon}$-law for nonthermal collisional rates of excitation by charged perturbers is presented.
Aims. Aim of this paper is to find a more general analytical expression for a surface value of the source function which can be used as an additional tool for verification of the non-LTE radiative transfer codes.
Methods. Under the impact approximation hypothesis, static, one-dimensional, plane-parallel atmosphere, constant magnetic field of arbitrary strength and direction, two-level atom model with unpolarized lower level and stimulated emission neglected, we introduce the unphysical terms into the equations of statistical equilibrium and solve the appropriate non-LTE integral equations.
Results. We derive a new analytical condition for the surface values of the source function components expressed on the basis of irreducible spherical tensors.

Key words. line: formation – polarization – radiative transfer

1. Introduction

In the series of papers of Landi Degl'Innocenti et al. (1991a,b), Landi Degl'Innocenti & Bommier (1994) (from now on referenced as Paper I), the general formalism of resonance line polarization scattering for a two-level atom has been developed. The non-LTE problem of the 2nd kind for an arbitrary magnetic field, three-dimensional geometry of the medium and arbitrary irradiation by external sources has been discussed. The effect of inelastic collisions with charged perturbers has been considered for the particular case of a relative Maxwellian velocity distribution.

Paper I analysed the analytical properties of the solutions in the particular case of a one-dimensional, semi-infinite, static atmosphere with a constant magnetic field of arbitrary strength and direction and assuming zero external irradiation of the atmosphere. They derived a generalization of the well known $\sqrt{\epsilon}$-law (e.g. Avrett & Hummer 1965; Mihalas 1970; Hubený 1987) for the case of polarized radiation and extended the previous results of Ivanov (1990) who studied scattering in a non-magnetic regime.

In most cases of practical interest the polarization degree is rather small. The purpose of this paper is to find a new analytical solution of the non-LTE problem in unphysical conditions in order to better verify the accuracy of the polarized radiation transfer codes. This is done by introduction of an unphysical source term in the polarization into the equations of statistical equilibrium. Such a generalization can be useful in testing the accuracy of the radiative transfer codes whose purpose is to deal with the non-thermal collisional processes (for instance in the impact polarization studies of solar flares).

Following the approach of the papers quoted above, we adopt the formalism of density matrix in the representation of irreducible tensorial operators (e.g. Fano 1957). We consider the lower level with total angular momentum j to be unpolarized. This level is completely described by the overall population which is set to 1 for normalization reasons. The upper level with angular momentum j' is described by the multipole components of the statistical tensor ρ^K_Q. Coherences between different levels j and j' are neglected but coherences between Zeeman sublevels of level j' are in general taken into account. The calculation is performed in the Wien limit of line frequency whose assumption makes it possible to neglect stimulated emission effect, and to preserve the linearity of the non-LTE problem.

2. Equations of statistical equilibrium

The suitable coordinate system Σ_0 for atomic state description is the one with the z-axis directed along the magnetic field (see Fig. 1).

Radiative rate contributions to the evolution of statistical operator ρ^K_Q are given by (Landi Degl'Innocenti 1985)

$$\left[\frac{\mathrm{d}\rho^K_Q}{\mathrm{d}t}\right]_{\mathrm{RAD}} = -\mathrm{i}A_{j'j}\Gamma Q\rho^K_Q - A_{j'j}\rho^K_Q + \frac{w^{(K)}_{j'j}(-1)^Q}{\sqrt{2j'+1}}B_{jj'}\overline{J}^K_{-Q}. \quad (1)$$

In this equation $A_{j'j}$ ($B_{jj'}$) is the Einstein coefficient of spontaneous emission (absorption) from level j' (j) to level j (j').

$\Gamma = 2\pi g_{j'} \nu_{\rm L}/A_{j'j}$ with $g_{j'}$ being the Landé factor of the level j' and $\nu_{\rm L}$ is the Larmor frequency. The transition-dependent numerical factor $w^{(K)}_{j'j}$ has been defined by Landi Degl'Innocenti (1984) as have the irreducible components of the mean radiation tensor $\bar{J}^K_Q$. Besides the radiative rates, collisional rates have to be considered in the statistical equilibrium, because the source of radiation in a semi-infinite atmosphere is the collisional excitation followed by radiative de-excitation. Thus, the source term of the radiative transfer equation originates in the inelastic collision effect. As the purpose of the present paper is to consider unphysical source terms in the non-zero ranks (K, Q) of the irreducible tensorial operator basis T^K_Q, we will introduce an unphysical (K, Q)-dependence to the inelastic collisional rates of the statistical equilibrium equation below. The purpose here is not to thus describe anisotropic collisions, which would require a proper formalism that is out of the scope of the present paper (see, for instance, Landi Degl'Innocenti & Landolfi 2004, for a two-level atom, and Derouich 2006, for polarization transfer rates in a multi-level atom due to isotropic collisions). We introduce as usual the depolarizing rate due to isotropic elastic collisions. Thus, the contribution of collisional rates reads

$$\left[\frac{{\rm d}\rho^K_Q}{{\rm d}t}\right]_{\rm COLL} = (C_{jj'})^K_Q - (C^{\rm R}_{j'j})^K_Q \rho^K_Q - D^{(K)}\rho^K_Q. \tag{2}$$

The terms $(C_{jj'})^K_Q$ and $(C^{\rm R}_{j'j})^K_Q$ on the right-hand side of Eq. (2) are the multipole components of collisional rates of excitation and relaxation respectively. $D^{(K)}$ is the depolarization rate due to elastic collisions[1].

The radiative and collisional rates can be added under the impact approximation hypothesis (Bommier & Sahal-Bréchot 1991) ${\rm d}\rho^K_Q/{\rm d}t = [{\rm d}\rho^K_Q/{\rm d}t]_{\rm RAD} + [{\rm d}\rho^K_Q/{\rm d}t]_{\rm COLL}$. Using Eqs. (1), (2), and the condition for static atmosphere, ${\rm d}\rho^K_Q/{\rm d}t = 0$, we obtain the equations of statistical equilibrium

$$\left[{\rm i}A_{j'j}\Gamma Q + A_{j'j} + (C^{\rm R}_{j'j})^K_Q + D^{(K)}\right]\rho^K_Q = \frac{w^{(K)}_{j'j}(-1)^Q}{\sqrt{2j'+1}} B_{jj'} \bar{J}^K_{-Q} + (C_{jj'})^K_Q. \tag{3}$$

By applying the relation between Einstein coefficients for spontaneous emission and absorption,

$$B_{jj'} = \frac{2j'+1}{2j+1}\frac{c^2}{2h\nu_0^3}A_{j'j}, \tag{4}$$

and dividing formula (4) by $A_{j'j}$, we obtain the equation

$$\left(1 + \epsilon^K_Q + \delta^{(K)} + {\rm i}\Gamma Q\right)\rho^K_Q = (-1)^Q w^{(K)}_{j'j}\bar{J}^K_{-Q}\frac{\sqrt{2j'+1}}{2j+1} \times \frac{c^2}{2h\nu_0^3} + \frac{(C_{jj'})^K_Q}{A_{j'j'}}. \tag{5}$$

One can introduce the dimensionless parameter of the depolarization rate

$$\delta^{(K)} = \frac{D^{(K)}}{A_{j'j}}, \tag{6}$$

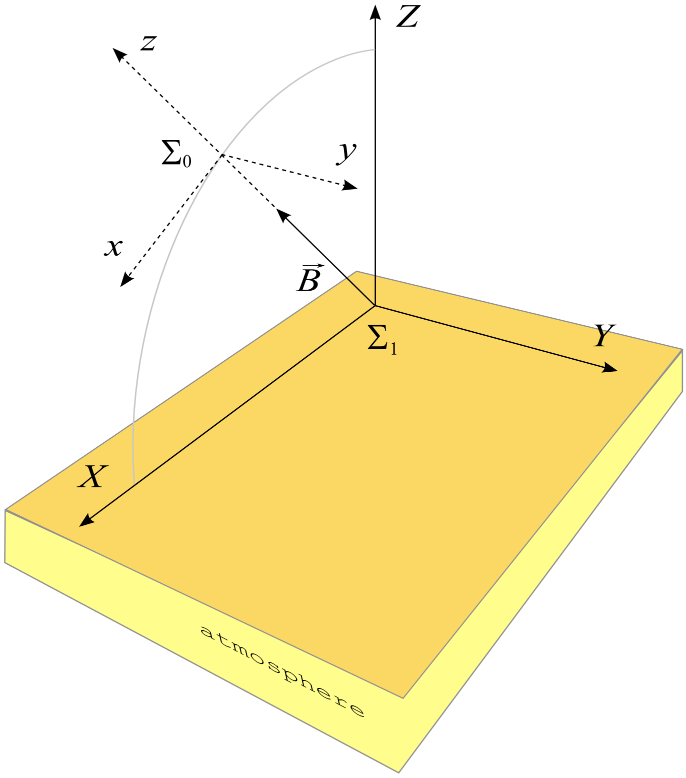

Fig. 1. The reference frame Σ_1 has its Z-axis oriented vertically with respect to the atmosphere, while the z-axis of the reference frame Σ_0 is parallel to the direction of magnetic field $\boldsymbol{B}$. The axes X and x lie in the same plane defined by Z-axis and $\boldsymbol{B}$; the axes Y and y are defined to complement the right-handed orthogonal coordinate systems.

and the irreducible tensor which plays the role of generalized photon destruction probability

$$\epsilon^K_Q = \frac{(C^{\rm R}_{j'j})^K_Q}{A_{j'j'}}. \tag{7}$$

If the relation $(C^{\rm R}_{j'j})^K_Q \neq 0$ is satisfied we may define the quantity

$$B^{(KQ)} = \frac{2h\nu_0^3}{c^2}\frac{2j+1}{\sqrt{2j'+1}}\frac{(C_{jj'})^K_Q}{(C^{\rm R}_{j'j})^K_Q}. \tag{8}$$

It is easy to show (see below) that in the particular case of a Maxwellian velocity distribution of colliders the relation $B^{(00)} = B_{\rm P}$ is satisfied, where $B_{\rm P}$ is the Planck function in the Wien limit at given temperature. Using the definition of irreducible components of the two-level source function (cf. Paper I)

$$S^K_Q = \frac{2h\nu_0^3}{c^2}\frac{2j+1}{\sqrt{2j'+1}}\rho^K_Q, \tag{9}$$

we obtain the statistical equilibrium equations in the compact form

$$\left(1 + \epsilon^K_Q + \delta^{(K)} + {\rm i}\Gamma Q\right) S^K_Q = w^{(K)}_{j'j}(-1)^Q \bar{J}^K_{-Q} + \epsilon^K_Q B^{(KQ)}. \tag{10}$$

3. Solution of the Wiener-Hopf equations

From now on we reduce our analysis to the case of semi-infinite, plane-parallel geometry with constant magnetic field along the

[1] This process cannot change a total population of the level. Therefore it is always $D^{(0)} = 0$. We take formally into account only the depolarization rate $D^{(K)}$ to use a formalism coherent with the previous papers. A general treatment of physically more relevant transfer of multipole components of the upper level is out of scope of this paper.

atmosphere. The velocity distribution and volume density of colliders is also constant along the atmosphere but it is in general non-thermal. The only position coordinate is the common line optical depth τ.

Following the procedure of Paper I a formal solution of radiative transfer equation is substituted into the definition of tensor $\overline{J}^K_Q$; after that we obtain a set of integral Wiener-Hopf equations of the 2nd kind,

$$(1 + \epsilon^K_Q + \delta^{(K)} + \mathrm{i}\Gamma Q) S^K_Q(\tau) = \sum_{K'Q'} \int_0^\infty \widetilde{K}_{KQ,K'Q'}(\tau,\tau') S^{K'}_{Q'}(\tau')\mathrm{d}\tau' + \epsilon^K_Q B^{(KQ)}, \tag{11}$$

which describe coupling of the tensors $\rho^K_Q(\tau)$ at different optical depths via radiation. Several important properties of kernels $\widetilde{K}_{KQ,K'Q'}(\tau,\tau')$ have been discussed by Landi Degl'Innocenti et al. (1990) and in Paper I. Using their indexing notation one can rewrite the Eq. (11) in the shorthanded form

$$a_i S_i(\tau) = \sum_j \int_0^\infty K_{ij}(|\tau-\tau'|) S_j(\tau')\mathrm{d}\tau' + b_i, \tag{12}$$

with

$$a_i = 1 + \epsilon^K_Q + \delta^{(K)} + \mathrm{i}\Gamma Q, \tag{13}$$

$$b_i = \epsilon^K_Q B^{(KQ)}. \tag{14}$$

The index i in these expressions runs between the limits 1 and N, where N is the number of ^K_Q-multipoles. In the following we briefly repeat the derivation performed by Frisch & Frisch (1975) emphasizing the differences due to presence of b_i terms.

Calculation of the derivative of (12) with respect to τ, splitting the integral on the right-hand side into two parts, multiplication of the equation by $S_i(\tau)$, summation over index i, and finally integration with respect to τ leads to the set of equations

$$\sum_i a_i \int_0^\infty S_i(\tau)\frac{\mathrm{d}S_i(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \sum_{i,j} S_j(0)\int_0^\infty K_{ij}(\tau)S_i(\tau)\mathrm{d}\tau + \sum_{i,j}\int_0^\infty \mathrm{d}\tau S_i(\tau)\int_0^\infty \mathrm{d}\tau' K_{ij}(|\tau'-\tau|)\frac{\mathrm{d}S_j(\tau')}{\mathrm{d}\tau'}. \tag{15}$$

The left-hand side of (15) is easily evaluated as

$$\frac{1}{2}\sum_i a_i\left[S_i(\infty)^2 - S_i(0)^2\right]. \tag{16}$$

The first term on the right-hand side of (15) is evaluated using the kernels symmetry $K_{ij}(t) = K_{ji}(t)$ and Eq. (12), so that we obtain

$$\sum_i S_i(0)\left[a_i S_i(0) - b_i\right], \tag{17}$$

while the second term equals

$$\frac{1}{2}\sum_i a_i\left[S_i(\infty)^2 - S_i(0)^2\right] - \sum_i b_i\left[S_i(\infty) - S_i(0)\right]. \tag{18}$$

We put these results into (15) to get

$$\sum_i a_i S_i(0)^2 = \sum_i b_i S_i(\infty). \tag{19}$$

Calculation of the limit $\tau \to \infty$ of both sides of Eq. (12) leads to the set of linear algebraic equations for the components of source function tensor in the infinite depth:

$$\sum_j\left[a_j\delta_{ij} - \int_{-\infty}^\infty K_{ij}(t)\mathrm{d}t\right]S_j(\infty) = b_i. \tag{20}$$

We can solve these equations and write

$$\boldsymbol{S}(\infty) = \boldsymbol{L}^{-1}\boldsymbol{b}, \tag{21}$$

where $\boldsymbol{S}$ is the formal vector of S_i components, $\boldsymbol{b}$ is the formal vector of b_i components, and the elements of matrix $\boldsymbol{L}$ are defined by relation

$$\{\boldsymbol{L}\}_{ij} = a_j\delta_{ij} - \int_{-\infty}^\infty K_{ij}(t)\mathrm{d}t. \tag{22}$$

Establishing a new matrix $\boldsymbol{\ell} = \boldsymbol{L}^{-1}$ and substituting (21) into (19) leads to the generalized form of the $\sqrt{\epsilon}$-law

$$\sum_i a_i S_i(0)^2 = \sum_{i,j} b_i b_j \ell_{ij}. \tag{23}$$

4. Particular solutions

Setting the special conditions for magnetic field and collisional rates, one recovers the less general but more common and explicit formulations of the $\sqrt{\epsilon}$-law than the one given by (23). In the following sections we will verify this result in the limiting conditions assumed in recent papers and we will analyse the simple examples of non-thermal collisional excitation.

4.1. Maxwellian velocity distribution of colliders

In the case of Maxwellian velocity distribution of colliders, relaxation rates of all multipole components ρ^K_Q are the same:

$$(C^{\mathrm{R}}_{j'j})^K_Q = C^{\mathrm{R}}_{j'j}, \tag{24}$$

where $C^{\mathrm{R}}_{j'j}$ is the usual relaxation rate for collisional deexcitation from j' to j. For excitation rates one has

$$(C_{jj'})^K_Q = \frac{C_{jj'}}{\sqrt{2j'+1}}\delta_{K0}\delta_{Q0}, \tag{25}$$

where the factor $(2j'+1)^{-1/2}$ has been introduced to make a connection with the usual collisional rate $C_{jj'}$ of standard unpolarized theory. In this isotropic case, there is no collisional excitation of higher ranks of density matrix. From the assumption of thermodynamic equilibrium one has

$$\frac{C_{jj'}}{C^{\mathrm{R}}_{j'j}} = \frac{2j'+1}{2j+1}\mathrm{e}^{-h\nu_0/k_{\mathrm{B}}T}, \tag{26}$$

where k_{B} stands for the Boltzmann constant and T for a temperature of the atmosphere. From (24) and (7) it is evident that $\epsilon^K_Q = \epsilon$ for all possible K and Q, where ϵ is the common photon destruction probability. Further

$$B^{(KQ)} = B_{\mathrm{P}}\delta_{K0}\delta_{Q0}. \tag{27}$$

Substituting the rates (24) and (25) into formula (22) and employing the general identity $\int_{-\infty}^\infty K_{i1}(t)\mathrm{d}t = \delta_{i1}$ (see Paper I) together with $b_i = \delta_{i1}$, we recover form (23) the formula (16) of the previously cited paper:

$$\sum_{KQ}(1+\epsilon+\delta^{(K)}+\mathrm{i}\Gamma Q)[S^K_Q(0)]^2 = \epsilon B_\mathrm{P}^2. \quad (28)$$

Assuming that there is zero magnetic field, i.e. $\Gamma = 0$, the source function tensor reduces due to symmetry reasons to the two non-vanishing components S^0_0 and S^2_0 in the reference frame Σ_1. This reference frame is suitable for descriptions of the atomic system under these conditions, so that we may identify $\Sigma_0 \equiv \Sigma_1$, with X and Y axes oriented arbitrary in the plane parallel to atmospheric surface. Further, assuming that there is no depolarization of the upper level ($\delta^{(K)} = 0$), we realize from (28):

$$\sqrt{\left[S^0_0(0)\right]^2+\left[S^2_0(0)\right]^2} = \sqrt{\frac{\epsilon}{1+\epsilon}}B_\mathrm{P} = \sqrt{\epsilon'}B_\mathrm{P}, \quad (29)$$

which is the same result derived in different notation by Ivanov (1990). For simplicity the common alternative to the photon destruction probability has been introduced: $\epsilon' = \epsilon/(1+\epsilon)$.

If depolarization of the upper level is high enough to destroy atomic level polarization ($\delta^{(K)} \to \infty$ for $K > 0$), or the upper level is unpolarizable, the common $\sqrt{\epsilon}$-law for scalar radiation is recovered,

$$S^0_0(0) = \sqrt{\frac{\epsilon}{1+\epsilon}}B_\mathrm{P} = \sqrt{\epsilon'}B_\mathrm{P}. \quad (30)$$

4.2. Anisotropic alignment (de)excitation

The relation $\epsilon^K_Q = \epsilon$ is not in general satisfied for all the multipoles because the relaxation of the ρ^K_Q state depends on the velocity distribution of colliders. In the following text we will neglect the effects of magnetic field.

Let us assume an example of a relative velocity distribution of particles that is axially symmetric with the axis of symmetry parallel to the vertical of the atmosphere (so that it is as in the former case $\Sigma_0 \equiv \Sigma_1$) and that the collisional interaction can be fully described by only the first two even multipole components of this distribution. Thanks to these assumptions the only non-vanishing excitation collisional rates are $(C_{jj'})^0_0$ and $(C_{jj'})^2_0$, the relaxation rates $(C^\mathrm{R}_{j'j})^0_0$ and $(C^\mathrm{R}_{j'j})^2_0$ and for the same reasons the only non-zero source function components are S^0_0 and S^2_0.

An explicit evaluation of the integrals of kernels $\int_{-\infty}^{\infty}\widetilde{K}_{KQ,K'Q'}(\tau,\tau')\mathrm{d}\tau'$ under these conditions shows that the only non-zero ones are given by (A5) and (A12) of Landi Degl'Innocenti et al. (1991b). In our notation they read

$$\int_{-\infty}^{\infty}\widetilde{K}_{00,00}(\tau,\tau')\mathrm{d}\tau' = 1, \quad (31)$$

$$\int_{-\infty}^{\infty}\widetilde{K}_{20,20}(\tau,\tau')\mathrm{d}\tau' = \frac{7}{10}W_2, \quad (32)$$

with $W_2 = (w^{(2)}_{j'j})^2$. Substituting these results into (23) we see that

$$\left(1+\epsilon^0_0\right)\left(S^0_0\right)^2 + \left(1+\epsilon^2_0\right)\left(S^2_0\right)^2 = \epsilon^0_0\left(B^{(00)}\right)^2 + \frac{\left(\epsilon^2_0 B^{(20)}\right)^2}{1+\epsilon^2_0-\frac{7}{10}W_2}. \quad (33)$$

To check the validity of polarized radiative transfer codes, it is advantageous if one can verify that the transfer of higher ranks of the radiation tensor is accurate enough. In the realistic scattering polarization models the polarization degree does not exceed a few percent so that $|S^0_0(0)| \gg |S^K_Q(0)|$. By setting arbitrary (even unphysical) collisional rates it is possible to verify transfer codes in conditions with $|S^0_0| \ll |S^K_Q|$.

To privilege transfer in higher ranks of the radiation tensor one can artificially suppress the excitation rate $(C_{jj'})^0_0$. In the extremal case one can set $(C_{jj'})^0_0 \to 0$. The easiest way to do this is the formal interchange of the role of excitation rates of population and alignment, i.e. $(C_{jj'})^0_0 \leftrightarrow (C_{jj'})^2_0$ of the original Maxwellian velocity distribution:

$$(C_{jj'})^0_0 = 0, \quad (C_{jj'})^2_0 = \frac{C_{jj'}}{\sqrt{2j'+1}} \quad (34)$$

(no collisional excitation to upper level population) and the relaxation rates set to the Maxwellian ones

$$(C^\mathrm{R}_{j'j})^0_0 = (C^\mathrm{R}_{j'j})^2_0 = C^\mathrm{R}_{j'j}. \quad (35)$$

In this case we have

$$B^{(00)} = 0, \quad B^{(20)} = B_\mathrm{P}, \quad (36)$$

and again

$$\epsilon^0_0 = \epsilon^2_0 = \epsilon. \quad (37)$$

Substituting this into (33) we find out the $\sqrt{\epsilon}$-law in the form

$$\sqrt{[S^0_0(0)]^2+[S^2_0(0)]^2} = \frac{\epsilon'}{\sqrt{1-\frac{7}{10}W_2(1-\epsilon')}}B_\mathrm{P}. \quad (38)$$

The particular collisional rates (34) are in fact arbitrary and have been chosen to obtain a formula similar to the one of the Maxwellian distribution case.

This relation is useful to test polarized radiative transfer codes, because in this unphysical case $S^2_0(0)$ is the largest term, unlike the physical case where the largest term is $S^0_0(0)$ and $S^2_0(0)$ is only a few percent of it. By applying Eq. (38) the test is much more sensitive to the polarization, and the polarization is better tested. We have thus successfully tested a multilevel non-LTE radiative transfer code that we are developing, but this code and its results are the subjects of a forthcoming paper.

5. Conclusions

We have derived a more general formulation of the so-called $\sqrt{\epsilon}$-law of radiation transfer. This analytical condition couples the value of source function tensor of a two-level atom with other physical properties of the atmosphere. The simplest result obtained in conditions of a non-magnetic, isothermal, plane-parallel, semi-infinite atmosphere with thermal velocity distribution of particles and unpolarized atomic levels (e.g. Mihalas 1970) has been generalized by Ivanov (1990) to account for scattering of polarized radiation and polarized upper atomic level. Further generalizations done in Paper I, which account for a magnetic field of arbitrary strength and direction, has been extended in the present paper to account for non-thermal collisional interactions. It was done by introducing the tensor of the photon destruction probability ϵ^K_Q and by defining the function $B^{(KQ)}$.

The resulting formula (23) reduces to the cases mentioned above if the physical conditions become more symmetric. On the other hand, situations with a high degree of perturbers velocity distribution anisotropy and especially ones with unphysical collisional rates result in a wide range of models which can be calculated both numerically and analytically. Thus they offer new possibilities for verification of the non-LTE radiation transfer codes.

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