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Confinement and anisotropy of ultrahigh-energy cosmic rays in isotropic plasma wave turbulence

I. Modification of the Hillas limit due to turbulence geometry*

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ABSTRACT

Context. The mean free path and anisotropy of galactic cosmic rays is calculated in weak plasma wave turbulence that is isotropically distributed with respect to the ordered uniform magnetic field.

Aims. The modifications on the value of the Hillas energy, above which cosmic rays are not confined to the Galaxy, are calculated. The original determination of the Hillas limit has been based on the case of slab turbulence where only parallel propagating plasma waves are allowed.

Methods. We use quasilinear cosmic ray Fokker-Planck coefficients to calculate the mean free path and the anisotropy in isotropic plasma wave turbulence.

Results. In isotropic plasma wave turbulence the Hillas limit is enhanced by about four orders of magnitude to $E_c = 2.03 \times 10^5 A n_e^{1/2} (L_{\max}/10\ \mathrm{pc})$ PeV resulting from the dominating influence of transit-time damping interactions of cosmic rays with obliquely propagating magnetosonic waves.

Conclusions. Below the energy E_c the cosmic ray mean free path and the anisotropy exhibit the well known $E^{1/3}$ energy dependence. At energies higher than E_c both transport parameters steepen to a E^3-dependence. This implies that cosmic rays even with ultrahigh energies of several hundreds of EeV can be rapidly pitch-angle scattered by interstellar plasma turbulence, and are thus confined to the Galaxy.

Key words. ISM: cosmic rays – ISM: magnetic fields – plasmas – scattering

1. Introduction

To unravel the nature of cosmic sources that accelerate cosmic rays to ultrahigh energies has been identified as one of the eleven fundamental science questions for the new century (Turner et al. 2002). Cosmic rays with energies up to at least 10^{14} eV are likely accelerated at the shock fronts associated with supernova remnants (for review see Blandford & Eichler 1987). Radio emissions and X-rays give conclusive evidence that electrons are accelerated there to near-light speed (Koyama et al. 1995, 1997; Tanimori et al. 2001; Allen et al. 1997; Slane et al. 1999; Borkowski et al. 2001). The HESS observations of supernova remnants up to ∼100 TeV provide direct evidence of very high energy particle acceleration in the shocks (Aharonian et al. 2004, 2005), while the leptonic or hadronic nature of these gamma-rays is currently being disputed (e.g. Enomoto et al. 2002; Reimer & Pohl 2002). The supernova remnant origin would be consistent with the observed GeV excess of diffuse galactic gamma radiation from the inner Galaxy (Büsching et al. 2001), although the GeV excess has been found to be present in all directions including galactic latitudes where no supernova remnants are present and the outer Galaxy (Strong et al. 2004). This indicates that the origin of the GeV excess is more complex and is not straightforwardly connected with supernova remnants in the inner Galaxy.

More puzzling are the much higher energy cosmic rays with energies as large as $10^{20.5}$ eV. It has been argued (Lucek & Bell 2000; Bell & Lucek 2001; Hillas 2006) that, due to the amplification of the magnetic field in the shock, the acceleration of cosmic rays in young supernova remnants is possible up to $\sim 10^{18}$ eV. This implies that such particles may have a Galactic origin. For ultrahigh-energy (10^{18}–$10^{20.5}$ eV) cosmic rays an extragalatic origin is favored by many researchers. Extragalactic ultrahigh-energy cosmic rays (UHECRs) coming from cosmological distances ≥50 Mpc should interact with the universal cosmic microwave background radiation (CMBR) and produce pions. For an extragalactic origin of UHECRs the detection or non-direction of the Greisen-Kuzmin-Zatsepin cutoff resulting from the photopion attenuation in the CMBR will have far-reaching consequences not only for astrophysics but also for fundamental particle physics as e.g. the breakup of Lorentz symmetry (Coleman & Glashow 1997) or the non-commutative quantum picture of spacetime (Amelio-Camelia et al. 1998).

Radio synchrotron radiation intensity and polarisation surveys of our own and external galaxies (for review see Sofue et al. 1986) have revealed that the interstellar medium is transversed by large-scale ordered magnetic fields with superposed plasma wave turbulence. The Galactic magnetic field has a regular and a random component of about equal strength. The

* Appendices are only available in electronic form at http://www.aanda.org

turbulent field has a broad spectrum of scales with the largest one being 10−100 pc (e.g. Beck 2007, and references therein). This could be compared with the gyroradius of ∼1 pc for 10^{15} eV particles, or ∼1 kpc for 10^{18} eV particles. The conventional size of the Galactic halo derived from abundances of radioactive isotopes in cosmic rays is about 4−6 kpc (Ptuskin & Soutoul 1998; Strong & Moskalenko 1998; Webber & Soutoul 1998). The turbulent magnetic field may thus present a mechanism for isotropization of Galactic cosmic rays up to 10^{17}−10^{18} eV (see, e.g., Candia et al. 2003).

According to the current understanding (reviewed in Schlickeiser 2002) the relativistic charged particles (hereafter referred to as cosmic ray particles) in these space plasmas are confined and accelerated by resonant interactions in these weakly random electromagnetic fields. In the presence of low-frequency magnetohydrodynamic plasma waves, whose magnetic field component is much larger than their electric field component, the particle's phase space distribution function adjusts rapidly to a quasi-equilibrium through pitch-angle diffusion, which is close to the isotropic distribution. The isotropic part of the phase space distribution function $F(z, p, t)$ obeys the *diffusion-convection-equation*

$$\frac{\partial F}{\partial t} - S_0 = \frac{\partial}{\partial z}\left[\kappa\frac{\partial F}{\partial z}\right] - V\frac{\partial F}{\partial z}$$
$$+\frac{p}{3}\frac{\partial V}{\partial z}\frac{\partial F}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 A_{\mathrm{M}}\frac{\partial F}{\partial p} - p^2\dot{p}_{\mathrm{Loss}}F\right] - \frac{F}{T_{\mathrm{c}}} \tag{1}$$

where the parallel spatial diffusion coefficient κ, the cosmic ray bulk speed V and the momentum diffusion coefficient A are determined by pitch-angle averages of three Fokker-Planck coefficients

$$\kappa = \frac{v}{3}\lambda = \frac{v^2}{8}\int_{-1}^{1}\mathrm{d}\mu\frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{2}$$

$$V = u + \frac{1}{3p^2}\frac{\partial}{\partial p}(p^3 D),\ D = \frac{3v}{4p}\int_{-1}^{1}\mathrm{d}\mu(1-\mu^2)\frac{D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}, \tag{3}$$

$$A_M = \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\left[D_{pp}(\mu) - \frac{D^2_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}\right]. \tag{4}$$

In Eq. (1) the space coordinate z is parallel to the uniform background magnetic field $\boldsymbol{B}_0$, S_0 is the source term, $\dot{p}_{\mathrm{Loss}}$ and T_{c} describe continuous and catastrophic momentum loss processes. See also Appendix A for a glossary and definitions of important symbols.

For many years the theoretical development of the resonant wave-particle interactions has mainly concentrated on the special case that the plasma waves propagate only parallel or antiparallel to the ordered magnetic field – the socalled slab turbulence. In this case only cosmic ray particles with gyroradii R_{L} smaller than the longest parallel wavelength $L_{\|,\max}$ of the plasma waves can resonantly interact. Obviously this condition is equivalent to a limit on the maximum particle rigidity R:

$$R = \frac{p}{Z} \le eB_0 L_{\|,\max}. \tag{5}$$

An alternative way to express the condition (5) is

$$E_{15}/Z \le 40\cdot\left(\frac{B_0}{4\,\mu\,\mathrm{G}}\right)\left(\frac{L_{\|,\max}}{10\ \mathrm{pc}}\right), \tag{6}$$

where E_{15} denotes the cosmic ray particle energy in units of 10^{15} eV. The limit set by the right hand side of Eq. (6) is referred to as Hillas limit (Hillas 1984). According to this limit, cosmic ray protons of energies larger than 40 PeV = 4×10^{16} eV cannot be confined or accelerated in the Milky Way, and an extragalactic origin for this cosmic ray component has to be invoked. Moreover, as the cosmic ray mean free path in case of spatial gradients is closely related to the cosmic ray anisotropy (Schlickeiser 1989, Eq. (94)), the Hillas limit (6) implies strong anisotropies at energies above 40 PeV which have not been observed by the KASKADE experiment (Antoni et al. 2004; Hörandel et al. 2006).

It is the purpose of this work to investigate how the Hillas limit (6) is affected if we discard the assumption of purely slab plasma waves, i.e. if we allow for oblique propagation angles θ of the plasma waves with respect to the ordered magnetic field component. There is ample observational evidence that obliquely propagating magnetohydrodynamic plasma waves exist in the interstellar medium (Armstrong et al. 1995; Lithwick & Goldreich 2001; Cho et al. 2002). In particular, we will consider the alternative extreme limit that the plasma waves propagation angles are isotropically distributed around the magnetic field direction. It has been emphasised before by Schlickeiser & Miller (1998) referred to as SM) that oblique propagation angles of fast magnetosonic waves leads to an order of magnitude quicker stochastic acceleration rate as compared to the slab case, since the compressional component of the obliquely propagating fast mode waves allows the effect of transit-time damping acceleration of cosmic ray particles. Here we will demonstrate that the obliqueness of fast mode and shear Alfven wave propagation also modifies the resulting parallel spatial diffusion coefficient and the Hillas limit.

2. Relevant magnetohydrodynamic plasma modes

Most cosmic plasmas have a small value of the plasma beta $\beta_{\mathrm{P}} = c_{\mathrm{S}}^2/V_{\mathrm{A}}^2$, which is defined by the ratio of the ion sound c_{S} to Alfven speed V_{A}, and thus indicates the ratio of thermal to magnetic pressure. For low-beta plasmas the two relevant magnetohydrodynamic wave modes are the

(1) incompressional *shear Alfven waves* with dispersion relation

$$\omega_{\mathrm{R}}^2 = V_{\mathrm{A}}^2 k_\|^2 \tag{7}$$

at parallel wavenumbers $|k_\||\ll\Omega_{\mathrm{p},0}/V_{\mathrm{A}}$, which have no magnetic field component along the ordered background magnetic field δB_z ($\|\ \boldsymbol{B}_0$) = 0,

(2) the *fast magnetosonic waves* with dispersion relation

$$\omega_{\mathrm{R}}^2 = V_{\mathrm{A}}^2 k^2,\quad k^2 = k_\|^2 + k_\perp^2 \tag{8}$$

for wavenumbers $|k|\ll\Omega_{\mathrm{p},0}/V_{\mathrm{A}}$, which have a compressive magnetic field component $\delta B_z\neq0$ for oblique propagation angles $\theta = \arccos(k_\|/k)\neq0$.

In the limiting case (commonly referred to as slab model) of parallel (to $\boldsymbol{B}_0$) propagation ($\theta = k_\perp = 0$) the shear Alfven waves become the left-handed circularly polarised Alfven-ion-cyclotron waves, whereas the fast magnetosonic waves become the right-handed circularly polarised Alfven-Whistler-electron-cyclotron waves.

Schlickeiser & Miller (1998) investigated the quasilinear interactions of charged particles with these two plasma waves. In case of negligible wave damping the interactions are of resonant nature: a cosmic ray particle of given velocity v, pitch angle cosine μ and gyrofrequency $\Omega_c = \Omega_{c,0}/\gamma$ interacts with waves whose wavenumber and real frequencies obey the condition

$$\omega_R(k) = v\mu k_{\parallel} + n\Omega_c, \tag{9}$$

for entire $n = 0, \pm 1, \pm 2, \ldots$

2.1. Resonant interactions of shear Alfven waves

For shear Alfven waves only interactions with $n \neq 0$ are possible. These are referred to as *gyroresonances* because inserting the dispersion relation (7) in the resonance condition (9) yields for the resonance parallel wavenumber

$$k_{\parallel,A} = \frac{n\Omega_c}{\pm V_A - v\mu}, \tag{10}$$

which apart from very small values of $|\mu| \leq V_A/v$ typically equals the inverse of the cosmic ray particle's gyroradius, $k_{\parallel,A} \simeq n/R_L$ and higher harmonics.

2.2. Resonant interactions of fast magnetosonic waves

In contrast, for fast magnetosonic waves the $n = 0$ resonance is possible for oblique propagation due its compressive magnetic field component. The $n = 0$ interactions are referred to as *transit-time damping*, hereafter TTD. Inserting the dispersion relation (8) into the resonance condition (9) in the case $n = 0$ yields

$$v\mu = \pm V_A / \cos\theta \tag{11}$$

as necessary condition which is independent from the wavenumber value k. Apparently all super-Alfvenic ($v \geq V_A$) cosmic ray particles are subject to TTD provided their parallel velocity $v\mu$ equals at least the wave speeds $\pm V_A$. Hence Eq. (11) is equivalent to the two conditions

$$|\mu| \geq V_A/v, \quad v \geq V_A. \tag{12}$$

Additionally, fast mode waves also allow gyroresonances ($n \neq 0$) at wavenumbers

$$k_F = \frac{n\Omega_c}{\pm V_A - v\mu\cos\theta}, \tag{13}$$

which is very similar to Eq. (10).

2.3. Implications for cosmic ray transport

The simple considerations of the last two subsections allow us the following immediate conclusions:

(1) With TTD-interactions alone, it would not be possible to scatter particles with $|\mu| \leq V_A/v$, i.e., particles with pitch angles near 90°. Obviously, these particles have basically no parallel velocity and cannot catch up with fast mode waves that propagate with the small but finite speeds $\pm V_A$. In particular this implies that with TTD alone it is not possible to establish an isotropic cosmic ray distribution function. Gyroresonances are needed to provide the crucial finite scattering at small values of μ.

(2) Conditions (11) and (12) reveal that TTD is no gyroradius effect. It involves fast mode waves at all wavenumbers provided the cosmic ray particles are super-Alfvenic and have large enough values of μ as required by Eq. (12). Because gyroresonances occur at single resonant wavenumbers only, see Eqs. (10) and (13), their contribution to the value of the Fokker-Planck coefficients in the interval $|\mu| \geq V_A/v$ is much smaller than the contribution from TTD. Therefore for comparable intensities of fast mode and shear Alfven waves, TTD will provide the overwhelming contribution to all Fokker-Planck coefficients $D_{\mu\mu}$, $D_{\mu p}$ and D_{pp} in the interval $|\mu| \geq V_A/v$. At small values of $|\mu| < V_A/v$ only gyroresonances contribute to the values of the Fokker-Planck coefficients involving according to Eqs. (10) and (13) wavenumbers at $k_{\parallel,A} = k_R \simeq \pm n\Omega_c/V_A$.

(3) The momentum diffusion coefficient (4)

$$A_M = \frac{1}{2}\int_{-1}^{1} d\mu\, [D_{pp}(\mu) - \frac{D_{\mu p}^2(\mu)}{D_{\mu\mu}(\mu)}] = A_T + A_2 \tag{14}$$

has contributions both from transit-time damping of fast mode waves,

$$A_T \simeq \int_{V_A/v}^{1} d\mu D_{pp}^{\mathrm{TTD}}(\mu), \tag{15}$$

and from second-order Fermi gyroresonant acceleration by shear Alfven waves (Schlickeiser 1989)

$$A_2 = \frac{1}{2}\int_{-1}^{1} d\mu \left[D_{pp}^{A}(\mu) - \frac{[D_{\mu p}^{A}(\mu)]^2}{D_{\mu\mu}^{A}(\mu)}\right]. \tag{16}$$

(4) On the other hand, the spatial diffusion coefficient (2)

$$\kappa = \frac{v^2}{8}\int_{-1}^{1} d\mu (1-\mu^2)^2\, D_{\mu\mu}^{-1}(\mu) \tag{17}$$

is given by the integral over the *inverse* of the Fokker-Planck coefficient $D_{\mu\mu}$, so here the small values of $D_{\mu\mu}$ due to gyroresonant interactions in the interval $|\mu| < V_A/v$ determine the spatial diffusion coefficient and the corresponding parallel mean free path

$$\kappa = v\lambda/3 \simeq \frac{v^2}{8}\int_{-V_A/v}^{V_A/v} \frac{d\mu}{D_{\mu\mu}^{G}(\mu)}. \tag{18}$$

The gyroresonances can be due to shear Alfven waves or fast magnetosonic waves. For relativistic cosmic rays the relevant range of pitch angle cosines $|\mu| \leq v_A/v$ is very small allowing us the approximation $D_{\mu\mu}^{G}(\mu) \simeq D_{\mu\mu}^{G}(0)$ so that

$$\kappa = v\lambda/3 \simeq \frac{v^2}{4}\frac{\epsilon}{D_{\mu\mu}^{G}(0)} = \frac{vV_A}{4D_{\mu\mu}^{G}(0)}. \tag{19}$$

(5) According to Eq. (90) of Schlickeiser (1989) the streaming cosmic ray anisotropy due to spatial gradients in the cosmic ray density is given by

$$\delta = \frac{F_{\max} - F_{\min}}{F_{\max} + F_{\min}} = \frac{1}{2F}\frac{v}{4}\frac{\partial F}{\partial z}\int_{-1}^{1} d\mu (1-\mu^2)\, D_{\mu\mu}^{-1}(\mu) \tag{20}$$

which also is determined by the smallest value of $D_{\mu\mu}$ around $\mu = 0$. Approximating again $D_{\mu\mu}(\mu) \simeq D_{\mu\mu}^{G}(0)$ for $|\mu| \leq \epsilon = V_A/v$ we derive with Eq. (19) the direct proportionality of the cosmic ray anisotropy with the parallel mean free path, i.e.

$$\delta \simeq \frac{v}{8}\frac{\partial F}{\partial \ln z}\frac{2V_A}{vD_{\mu\mu}^{G}(0)} = \frac{V_A}{4}\frac{1}{D_{\mu\mu}^{G}(0)}\frac{\partial F}{\partial \ln z} = \frac{1}{3}\lambda\frac{\partial F}{\partial \ln z}. \tag{21}$$

Introducing the characteristic spatial gradient of the cosmic ray density $\langle z\rangle^{-1} \equiv (1/F)|\partial F/\partial z|$ Eq. (21) reads

$$\delta = \frac{\lambda}{3\langle z\rangle}. \tag{22}$$

Cosmic ray gradients derived from diffuse galactic GeV gamma-ray emissivities (Strong & Mattox 1996) suggest a value of $\langle z\rangle \simeq 2$ kpc.

3. Quasilinear cosmic ray mean free path and anisotropy isotropic plasma wave turbulence

Throughout this work we consider isotropic linearly polarised magnetohydrodynamic turbulence so that the components of the magnetic turbulence tensor for plasma mode j is

$$P^j_{lm}(\boldsymbol{k}) = \frac{g^j(k)}{8\pi k^2}\left(\delta_{lm} - \frac{k_k k_m}{k^2}\right). \tag{23}$$

The magetic energy density in wave component j then is

$$(\delta B)^2_j = \int \mathrm{d}^3k \sum_{i=1}^{3} P_{ii}(\boldsymbol{k}) = \int_0^\infty \mathrm{d}k g^j(k). \tag{24}$$

We adopt a Kolmogorov-like power law dependence (index $q > 1$) of $g^j(k)$ above the minimum wavenumber $k_{\min}$

$$g^j(k) = g^j_0 k^{-q} \quad \text{for } k > k_{\min}. \tag{25}$$

The normalisation (24) then implies

$$g^j_0 = (q-1)(\delta B)^2_j k^{q-1}_{\min}. \tag{26}$$

Moreover we adopt a vanishing cross helicity of each plasma mode, i.e. equal intensity of forward and backward moving waves, so that g^j_0 refers to the total energy density of each mode.

According to Eq. (30) of SM the Fokker-Planck coefficients $D^{\mathrm{F}}_{\mu\mu}$ and $D^{\mathrm{F}}_{pp} = \epsilon^2 p^2 D_{\mu\mu}$ with $\epsilon = V_{\mathrm{A}}/v$ for fast mode waves are the sum of contributions from transit-time damping (T) and gyroresonant interactions (G):

$$D^{\mathrm{F}}_{\mu\mu}(\mu) = \frac{\pi\Omega^2(1-\mu^2)}{4B_0^2}[D_{\mathrm{T}}(\mu) + D_{\mathrm{G}}(\mu)] \tag{27}$$

with

$$\begin{aligned} D_{\mathrm{T}}(\mu) &= (q-1)(\delta B)^2_{\mathrm{F}}|\Omega|^{-1}(R_{\mathrm{L}}k_{\min})^{q-1}H[|\mu|-\epsilon] \\ &\times \frac{1+(\epsilon/\mu)^2}{|\mu|}\left[(1-\mu^2)(1-(\epsilon/\mu)^2)\right]^{q/2} \\ &\times \int_U^\infty \mathrm{d}s\, s^{-(1+q)} J_1^2(s), \end{aligned} \tag{28}$$

where the lower integration boundary is

$$U = k_{\min}R_{\mathrm{L}}\sqrt{(1-\mu^2)(1-(\epsilon/\mu)^2)}, \tag{29}$$

and $\eta = \cos\theta$. $R_{\mathrm{L}} = v/|\Omega|$ denotes the gyrofrequency of the cosmic ray particle, H is the Heaviside' step function and $J_1(s)$ is the Bessel function of the first kind.

The gyroresonant contribution from fast mode waves is

$$\begin{aligned} D_{\mathrm{G}}(\mu) &= \frac{q-1}{2}(\delta B)^2_F k^{q-1}_{\min} \sum_{n=1}^{\infty}\sum_{j=\pm1}\int_{-1}^{1}\mathrm{d}\eta(1+\eta^2) \\ &\times \int_{k_{\min}}^\infty \mathrm{d}k k^{-q}[J_n'(kR_{\mathrm{L}}\sqrt{(1-\eta^2)(1-\mu^2)})]^2 \\ &\times \Big[\delta(k[v\mu\eta - jV_{\mathrm{A}}]+n\Omega)+\delta(k[v\mu\eta - jV_{\mathrm{A}}]-n\Omega)\Big]. \end{aligned} \tag{30}$$

On the other hand shear Alfven waves provide only gyroresonant ($n \neq 1$) interactions yielding

$$\begin{aligned} &\left(D^{\mathrm{A}}_{\mu\mu}, D^{\mathrm{A}}_{\mu p}, D^{\mathrm{A}}_{pp}\right) = \pi(q-1)\Omega^2(1-\mu^2)k^{q-1}_{\min}\frac{(\delta B)^2_{\mathrm{A}}}{32B_0^2}\sum_{n=1}^{\infty} \\ &\times \sum_{j=\pm1}\left([1-j\mu\epsilon]^2, j\epsilon p[1-j\mu\epsilon], (\epsilon p)^2\right)\int_{-1}^{1}\mathrm{d}\eta(1+\eta^2) \\ &\times \int_{k_{\min}}^\infty \mathrm{d}k\, k^{-q}\Big[\delta([v\mu - jV_{\mathrm{A}}]\eta k + n\Omega) \\ &+\delta([v\mu - jV_{\mathrm{A}}]\eta k - n\Omega)\Big]\Big[(J_{n-1}(kR_{\mathrm{L}}\sqrt{(1-\mu^2)(1-\eta^2)}) \\ &+J_{n+1}(kR_{\mathrm{L}}\sqrt{(1-\mu^2)(1-\eta^2)}\Big]^2. \end{aligned} \tag{31}$$

According to SM at particle pitch-angles outside the interval $|\mu| \geq \epsilon$ transit-time damping provides the dominant and overwhelming contribution to these Fokker-Planck coefficients. This justifies the approximations to derive Eqs. (19) and (21) for the cosmic ray mean free path and anisotropy, respectively, Both transport parameters are primarily fixed by the small but finite scattering due to gyroresonant interactions in the interval $|\mu| < \epsilon$. We then derive

$$\begin{aligned} \lambda &\simeq \frac{3v}{8}\int_{-\epsilon}^{\epsilon}\mathrm{d}\mu(1-\mu^2)^2\,[D^F_{\mu\mu}(\mu) + D^{\mathrm{A}}_{\mu\mu}(\mu)]^{-1} \\ &\simeq \frac{3v\epsilon}{4[D^F_{\mu\mu}(\mu=0) + D^{\mathrm{A}}_{\mu\mu}(\mu=0)]}, \end{aligned} \tag{32}$$

and

$$\delta = \frac{1}{3}\lambda\frac{\partial F}{\partial \ln z} \simeq \frac{v\epsilon}{4[D^{\mathrm{F}}_{\mu\mu}(\mu=0) + D^{\mathrm{A}}_{\mu\mu}(\mu=0)]}\frac{\partial F}{\partial \ln z}. \tag{33}$$

In the following, we consider both transport coefficients for positively charged cosmic ray particles with $\Omega > 0$ especially in the limit $k_{\min}R_{\mathrm{L}} \gg 1$.

3.1. Gyroresonant Fokker-Planck coefficients at $\mu = 0$

At $\mu = 0$ the contribution from shear Alfven waves to the pitch-angle Fokker-Planck coefficient is according to Eq. (23)

$$\begin{aligned} &D^{\mathrm{A}}_{\mu\mu}(\mu=0) \simeq \frac{\pi(q-1)\Omega^2 k^{q-1}_{\min}(\delta B)^2_{\mathrm{A}}}{16B_0^2}\sum_{n=1}^{\infty}\int_{k_{\min}}^{\infty}\mathrm{d}k\, k^{-q-1} \\ &\times\left(1+\frac{n^2\Omega^2}{V^2_{\mathrm{A}}k^2}\right)H\left[k-\frac{n\Omega}{V_{\mathrm{A}}}\right]\left[J_{n-1}\left(R_{\mathrm{L}}\sqrt{k^2-\frac{n^2\Omega^2}{V^2_{\mathrm{A}}}}\right)\right. \\ &\left.+J_{n+1}\left(R_{\mathrm{L}}\sqrt{k^2-\frac{n^2\Omega^2}{V^2_{\mathrm{A}}}}\right)\right]^2, \end{aligned} \tag{34}$$

where we readily performed the η-integration. Substituting $t = R_{\mathrm{L}}[k^2 - (n^2\Omega^2/V^2_{\mathrm{A}})]^{1/2}$, and using $V_{\mathrm{A}}/\Omega = \epsilon R_{\mathrm{L}}$, Eq. (34) reduces to

$$\begin{aligned} &D^{\mathrm{A}}_{\mu\mu}(\mu=0) \simeq \frac{\pi(q-1)\Omega(\delta B)^2_{\mathrm{A}}}{16\epsilon B_0^2}[k_{\min}R_{\mathrm{L}}]^{q-1}\sum_{n=1}^{\infty}\int_{U_{\mathrm{A}}}^{\infty}\mathrm{d}t\, t \\ &\times\left(t^2+\frac{2n^2}{\epsilon^2}\right)\left[t^2+\frac{n^2}{\epsilon^2}\right]^{-(q+4)/2}(J_{n-1}(t)+J_{n+1}(t))^2 \end{aligned} \tag{35}$$

where

$$U_{\rm A} = \max\left(0, \left[R_{\rm L}^2 k_{\rm min}^2 - \frac{n^2}{\epsilon^2}\right]^{1/2}\right). \tag{36}$$

Likewise the contribution from gyroresonant interactions with fast mode waves is according to Eqs. (27) and (30)

$$D_{\mu\mu}^{\rm F}(\mu = 0) \simeq \frac{\pi(q-1)\Omega^2 k_{\rm min}^{q-1}(\delta B)_{\rm F}^2}{4V_{\rm A}B_0^2}\left[\frac{V_{\rm A}}{\Omega}\right]^q \sum_{n=1}^{\infty}$$
$$\times n^{-q} H\left[n - \frac{k_{\rm min}V_{\rm A}}{\Omega}\right]\int_{-1}^{1} {\rm d}\eta(1+\eta^2)\left(J_n^{'}\left(\frac{n}{\epsilon}\sqrt{1-\eta^2}\right)\right)^2 \tag{37}$$

where we performed the k-integration. With $V_{\rm A}/\Omega = \epsilon R_{\rm L}$, Eq. (37) becomes

$$D_{\mu\mu}^{\rm F}(\mu = 0) \simeq \frac{\pi(q-1)\Omega(\delta B)_{\rm F}^2}{4B_0^2}[k_{\rm min}R_{\rm L}\epsilon]^{q-1}\sum_{n=1}^{\infty} n^{-q}$$
$$\times H[n - \epsilon R_{\rm L} k_{\rm min}]\int_{-1}^{1} {\rm d}\eta(1+\eta^2)\left(J_n^{'}\left(\frac{n}{\epsilon}\sqrt{1-\eta^2}\right)\right)^2. \tag{38}$$

The Bessel function integral in Eq. (38)

$$I_1 = \int_{-1}^{1} {\rm d}\eta(1+\eta^2)\left(J_n^{'}\left(\frac{n}{\epsilon}\sqrt{1-\eta^2}\right)\right)^2 \tag{39}$$

has been calculated asymptotically by SM to lowest order in the small quantity $\epsilon = V_{\rm A}/v \ll 1$ as

$$I_1 \simeq \frac{3}{2}\frac{\epsilon}{n} \tag{40}$$

yielding

$$D_{\mu\mu}^{\rm F}(\mu = 0) \simeq \frac{3\pi(q-1)\Omega\epsilon(\delta B)_{\rm F}^2}{4B_0^2}[k_{\rm min}R_{\rm L}\epsilon]^{q-1}$$
$$\times\sum_{n=1}^{\infty} n^{-(q+1)} H[n - \epsilon R_{\rm L}k_{\rm min}]. \tag{41}$$

In Appendix B we evaluate the Bessel function integral in Eq. (35)

$$I_2 = \int_{U_{\rm A}}^{\infty} {\rm d}t\, t\left(t^2 + \frac{2n^2}{\epsilon^2}\right)\left[t^2 + \frac{n^2}{\epsilon^2}\right]^{-(q+4)/2} \times (J_{n-1}(t) + J_{n+1}(t))^2 \tag{42}$$

for small and large values of $k_{\rm min}R_{\rm L}\epsilon$.

For values $k_{\rm min}R_{\rm L}\epsilon \le 1$ we obtain approximately

$$I_2(k_{\rm min}R_{\rm L}\epsilon \le 1) \simeq \frac{8}{\pi}\epsilon^{q+2}n^{-q}\Big[1 + (-1)^n 1.00813\Big] \tag{43}$$

yielding

$$D_{\mu\mu}^{\rm A}(\mu = 0, k_{\rm min}R_{\rm L}\epsilon \le 1) \simeq \frac{(q-1)\Omega\epsilon^2(\delta B)_{\rm A}^2}{2^{1+q}B_0^2}$$
$$\times[k_{\rm min}R_{\rm L}\epsilon]^{q-1}\Big[2.00813\zeta(q) + 0.00813\zeta(q, 0.5)\Big], \tag{44}$$

in terms of the zeta and the generalised zeta functions of Riemann (Whittaker & Watson 1978).

For values of $k_{\rm min}R_{\rm L}\epsilon > 1$ we obtain Eq. (43) for values of $n \ge N+1$, where $N = \inf[k_{\rm min}R_{\rm L}\epsilon]$ is the largest integer smaller than $\epsilon R_{\rm L}k_{\rm min}$, while for smaller n

$$I_2(k_{\rm min}R_{\rm L}\epsilon > 1, n = N) \simeq 4\epsilon^{q+2}N^{-(q+1)} \tag{45}$$

and

$$I_2(k_{\rm min}R_{\rm L}\epsilon > 1, n \le N-1) \simeq \frac{4n^2}{\pi(q+3)}U_{\rm A}^{-(q+3)}. \tag{46}$$

According to Eq. (35) this yields

$$D_{\mu\mu}^{\rm A}(\mu = 0, k_{\rm min}R_{\rm L}\epsilon > 1) \simeq \frac{(q-1)\Omega\epsilon^2(\delta B)_{\rm A}^2}{2B_0^2}[k_{\rm min}R_{\rm L}\epsilon]^{q-1}$$
$$\left[\frac{\pi}{2N^{q+1}} + \frac{\epsilon}{2(q+3)}\sum_{n=1}^{N-1} n^{-(q+1)}\left[\left(\frac{R_{\rm L}k_{\rm min}\epsilon}{n}\right)^2 - 1\right]^{-(q+3)/2}\right.$$
$$\left. + \sum_{n=N+1}^{\infty} n^{-q}[1 + (-1)^n 1.00813]\right]. \tag{47}$$

Comparing the Fokker-Planck coefficients from fast mode waves (41) and Alfven waves (Eqs. (44) and (47)) we note that the latter one is always smaller by the small ratio $\epsilon = V_{\rm A}/v$ than the first one:

$$D_{\mu\mu}^{\rm A}(\mu = 0) \simeq \epsilon D_{\mu\mu}^{\rm F}(\mu = 0) \tag{48}$$

so that the gyroresonant contribution from Alfven waves can be neglected in comparison to the gyroresonant contribution from fast mode waves.

3.2. Cosmic ray mean free path

Neglecting $D_{\mu\mu}^{\rm A}(\mu = 0)$ we obtain for the cosmic ray mean free path (32)

$$\lambda(\gamma) \simeq \frac{3v\epsilon}{4D_{\mu\mu}^{\rm F}(\mu = 0)}$$
$$= \frac{1}{\pi(q-1)}\frac{B_0^2}{(\delta B)_{\rm F}^2}\frac{R_{\rm L}(k_{\rm min}R_{\rm L}\epsilon)^{1-q}}{\sum_{n=1}^{\infty} n^{-(q+1)}H[n - \epsilon R_{\rm L}k_{\rm min}]}, \tag{49}$$

which exhibits the familiar Lorentzfactor dependence $\propto\beta\gamma^{2-q} \simeq \gamma^{2-q}$ at Lorentzfactors $\gamma \le \gamma_{\rm c}$ below a critical Lorentz factor defined by

$$\gamma_{\rm c} = k_{\rm c}/k_{\rm min} \tag{50}$$

with $k_{\rm c} = \Omega_{0,p}/V_{\rm A} = \omega_{\rm p,i}/c$ being the inverse ion skin length. The Lorentzfactor dependence $\lambda \propto \gamma^{2-q}$ especially holds at rigidities $1 \le k_{\rm min}R_{\rm L} \le \epsilon = c/V_{\rm A}$, in a rigidity range where the slab turbulence model would predict an infinitely large mean free path.

Expresing $k_{\rm min} = 2\pi/L_{\rm max}$ in terms of the longest wavelength of isotropic fast mode waves $L_{\rm max} = 10$ pc yields

$$\gamma_{\rm c} = \frac{\omega_{\rm p,i}L_{\rm max}}{2\pi c} = 2.16\times 10^{11} n_{\rm e}^{1/2}\left(\frac{L_{\rm max}}{10\ {\rm pc}}\right). \tag{51}$$

The corresponding cosmic ray hadron energy is

$$E_{\rm c} = A\gamma_{\rm c}m_{\rm p}c^2 = 2.03\times 10^5 A n_{\rm e}^{1/2}\left(\frac{L_{\rm max}}{10\ {\rm pc}}\right)\ {\rm PeV} \tag{52}$$

which is four orders of magnitude larger than the Hillas limit (6) for equal values of the maximum wavelength. This difference demonstrates the dramatic influence of the plasma turbulence geometry (slab versus isotropically distributed waves) on the confinement of cosmic rays in the Galaxy. With isotropically distributed fast mode waves, even ultrahigh energy cosmic rays obey the scaling $\lambda\gamma^{q-2} = $ const.

Only, at ultrahigh Lorentzfactors $\gamma > \gamma_c$ or energies $E > E_c$ the mean free path (49) approaches the much steeper dependence

$$\lambda(\gamma > \gamma_c) \simeq \frac{1}{\pi(q-1)} \frac{B_0^2}{(\delta B)_F^2} R_L (k_{\min} R_L \epsilon)^2 \propto \beta\gamma^3 \simeq \gamma^3, \quad (53)$$

independent from the turbulence spectral index q. Here the mean free path quickly attains very large values gretaer than the typical scales of the Galaxy.

3.3. Anisotropy

Because of the direct proportionality between mean free path and anisotropy, the cosmic ray anisotropy (33) shows the same behaviour as a function of energy:

$$\delta(E) \simeq \frac{1}{3\pi(q-1)} \frac{B_0^2}{(\delta B)_F^2} \frac{\partial F}{\partial \ln z} \times \frac{R_L (k_{\min} R_L \epsilon)^{1-q}}{\sum_{n=1}^{\infty} n^{-(q+1)} H[n - \epsilon R_L k_{\min}]} \quad (54)$$

which is proportional $\delta(E \le E_c) \propto E^{2-q}$ at energies below E_c and $\delta(E > E_c) \propto E^3$ at energies above E_c. In particular we obtain no drastic change in the energy dependence of the anisotropy at PeV energies. Quantitatively, with Eq. (22), $q = 5/3$ and $V_A = 20\ \mathrm{km\,s^{-1}}$ we find

$$\delta(E) = 0.152 \left(\frac{L_{\max}}{10\ \mathrm{pc}}\right) \left(\frac{\langle z \rangle}{2\ \mathrm{kpc}}\right)^{-1} \left(\frac{(B_0/\delta B)_F}{10}\right)^2 \times \frac{(E/E_c)^{1/3}}{\sum_{n=1}^{\infty} n^{-(8/3)} H[n - (E/E_c)]}. \quad (55)$$

At $E_c = 20$ EeV energies we calculate an anisotropy of less than 15 percent, whereas at smaller energies the anisotropy values decrease proportional to $(E/E_c)^{1/3}$.

4. Summary and conclusions

We have investigated the implications of isotropically distributed interstellar magnetohydrodynamic plasma waves on the scattering mean free path and the spatial anisotropy of high-energy cosmic rays. We demonstrate a drastic modification of the energy dependence of both cosmic ray transport parameters compared to previous calculations that have assumed that the plasma waves propagate only parallel or antiparallel to the ordered magnetic field (slab turbulence). In case of slab turbulence cosmic rays with Larmor radius R_L resonantly interact with plasma waves with wave vectors at $k_{\mathrm{res}} = R_L^{-1}$. If the slab wave turbulence power spectrum vanishes for wavenumbers less than $k_{\min}$, as a consequence then cosmic rays with Larmor radii larger than $k_{\min}^{-1}$ cannot be scattered in pitch-angle, causing the so-called Hillas limit for the maximum energy $E_{15}^{H} = 40Z \cdot (B_0/4\ \mu\mathrm{G})(L_{\|,\max}/10\ \mathrm{pc})$ of cosmic rays being confined in the Galaxy. At about these energies this would imply a drastic increase in the spatial anisotropy of cosmic rays that has not been detected by KASKADE and other air shower experiments.

In case of isotropically distributed interstellar magnetohydrodynamic waves we demonstrated that the Hillas energy E^H is modified to a limiting total energy that is about 4 orders of magnitude larger $E_c = 2.03 \times 10^5 A n_e^{1/2} (L_{\max}/10\ \mathrm{pc})$ PeV, where A denotes the mass number and $L_{\max}$ the maximum wavenumber of isotropic plasma waves. Below this energy the cosmic ray mean free path and the anisotropy exhibit the well known E^{2-q} energy dependence, where $q = 5/3$ denotes the spectral index of the Kolmogorov spectrum. At energies higher than E_c both transport parameters steepen to a E^3-dependence. This implies that cosmic rays even with ultrahigh energies of several tens of EeV can be rapidly pitch-angle scattered by interstellar plasma turbulence, and are thus confined to the Galaxy.

The physical reason for the four orders of magnitude higher value of the limiting energy is the occurrence of dominating transit-time damping interactions of cosmic rays with magnetosonic plasma waves due to their compressive magnetic field component along the ordered magnetic field. This $n = 0$ resonance is not a gyroresonance implying that cosmic rays interact with plasma waves at all wavenumbers provided that the cosmic ray parallel speed (transit speed) equals the parallel phase speed of magnetosonic waves. Only at small values of the cosmic ray pitch-angle cosine $|\mu| \le \epsilon = V_A/v$, where the cosmic ray particles spiral at nearly ninety degrees with very small parallel speeds less than the minimum magnetosonic phase speed V_A, gyroresonant interactions are necessary to scatter csomic rays. However, the gyroresonance condition of cosmic rays at $\mu = 0$ reads $k_{\mathrm{res}} = (R_L \epsilon)^{-1}$ instead of the slab condition $k_{\mathrm{res}} = (R_L)^{-1}$ causing the limiting energy enhancement from E^H to E_c by the large factor $\epsilon^{-1} = c/V_A \simeq O(10^4)$.

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References

Abramowitz, M., & Stegun, I. A. 1972, Handbook of Mathematical Functions (Washington: National Bureau of Standards)
Aharonian, F., Akhperjanian, A. G., Aye, K.-M., et al. 2004, Nature, 432, 75
Aharonian, F., Akhperjanian, A. G., Aye, K.-M., et al. 2005, A&A, 437, L7
Allen, G. E., Keohane, J. W., Gotthelf, E. V., et al. 1997, ApJ, 487, L97
Amelio-Camelia, G., Ellis, J., Mavromatos, N. E., Nanopoulos, D. V., & Srakar, S. 1998, Nature, 393, 763
Antoni, T., Apel, W. D., Badea, A. F., et al. 2004, ApJ, 604, 687
Armstrong, J. W., Rickett, B. J., & Spangler, S. R. 1995, ApJ, 443, 209
Beck, R. 2007, EAS Publ. Ser., 23, 19 [arXiv:astro-ph/0603531]
Bell, A. R., & Lucek, S. G. 2001, MNRAS, 321, 433
Blandford, R. D., & Eichler, D. 1987, Phys. Rep., 154, 1
Borkowski, K. J., Rho, J., Reynolds, S. P., & Dyer, K. K. 2001, ApJ, 550, 334
Büsching, I., Pohl, M., & Schlickeiser, R. 2001, A&A, 377, 1056
Candia, J., Mollerach, S., & Roulet, E. 2003, JCAP, 5, 3
Coleman, S., & Glashow, S. L. 1997, Phys. Lett. B, 405, 249
Cho, Y., Lazarian, A., & Vishniac, E. 2002, ApJ, 566, 49
Enomoto, R., Tanimori, T., Naito, T., et al. 2002, Nature, 416, 823
Hillas, A. M. 1984, ARA&A, 22, 425
Hillas, A. M. 2006, J. Phys.: Conf. Ser., 47, 168
Hörandel, J. R., Kalmykov, N. N., Timokhin, A. V. 2006, J. Phys. Conf. Ser., 47, 132
Koyama, K., Petre, R., Gotthelf, E. V., et al. 1995, Nature, 378, 255
Koyama, K., Kinugasa, K., Matsuzaki, K., et al. 1997, PASJ, 49, L7
Lithwick, Y., & Goldreich, P. 2001, ApJ, 567, 479
Lucek, S. G., & Bell, A. R. 2000, MNRAS, 314, 65
Ptuskin, V. S., & Soutoul, A. 1998, A&A, 337, 859
Reimer, O., & Pohl, M. 2002, A&A, 390, L43
Schlickeiser, R. 1989, ApJ, 336, 243
Schlickeiser, R. 2002, Cosmic Ray Astrophysics (Berlin: Springer)
Schlickeiser, R., & Miller, J. A. 1998, ApJ, 492, 352 (SM)
Slane, P., Gaensler, B. M., Dame, T. M., et al. 1999, ApJ, 525, 357
Sofue, Y., Fujimoto, M., & Wielebinski, R. 1986, ARA&A, 24, 459
Strong, A. W., & Mattox, J. R. 1996, A&A, 308, L21
Strong, A. W., & Moskalenko, I. V. 1998, ApJ, 509, 212
Strong, A. W., Moskalenko, I. V., & Reimer, O. 2004, ApJ, 613, 962
Tanimori, T., Hayami, Y., Kamei, S., et al. 1998, ApJ, 497, L25
Turner, M. S., et al. 2002, Report to the National Academy of Science
Webber, W. R., & Soutoul, A. 1998, ApJ, 506, 335
Whittaker, E. T., & Watson, G. N. 1978, A Course of Modern Analysis (Cambridge: Cambridge University Press)

Online Material

Appendix A: Glossary and definitions of important symbols

$A = m/m_{\rm p}$: cosmic ray particle mass or nucleon number

$A_{\rm M}$: momentum diffusion coefficient os cosmic rays

β: cosmic ray velocity in units of c

$\beta_{\rm P} = c_{\rm S}^2/V_{\rm A}^2$: plasma beta

B_0: uniform magnetic field strength

δB: strength of total fluctuating magnetic fields

$\delta B_{\rm F}$: strength of fast magnetosonic plasma wave magnetic fields

$\delta B_{\rm A}$: strength of shear Alfven plasma wave magnetic fields

c: vacuum speed of light

$c_{\rm S} = \sqrt{2k_{\rm B}T/m_{\rm p}}$: ion sound speed

$\gamma = E/mc^2 = (1-\beta^2)^{-1/2}$: cosmic ray Lorentz factor

$\gamma_{\rm c} = E_{\rm c}/mc^2$: critical cosmic ray Lorentz factor where the energy dependence of the mean free path changes

D_{ij}: Fokker-Planck coefficient

$\delta(p)$: cosmic ray anisotropy

$E = \gamma mc^2$: total kinetic energy of cosmic ray particle

$E_{\rm c} = \gamma_{\rm c} mc^2$: critical cosmic ray total kinetic energy where the energy dependence of the mean free path changes

$\epsilon = V_{\rm A}/c$: ratio of Alfven speed to speed of light

$F(z, p, t)$: isotropic part of cosmic ray phase space density

$g^j(k) \propto k^{-q}$: magnetic field turbulence spectrum of plasma wave mode j

$J_n(x)$: Bessel function of first kind and order n

$\boldsymbol{k} = (k_x, k_y, k_z)$: plasma wave vector and its cartesian components

$k_\| = k_z = k\cos\theta$: component of plasma wave vector parallel to uniform magnetic field

$k_\perp = \sqrt{k_x^2 + k_y^2} = k\sin\theta$: component of plasma wave vector perpendicular to uniform magnetic field

$k_{\rm min} = 2\pi/\lambda_{\rm max}$: minimum wavenumber of plasma waves

$k_{\rm c} = \omega_{\rm p,i}/c$: inverse ion skin length

$\kappa = v\lambda/3$: spatial diffusion coefficient of cosmic rays parallel to uniform magnetic field

$\lambda = 3\kappa/v$: parallel mean free path of cosmic rays

$\lambda_{\rm max} = 2\pi/k_{\rm min}$: maximum wavenumber of plasma waves

$L_{\rm max}$: maximum wavenumber of isotropic fast magnetosonic waves

$Ł_{\|,\rm max}$: maximum wavenumber of parallel propagating (slab) plasma waves

$m = Am_{\rm p}$: mass of cosmic ray particle

$m_{\rm p}$: proton mass

$\mu = p_\|/p$: pitch angle cosine of cosmic ray particle

$n_{\rm e}$: number density of electrons in interstellar medium

$\omega_{\rm R}$: real part of plasma wave frequency

$\omega_{\rm p,i} = \sqrt{4\pi n_{\rm e} e^2/m_{\rm p}}$: proton plasma frequency in interstellar ionized gas

$\Omega_{\rm c,0} = |ZeB_0/mc|$: nonrelativistic gyrofrequency of cosmic ray particle in uniform magnetic field B_0

$\Omega_{\rm c} = \Omega_{\rm c,0}/\gamma$: relativistic gyrofrequency of cosmic ray particle in uniform magnetic field B_0

$\Omega_{\rm p,0} = eB_0/m_{\rm p}c$: nonrelativistic gyrofrequency of proton in uniform magnetic field B_0

p: total momentum of cosmic ray particle

$\dot{p}_{\rm Loss}$: continuous momentum loss rate of cosmic ray particle

$P^j_{lm}(\boldsymbol{k})$: magnetic turbulence tensor for plasma mode j

q: spectral index of turbulence power law spectrum

$R = p/Z$: rigidity of cosmic ray particle

$R_{\rm L} = v/\Omega_{\rm c}$: gyroradius of cosmic ray particle in uniform magnetic field B_0

T: temperature of interstellar gas

$T_{\rm c}$: catastrophic loss time of cosmic ray particle

$\theta = \arccos(k_\|/k)$: propagation angle of plasma wave with respect to uniform magnetic field direction

u: velocity of plasma wave-carrying interstellar gas

$v = \beta c$: velocity of cosmic ray particle

V: cosmic ray bulk speed

$V_{\rm A} = B_0/\sqrt{4\pi m_{\rm p} n_{\rm e}}$: Alfven velocity

Z: cosmic ray particle charge or atomic number

Appendix B: Asymptotic calculation of the integral (42)

The task is to calculate the integral (42)

$$I_2 = \int_{U_\mathrm{A}}^\infty \mathrm{d}t\, t\left(t^2+\frac{2n^2}{\epsilon^2}\right)\left[t^2+\frac{n^2}{\epsilon^2}\right]^{-(q+4)/2}\left[(J_{n-1}(t)+J_{n+1}(t)\right]^2, \quad (56)$$

for small and large values of $k_\mathrm{min}R_\mathrm{L}$ using the approximations of Bessel functions for small and large arguments (Abramowitz & Stegun 1972), yielding

$$J_n^2(t \ll 1) \simeq \frac{t^{2n}}{2^{2n}\Gamma^2[n+1]}, \quad (57)$$

and

$$J_n^2(t \gg 1) \simeq \frac{1}{\pi t}[1+(-1)^n \sin(2t)]. \quad (58)$$

According to Eq. (36)

$$U_\mathrm{A} = \max\left(0, \left[R_\mathrm{L}^2 k_\mathrm{min}^2 - \frac{n^2}{\epsilon^2}\right]^{1/2}\right),$$

the lower integration boundary $U_\mathrm{A} = 0$ in the case $k_\mathrm{min}R_\mathrm{L}\epsilon \le 1$ which includes in particular the limit $k_\mathrm{min}R_\mathrm{L} \ll 1$ because $\epsilon \ll 1$.

4.1. Case $k_{min}R_L\epsilon \le 1$

With the identity

$$J_{n-1}(t) + J_{n+1}(t) = \frac{2nJ_n(t)}{t} \quad (59)$$

we obtain

$$I_2(k_\mathrm{min}R_\mathrm{L}\epsilon \le 1) = 4n^2\left[W\left[\frac{q+2}{2}\right] + \frac{n^2}{\epsilon^2}W\left[\frac{q+4}{2}\right]\right] \quad (60)$$

where

$$W[\alpha] \equiv \int_0^\infty \mathrm{d}t\, t^{-1}\frac{J_n^2(t)}{\left[t^2+\frac{n^2}{\epsilon^2}\right]^\alpha}. \quad (61)$$

With the asymptotics (57) and (58) we obtain

$$W[\alpha] \simeq \left(\frac{\epsilon}{n}\right)^{2\alpha}\left[\frac{1}{2^{2n}\Gamma^2[n+1]}\int_0^1 \mathrm{d}t t^{2n-1} + \frac{1}{\pi}\int_1^{n/\epsilon}\mathrm{d}t t^{-2}[1+(-1)^n\sin(2t)]\right] + \frac{1}{\pi}\int_{n/\epsilon}^\infty \mathrm{d}t t^{-2(1+\alpha)}[1+(-1)^n\sin(2t)]$$

$$\simeq \left(\frac{\epsilon}{n}\right)^{2\alpha}\left[\frac{1}{\pi}\left[1+(-1)^n 1.00813 - \frac{\epsilon}{n} - \frac{(-1)^n}{2}\left(\frac{\epsilon}{n}\right)^2\cos\left(\frac{2n}{\epsilon}\right)\right] + \frac{1}{n2^{2n+1}\Gamma^2[n+1]}\right] + \frac{1}{\pi(1+2\alpha)}\left(\frac{\epsilon}{n}\right)^{1+2\alpha} + \frac{(-1)^n}{\pi}j_1, \quad (62)$$

where we use

$$2\int_1^\infty \mathrm{d}x\, x^{-2}\, \sin x = 2(\sin(1) - Ci(1)) = 1.00813$$

and where

$$j_1 = \int_{n/\epsilon}^\infty \mathrm{d}t t^{-2-2\alpha}\sin 2t = 2^{2\alpha}\left[\imath^{-2-2\alpha}\Gamma\left[-(1+2\alpha), -2\imath\frac{n}{\epsilon}\right] + (-\imath)^{-2-2\alpha}\Gamma\left[-(1+2\alpha), 2\imath\frac{n}{\epsilon}\right]\right] \quad (63)$$

in terms of the incomplete gamma function. For large arguments $(n/\epsilon) \gg 1$ we obtain asymptotically

$$j_1 \simeq \frac{1}{2}\left(\frac{\epsilon}{n}\right)^{2+2\alpha}\cos\left(\frac{2n}{\epsilon}\right). \quad (64)$$

Collecting terms we find to lowest order in $\frac{\epsilon}{n} \ll 1$

$$W[\alpha] \simeq \frac{1}{\pi}\left(\frac{\epsilon}{n}\right)^{2\alpha}\left[1+(-1)^n 1.00813 + \frac{\pi}{n2^{2n+1}\Gamma^2[n+1]}\right] \quad (65)$$

so that

$$I_2(k_\mathrm{min}R_\mathrm{L}\epsilon \le 1) \simeq \frac{8}{\pi}\epsilon^{q+2}n^{-q} \times\left[1+(-1)^n 1.00813 + \frac{\pi}{n2^{2n+1}\Gamma^2[n+1]}\right]. \quad (66)$$

4.2. Case $k_{min}R_L\epsilon > 1$

In this case $U_\mathrm{A} = 0$ for $n \ge N+1$, and $U_\mathrm{A} = \sqrt{(R_\mathrm{L}k_\mathrm{min})^2 - (n/\epsilon)^2}$ for $n \le N$, where

$$N = \inf[\epsilon R_\mathrm{L}k_\mathrm{min}] \quad (67)$$

denotes the largest integer smaller than $\epsilon R_\mathrm{L}k_\mathrm{min}$. Hence we obtain again Eq. (66) for $n \ge N+1$

$$I_2(k_\mathrm{min}R_\mathrm{L}\epsilon > 1, n \ge N+1) \simeq \frac{8}{\pi}\epsilon^{q+2}n^{-q} \times\left[1+(-1)^n 1.00813 + \frac{\pi}{n2^{2n+1}\Gamma^2[n+1]}\right]. \quad (68)$$

For values of $n \le N$ we find that

$$I_2(k_\mathrm{min}R_\mathrm{L}\epsilon > 1, n \le N) = 4n^2\left[V\left[\frac{q+2}{2}\right] + \frac{n^2}{\epsilon^2}V\left[\frac{q+4}{2}\right]\right] \quad (69)$$

where

$$V[\alpha] \equiv \int_{U_\mathrm{A}}^\infty \mathrm{d}t\, t^{-1}\frac{J_n^2(t)}{[t^2+\frac{n^2}{\epsilon^2}]^\alpha} = \left(\frac{\epsilon}{n}\right)^{2\alpha}\int_{\epsilon U_\mathrm{A}/n}^\infty \mathrm{d}t\, t^{-1}\frac{J_n^2(nt/\epsilon)}{[1+t^2]^\alpha}. \quad (70)$$

We may express

$$k_\mathrm{min}R_\mathrm{L}\epsilon = N(1+\phi) \quad (71)$$

with $\phi < 1/N$, so that the lower integration boundary in (70) is

$$\frac{\epsilon}{n}U_\mathrm{A} = \left[\left(\frac{k_\mathrm{min}R_\mathrm{L}\epsilon}{n} - 1\right)\left(\frac{k_\mathrm{min}R_\mathrm{L}\epsilon}{n} + 1\right)\right]^{1/2} = \frac{N}{n}\left[\left(1+\phi-\frac{n}{N}\right)\left(1+\phi+\frac{n}{N}\right)\right]^{1/2}. \quad (72)$$

In cases where $N \geq 2$, Eq. (72) yields that for all values of n such that $1 \leq n \leq N-1$ the lower integration boundary $\frac{\epsilon}{n} U_{\rm A}$ is greater unity. Using the expansion (58) in this case we find that

$$V[\alpha, n \leq N-1] \simeq \frac{1}{\pi}\left(\frac{\epsilon}{n}\right)^{2\alpha+1} \int_{\epsilon U_{\rm A}/n}^{\infty} {\rm d}t\, t^{-2-2\alpha}$$
$$\times\left[1+(-1)^n \sin\left(\frac{2nt}{\epsilon}\right)\right] \simeq \frac{1}{\pi(1+2\alpha)} U_{\rm A}^{-(2\alpha+1)}$$
$$\times\left[1+(-1)^n \frac{1+2\alpha}{2U_{\rm A}}\cos(2U_{\rm A})\right] \simeq \frac{U_{\rm A}^{-(2\alpha+1)}}{\pi(1+2\alpha)} \tag{73}$$

In the remaining case $n = N$ the lower integration boundary (72)

$$\frac{\epsilon}{N} U_{\rm A} = \sqrt{\phi(2+\phi)} \leq \sqrt{2.5\phi} < 1 \tag{74}$$

is smaller unity, so that we approximate Eq. (70) in this case by

$$V[\alpha, n = N] \simeq \left(\frac{\epsilon}{N}\right)^{2\alpha}\left[\int_{\epsilon U_{\rm A}/N}^{1} {\rm d}t\, t^{-1} J_N^2\left(\frac{Nt}{\epsilon}\right)\right.$$
$$\left. + \int_1^{\infty} {\rm d}t\, t^{-1-2\alpha} J_N^2\left(\frac{Nt}{\epsilon}\right)\right] \simeq \left(\frac{\epsilon}{N}\right)^{2\alpha}$$
$$\times\left[j_2 + \frac{\epsilon}{\pi N(1+2\alpha)}\left(1+(-1)^n(1+2\alpha)\frac{\epsilon}{2N}\cos\left(\frac{2N}{\epsilon}\right)\right)\right] \tag{75}$$

where we approximate

$$j_2 = \int_{\epsilon U_{\rm A}/N}^{1} {\rm d}t\, t^{-1} J_N^2(\frac{Nt}{\epsilon}) <$$
$$\int_0^{\infty} {\rm d}t\, t^{-1} J_N^2\left(\frac{Nt}{\epsilon}\right) = \frac{1}{2N} \tag{76}$$

by its upper limit to obtain

$$V[\alpha, n = N] \simeq \frac{\left(\frac{\epsilon}{N}\right)^{2\alpha}}{2N}\cdot \tag{77}$$

Collecting terms in Eq. (69) we derive

$$I_2(k_{\rm min} R_{\rm L} \epsilon > 1, n = N) \simeq 4\epsilon^{q+2} N^{-(q+1)} \tag{78}$$

and

$$I_2(k_{\rm min} R_{\rm L} \epsilon > 1, n \leq N-1) \simeq \frac{4n^2}{\pi(q+3)} U_{\rm A}^{-(q+3)} \tag{79}$$