A&A 465, 509–524 (2007)
DOI: 10.1051/0004-6361:20066209

Astronomy & Astrophysics

The asymptotic representation of higher-order g^+-modes in stars with a convective core

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Received 8 August 2006 / Accepted 4 December 2006

ABSTRACT

Aims. A first-order asymptotic representation of higher-order non-radial g^+-modes in spherically symmetric stars with a convective core is constructed from the full fourth-order system of governing equations. Stars are considered that, besides their convective core, also contain a radiative envelope, or both an intermediate radiative zone and a convective envelope. At the same time, the earlier asymptotic theory of Willems et al. (1997, A&A, 318, 99) relative to stars consisting of a convective core and a radiative envelope is made more transparent.

Methods. As in the asymptotic theory of Smeyers (2006, A&A, 451, 223) for low-degree, higher-order p-modes, two-variable expansion procedures and boundary-layer theory are applied to the fourth-order system of differential equations established by Pekeris (1938, ApJ, 88, 189).

Results. Eigenfrequency equations are derived in terms of the radial order n of the g^+-mode. The first $n-1$ nodes of the radial component of the Lagrangian displacement coincide with the $n-1$ nodes of the divergence of the Lagrangian displacement, and are situated in the radiative envelope or in the intermediate radiative zone according to the type of star considered. The radial displacement displays an nth node near the surface. In stars containing an intermediate radiative zone and a convective envelope, the nth node is situated in the envelope.

Conclusions. As well as for higher-order p-modes of spherically symmetric stars, the divergence of the Lagrangian displacement plays a basic role in the development of the asymptotic theory.

Key words. stars: oscillations – methods: analytical

1. Introduction

The viewpoint that γ Doradus stars, which are intermediate mass stars oscillating with several periods in the range of 0.5 to 3 days and with amplitudes of a few hundredths of magnitude, are non-radial gravity-mode pulsators of low degree and high radial order, has revived the interest in the asymptotic theory of low-degree, higher-order g^+-modes in stars (Kaye et al. 1999; Aerts et al. 2004; Moya et al. 2005; Dupret et al. 2005).

In the past, Tassoul (1980) developed an asymptotic representation of higher-order g^+-modes in the Cowling approximation, in which the Eulerian perturbation of the gravitational potential is neglected. For stars composed of a convective core and a radiative envelope, Willems et al. (1997) redeveloped the asymptotic theory without neglecting that perturbation. These authors started from the full fourth-order system of differential equations in the divergence and the radial component of the Lagrangian displacement that stems from Pekeris (1938) and was reintroduced by Tassoul (1990) in an asymptotic treatment of higher-order p-modes. To the system of equations, they applied perturbation methods that are adequate for singular perturbation problems: two-variable expansion procedures at larger distances from the boundary and the turning points, and boundary-layer theory near these points (Kevorkian & Cole 1968, 1996). However, the asymptotic theory of Willems et al. is partly obscured by the choice of boundary-layer coordinates which are identical to the fast variables used at larger distances.

Our aim is to develop an asymptotic representation of higher-order g^+-modes in stars with a convective core. We consider stars that contain a radiative envelope as well as stars that contain a convective envelope and an intermediate radiative zone. We also start from Pekeris' system of differential equations and apply two-variable expansion procedures and boundary-layer theory, but we define the boundary-layer coordinates in a regular way. In our asymptotic treatment, we consider the convective core, where the square of the Brunt-Väisälä frequency is generally small in absolute value, to be in adiabatic equilibrium.

The outline of the paper is as follows. In Sect. 2, we recall the basic equations. In Sect. 3, we construct the asymptotic representation of higher-order g^+-modes for stars composed of a convective core and a radiative envelope, and in Sect. 4, that for stars composed of a convective core, an intermediate radiative zone, and a convective envelope. Section 5 is devoted to concluding remarks.

2. Basic equations

Consider a non-rotating spherically symmetric star in hydrostatic equilibrium with mass M and radius R that is subject to a linear, isentropic g^+-oscillation depending on time by a factor

$\exp(\mathrm{i}\,\sigma\,t)$ and belonging to a spherical harmonic $Y_\ell^m(\theta,\phi)$. With respect to a system of spherical coordinates r, θ, ϕ whose origin coincides with the star's mass centre, the Lagrangian displacement and its divergence can be represented as

$$\boldsymbol{\xi}(r,\theta,\phi) = \left[\xi(r)\,\mathbf{1}_r + \frac{\eta(r)}{r}\left(\mathbf{1}_\theta\,\frac{\partial}{\partial\theta} + \mathbf{1}_\phi\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)\right]Y_\ell^m(\theta,\phi), \quad (1)$$

$$\alpha(r,\theta,\phi) = \alpha(r)\,Y_\ell^m(\theta,\phi) \equiv \left[\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\xi(r)\right) - \frac{\ell(\ell+1)}{r^2}\,\eta(r)\right]Y_\ell^m(\theta,\phi). \quad (2)$$

For the construction of the asymptotic representation of higher-order g^+-modes, we start from the following fourth-order system of two differential equations for the radial functions $\alpha(r)$ and $\xi(r)$:

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} + K_2(r)\,\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \left[\frac{K_1(r)}{\sigma^2} + K_3(r) + \frac{\sigma^2}{c^2(r)}\right]\alpha = -K_4(r)\,\frac{\mathrm{d}\xi}{\mathrm{d}r}, \quad (3)$$

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}r^2} + \frac{4}{r}\,\frac{\mathrm{d}\xi}{\mathrm{d}r} - \frac{\ell(\ell+1)-2}{r^2}\,\xi = \frac{\mathrm{d}\alpha}{\mathrm{d}r} - \left[\frac{c^2(r)}{g(r)}\,\frac{K_1(r)}{\sigma^2} - \frac{2}{r}\right]\alpha. \quad (4)$$

The coefficients $K_1(r)$, $K_2(r)$, $K_3(r)$, $K_4(r)$ are defined as

$$K_1(r) = \ell(\ell+1)\,\frac{N^2}{r^2}, \quad (5)$$

$$K_2(r) = \frac{2}{r} + \frac{2}{\rho\,c^2}\,\frac{\mathrm{d}\left(\rho\,c^2\right)}{\mathrm{d}r} - \frac{1}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}, \quad (6)$$

$$K_3(r) = -\frac{\ell(\ell+1)}{r^2} + \frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{1}{r}\right) + \frac{1}{\rho\,c^2}\,\frac{\mathrm{d}\left(\rho\,c^2\right)}{\mathrm{d}r}\left(\frac{2}{r} - \frac{1}{\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) + \frac{1}{\rho\,c^2}\,\frac{\mathrm{d}^2\left(\rho\,c^2\right)}{\mathrm{d}r^2}, \quad (7)$$

$$K_4(r) = -\frac{2\,g}{c^2}\left(\frac{1}{g}\,\frac{\mathrm{d}g}{\mathrm{d}r} - \frac{1}{r}\right). \quad (8)$$

The variables have their usual meaning: $\rho(r)$ is the mass density, $g(r)$ the gravity, $c(r)$ the isentropic sound velocity, and $N^2(r)$ the square of the Brunt-Väisälä frequency.

The solutions must satisfy boundary conditions. At $r = 0$, the radial component of the Lagrangian displacement must remain finite. At $r = R$, the Lagrangian perturbation of the pressure must be zero. The condition implies that the divergence of the Lagrangian displacement must be finite at that point, since the equilibrium pressure vanishes there, and the Lagrangian perturbation of the pressure is related to the divergence of the displacement as

$$\delta P = -\Gamma_1\,P\,\alpha, \quad (9)$$

where $P(r)$ is the equilibrium pressure, and $\Gamma_1(r)$ the generalized isentropic coefficient $\Gamma_1 \equiv (\partial\ln P/\partial\ln\rho)_S$. Finally, the continuity of the gravitational potential and its gradient at the star's perturbed surface requires that

$$\left(\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right)_R + \frac{\ell+1}{R}\,\Phi'_R = -(4\,\pi\,G\,\rho\,\xi)_R. \quad (10)$$

Here $\Phi'(r)$ is the Eulerian perturbation of the gravitational potential.

We make the differential equations and the boundary condition dimensionless by expressing the time t, the radial coordinate r, the pressure $P(r)$, the mass density $\rho(r)$, the gravity $g(r)$, the isentropic sound velocity $c(r)$, the gravitational potential $\Phi(r)$, and both the radial component $\xi(r)$ and the transverse component $\eta(r)/r$ of the Lagrangian displacement respectively in the units $\left[R^3/(GM)\right]^{1/2}$, R, $GM^2/\left(4\pi R^4\right)$, $M/\left(4\pi R^3\right)$, GM/R^2, $(GM/R)^{1/2}$, GM/R, R. We suppose that the angular frequency σ expressed in the unit $\left(GM/R^3\right)^{1/2}$ is a small quantity and denote it as ε. With this definition, ε is a small dimensionless quantity that corresponds to the ratio of 2π times the star's dynamic time scale to the oscillation period.

At the boundaries between a radiative region and a convective region, $N^2(r)$ vanishes in the term of Eq. (3) that contains the large parameter, so that each of these boundaries introduces a turning point into the equation. We suppose the mass density to be continuous even at the turning points.

Several regions must be distinguished in the radial direction: regions at larger distances from the boundary and the turning points, and regions near a boundary or a turning point. In the various regions, we start from a *homogeneous* second-order differential equation for the lowest-order asymptotic approximation of $\alpha(r)$, which is derived from Eq. (3). Next, we derive an *inhomogeneous* second-order differential equation for the lowest-order asymptotic approximation of $\xi(r)$, which involves the lowest-order asymptotic approximation of $\alpha(r)$ in its inhomogeneous part.

3. Stars with a radiative envelope

For stars composed of a convective core and a radiative envelope, a turning point appears in Eq. (3) at the boundary between the two regions.

We start the construction of the asymptotic solutions from the region in the radiative envelope that is situated at larger distances from the turning point and the star's surface at $r = R$. Secondly, we construct boundary-layer solutions near the turning point and match them to the asymptotic solutions valid at larger distances. We also connect the boundary-layer solutions to the solutions valid in the adiabatic core. Thirdly, we construct boundary-layer solutions from the star's surface at $r = R$. After the matching of these solutions to the asymptotic solutions valid at larger distances, the eigenfrequency equation can be derived.

By imposing the boundary condition relative to the Eulerian perturbation of the gravitational potential at the star's surface, we fix the last undetermined constant in the asymptotic solutions. We then construct the asymptotic solutions that are uniformly valid from the lower boundary of the radiative envelope and from the star's surface respectively. Finally, we identify the radial order of a g^+-mode associated with a given eigenfrequency.

3.1. *Asymptotic solutions in the radiative envelope at larger distances from its boundaries*

For the region in the radiative envelope that is situated at larger distances from the boundary of the convective core at $r = r_\mathrm{a}$ and the star's surface at $r = R$, we adopt asymptotic solutions similar to those constructed by Smeyers et al. (1995) by means of an expansion procedure in terms of a fast and a slow variable. The slow variable is still the radial coordinate r, but the fast variable is defined as

$$\tau(r) = \frac{1}{\varepsilon}\int_{r_\mathrm{a}}^{r} K_1^{1/2}\,(r')\,\mathrm{d}r'. \quad (11)$$

At a given degree ℓ, the rate of increase of the fast variable as a function of the radial coordinate depends on the variation of the integrand $\left(N^2/r^2\right)^{1/2}$ in the radiative envelope.

The lowest-order asymptotic solutions for $\alpha(r)$ and $\xi(r)$ are given by a linear combination of trigonometric functions of the fast variable whose amplitudes vary as functions of the slow variable. The lowest-order asymptotic solution for $\xi(r)$ also contains a non-oscillatory part. The solutions can be written as

$$\left.\begin{aligned}\alpha^{(\mathrm{o})}(r,\varepsilon) &= K_5(r)\left(A_0^*\cos\tau + B_0^*\sin\tau\right),\\ \xi^{(\mathrm{o})}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(\mathrm{o})}(r,\varepsilon) + G_0^{(\mathrm{o})}(r),\end{aligned}\right\} \tag{12}$$

where

$$K_5(r) = g(r)\left[N^2(r)\,r^6\,c^8(r)\rho^2(r)\right]^{-1/4}. \tag{13}$$

The function $G_0^{(\mathrm{o})}(r)$ is a general solution of the second-order Clairaut equation

$$\frac{\mathrm{d}^2G_0^{(\mathrm{o})}}{\mathrm{d}r^2} + 2\left(\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}r} + \frac{1}{r}\right)\frac{\mathrm{d}G_0^{(\mathrm{o})}}{\mathrm{d}r} - \frac{\ell(\ell+1)-2}{r^2}\,G_0^{(\mathrm{o})} = 0 \tag{14}$$

and can be expressed as

$$G_0^{(\mathrm{o})}(r) = C_0^*\,y_1(r) + D_0^*\,y_2(r). \tag{15}$$

The particular solutions $y_1(r)$ and $y_2(r)$ are chosen in such a way that they behave respectively as $r^{\ell-1}$ and as $r^{-(\ell+2)}$, as $r \to 0$. Furthermore, A_0^*, B_0^*, C_0^*, D_0^* are general constants.

3.2. Boundary-layer solutions from the boundary between the convective core and the radiative envelope

Near the boundary between the convective core and the radiative envelope situated at the radial distance $r = r_\mathrm{a}$, we construct boundary-layer solutions towards the star's surface.

Because of Solutions (12), we pass on from the function $\xi(r)$ to the function $w(r)$ by means of the transformation

$$\xi(r) = \frac{c^2(r)}{g(r)}\,w(r), \tag{16}$$

so that Eqs. (3) and (4) become

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} + K_2(r)\frac{\mathrm{d}\alpha}{\mathrm{d}r} + \left[\frac{K_1(r)}{\varepsilon^2} + K_3(r) + \frac{\varepsilon^2}{c^2(r)}\right]\alpha = -K_4(r)\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{c^2(r)}{g(r)}\,w\right], \tag{17}$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2} + \left[\frac{4}{r} + 2\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{c^2(r)}{g(r)}\right]\frac{\mathrm{d}w}{\mathrm{d}r} + \left[-\frac{\ell(\ell+1)-2}{r^2} + \frac{4}{r}\frac{\mathrm{d}}{\mathrm{d}r}\ln\frac{c^2(r)}{g(r)} + \frac{g(r)}{c^2(r)}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\frac{c^2(r)}{g(r)}\right]w = \frac{g(r)}{c^2(r)}\frac{\mathrm{d}\alpha}{\mathrm{d}r} - \left[\frac{K_1(r)}{\varepsilon^2} - \frac{2}{r}\frac{g(r)}{c^2(r)}\right]\alpha. \tag{18}$$

As $r \to r_\mathrm{a}$, the functions appearing in the coefficients of the equations behave as

$$\left.\begin{aligned}\rho(r) &= \rho\left(r_\mathrm{a}\right)\left[1 + O\left(s_\mathrm{a}\right)\right],\\ g(r) &= g\left(r_\mathrm{a}\right)\left[1 + O\left(s_\mathrm{a}\right)\right],\\ c(r) &= c\left(r_\mathrm{a}\right)\left[1 + O\left(s_\mathrm{a}\right)\right],\\ N^2(r) &= N_\mathrm{a}^2\,s_\mathrm{a}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ K_1(r) &= \frac{\ell(\ell+1)}{r_\mathrm{a}^2}\,N_\mathrm{a}^2\,s_\mathrm{a}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ &\equiv K_{1,\mathrm{a}}\,s_\mathrm{a}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ K_2(r) &= K_{2,\mathrm{a}}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ K_3(r) &= K_{3,\mathrm{a}}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ K_4(r) &= K_{4,\mathrm{a}}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\\ K_5(r) &= K_{5,\mathrm{a}}\,s_\mathrm{a}^{-1/4}\,\left[1 + O\left(s_\mathrm{a}\right)\right],\end{aligned}\right\} \tag{19}$$

with $s_\mathrm{a} = r - r_\mathrm{a}$.

We adopt the boundary-layer coordinate

$$s_\mathrm{a}^*(r) = \frac{s_\mathrm{a}(r)}{\delta(\varepsilon)}, \tag{20}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, transform the differential operators in Eqs. (17) and (18) into differential operators in terms of the boundary-layer coordinate $s_\mathrm{a}^*(r)$, introduce asymptotic expansions of the form

$$\left.\begin{aligned}\alpha^{(\mathrm{a})}(r;\varepsilon) &= \mu_0^{(\mathrm{a})}(\varepsilon)\,\alpha_0^{(\mathrm{a})}\left(s_\mathrm{a}^*\right) + \mu_1^{(\mathrm{a})}(\varepsilon)\,\alpha_1^{(\mathrm{a})}\left(s_\mathrm{a}^*\right)\\ &\quad + \ldots,\\ w^{(\mathrm{a})}(r;\varepsilon) &= \nu_0^{(\mathrm{a})}(\varepsilon)\,w_0^{(\mathrm{a})}\left(s_\mathrm{a}^*\right) + \nu_1^{(\mathrm{a})}(\varepsilon)\,w_1^{(\mathrm{a})}\left(s_\mathrm{a}^*\right)\\ &\quad + \ldots,\end{aligned}\right\} \tag{21}$$

and use Taylor Series (19).

Equation (17) can then be brought in the form

$$\begin{aligned}&\mu_0^{(\mathrm{a})}(\varepsilon)\left\{\frac{1}{\delta^2(\varepsilon)}\frac{\mathrm{d}^2\alpha_0^{(\mathrm{a})}}{\mathrm{d}s_\mathrm{a}^{*2}} + \frac{K_{2,\mathrm{a}}}{\delta(\varepsilon)}\frac{\mathrm{d}\alpha_0^{(\mathrm{a})}}{\mathrm{d}s_\mathrm{a}^*}\right.\\ &\qquad\left. + \left[\frac{\delta(\varepsilon)}{\varepsilon^2}\,K_{1,\mathrm{a}}\,s_\mathrm{a}^* + K_{3,\mathrm{a}} + \frac{\varepsilon^2}{c_\mathrm{a}^2}\right]\alpha_0^{(\mathrm{a})} + \ldots\right\}\\ &\qquad + \mu_1^{(\mathrm{a})}(\varepsilon)\left\{\ldots\right\} + \ldots =\\ &\qquad \nu_0^{(\mathrm{a})}(\varepsilon)\left(-\frac{K_{4,\mathrm{a}}}{\delta(\varepsilon)}\frac{c_\mathrm{a}^2}{g_\mathrm{a}}\frac{\mathrm{d}w_0^{(\mathrm{a})}}{\mathrm{d}s_\mathrm{a}^*} + \ldots\right) + \nu_1^{(\mathrm{a})}(\varepsilon)\left(\ldots\right) + \ldots\end{aligned} \tag{22}$$

We start from a first dominant boundary-layer equation that is homogeneous. Therefore, we assume that $\nu_0^{(\mathrm{a})}(\varepsilon)$ is of a higher order in ε than $\mu_0^{(\mathrm{a})}(\varepsilon)/\delta(\varepsilon)$ and $\mu_0^{(\mathrm{a})}(\varepsilon)\,\delta^2(\varepsilon)/\varepsilon^2$. The equation then takes the form

$$\frac{1}{\delta^2(\varepsilon)}\frac{\mathrm{d}^2\alpha_0^{(\mathrm{a})}}{\mathrm{d}s_\mathrm{a}^{*2}} + \frac{\delta(\varepsilon)}{\varepsilon^2}\,K_{1,\mathrm{a}}\,s_\mathrm{a}^*\,\alpha_0^{(\mathrm{a})} = 0. \tag{23}$$

The term involving the second derivative is of the same order in ε as the term with the large parameter, when

$$\delta(\varepsilon) = \varepsilon^{2/3}. \tag{24}$$

A general solution of the boundary-layer equation in terms of Bessel functions of the first kind is given by

$$\alpha_0^{(a)}(s_a^*) = \sqrt{s_a^*}\left[A_{0,a}\, J_{1/3}\left(\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{*3/2}\right)\right.$$
$$\left. +B_{0,a}\, J_{-1/3}\left(\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{*3/2}\right)\right], \tag{25}$$

where $A_{0,a}$ and $B_{0,a}$ are general constants.

Next, Eq. (18) can be brought in the form

$$\nu_0^{(a)}(\varepsilon)\left[\frac{1}{\varepsilon^{4/3}}\,\frac{d^2 w_0^{(a)}}{ds_a^{*2}} + \left(\varepsilon^{-2/3}\right)\right] + \nu_1^{(a)}(\varepsilon)(\ldots) + \ldots =$$
$$\mu_0^{(a)}(\varepsilon)\left[-\frac{K_{1,a}}{\varepsilon^{4/3}}\, s_a^*\,\alpha_0^{(a)} + O\left(\varepsilon^{-2/3}\right)\right] + \mu_1^{(a)}(\varepsilon)(\ldots) + \ldots \tag{26}$$

Since we are dealing with a *fourth-order system* of differential equations for $\alpha_0^{(a)}(s_a^*)$ and $w_0^{(a)}(s_a^*)$, we derive a second dominant boundary-layer equation that is inhomogeneous. Therefore, we set

$$\nu_0^{(a)}(\varepsilon) = \mu_0^{(a)}(\varepsilon). \tag{27}$$

This equality is compatible with the suppositions made above in the context of the derivation of the first dominant boundary-layer equation. The second dominant boundary-layer equation then takes the form

$$\frac{d^2 w_0^{(a)}}{ds_a^{*2}} = -K_{1,a}\, s_a^*\,\alpha_0^{(a)}. \tag{28}$$

By subtracting the first dominant boundary-layer equation, one obtains

$$\frac{d^2}{ds_a^{*2}}\left(w_0^{(a)} - \alpha_0^{(a)}\right) = 0, \tag{29}$$

so that, after integration,

$$w_0^{(a)}(s_a^*) = \alpha_0^{(a)}(s_a^*) + C_{0,a}\, s_a^* + D_{0,a}, \tag{30}$$

where $C_{0,a}\, s_a^*$ and $D_{0,a}$ are particular solutions of the homogeneous equation, and $C_{0,a}$ and $D_{0,a}$ general constants.

When one returns to the equation involving the parameter ε

$$\nu_0^{(a)}(\varepsilon)\,\frac{d^2 w_0^{(a)}}{ds_a^{*2}} = -\mu_0^{(a)}(\varepsilon)\, K_{1,a}\, s_a^*\,\alpha_0^{(a)}, \tag{31}$$

one sees that, with the particular solutions $C_{0,a}\, s_a^*$ and $D_{0,a}$, functions $\nu_0^{(a)}(\varepsilon)$ may be associated that differ from the function $\mu_0^{(a)}(\varepsilon)$. Therefore, in view of the matching to the asymptotic solutions valid at larger radial distances from the turning point, we write the boundary-layer solutions $\alpha^{(a)}(r;\varepsilon)$ and $\xi^{(a)}(r;\varepsilon)$ in the more general form

$$\left.\begin{aligned}
\alpha^{(a)}(r;\varepsilon) &= \mu_0^{(a)}(\varepsilon)\,\sqrt{s_a^*}\left[A_{0,a}\, J_{1/3}\left(\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{*3/2}\right)\right.\\
&\qquad\left. +B_{0,a}\, J_{-1/3}\left(\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{*3/2}\right)\right],\\
\xi^{(a)}(r;\varepsilon) &= \frac{c^2(r_a)}{g(r_a)}\left[\alpha^{(a)}(r;\varepsilon)\right.\\
&\qquad\left. +\nu_0^{(a,2)}(\varepsilon)\, C_{0,a}\, s_a^* + \nu_0^{(a,3)}(\varepsilon)\, D_{0,a}\right].
\end{aligned}\right\} \tag{32}$$

The matching condition relative to the divergence of the Lagrangian displacement is

$$\lim_{s_a\to\infty}\alpha^{(a)}(r;\varepsilon) = \lim_{s_a\to 0}\alpha^{(o)}(r,\varepsilon). \tag{33}$$

As $s_a \to 0$, the fast variable $\tau(r)$ reduces to

$$\tau(r) = \frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{3/2}, \tag{34}$$

so that

$$\lim_{s_a\to 0}\alpha^{(o)}(r,\varepsilon) = \frac{K_{5,a}}{s_a^{1/4}}\left[A_0^*\,\cos\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{3/2}\right)\right.$$
$$\left. +B_0^*\,\sin\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{3/2}\right)\right]. \tag{35}$$

As $s_a \to \infty$, the use of the first asymptotic approximations of Bessel functions with large arguments yields

$$\lim_{s_a\to\infty}\alpha^{(a)}(r;\varepsilon) = \mu_0^{(a)}(\varepsilon)\,\varepsilon^{1/6}\,\frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,a}^{1/4}}\,\frac{1}{s_a^{1/4}}$$
$$\left[A_{0,a}\,\sin\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{3/2} + \frac{\pi}{12}\right)\right.$$
$$\left. +B_{0,a}\,\sin\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,a}}\, s_a^{3/2} + \frac{5\pi}{12}\right)\right]. \tag{36}$$

The matching condition is satisfied when

$$\mu_0^{(a)}(\varepsilon) = \varepsilon^{-1/6}, \tag{37}$$

and

$$\left.\begin{aligned}
A_0^* &= \frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,a}^{1/4}\, K_{5,a}}\left(A_{0,a}\,\sin\frac{\pi}{12} + B_{0,a}\,\cos\frac{\pi}{12}\right),\\
B_0^* &= \frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,a}^{1/4}\, K_{5,a}}\left(A_{0,a}\,\cos\frac{\pi}{12} + B_{0,a}\,\sin\frac{\pi}{12}\right).
\end{aligned}\right\} \tag{38}$$

Next, the matching condition relative to the radial component of the Lagrangian displacement is

$$\lim_{s_a\to\infty}\xi^{(a)}(r;\varepsilon) = \lim_{s_a\to 0}\xi^{(o)}(r,\varepsilon). \tag{39}$$

The condition is automatically satisfied for the oscillatory parts of the functions $\xi^{(a)}(r;\varepsilon)$ and $\xi^{(o)}(r;\varepsilon)$. For the non-oscillatory parts, it leads to the equalities

$$\left.\begin{aligned}
&C_{0,a} = 0,\\
&\nu_0^{(a,3)}(\varepsilon) = \varepsilon^0, \qquad D_{0,a} = \frac{g(r_a)}{c^2(r_a)}\, G_0^{(o)}(r_a)\,.
\end{aligned}\right\} \tag{40}$$

3.3. Continuity with the solutions valid in the adiabatic core

In the convective core, which is considered to be in adiabatic equilibrium, $N^2(r) = 0$, so that $K_1(r) = 0$. Consequently, the system of the Eqs. (3) and (4) contains no term of order $1/\varepsilon^2$. At order ε^0, the system of equations takes the form

$$\left.\begin{aligned}
&\frac{d^2\alpha}{dr^2} + K_2(r)\,\frac{d\alpha}{dr} + K_3(r)\,\alpha = -K_4(r)\,\frac{d\xi}{dr},\\
&\frac{d^2\xi}{dr^2} + \frac{4}{r}\,\frac{d\xi}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi = \frac{d\alpha}{dr} + \frac{2}{r}\,\alpha.
\end{aligned}\right\} \tag{41}$$

The variations of pressure and mass density to which an adiabatically moving mass element is subject in a region in adiabatic equilibrium, result exclusively from the stratification of the equilibrium quantities in the radial direction. Therefore, the Lagrangian perturbations of pressure and mass density are given by

$$\delta P = \frac{\mathrm{d}P}{\mathrm{d}r}\,\xi, \qquad \delta\rho = \frac{\mathrm{d}\rho}{\mathrm{d}r}\,\xi. \tag{42}$$

Since

$$\alpha = -\frac{\delta\rho}{\rho} = -\frac{\delta P}{\rho\, c^2}, \tag{43}$$

it follows, by the condition of hydrostatic equilibrium, that the divergence and the radial component of the Lagrangian displacement are related as

$$\alpha = \frac{g}{c^2}\,\xi. \tag{44}$$

One verifies that this relation satisfies Eqs. (41), by taking into account that

$$\left.\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\rho\,\frac{g}{c^2},\\ \frac{\mathrm{d}g}{\mathrm{d}r} &= -\frac{2}{r}\,g + 4\,\pi\,G\,\rho.\end{aligned}\right\} \tag{45}$$

The fourth-order system of differential Eqs. (41) admits of a linear combination of two particular solutions satisfying the requirement that the radial displacement must remain finite at $r = 0$. Correspondingly, the admissible solutions for the divergence and the radial component of the Lagrangian displacement involve two general constants. We represent these solutions by $\mu^{(c)}(\varepsilon)\,\alpha_0^{(c)}(r)$ and $\mu^{(c)}(\varepsilon)\,\xi_0^{(c)}(r)$, where $\mu^{(c)}(\varepsilon)$ is a yet undetermined function.

At the boundary between the convective core and the radiative envelope, both the Lagrangian displacement and the Lagrangian perturbation of the pressure must be continuous. The continuity of the Lagrangian perturbation of the pressure implies the continuity of the divergence of the Lagrangian displacement. Because of Eq. (2), the latter condition requires that, in addition to the components of the Lagrangian displacement, the first derivative of the radial component must be continuous.

The continuity of the divergence of the Lagrangian displacement, and that of the radial component and its first derivative demand that

$$\left.\begin{aligned}\mu^{(c)}(\varepsilon)\,\alpha_0^{(c)}(r_\mathrm{a}) &= \alpha_0^{(a)}(r_\mathrm{a}) \equiv \varepsilon^{-1/6}\,\frac{3^{1/3}}{\Gamma(2/3)\,K_{1,\mathrm{a}}^{1/6}}\,B_{0,\mathrm{a}},\\ \mu^{(c)}(\varepsilon)\,\xi_0^{(c)}(r_\mathrm{a}) &= \frac{c^2(r_\mathrm{a})}{g(r_\mathrm{a})}\,\alpha_0^{(a)}(r_\mathrm{a}) + G_0^{(o)}(r_\mathrm{a}),\\ \mu^{(c)}(\varepsilon)\left(\frac{\mathrm{d}\xi_0^{(c)}}{\mathrm{d}r}\right)_{r=r_\mathrm{a}} &= \varepsilon^{-5/6}\,\frac{3^{7/6}\,\Gamma(2/3)\,K_{1,\mathrm{a}}^{1/6}}{2\,\pi}\,\frac{c^2(r_\mathrm{a})}{g(r_\mathrm{a})}\,A_{0,\mathrm{a}}\\ &\quad+\varepsilon^{-1/6}\,\frac{3^{1/3}}{\Gamma(2/3)\,K_{1,\mathrm{a}}^{1/6}}\left[\frac{\mathrm{d}}{\mathrm{d}r}\,\frac{c^2(r)}{g(r)}\right]_{r=r_\mathrm{a}} B_{0,\mathrm{a}}.\end{aligned}\right\} \tag{46}$$

From the first condition, it follows that

$$\mu^{(c)}(\varepsilon) = \varepsilon^{-1/6}, \tag{47}$$

and from the second condition, that

$$G_0^{(o)}(r_\mathrm{a}) = 0. \tag{48}$$

The second condition then imposes that

$$\xi_0^{(c)}(r_\mathrm{a}) = \frac{c^2(r_\mathrm{a})}{g(r_\mathrm{a})}\,\alpha_0^{(c)}(r_\mathrm{a}) \tag{49}$$

in accordance with Relation (44).

The third condition requires that

$$A_{0,\mathrm{a}} = 0. \tag{50}$$

By means of Condition (49) and the third Condition (46), the two constants involved in the solutions that are valid in the convective core can be fixed.

On the basis of the conditions of continuity at the boundary between the convective core and the radiative envelope, one arrives at the conclusion that the boundary-layer solutions constructed in the radiative envelope involve only the constant $B_{0,\mathrm{a}}$, so that these solutions reduce to

$$\left.\begin{aligned}\alpha^{(a)}(r;\varepsilon) &= \varepsilon^{-1/6}\,B_{0,\mathrm{a}}\,\sqrt{s_\mathrm{a}^*}\,J_{-1/3}\left(\frac{2}{3}\,\sqrt{K_{1,\mathrm{a}}}\,s_\mathrm{a}^{*3/2}\right),\\ w^{(a)}(r;\varepsilon) &= \alpha^{(a)}(r;\varepsilon).\end{aligned}\right\} \tag{51}$$

Correspondingly, Eqs. (38) reduce to

$$\left.\begin{aligned}A_0^* &= B_{0,\mathrm{a}}\,\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,\mathrm{a}}^{1/4}\,K_{5,\mathrm{a}}}\,\cos\frac{\pi}{12},\\ B_0^* &= B_{0,\mathrm{a}}\,\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,\mathrm{a}}^{1/4}\,K_{5,\mathrm{a}}}\,\sin\frac{\pi}{12}.\end{aligned}\right\} \tag{52}$$

3.4. Boundary-layer solutions near the surface

Since the point $r = R$ is a singular point of Eqs. (17) and (18), we treat the small region near the star's surface as a boundary layer. For the construction of the solutions, it is convenient to introduce the coordinate

$$z = R - r. \tag{53}$$

As in Smeyers et al. (1995), we assume that, near the surface, the mass density tends to zero as

$$\rho(r) = \rho_\mathrm{s}\,z^{n_\mathrm{e}}\,[1 + O(z)], \tag{54}$$

with $n_\mathrm{e} > 0$. When furthermore the mass contained inside a sphere with radius r is considered to be equal to the star's total mass M, the gravity behaves as

$$g(r) = g_\mathrm{s}\,[1 + O(z)] \tag{55}$$

with $g_\mathrm{s} = G\,M/R^2$, and it results from the condition of hydrostatic equilibrium that

$$\begin{aligned}P(r) &= g_\mathrm{s}\,\frac{\rho_\mathrm{s}}{n_\mathrm{e}+1}\,z^{n_\mathrm{e}+1}\,[1 + O(z)]\\ &\equiv P_\mathrm{s}\,z^{n_\mathrm{e}+1}\,[1 + O(z)].\end{aligned} \tag{56}$$

The isentropic sound velocity and the square of the Brunt-Väisälä frequency can then be expressed as

$$\left.\begin{aligned}c(r) &= \left(g_\mathrm{s}\,\frac{\Gamma_{1,\mathrm{s}}}{n_\mathrm{e}+1}\right)^{1/2} z^{1/2}\,[1 + O(z)],\\ &\equiv c_\mathrm{s}\,z^{1/2}\,[1 + O(z)],\\ N^2(r) &= -g_\mathrm{s}\left(\frac{g_\mathrm{s}}{c_\mathrm{s}^2} - n_\mathrm{e}\right) z^{-1}\,[1 + O(z)]\\ &\equiv N_\mathrm{s}^2\,z^{-1}[1 + O(z)].\end{aligned}\right\} \tag{57}$$

Moreover, the coefficients $K_1(r)$, $K_2(r)$, $K_3(r)$, $K_4(r)$, $K_5(r)$ behave as

$$\left.\begin{aligned} K_1(r) &= \frac{\ell(\ell+1)}{R^2} N_s^2 z^{-1} [1 + O(z)] \\ &\equiv K_{1,s} z^{-1} [1 + O(z)], \\ K_2(r) &= -(n_e + 2) z^{-1} [1 + O(z)], \\ K_3(r) &= K_{3,s} z^{-1} [1 + O(z)], \\ K_4(r) &= K_{4,s} z^{-1} [1 + O(z)], \\ K_5(r) &= K_{5,s} z^{-(n_e+3/2)/2} [1 + O(z)]. \end{aligned}\right\} \tag{58}$$

In Eq. (17), the second derivative of α with respect to z and the singular terms are of the same order in ε as the term with the large parameter, when a boundary-layer coordinate $z^*(z)$ is introduced by means of the equation

$$\left(\frac{dz^*}{dz}\right)^2 = \frac{1}{\varepsilon^2} \frac{K_{1,s}}{z}. \tag{59}$$

If $z^*(0) = 0$, the non-negative boundary-layer coordinate is given by

$$z^*(z) = \frac{1}{\varepsilon} 2 K_{1,s}^{1/2} z^{1/2}. \tag{60}$$

Next, we transform the differential operators in the equations and introduce asymptotic expansions of the form

$$\left.\begin{aligned} \alpha^{(s)}(r;\varepsilon) &= \mu_0^{(s)}(\varepsilon)\, \alpha_0^{(s)}(z^*) + \mu_1^{(s)}(\varepsilon)\, \alpha_1^{(s)}(z^*) \\ &\quad + \ldots, \\ w^{(s)}(r;\varepsilon) &= \nu_0^{(s)}(\varepsilon)\, w_0^{(s)}(z^*) + \nu_1^{(s)}(\varepsilon)\, w_1^{(s)}(z^*) \\ &\quad + \ldots \end{aligned}\right\} \tag{61}$$

Equation (17) can then be brought in the form

$$\begin{aligned} &\mu_0^{(s)}(\varepsilon)\left[\frac{1}{\varepsilon^4}\left(\frac{d^2\alpha_0^{(s)}}{dz^{*2}} + \frac{2n_e+3}{z^*}\frac{d\alpha_0^{(s)}}{dz^*} + \alpha_0^{(s)}\right)\right. \\ &\quad \left. + O\left(\varepsilon^{-2}\right)\right] + \mu_1^{(s)}(\varepsilon)[\ldots] + \ldots = \\ &\quad \nu_0^{(s)}(\varepsilon)\left[O\left(\varepsilon^{-2}\right)\right] + \nu_1^{(s)}(\varepsilon)[\ldots] + \ldots \end{aligned} \tag{62}$$

If $\nu_0^{(s)}(\varepsilon)$ is of a higher order in ε than $\mu_0^{(s)}(\varepsilon)\,\varepsilon^{-2}$, the first dominant boundary-layer equation is homogeneous and given by

$$\frac{d^2\alpha_0^{(s)}}{dz^{*2}} + \frac{2n_e+3}{z^*}\frac{d\alpha_0^{(s)}}{dz^*} + \alpha_0^{(s)} = 0. \tag{63}$$

The equation admits of the solution that satisfies the requirement that the divergence of the Lagrangian displacement must remain finite at $r = R$

$$\alpha_0^{(s)}(z^*) = A_{0,s}\, z^{*-(n_e+1)} J_{n_e+1}(z^*), \tag{64}$$

where $A_{0,s}$ is a general constant.

Next, Eq. (18) can be brought in the form

$$\begin{aligned} &\nu_0^{(s)}(\varepsilon)\left[\frac{1}{\varepsilon^4}\left(\frac{d^2 w_0^{(s)}}{dz^{*2}} + \frac{3}{z^*}\frac{dw_0^{(s)}}{dz^*}\right) + O\left(\varepsilon^{-2}\right)\right] \\ &\quad + \nu_1^{(s)}(\varepsilon)[\ldots] + \ldots = \\ &\quad \mu_0^{(s)}(\varepsilon)\left[-\frac{1}{\varepsilon^4}\left(2\frac{g_s}{c_s^2}\frac{1}{z^*}\frac{d\alpha_0^{(s)}}{dz^*} + \alpha_0^{(s)}\right) + O\left(\varepsilon^{-2}\right)\right] \\ &\quad + \mu_1^{(s)}(\varepsilon)[\ldots] + \ldots \end{aligned} \tag{65}$$

The second dominant boundary-layer equation is inhomogeneous when

$$\nu_0^{(s)}(\varepsilon) = \mu_0^{(s)}(\varepsilon), \tag{66}$$

and takes the form

$$\frac{d^2 w_0^{(s)}}{dz^{*2}} + \frac{3}{z^*}\frac{dw_0^{(s)}}{dz^*} = -2\frac{g_s}{c_s^2}\frac{1}{z^*}\frac{d\alpha_0^{(s)}}{dz^*} - \alpha_0^{(s)}. \tag{67}$$

By subtracting the first dominant boundary-layer equation and introducing the function

$$w_0^*(z^*) = w_0^{(s)}(z^*) - \alpha_0^{(s)}(z^*), \tag{68}$$

one obtains the inhomogeneous differential equation

$$\frac{d^2 w_0^*}{dz^{*2}} + \frac{3}{z^*}\frac{dw_0^*}{dz^*} = 2\frac{N_s^2}{g_s}\frac{1}{z^*}\frac{d\alpha_0^{(s)}}{dz^*}. \tag{69}$$

A general solution of it is given by

$$\begin{aligned} w_0^*(z^*) &= C_{0,s}\, z^{*-2} + D_{0,s} \\ &\quad - \frac{N_s^2}{g_s}\left[z^{*-2}\int_0^{z^*} z'^2 \frac{d\alpha_0^{(s)}(z')}{dz'}\,dz' - \alpha_0^{(s)}(z^*)\right], \end{aligned} \tag{70}$$

where $C_{0,s}$ and $D_{0,s}$ are general constants.

After partial integration and use of the recurrence relation between Bessel functions

$$z'^{-n_e} J_{n_e+1}(z') = -\frac{d}{dz'}\left[z'^{-n_e} J_{n_e}(z')\right], \tag{71}$$

the solution becomes

$$\begin{aligned} w_0^*(z^*) &= -2\frac{N_s^2}{g_s} A_{0,s}\, z^{*-(n_e+2)} J_{n_e}(z^*) \\ &\quad + C_{0,s}\, z^{*-2} + D_{0,s}. \end{aligned} \tag{72}$$

By use of the second recurrence relation between Bessel functions

$$J_{n_e}(z^*) = \frac{dJ_{n_e+1}(z^*)}{dz^*} + \frac{n_e+1}{z^*} J_{n_e+1}(z^*), \tag{73}$$

the solution can be rewritten as

$$\begin{aligned} w_0^*(z^*) &= -\alpha_0^{(s)}(z^*)\, 2\frac{N_s^2}{g_s}\frac{1}{z^{*2}}\left[\frac{d\ln J_{n_e+1}(z^*)}{d\ln z^*} + (n_e+1)\right] \\ &\quad + C_{0,s}\, z^{*-2} + D_{0,s}. \end{aligned} \tag{74}$$

Finally, one gets

$$\begin{aligned} w_0^{(s)}(z^*) &= \alpha_0^{(s)}(z^*)\left\{1 - 2\frac{N_s^2}{g_s}\frac{1}{z^{*2}}\left[\frac{d\ln J_{n_e+1}(z^*)}{d\ln z^*} + (n_e+1)\right]\right\} \\ &\quad + C_{0,s}\, z^{*-2} + D_{0,s}. \end{aligned} \tag{75}$$

By associating different functions $\nu_0^{(s)}(\varepsilon)$ with the particular solutions of the homogeneous part of Eq. (67), we write the boundary-layer solutions in the more general form

$$\left.\begin{aligned} \alpha^{(s)}(r;\varepsilon) &= \mu_0^{(s)}(\varepsilon)\, A_{0,s}\, z^{*-(n_e+1)} J_{n_e+1}(z^*), \\ \xi^{(s)}(r;\varepsilon) &= \frac{c_s^2}{g_s} z\left\{\alpha^{(s)}(r;\varepsilon)\right. \\ &\quad \left\{1 - 2\frac{N_s^2}{g_s}\frac{1}{z^{*2}}\left[\frac{d\ln J_{n_e+1}(z^*)}{d\ln z^*} + (n_e+1)\right]\right\} \\ &\quad \left. + \nu_0^{(s,2)}(\varepsilon)\, C_{0,s}\, z^{*-2} + \nu_0^{(s,3)}(\varepsilon)\, D_{0,s}\right\}. \end{aligned}\right\} \tag{76}$$

The conditions for the matching of these boundary-layer solutions to the asymptotic solutions valid at larger distances are similar to Conditions (33) and (39).

Here it is convenient to introduce the fast variable $\tau_s(r)$ as

$$\tau_s(r) \equiv \tau_R - \tau(r) = \frac{1}{\varepsilon}\int_0^z K_1^{1/2}(r')\,\mathrm{d}z' \tag{77}$$

with $\tau_R = \tau(R)$. As $z \to 0$, $\tau_s(r) = z^*$, so that for the solution $\alpha^{(o)}(r;\varepsilon)$, which is valid at larger distances from the singular point and is given by the first Eq. (12), one has that

$$\lim_{z\to 0}\alpha^{(o)}(r;\varepsilon) = K_{5,s}\, z^{-(n_e+3/2)/2} \times\Big[\Big(A_0^*\cos\tau_R + B_0^*\sin\tau_R\Big)\cos z^* + \Big(A_0^*\sin\tau_R - B_0^*\cos\tau_R\Big)\sin z^*\Big]. \tag{78}$$

On the other hand, for the boundary-layer solution $\alpha^{(s)}(r;\varepsilon)$, one observes that

$$\lim_{z\to\infty}\alpha^{(s)}(r;\varepsilon) = \mu_0^{(s)}(\varepsilon)\,\varepsilon^{n_e+3/2}A_{0,s} \times\left(\frac{2}{\pi}\right)^{1/2}\left(2K_{1,s}^{1/2}\right)^{-(n_e+3/2)} z^{-(n_e+3/2)/2} \times\sin\left[z^* - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right]. \tag{79}$$

The matching condition relative to the divergence of the Lagrangian displacement is satisfied when

$$\mu_0^{(s)}(\varepsilon) = \varepsilon^{-(n_e+3/2)} \tag{80}$$

and

$$\left.\begin{aligned} A_0^* &= A_{0,s}\,F\sqrt{\frac{2}{\pi}}\sin\left[\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right],\\ B_0^* &= -A_{0,s}\,F\sqrt{\frac{2}{\pi}}\cos\left[\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right],\end{aligned}\right\} \tag{81}$$

with

$$F = \left(2K_{1,s}^{1/2}\right)^{-(n_e+3/2)} K_{5,s}^{-1}. \tag{82}$$

By the foregoing equalities, the oscillatory parts of the functions $\xi^{(s)}(r;\varepsilon)$ and $\xi^{(o)}(r;\varepsilon)$ are also matched. The non-oscillatory parts are matched when

$$\left.\begin{aligned} &D_{0,s} = 0,\\ &\nu_0^{(s,2)}(\varepsilon) = \varepsilon^{-2}, \qquad C_{0,s} = \frac{g_s}{c_s^2}\,4K_{1,s}\,G_0(R).\end{aligned}\right\} \tag{83}$$

3.5. The eigenfrequency equation

The constants A_0^* and B_0^* appear in both Eqs. (52) and (81), which result from the matchings of the boundary-layer solutions to the solutions valid at larger distances. The elimination of these constants leads to the system of two algebraic, linear, homogeneous equations

$$\left.\begin{aligned} &B_{0,a}\frac{\sqrt{3}}{\sqrt{\pi}K_{1,a}^{1/4}K_{5,a}}\cos\frac{\pi}{12} - A_{0,s}\,F\sqrt{\frac{2}{\pi}}\sin\left[\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right] = 0,\\ &B_{0,a}\frac{\sqrt{3}}{\sqrt{\pi}K_{1,a}^{1/4}K_{5,a}}\sin\frac{\pi}{12} + A_{0,s}\,F\sqrt{\frac{2}{\pi}}\cos\left[\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2}\right] = 0.\end{aligned}\right\} \tag{84}$$

The condition for the system of equations to admit of a non-trivial solution for the constants $B_{0,a}$ and $A_{0,s}$, is

$$\cos\left[\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2} - \frac{\pi}{12}\right] = 0, \tag{85}$$

so that the eigenfrequency equation is given by

$$\tau_R - \left(n_e+\frac{1}{2}\right)\frac{\pi}{2} - \frac{\pi}{12} = (2n-1)\frac{\pi}{2}, \quad n = 1,2,3,\ldots \tag{86}$$

or, more explicitly, by

$$\tau_R \equiv \frac{[\ell(\ell+1)]^{1/2}}{|\sigma|}\int_{r_a}^{R}\left(\frac{N^2(r)}{r^2}\right)^{1/2}\mathrm{d}r = \left(2n + n_e - \frac{1}{3}\right)\frac{\pi}{2}. \tag{87}$$

We show below in Sect. 3.8 that the number n corresponds to the radial order of the g^+-mode as defined in the Cowling classification of non-radial oscillations.

Eigenfrequency Eq. (86) corresponds to the eigenfrequency equation derived by Willems et al. (1997). As already observed by the latter authors, Eq. (A12) of Tassoul (1980) obtained in the Cowling approximation agrees with the eigenfrequency equation derived from the full fourth-order system of equations.

For any asymptotic eigenfrequency, a non-trivial solution for the constants $B_{0,a}$ and $A_{0,s}$ exists. With this solution, the constants A_0^* and B_0^* are fixed by means of Eqs. (52) or (81).

Thus far, all constants involved in the asymptotic solutions have been determined, apart from the constant $C_{0,s}$ and the two constants involved in the function $G_0^{(o)}(r)$. In the next section, we show that the constant $C_{0,s}$ is equal to zero. From Eq. (48) and the third Eq. (83), it then follows that the function $G_0^{(o)}(r)$ is identically zero.

3.6. The boundary condition at the surface

At $r = R$, Condition (10) relative to the Eulerian perturbation of the gravitational potential $\Phi'(r)$ must be imposed. To this end, we observe that, at any point, the Eulerian perturbation of the gravitational potential and its first derivative are related to the functions $\alpha(r)$ and $\xi(r)$ as

$$\left.\begin{aligned} \Phi' &= \left(c^2\alpha - g\,\xi\right) + \varepsilon^2\frac{r^2}{\ell(\ell+1)}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\xi\right) - \alpha\right],\\ \frac{\mathrm{d}\Phi'}{\mathrm{d}r} &= \frac{\mathrm{d}\left(c^2\alpha\right)}{\mathrm{d}r} - \frac{N^2}{g}c^2\alpha - g\frac{\mathrm{d}\xi}{\mathrm{d}r} + \left(2\frac{g}{r} - 4\pi G\rho\right)\xi + \varepsilon^2\xi.\end{aligned}\right\} \tag{88}$$

From boundary-layer Solutions (76), one derives that the divergence and the radial component of the Lagrangian displacement at $r = R$ are given by

$$\left.\begin{aligned} \alpha_R &= \varepsilon^{-(n_e+3/2)}A_{0,s}\frac{2^{-(n_e+1)}}{\Gamma(n_e+2)},\\ \xi_R &= -\frac{c_s^2}{g_s}\frac{1}{4K_{1,s}} \times\left[\varepsilon^{-(n_e-1/2)}A_{0,s}\frac{2^{-(n_e-1)}}{\Gamma(n_e+1)}\frac{N_s^2}{g_s} - C_{0,s}\right],\end{aligned}\right\} \tag{89}$$

and their first derivatives, by

$$\left.\begin{aligned} \left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right)_R &= \varepsilon^{-(n_\mathrm{e}+7/2)}\,A_{0,\mathrm{s}}\,\frac{2^{-(n_\mathrm{e}+1)}}{\Gamma\left(n_\mathrm{e}+3\right)}\,K_{1,\mathrm{s}},\\ \left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)_R &= -\varepsilon^{-(n_\mathrm{e}+3/2)}\,A_{0,\mathrm{s}}\,\frac{2^{-(n_\mathrm{e}+1)}}{\Gamma\left(n_\mathrm{e}+2\right)}\\ &\quad\times\frac{c_\mathrm{s}^2}{g_\mathrm{s}}\left(\frac{N_\mathrm{s}^2}{g_\mathrm{s}}+1\right). \end{aligned}\right\} \tag{90}$$

In boundary Condition (10), terms involving the constant $A_{0,\mathrm{s}}$ as well as terms involving the constant $C_{0,\mathrm{s}}$ appear. The leading order in ε of the terms involving the constant $A_{0,\mathrm{s}}$ is $-\left(n_\mathrm{e}+3/2\right)$, while the terms involving the constant $C_{0,\mathrm{s}}$ are of the order ε^0. The sum of the terms involving the constant $A_{0,\mathrm{s}}$ that are of the leading order turns out to be equal to zero, so that the boundary condition is automatically satisfied by the boundary-layer solution $\alpha^{(\mathrm{s})}(r;\varepsilon)$ and the oscillatory part of the boundary-layer solution $\xi^{(\mathrm{s})}(r;\varepsilon)$. For the boundary condition to be also satisfied by the terms involving the constant $C_{0,\mathrm{s}}$, one must set

$$C_{0,\mathrm{s}} = 0. \tag{91}$$

From the third Eq. (83), it then follows that

$$G_0^{(\mathrm{o})}(R) = 0. \tag{92}$$

In combination with Eq. (48), it results that, in Solution (15) for the function $G_0^{(0)}(r)$,

$$C_0^* = 0 \quad\text{and}\quad D_0^* = 0, \tag{93}$$

so that the function is identically zero at all points of the radiative envelope. Hence, one arrives at the conclusion that the asymptotic solution for $\xi(r)$, as well as the asymptotic solution for $\alpha(r)$, is purely oscillatory in the whole radiative envelope.

The boundary-layer solutions $\alpha^{(\mathrm{s})}(r;\varepsilon)$ and $\xi^{(\mathrm{s})}(r;\varepsilon)$ have opposite signs at $r = R$. This difference in sign results from the fact that the radial component of the Lagrangian displacement displays one node more than the divergence of the Lagrangian displacement in the outermost layers of the radiative envelope, while deeper in the envelope these two functions are related to each other by the simple relation $\xi(r) = \left[c^2(r)/g(r)\right]\alpha(r)$. We refer to Sect. 3.8 for more details.

3.7. Uniformly valid asymptotic solutions

Since all constants involved in the various asymptotic solutions have now been determined, uniformly valid asymptotic solutions for the divergence and the radial component of the Lagrangian displacement can be constructed in a final form. Uniformly valid asymptotic solutions from the boundary between the convective core and the radiative envelope and from the star's surface are given by the sum of the boundary-layer solution and the solution valid at larger distances, minus the part common to both solutions. We here present the uniformly valid asymptotic solutions in terms of the constants $B_{0,\mathrm{a}}$ and $A_{0,\mathrm{s}}$.

The asymptotic solutions that are uniformly valid from the boundary between the convective core and the radiative envelope to a distance sufficiently large from the star's surface take the form

$$\left.\begin{aligned} \alpha^{(\mathrm{a,u})}(r;\varepsilon) &= B_{0,\mathrm{a}}\left[\varepsilon^{-1/2}\,\sqrt{s_\mathrm{a}}\,J_{-1/3}\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,\mathrm{a}}}\,s_\mathrm{a}^{3/2}\right)\right.\\ &\quad+\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,\mathrm{a}}^{1/4}\,K_{5,\mathrm{a}}}\,K_5(r)\,\cos\left(\tau-\frac{\pi}{12}\right)\\ &\quad\left.-\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,\mathrm{a}}^{1/4}}\,\frac{1}{s_\mathrm{a}^{1/4}}\,\cos\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,\mathrm{a}}}\,s_\mathrm{a}^{3/2}-\frac{\pi}{12}\right)\right],\\ \xi^{(\mathrm{a,u})}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(\mathrm{a,u})}(r;\varepsilon). \end{aligned}\right\} \tag{94}$$

The uniformly valid asymptotic solution $\alpha^{(\mathrm{a,u})}(r;\varepsilon)$ can be expressed in the compact form

$$\alpha^{(\mathrm{a,u})}(r;\varepsilon) = B_{0,\mathrm{a}}\,\frac{\sqrt{3}}{\sqrt{2}\,K_{1,\mathrm{a}}^{1/4}\,K_{5,\mathrm{a}}}\,K_5(r)\,\tau^{1/2}\,J_{-1/3}(\tau), \tag{95}$$

which is similar to the form of the asymptotic solution given in Willems et al. (1997).

Next, the asymptotic solutions for the divergence and the radial component of the Lagrangian displacement that are uniformly valid from the star's surface to a distance sufficiently large from the boundary between the convective core and the radiative envelope take the form

$$\left.\begin{aligned} \alpha^{(\mathrm{s,u})}(r;\varepsilon) &= A_{0,\mathrm{s}}\left\{\varepsilon^{-(n_\mathrm{e}+3/2)}\,z^{*-(n_\mathrm{e}+1)}\,J_{n_\mathrm{e}+1}\left(z^*\right)\right.\\ &\quad+F\,K_5(r)\,\sqrt{\frac{2}{\pi}}\,\sin\left[\tau_\mathrm{s}-\left(n_\mathrm{e}+\frac{1}{2}\right)\frac{\pi}{2}\right]\\ &\quad-F\,K_{5,\mathrm{s}}\,z^{-(n_\mathrm{e}+3/2)/2}\\ &\quad\left.\times\sqrt{\frac{2}{\pi}}\,\sin\left[\frac{1}{\varepsilon}\,2\,K_{1,\mathrm{s}}^{1/2}\,z^{1/2}-\left(n_\mathrm{e}+\frac{1}{2}\right)\frac{\pi}{2}\right]\right\},\\ \xi^{(\mathrm{s,u})}(r;\varepsilon) &= A_{0,\mathrm{s}}\left\{\varepsilon^{-(n_\mathrm{e}+3/2)}\,\frac{c_\mathrm{s}^2}{g_\mathrm{s}}\,z\,z^{*-(n_\mathrm{e}+1)}\,J_{n_\mathrm{e}+1}\left(z^*\right)\right.\\ &\quad\times\left\{1-2\,\frac{N_\mathrm{s}^2}{g_\mathrm{s}}\,\frac{1}{z^{*2}}\left[\frac{\mathrm{d}\ln J_{n_\mathrm{e}+1}\left(z^*\right)}{\mathrm{d}\ln z^*}+\left(n_\mathrm{e}+1\right)\right]\right\}\\ &\quad+F\,\frac{c^2(r)}{g(r)}\,K_5(r)\,\sqrt{\frac{2}{\pi}}\,\sin\left[\tau_\mathrm{s}-\left(n_\mathrm{e}+\frac{1}{2}\right)\frac{\pi}{2}\right]\\ &\quad-F\,\frac{c_\mathrm{s}^2}{g_\mathrm{s}}\,K_{5,\mathrm{s}}\,z^{-(n_\mathrm{e}-1/2)/2}\\ &\quad\left.\times\sqrt{\frac{2}{\pi}}\,\sin\left[\frac{1}{\varepsilon}\,2\,K_{1,\mathrm{s}}^{1/2}\,z^{1/2}-\left(n_\mathrm{e}+\frac{1}{2}\right)\frac{\pi}{2}\right]\right\}. \end{aligned}\right\} \tag{96}$$

These uniformly valid asymptotic solutions can be expressed in the compact forms

$$\left.\begin{aligned} \alpha^{(\mathrm{s,u})}(r;\varepsilon) &= A_{0,\mathrm{s}}\,F\,K_5(r)\,\tau_\mathrm{s}^{1/2}\,J_{n_\mathrm{e}+1}\left(\tau_\mathrm{s}\right),\\ \xi^{(\mathrm{s,u})}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(\mathrm{s,u})}(r;\varepsilon)\\ &\quad\times\left\{1-2\,\frac{N_\mathrm{s}^2}{g_\mathrm{s}}\,\frac{1}{\tau_\mathrm{s}^2}\left[\frac{\mathrm{d}\ln J_{n_\mathrm{e}+1}\left(\tau_\mathrm{s}\right)}{\mathrm{d}\ln\tau_\mathrm{s}}+\left(n_\mathrm{e}+1\right)\right]\right\}, \end{aligned}\right\} \tag{97}$$

which are similar to the forms of the asymptotic solutions given in Willems et al. (1997).

The orders in ε of the eigenfunctions $\alpha(r)$, $\xi(r)$, and $\eta(r)$, respectively, in the adiabatic core, in the radiative envelope, and at the surface are presented in Table 1.

Table 1. The orders in ε of the divergence, the radial component, and the transverse component of the Lagrangian displacement for different regions in a star composed of a convective core in adiabatic equilibrium and a radiative envelope.

	Adiab. core	Rad. envelope	Surface
$\alpha(r)$	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-(n_e+3/2)}$
$\xi(r)$	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-(n_e-1/2)}$
$\eta(r)$	$\varepsilon^{-1/6}$	ε^{-1}	$\varepsilon^{-(n_e+3/2)}$

The order in ε of the Eulerian perturbation of the pressure, $P'(r)$, can be determined by means of the equation

$$P'(r) = \rho(r)\, g(r) \left(\xi(r) - \frac{c^2(r)}{g(r)}\, \alpha(r) \right). \qquad (98)$$

Since $\xi(r) = \left[c^2(r)/g(r)\right] \alpha(r)$ in the adiabatic core and in the radiative envelope, the lowest-order asymptotic approximation of the Eulerian perturbation of the pressure is identically zero in these regions. In the boundary layer near the surface, $P'(r)$ is of the order of $\varepsilon^{-(n_e-1/2)}$.

Next, the order in ε of the Eulerian perturbation of the gravitational potential, $\Phi'(r)$, can be determined by means of the equation

$$\Phi'(r) = \varepsilon^2\, \eta(r) - \frac{P'(r)}{\rho(r)} \cdot \qquad (99)$$

Hence, in the adiabatic core, $\Phi'(r)$ is of the order of $\varepsilon^{5/6}$, and in the radiative envelope, of the order of ε. In the boundary layer near the surface, $\Phi'(r)$ is given by a difference of two terms of the same order, so that its lowest possible order is $\varepsilon^{-(n_e-1/2)}$.

3.8. Identification of the radial order of a g^+-mode

In accordance with Cowling's classification of non-radial oscillations of stars, the radial order of the g^+-mode that is associated with a number n in eigenfrequency Eq. (87) corresponds to the number of nodes the asymptotic solution for $\xi(r)$ displays between $r = 0$ and $r = R$. In the case of a star composed of a convective core and a radiative envelope, all the nodes are situated in the radiative envelope. In order to count them, we start from the asymptotic solution $\xi^{(o)}(r;\varepsilon)$, which is valid at distances sufficiently large from the lower boundary of the radiative envelope, and the star's surface. We then examine to which extent the nodes of the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ can be connected with those of the uniformly valid solutions $\xi^{(a,u)}(r;\varepsilon)$ and $\xi^{(s,u)}(r;\varepsilon)$.

First, by means of Eqs. (52), the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ can be transformed into

$$\xi^{(o)}(r;\varepsilon) = B_{0,a}\, \frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,a}^{1/4}\, K_{5,a}}\, \frac{c^2(r)}{g(r)}\, K_5(r) \cos\left(\tau - \frac{\pi}{12}\right), \qquad (100)$$

so that the positions of its nodes are given by

$$\tau^0 = \left(2j - \frac{5}{6}\right) \frac{\pi}{2}, \qquad j = 1, 2, 3, \ldots \qquad (101)$$

The node associated with $j = 1$ is located at $\tau^0 = 1.83$, while the first node of the asymptotic solution $\xi^{(a,u)}(r;\varepsilon)$, which is uniformly valid from the turning point at $r = r_a$, is determined by the first zero of the Bessel function $J_{-1/3}(\tau)$ and is situated at $\tau = 1.87$. Hence, the node of the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ that is associated with $j = 1$ can be related to the first node of the uniformly valid solution $\xi^{(a,u)}(r;\varepsilon)$ counted from $r = r_a$.

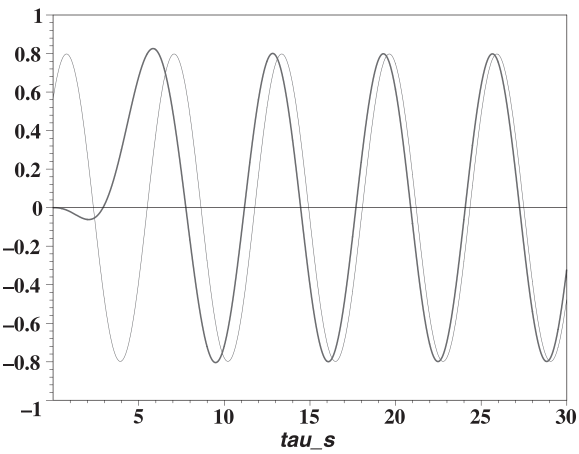

Fig. 1. The functions $H_1(\tau_s)$ (thick line) and $H_2(\tau_s)$ (thin line) for $n_e = 3$.

Next, by means of Eqs. (81), the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ can be transformed into

$$\xi^{(o)}(r;\varepsilon) = A_{0,s}\, F\, \sqrt{\frac{2}{\pi}\, \frac{c^2(r)}{g(r)}}\, K_5(r) \sin\left[\tau_s - \left(n_e + \frac{1}{2}\right)\frac{\pi}{2}\right] \cdot \qquad (102)$$

We here consider the case $n_e = 3$ as an example. The positions of the nodes of the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ are then given by

$$\tau_s^0 = \left(2k - \frac{1}{2}\right) \frac{\pi}{2}, \qquad k = 1, 2, 3, \ldots \qquad (103)$$

In order to relate the nodes of the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ to those of the asymptotic solution $\xi^{(s,u)}(r;\varepsilon)$, which is uniformly valid from $r = R$, we consider the parts of the solutions depending on the fast variable τ_s:

$$H_1(\tau_s) = \tau_s^{1/2}\, J_4(\tau_s) \left\{ 1 - 2\, \frac{N_s^2}{g_s}\, \frac{1}{\tau_s^2} \left[\frac{\mathrm{d} \ln J_4(\tau_s)}{\mathrm{d} \ln \tau_s} + 4 \right] \right\},$$

for $\xi^{(s,u)}(r;\varepsilon)$, and

$$H_2(\tau_s) = \sqrt{\frac{2}{\pi}} \sin\left(\tau_s - \frac{7\pi}{4}\right)$$

for $\xi^{(o)}(r;\varepsilon)$. The functions $H_1(\tau_s)$ and $H_2(\tau_s)$ are represented in Fig. 1 for $\Gamma_{1,s} = 5/3$ and $N_s^2/g_s = 3/5$. It appears that the node of the asymptotic solution $\xi^{(o)}(r;\varepsilon)$ that is associated with $k = 2$ must be related to the first node of the asymptotic solution $\xi^{(s,u)}(r;\varepsilon)$. The node associated with $k = 2$ is located at $\tau_s^0(\text{first}) = 7\pi/4$ or, equivalently, at

$$\tau^0(\text{last}) = \tau_R - \tau_s^0(\text{first}) = \left(2n - \frac{5}{6}\right) \frac{\pi}{2} \cdot$$

Hence, according to Eq. (101), the last node counted from $r = r_a$ is the nth node. The asymptotic solution for the radial component of the Lagrangian displacement thus displays n nodes between the lower boundary of the radiative envelope and the star's surface, so that the g^+-mode considered has the radial order n.

The number of nodes of the asymptotic solution for the divergence of the Lagrangian displacement can be determined in a similar way. The nodes of the solution $\alpha^{(a,u)}(r;\varepsilon)$, which is uniformly valid from $r = r_a$, coincide with those of the uniformly

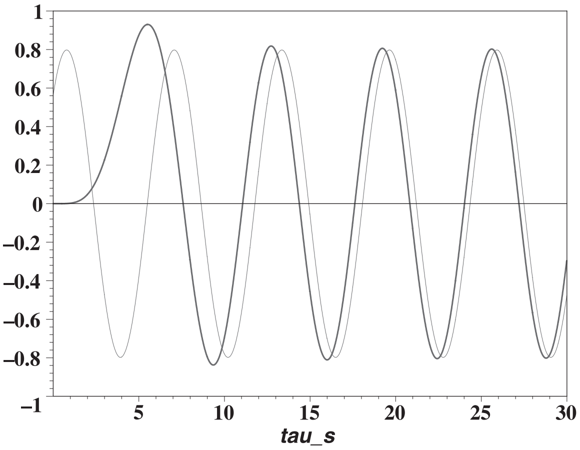

Fig. 2. The functions $H_2(\tau_s)$ (thin line) and $H_3(\tau_s)$ (thick line) for $n_e = 3$.

valid solution $\xi^{(a,u)}(r;\varepsilon)$. In order to relate the nodes of the asymptotic solution $\alpha^{(o)}(r;\varepsilon)$ to those of the solution $\alpha^{(s,u)}(r;\varepsilon)$, which is uniformly valid from $r = R$, it is convenient to compare the function $H_2(\tau_s)$ with the function $H_3(\tau_s) = \tau_s^{1/2} J_4(\tau_s)$. From the representation of these functions in Fig. 2, it appears that the node of the asymptotic solution $\alpha^{(o)}(r;\varepsilon)$ that is associated with $k = 3$ must be related to the first node of the uniformly valid solution $\alpha^{(s,u)}(r;\varepsilon)$. Consequently, compared with the asymptotic solution for the radial component of the Lagrangian displacement, the asymptotic solution for the divergence of the Lagrangian displacement displays one node less, i.e. only $(n-1)$ nodes, between the lower boundary of the radiative envelope and the star's surface. The additional node of the radial component is located close to the star's surface at the point whose coordinate τ_s is solution of the equation

$$1 - 2\,\frac{N_s^2}{g_s}\,\frac{1}{\tau_s^2}\left[\frac{\mathrm{d}\ln J_4(\tau_s)}{\mathrm{d}\ln\tau_s} + 4\right] = 0.$$

4. Stars consisting of a convective core, a radiative zone, and a convective envelope

For stars consisting of a convective core, an intermediate radiative zone, and a convective envelope, a second turning point appears in Eq. (3) at the point $r = r_b$ corresponding to the radial distance of the boundary between the radiative zone and the convective envelope. The asymptotic solutions constructed in the previous section for a star composed of a convective core and a radiative envelope remain valid, except those constructed from the surface of the radiative envelope.

In this section, we start constructing boundary-layer solutions on the inner side of the boundary between the radiative zone and the convective envelope, and match them to the asymptotic solutions valid at larger distances from the boundaries of the radiative zone.

Then we turn to the construction of asymptotic solutions in the convective envelope. Here we successively construct two-variable solutions in the central part of the envelope, boundary-layer solutions near the boundary between the radiative zone and the convective envelope, and boundary-layer solutions near the star's surface.

Next, we derive the eigenfrequency equation, impose the condition relative to the Eulerian perturbation of the gravitational potential at the star's surface, and construct uniformly valid asymptotic solutions.

Finally, we determine the radial order of the g^+-mode associated with a given eigenfrequency in the case in which the slope of the Brunt-Väisälä is continuous at the boundary beween the intermediate radiative zone and the convective envelope.

4.1. Boundary-layer solutions on the inner side of the boundary between the radiative zone and the convective envelope

On the inner side of the boundary between the radiative zone and the convective envelope, the use of the coordinate $s_b = r_b - r$ is convenient. When the coefficients of Eqs. (17) and (18) behave similarly as on the outer side of the turning point at $r = r_a$, it formally suffices to replace the subscript a by the subscript b in Taylor Expansions (19) to get the appropriate Taylor expansions.

Proceeding as in Sect. 3.2, we introduce the boundary-layer coordinate

$$s_b^*(r) = \frac{s_b(r)}{\varepsilon^{2/3}} \tag{104}$$

and derive the boundary-layer solutions

$$\left.\begin{aligned}
\alpha^{(b)}(r;\varepsilon) &= \mu_0^{(b)}(\varepsilon)\,\sqrt{s_b^*}\Bigg[A_{0,b}\,J_{1/3}\left(\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{*3/2}\right)\\
&\quad + B_{0,b}\,J_{-1/3}\left(\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{*3/2}\right)\Bigg],\\
\xi^{(b)}(r;\varepsilon) &= \frac{c^2(r_b)}{g(r_b)}\Bigg[\alpha^{(b)}(r;\varepsilon)\\
&\quad + \nu_0^{(b,2)}(\varepsilon)\,C_{0,b}\,s_b^* + \nu_0^{(b,3)}(\varepsilon)\,D_{0,b}\Bigg],
\end{aligned}\right\} \tag{105}$$

where $A_{0,b}$, $B_{0,b}$, $C_{0,b}$, $D_{0,b}$ are general constants, and $\mu_0^{(b)}(\varepsilon)$, $\nu_0^{(b,2)}(\varepsilon)$, $\nu_0^{(b,3)}(\varepsilon)$ yet undetermined functions.

For the matching of the boundary-layer solution $\alpha^{(b)}(r;\varepsilon)$ to the asymptotic solution $\alpha^{(o)}(r;\varepsilon)$ valid at larger distances from the turning point, we observe that, as $s_b \to 0$, the fast variable $\tau(r)$ defined by Eq. (11) behaves as

$$\tau(r) = \tau_{\mathrm{Rad}} - \frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{3/2} \tag{106}$$

with

$$\tau_{\mathrm{Rad}} = \frac{1}{\varepsilon}\int_{r_a}^{r_b} K_1^{1/2}(r')\,\mathrm{d}r', \tag{107}$$

so that

$$\begin{aligned}
\lim_{s_b\to 0}\alpha^{(o)}(r,\varepsilon) &= \frac{K_{5,b}}{s_b^{1/4}}\Bigg[A_0^*\,\cos\left(\tau_{\mathrm{Rad}} - \frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{3/2}\right)\\
&\quad + B_0^*\,\sin\left(\tau_{\mathrm{Rad}} - \frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{3/2}\right)\Bigg].
\end{aligned} \tag{108}$$

On the other hand, for $s_b \to \infty$, one has that

$$\begin{aligned}
\lim_{s_b\to\infty}\alpha^{(b)}(r;\varepsilon) &= \mu_0^{(b)}(\varepsilon)\,\varepsilon^{1/6}\,\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,b}^{1/4}}\,\frac{1}{s_b^{1/4}}\\
&\quad\times\Bigg[A_{0,b}\,\sin\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{3/2} + \frac{\pi}{12}\right)\\
&\quad + B_{0,b}\,\cos\left(\frac{1}{\varepsilon}\,\frac{2}{3}\,\sqrt{K_{1,b}}\,s_b^{3/2} - \frac{\pi}{12}\right)\Bigg].
\end{aligned} \tag{109}$$

The matching relative to the divergence of the Lagrangian displacement then leads to

$$\mu_0^{(b)}(\varepsilon) = \varepsilon^{-1/6} \tag{110}$$

and

$$\left.\begin{aligned} A_0^* &= \frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,b}^{1/4}\, K_{5,b}} \left[A_{0,b}\, \sin\left(\tau_{\rm Rad} + \frac{\pi}{12}\right)\right. \\ &\quad \left. +B_{0,b}\, \cos\left(\tau_{\rm Rad} - \frac{\pi}{12}\right)\right], \\ B_0^* &= -\frac{\sqrt{3}}{\sqrt{\pi}\, K_{1,b}^{1/4}\, K_{5,b}} \left[A_{0,b}\, \cos\left(\tau_{\rm Rad} + \frac{\pi}{12}\right)\right. \\ &\quad \left. -B_{0,b}\, \sin\left(\tau_{\rm Rad} - \frac{\pi}{12}\right)\right]. \end{aligned}\right\} \tag{111}$$

Moreover, the matching relative to the non-oscillatory parts in the radial component of the Lagrangian displacement leads to

$$\left.\begin{aligned} &C_{0,b} = 0, \\ &\nu_0^{(b,3)}(\varepsilon) = \varepsilon^0, \qquad D_{0,b} = \frac{g\,(r_b)}{c^2\,(r_b)}\, G_0^{(o)}\,(r_b)\,. \end{aligned}\right\} \tag{112}$$

At the turning point at $r = r_b$, the divergence of the Lagrangian displacement, and the radial component of the Lagrangian displacement and its first derivative have the values

$$\left.\begin{aligned} &\alpha^{(b)}\,(r_b;\varepsilon) = \varepsilon^{-1/6}\, \frac{3^{1/3}}{\Gamma(2/3)\, K_{1,b}^{1/6}}\, B_{0,b} \\ &\xi^{(b)}\,(r_b;\varepsilon) = \frac{c^2\,(r_b)}{g\,(r_b)}\, \alpha^{(b)}\,(r_b;\varepsilon) + G_0^{(o)}\,(r_b)\,, \\ &\left(\frac{d\xi^{(b)}(r;\varepsilon)}{dr}\right)_{r=r_b} = -\varepsilon^{-5/6}\, \frac{3^{7/6}\,\Gamma(2/3)\, K_{1,b}^{1/6}}{2\pi}\, \frac{c^2\,(r_b)}{g\,(r_b)}\, A_{0,b}. \end{aligned}\right\} \tag{113}$$

So far the two general constants A_0^* and B_0^*, which appear in the two-variable solutions $\alpha^{(o)}(r,\varepsilon)$ and $\xi^{(o)}(r,\varepsilon)$ given by Eqs. (12) and valid in the radiative zone at larger distances from its boundaries, are related to the constant $B_{0,a}$ by Eqs. (52), and to the two constants $A_{0,b}$, and $B_{0,b}$ by Eqs. (111). From the asymptotic solutions in the convective envelope, which are constructed hereafter, it will result that the two constants $A_{0,b}$, and $B_{0,b}$ are related to a single constant denoted as $B_{0,c}$. Therefore, Eqs. (111) will be replaced by two equations that relate the constants A_0^* and B_0^* to the new constant $B_{0,c}$ (see Sect. 4.2.4).

4.2. Asymptotic solutions in the convective envelope

Since $N^2 < 0$ in the convective envelope, we replace the coefficient of the term with the large parameter in Eq. (3) by the positive coefficient

$$K_1'(r) \equiv -K_1(r) = -\ell(\ell+1)\,\frac{N^2}{r^2}. \tag{114}$$

With this modification, Eqs. (3) and (4) become

$$\left.\begin{aligned} &\frac{d^2\alpha}{dr^2} + K_2(r)\,\frac{d\alpha}{dr} \\ &\quad + \left[-\frac{K_1'(r)}{\varepsilon^2} + K_3(r) + \frac{\varepsilon^2}{c^2(r)}\right]\alpha = -K_4(r)\,\frac{d\xi}{dr}, \\ &\frac{d^2\xi}{dr^2} + \frac{4}{r}\,\frac{d\xi}{dr} - \frac{\ell(\ell+1)-2}{r^2}\,\xi = \\ &\quad \frac{d\alpha}{dr} + \left[\frac{c^2(r)}{g(r)}\,\frac{K_1'(r)}{\varepsilon^2} + \frac{2}{r}\right]\alpha. \end{aligned}\right\} \tag{115}$$

4.2.1. Asymptotic solutions valid in the convective envelope at larger distances from its boundaries

For the construction of asymptotic solutions valid in the convective envelope at larger distances from the boundaries, we use a two-variable expansion procedure similar to that applied for a radiative envelope in Sect. 3.1. We still adopt the radial coordinate r as the slow variable but define the fast variable as

$$\tau_e(r) = \frac{1}{\varepsilon}\int_{r_b}^{r} K_1'^{1/2}\,(r')\,dr'. \tag{116}$$

In terms of these variables, the asymptotic solutions for the divergence and the radial component of the Lagrangian displacement take the form

$$\left.\begin{aligned} &\alpha^{(e)}(r;\varepsilon) = \mu_0^{(e)}(\varepsilon)\, K_5'(r)\left[A_0^{**}\, \exp\tau_e + B_0^{**}\, \exp\left(-\tau_e\right)\right], \\ &\xi^{(e)}(r;\varepsilon) = \frac{c^2(r)}{g(r)}\,\alpha^{(e)}(r;\varepsilon) + \mu_0^{(e)}(\varepsilon)\, G_0^{(e)}(r), \end{aligned}\right\} \tag{117}$$

where the function $K_5'(r)$ is defined as

$$K_5'(r) = g(r)\left[-N^2(r)\, r^6\, c^8(r)\,\rho^2(r)\right]^{-1/4}. \tag{118}$$

The function $G_0^{(e)}(r)$ is the general solution of Clairaut's second-order differential equation

$$G_0^{(e)}(r) = C_0^{**}\, y_1(r) + D_0^{**}\, y_2(r). \tag{119}$$

We let the particular solutions $y_1(r)$ and $y_2(r)$ correspond to those appearing in Solution (15) for $G_0^{(o)}(r)$. The function $\mu_0^{(e)}(\varepsilon)$ is yet undetermined, and A_0^{**}, B_0^{**}, C_0^{**}, D_0^{**} are general constants.

The main differences between asymptotic Solutions (117) valid in a convective envelope and asymptotic Solutions (12) valid in a radiative envelope are that the trigonometric functions of the fast variable $\tau(r)$ are replaced by exponential functions of the fast variable $\tau_e(r)$, and that the function $K_5(r)$ is replaced by the function $K_5'(r)$.

To asymptotic Solutions (117), boundary-layer solutions valid respectively near the upper boundary of the radiative zone and near the star's surface must be matched.

4.2.2. Boundary-layer solutions on the outer side of the boundary between the radiative zone and the convective envelope

We allow for the possibility that the first derivative of the square of the Brunt-Väisälä frequency is not continuous at the boundary between the radiative zone and the convective envelope. Therefore, on the outer side of this boundary, we use Taylor series of the form

$$\left.\begin{aligned} \rho(r) &= \rho\,(r_b)\,[1 + O\,(s_c)]\,, \\ g(r) &= g\,(r_b)\,[1 + O\,(s_c)]\,, \\ c(r) &= c\,(r_b)\,[1 + O\,(s_c)]\,, \\ N^2(r) &= N_c^2\, s_c\,[1 + O\,(s_c)]\,, \\ K_1'(r) &= -\frac{\ell(\ell+1)}{r_b^2}\, N_c^2\, s_c\,[1 + O\,(s_c)]\,, \\ &\equiv K_{1,c}'\, s_c\,[1 + O\,(s_c)]\,, \\ K_2(r) &= K_{2,c}\,[1 + O\,(s_c)]\,, \\ K_3(r) &= K_{3,c}\,[1 + O\,(s_c)]\,, \\ K_4(r) &= K_{4,c}\,[1 + O\,(s_c)]\,, \\ K_5'(r) &= K_{5,c}'\, s_c^{-1/4}\,[1 + O\,(s_c)]\,, \end{aligned}\right\} \tag{120}$$

with $s_c(r) = r - r_b$. For the sake of clearness, the coefficient N_c^2 in the Taylor series of $N^2(r)$ may be different from the coefficient N_b^2, which appears in the Taylor series of $N^2(r)$ valid on the inner side of the boundary between the radiative zone and the convective envelope

The boundary-layer solutions can be constructed in a way similar to that followed in Sect. 3.2, but they now involve modified Bessel functions I and K instead of Bessel functions of the first kind. In terms of the boundary-layer coordinate

$$s_{\mathrm{c}}^*(r) = \frac{s_{\mathrm{c}}(r)}{\varepsilon^{2/3}}, \tag{121}$$

the boundary-layer solutions can be written as

$$\left.\begin{aligned} \alpha^{(\mathrm{c})}(r;\varepsilon) &= \mu_0^{(\mathrm{c})}(\varepsilon)\,\sqrt{s_{\mathrm{c}}^*}\Bigg[A_{0,\mathrm{c}}\,I_{1/3}\left(\frac{2}{3}\,\sqrt{K'_{1,\mathrm{c}}}\,s_{\mathrm{c}}^{*3/2}\right)\\ &\quad +B_{0,\mathrm{c}}\,K_{1/3}\left(\frac{2}{3}\,\sqrt{K'_{1,\mathrm{c}}}\,s_{\mathrm{c}}^{*3/2}\right)\Bigg],\\ \xi^{(\mathrm{c})}(r;\varepsilon) &= \frac{c^2(r_{\mathrm{b}})}{g(r_{\mathrm{b}})}\Bigg[\alpha^{(\mathrm{c})}(r;\varepsilon)\\ &\quad +\nu_0^{(\mathrm{c},2)}(\varepsilon)\,C_{0,\mathrm{c}}\,s_{\mathrm{c}}^* + \nu_0^{(\mathrm{c},3)}(\varepsilon)\,D_{0,\mathrm{c}}\Bigg], \end{aligned}\right\} \tag{122}$$

where $\mu_0^{(\mathrm{c})}(\varepsilon)$, $\nu_0^{(\mathrm{c},2)}(\varepsilon)$, $\nu_0^{(\mathrm{c},3)}(\varepsilon)$ are yet undetermined functions, and $A_{0,\mathrm{c}}$, $B_{0,\mathrm{c}}$, $C_{0,\mathrm{c}}$, $D_{0,\mathrm{c}}$ general constants.

The boundary-layer solutions $\alpha^{(\mathrm{c})}(r;\varepsilon)$ and $\xi^{(\mathrm{c})}(r;\varepsilon)$ are matched to the asymptotic solutions $\alpha^{(\mathrm{e})}(r;\varepsilon)$ and $\xi^{(\mathrm{e})}(r;\varepsilon)$ valid at larger distances in the convective envelope.

The matching relative to the divergence of the Lagrangian displacement leads to

$$\mu_0^{(\mathrm{e})}(\varepsilon) = \varepsilon^{1/6}\,\mu_0^{(\mathrm{c})}(\varepsilon) \tag{123}$$

and

$$\left.\begin{aligned} A_{0,\mathrm{c}} &= \frac{2\,\sqrt{\pi}}{\sqrt{3}}\,K'^{1/4}_{1,\mathrm{c}}\,K'_{5,\mathrm{c}}\,A_0^{**},\\ B_{0,\mathrm{c}} &= \frac{2}{\sqrt{3\,\pi}}\,K'^{1/4}_{1,\mathrm{c}}\,K'_{5,\mathrm{c}}\,B_0^{**}, \end{aligned}\right\} \tag{124}$$

and the matching relative to the non-oscillatory parts in the radial component of the Lagrangian displacement, to

$$\left.\begin{aligned} &C_{0,\mathrm{c}} = 0,\\ &\nu_0^{(\mathrm{c},3)}(\varepsilon) = \mu_0^{(\mathrm{e})}(\varepsilon), \quad D_{0,\mathrm{c}} = \frac{g(r_{\mathrm{b}})}{c^2(r_{\mathrm{b}})}\,G_0^{(\mathrm{e})}(r_{\mathrm{b}})\,. \end{aligned}\right\} \tag{125}$$

At the turning point at $r = r_{\mathrm{b}}$, the divergence of the Lagrangian displacement, and the radial component of the Lagrangian displacement and its first derivative have the values

$$\left.\begin{aligned} \alpha^{(\mathrm{c})}(r_{\mathrm{b}};\varepsilon) &= \mu_0^{(\mathrm{c})}(\varepsilon)\,\frac{3^{-1/6}\,\pi}{\Gamma(2/3)\,K'^{1/6}_{1,\mathrm{c}}}\,B_{0,\mathrm{c}},\\ \xi^{(\mathrm{c})}(r_{\mathrm{b}};\varepsilon) &= \frac{c^2(r_{\mathrm{b}})}{g(r_{\mathrm{b}})}\,\alpha^{(\mathrm{c})}(r_{\mathrm{b}};\varepsilon) + \nu_0^{(\mathrm{c},3)}(\varepsilon)\,G_0^{(\mathrm{e})}(r_{\mathrm{b}}),\\ \left(\frac{\mathrm{d}\xi^{(\mathrm{c})}(r;\varepsilon)}{\mathrm{d}r}\right)_{r_{\mathrm{b}}} &= \mu_0^{(\mathrm{c})}(\varepsilon)\,\varepsilon^{-2/3}\,\frac{3^{7/6}\,\Gamma(2/3)}{2\,\pi}\\ &\quad K'^{1/6}_{1,\mathrm{c}}\,\frac{c^2(r_{\mathrm{b}})}{g(r_{\mathrm{b}})}\left(A_{0,\mathrm{c}} - \frac{\pi}{\sqrt{3}}\,B_{0,\mathrm{c}}\right). \end{aligned}\right\} \tag{126}$$

The continuity of the Lagrangian displacement and the Lagrangian perturbation of the pressure at the boundary between the radiative zone and the convective envelope requires that the divergence, the radial component, and the first derivative of the radial component of the Lagrangian displacement be continuous at $r = r_{\mathrm{b}}$. On the grounds of Equalities (113) and (126), it results that

$$\left.\begin{aligned} &\mu_0^{(\mathrm{c})}(\varepsilon) = \varepsilon^{-1/6},\\ &B_{0,\mathrm{b}} = \frac{\pi}{\sqrt{3}}\left(\frac{K_{1,\mathrm{b}}}{K'_{1,\mathrm{c}}}\right)^{1/6}\,B_{0,\mathrm{c}},\\ &G_0^{(\mathrm{e})}(r_{\mathrm{b}}) = G_0^{(\mathrm{o})}(r_{\mathrm{b}}),\\ &A_{0,\mathrm{b}} = -\left(\frac{K'_{1,\mathrm{c}}}{K_{1,\mathrm{b}}}\right)^{1/6}\left(A_{0,\mathrm{c}} - \frac{\pi}{\sqrt{3}}\,B_{0,\mathrm{c}}\right). \end{aligned}\right\} \tag{127}$$

The constant $B_{0,\mathrm{b}}$ is related to the constant $B_{0,\mathrm{c}}$, and the constant $A_{0,\mathrm{b}}$ to the constants $A_{0,\mathrm{c}}$ and $B_{0,\mathrm{c}}$. Hereafter, it will result from the matching of the boundary-layer solutions valid near the surface that the constant $A_{0,\mathrm{c}}$ is equal to zero, so that constants $A_{0,\mathrm{b}}$ and $B_{0,\mathrm{b}}$ will just be related to constant $B_{0,\mathrm{c}}$.

From the combination of the first Eqs. (127) with (123), it follows that

$$\mu_0^{(\mathrm{e})}(\varepsilon) = \varepsilon^0. \tag{128}$$

4.2.3. Boundary-layer solutions near the surface

For the construction of boundary-layer solutions near the surface of a star with a convective envelope, we proceed in a similar way as for that of boundary-layer solutions near the surface of a star with a radiative envelope.

We now introduce the boundary-layer coordinate

$$z_{\mathrm{e}}^*(z) = \frac{1}{\varepsilon}\,2\,K'^{1/2}_{1,\mathrm{s}}\,z^{1/2}. \tag{129}$$

The first dominant boundary-layer equation is

$$\frac{\mathrm{d}^2\alpha_0^{(\mathrm{s})}}{\mathrm{d}z_{\mathrm{e}}^{*2}} + \frac{2\,n_{\mathrm{e}}+3}{z_{\mathrm{e}}^*}\,\frac{\mathrm{d}\alpha_0^{(\mathrm{s})}}{\mathrm{d}z_{\mathrm{e}}^*} - \alpha_0^{(\mathrm{s})} = 0 \tag{130}$$

and admits of the solution that remains finite at $r = R$

$$\alpha_0^{(\mathrm{s})}(z_{\mathrm{e}}^*) = A_{0,\mathrm{s}}\,z_{\mathrm{e}}^{*-(n_{\mathrm{e}}+1)}\,I_{n_{\mathrm{e}}+1}(z_{\mathrm{e}}^*), \tag{131}$$

where $A_{0,\mathrm{s}}$ is a general constant.

The second dominant boundary-layer equation is

$$\frac{\mathrm{d}^2 w_0^{(\mathrm{s})}}{\mathrm{d}z_{\mathrm{e}}^{*2}} + \frac{3}{z_{\mathrm{e}}^*}\,\frac{\mathrm{d}w_0^{(\mathrm{s})}}{\mathrm{d}z_{\mathrm{e}}^*} = -2\,\frac{g_{\mathrm{s}}}{c_{\mathrm{s}}^2}\,\frac{1}{z_{\mathrm{e}}^*}\,\frac{\mathrm{d}\alpha_0^{(\mathrm{s})}}{\mathrm{d}z_{\mathrm{e}}^*} + \alpha_0^{(\mathrm{s})}. \tag{132}$$

By subtracting the first dominant boundary-layer equation and introducing the function

$$w_0^*(z_{\mathrm{e}}^*) = w_0^{(\mathrm{s})}(z_{\mathrm{e}}^*) - \alpha_0^{(\mathrm{s})}(z_{\mathrm{e}}^*), \tag{133}$$

one obtains an inhomogeneous differential equation of the same form as Eq. (69), with a solution of the same form as Solution (70). The integral in the solution is transformed by partial integration and use of a recurrence relation between modified Bessel functions I, so that

$$\begin{aligned} w_0^*(z_{\mathrm{e}}^*) &= 2\,A_{0,\mathrm{s}}\,\frac{N_{\mathrm{s}}^2}{g_{\mathrm{s}}}\,z_{\mathrm{e}}^{*-(n_{\mathrm{e}}+2)}\,I_{n_{\mathrm{e}}}(z_{\mathrm{e}}^*)\\ &\quad +C_{0,\mathrm{s}}\,z_{\mathrm{e}}^{*-2} + D_{0,\mathrm{s}}. \end{aligned} \tag{134}$$

By the use of a second recurrence relation between modified Bessel functions I, the solution is transformed into

$$\begin{aligned} w_0^*(z_{\mathrm{e}}^*) &= \alpha_0^{(\mathrm{s})}(z_{\mathrm{e}}^*)\,2\,\frac{N_{\mathrm{s}}^2}{g_{\mathrm{s}}}\,\frac{1}{z_{\mathrm{e}}^{*2}}\left[\frac{\mathrm{d}\ln I_{n_{\mathrm{e}}+1}(z_{\mathrm{e}}^*)}{\mathrm{d}\ln z_{\mathrm{e}}^*} + (n_{\mathrm{e}}+1)\right]\\ &\quad +C_{0,\mathrm{s}}\,z_{\mathrm{e}}^{*-2} + D_{0,\mathrm{s}}. \end{aligned} \tag{135}$$

Consequently, the boundary-layer solution for $w_0^{(s)}(z_e^*)$ is given by

$$w_0^{(s)}(z_e^*) = \alpha_0^{(s)}(z_e^*)\left\{1 + 2\,\frac{N_s^2}{g_s}\,\frac{1}{z_e^{*2}}\left[\frac{\mathrm{d}\ln I_{n_e+1}(z_e^*)}{\mathrm{d}\ln z_e^*} + (n_e+1)\right]\right\} + C_{0,s}\,z_e^{*-2} + D_{0,s}. \tag{136}$$

The boundary-layer solutions for the divergence and the radial component of the Lagrangian displacement can then be written in the more general form

$$\left.\begin{aligned}
&\alpha^{(s)}(r;\varepsilon) = \mu_0^{(s)}(\varepsilon)\,A_{0,s}\,z_e^{*-(n_e+1)}\,I_{n_e+1}(z_e^*),\\
&\xi^{(s)}(r;\varepsilon) = \frac{c_s^2}{g_s}\,z\left\{\alpha^{(s)}(r;\varepsilon)\right.\\
&\qquad\times\left\{1 + 2\,\frac{N_s^2}{g_s}\,\frac{1}{z_e^{*2}}\left[\frac{\mathrm{d}\ln I_{n_e+1}(z_e^*)}{\mathrm{d}\ln z_e^*} + (n_e+1)\right]\right\}\\
&\qquad\left.+\nu_0^{(s,2)}(\varepsilon)\,C_{0,s}\,z_e^{*-2} + \nu_0^{(s,3)}(\varepsilon)\,D_{0,s}\right\},
\end{aligned}\right\} \tag{137}$$

where $\mu_0^{(s)}(\varepsilon)$, $\nu_0^{(s,2)}(\varepsilon)$, $\nu_0^{(s,3)}(\varepsilon)$ are yet undetermined functions.

The matching relative to the divergence of the Lagrangian displacement leads to

$$\mu_0^{(s)}(\varepsilon) = \varepsilon^{-(n_e+3/2)} \tag{138}$$

and

$$\left.\begin{aligned}
&A_0^{**} = 0,\\
&B_0^{**} = A_{0,s}\,\frac{\exp\left[\tau_e(R)\right]}{\sqrt{2\pi}\,\left(2\,K_{1,s}'^{1/2}\right)^{n_e+3/2}\,K_{5,s}'},
\end{aligned}\right\} \tag{139}$$

and the matching relative to the non-oscillatory parts in the radial component of the Lagrangian displacement, to

$$\left.\begin{aligned}
&D_{0,s} = 0,\\
&\nu_0^{(s,2)}(\varepsilon) = \varepsilon^{-2}\,\mu_0^{(e)}(\varepsilon),\\
&C_{0,s} = \frac{g_s}{c_s^2}\,4\,K_{1,s}'\,G_0^{(e)}(R).
\end{aligned}\right\} \tag{140}$$

4.2.4. Main result of the construction of asymptotic solutions in the convective envelope

According to Eqs. (124) and (139), it follows from the construction of the asymptotic solutions in the convective envelope that

$$A_{0,c} = 0, \tag{141}$$

so that the last Eq. (127) reduces to

$$A_{0,b} = \frac{\pi}{\sqrt{3}}\left(\frac{K_{1,c}'}{K_{1,b}}\right)^{1/6} B_{0,c}. \tag{142}$$

By this equation and one of Eqs. (127), the constants $A_{0,b}$ and $B_{0,b}$ are now related to the single constant $B_{0,c}$.

For the sake of simplification of notation, we introduce the quantity

$$K = \frac{K_{1,c}'}{K_{1,b}}. \tag{143}$$

One then has that

$$K = \left[\frac{\left[\mathrm{d}K_1'(r)/\mathrm{d}s_c\right]_{\mathrm{rh}}}{\left[\mathrm{d}K_1(r)/\mathrm{d}s_b\right]_{\mathrm{lh}}}\right]_{r=r_b} = \left[\frac{\left(\mathrm{d}N^2/\mathrm{d}r\right)_{\mathrm{rh}}}{\left(\mathrm{d}N^2/\mathrm{d}r\right)_{\mathrm{lh}}}\right]_{r=r_b}. \tag{144}$$

Hence, K is equal to the ratio of the right-hand first derivative of $N^2(r)$ to the left-hand first derivative of $N^2(r)$, considered at $r = r_b$. It may be noted that K is a positive quantity.

Equations (111) can then be brought in the form

$$\left.\begin{aligned}
&A_0^* = B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\left[K^{1/6}\,\sin\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right)\right.\\
&\qquad\left.+K^{-1/6}\,\cos\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right],\\
&B_0^* = -B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\left[K^{1/6}\,\cos\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right)\right.\\
&\qquad\left.-K^{-1/6}\,\sin\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right].
\end{aligned}\right\} \tag{145}$$

4.3. The eigenfrequency equation

Elimination of the constants A_0* and B_0^* from the foregoing equations and Eqs. (52) leads to the system of two algebraic, linear, homogeneous equations

$$\left.\begin{aligned}
&B_{0,a}\,\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,a}^{1/4}\,K_{5,a}}\,\cos\frac{\pi}{12}\\
&-B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\left[K^{1/6}\,\sin\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right)\right.\\
&\qquad\left.+K^{-1/6}\,\cos\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right] = 0,\\
&B_{0,a}\,\frac{\sqrt{3}}{\sqrt{\pi}\,K_{1,a}^{1/4}\,K_{5,a}}\,\sin\frac{\pi}{12}\\
&+B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\left[K^{1/6}\,\cos\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right)\right.\\
&\qquad\left.-K^{-1/6}\,\sin\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right] = 0.
\end{aligned}\right\} \tag{146}$$

The necessary and sufficient condition for the system of equations to admit of a non-trivial solution for the constants $B_{0,a}$ and $B_{0,c}$ yields the eigenfrequency equation

$$\begin{aligned}
&\left[K^{1/6}\,\cos\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right) - K^{-1/6}\,\sin\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right]\cos\frac{\pi}{12}\\
&\quad+\left[K^{1/6}\,\sin\left(\tau_{\mathrm{Rad}} + \frac{\pi}{12}\right)\right.\\
&\quad\left.+K^{-1/6}\,\cos\left(\tau_{\mathrm{Rad}} - \frac{\pi}{12}\right)\right]\sin\frac{\pi}{12} = 0.
\end{aligned} \tag{147}$$

In the supposition that $K \neq 0$, the eigenfrequency equation can be transformed into

$$\cos\left(\tau_{\mathrm{Rad}}\right) - K^{-1/3}\,\sin\left(\tau_{\mathrm{Rad}} - \frac{\pi}{6}\right) = 0. \tag{148}$$

Next, proceeding as Tassoul (1980, Appendix), we set

$$K^{-1/3} = \frac{\sin\left(\pi/6 - \theta_2\right)}{\sin\left(\pi/6 + \theta_2\right)}, \tag{149}$$

so that the eigenfrequency equation can be written as

$$\begin{aligned}\tau_{\mathrm{Rad}} &\equiv \frac{[\ell(\ell+1)]^{1/2}}{|\sigma|} \int_{r_a}^{r_b} \left(\frac{N^2(r)}{r^2}\right)^{1/2} \mathrm{d}r \\ &= \left(2\,n - \frac{4}{3}\right)\frac{\pi}{2} + \theta_2,\end{aligned} \tag{150}$$

where θ_2 can be expressed in terms of K as

$$\theta_2 = -\frac{\pi}{6} + \arctan \frac{2\,K^{1/3}\,\cos\pi/6}{2 + K^{1/3}}\cdot \tag{151}$$

In the particular case in which the slope of N^2 is continuous at the turning point at $r = r_b$, $K'_{1,c} = K_{1,b}$, so that $\theta_2 = 0$. For this case, we show below in Sect. 4.6 that n corresponds to the radial order of the g^+-mode.

Eigenfrequency Eq. (A.13) of Tassoul (1980), which has been derived in the Cowling approximation, agrees with our Eq. (150), when, in that equation, θ_1 is set equal to $\pi/6$ because of the adiabatic equilibrium adopted in the convective core (see Tassoul's comment above her Eq. (A9)), and κ is replaced by $n - 1$.

4.4. The boundary condition at the surface

In order to impose boundary Condition (10) relative to the Eulerian perturbation of the gravitational potential at $r = R$, we again use Eqs. (88). The divergence and the radial component of the Lagrangian displacement at $r = R$ are given by

$$\left.\begin{aligned}\alpha_R &= \varepsilon^{-(n_e+3/2)}\,A_{0,s}\,\frac{2^{-(n_e+1)}}{\Gamma\,(n_e+2)}, \\ \xi_R &= \frac{c_s^2}{g_s}\,\frac{1}{4\,K'_{1,s}}\Bigg[\varepsilon^{-(n_e-1/2)}\,A_{0,s}\,\frac{2^{-(n_e-1)}}{\Gamma\,(n_e+1)}\,\frac{N_s^2}{g_s} \\ &\quad +\mu_0^{(e)}(\varepsilon)\,C_{0,s}\Bigg],\end{aligned}\right\} \tag{152}$$

and their first derivatives, by

$$\left.\begin{aligned}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}r}\right)_R &= -\varepsilon^{-(n_e+7/2)}\,A_{0,s}\,\frac{2^{-(n_e+1)}}{\Gamma\,(n_e+3)}\,K'_{1,s}, \\ \left(\frac{\mathrm{d}\xi}{\mathrm{d}r}\right)_R &= -\varepsilon^{-(n_e+3/2)}\,A_{0,s}\,\frac{2^{-(n_e+1)}}{\Gamma\,(n_e+2)} \\ &\quad \times\frac{c_s^2}{g_s}\left(\frac{N_s^2}{g_s}+1\right).\end{aligned}\right\} \tag{153}$$

As in the case of a star with a radiative envelope, the boundary condition is automatically satisfied by the boundary-layer solution $\alpha^{(s)}(r;\varepsilon)$ and the oscillatory part of the boundary-layer solution $\xi^{(s)}(r;\varepsilon)$. The terms involving the non-oscillatory part of the boundary-layer solution $\xi^{(s)}(r;\varepsilon)$ are of the order of ε^0. For the boundary condition to be satisfied also for these terms, one must set

$$C_{0,s} = 0. \tag{154}$$

From the third Eq. (140), it then follows that

$$G_0^{(e)}(R) = 0. \tag{155}$$

This equation, Eq. (48), and the third Eq. (127) generally imply that

$$G_0^{(o)}(r) \equiv 0, \quad G_0^{(e)}(r) \equiv 0, \tag{156}$$

so that

$$C_0^* = 0, \quad D_0^* = 0, \quad C_0^{**} = 0, \quad D_0^{**} = 0. \tag{157}$$

As well as in the case of a star with a radiative envelope, the boundary-layer solutions $\alpha^{(s)}(r;\varepsilon)$ and $\xi^{(s)}(r;\varepsilon)$ have opposite signs at the surface of a star with a convective envelope.

4.5. Uniformly valid asymptotic solutions

Since all constants involved in the asymptotic solutions are now fixed, the uniformly valid asymptotic solutions for the divergence and the radial component of the Lagrangian displacement can be presented in final forms:

1. the uniformly valid solutions $\alpha^{(a,u)}(r;\varepsilon)$ and $\xi^{(a,u)}(r;\varepsilon)$ given by Eqs. (94) remain uniformly valid in the intermediate radiative zone, from the upper boundary of the convective core to a distance sufficiently large from the lower boundary of the convective envelope;
2. the asymptotic solutions that are uniformly valid in the intermediate radiative zone, from the lower boundary of the convective envelope to a distance sufficiently large from the upper boundary of the convective core, can be expressed in the compact form

$$\left.\begin{aligned}\alpha^{(b,u)}(r;\varepsilon) &= B_{0,c}\,\frac{\pi}{\sqrt{2}\,K_{1,b}^{1/4}\,K_{5,b}}\,K_5(r) \\ &\quad \tau_b^{1/2}\left[K^{1/6}\,J_{1/3}\,(\tau_b) + K^{-1/6}\,J_{-1/3}\,(\tau_b)\right], \\ \xi^{(b,u)}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(b,u)}(r;\varepsilon),\end{aligned}\right\} \tag{158}$$

with $\tau_b = \tau_{\mathrm{Rad}} - \tau$;
3. the asymptotic solutions that are uniformly valid in the convective envelope, from the upper boundary of the radiative zone to a distance sufficiently large from the star's surface, can be expressed in the compact form

$$\left.\begin{aligned}\alpha^{(c,u)}(r;\varepsilon) &= B_{0,c}\,\frac{\sqrt{3}}{\sqrt{2}\,K'^{1/4}_{1,c}\,K'_{5,c}}\,K'_5(r)\,\tau_e^{1/2}\,K_{1/3}\,(\tau_e), \\ \xi^{(c,u)}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(c,u)}(r;\varepsilon);\end{aligned}\right\} \tag{159}$$

4. the asymptotic solutions that are uniformly valid in the convective envelope, from the star's surface to a distance sufficiently large distance from the lower boundary of the envelope, can be expressed in the compact form

$$\left.\begin{aligned}\alpha^{(s,u)}(r;\varepsilon) &= B_{0,c} \\ &\quad \times\frac{\sqrt{3}\,\pi\,\exp\left[-\tau_e(R)\right]}{\sqrt{2}\,K'^{1/4}_{1,c}\,K'_{5,c}}\,K'_5(r)\,\tau_s^{1/2}\,I_{n_e+1}\,(\tau_s), \\ \xi^{(s,u)}(r;\varepsilon) &= \frac{c^2(r)}{g(r)}\,\alpha^{(s,u)}(r;\varepsilon)\Bigg\{1 \\ &\quad +2\,\frac{N_s^2}{g_s}\,\frac{1}{\tau_s^2}\left[\frac{\mathrm{d}\ln I_{n_e+1}\,(\tau_s)}{\mathrm{d}\ln\tau_s} + (n_e+1)\right]\Bigg\},\end{aligned}\right\} \tag{160}$$

with $\tau_s = \tau_e(R) - \tau_e$.

The orders in ε of the eigenfunctions $\alpha(r)$, $\xi(r)$, and $\eta(r)$ in various regions of the star are presented in Table 2. The orders in ε of the eigenfunctions $P'(r)$ and $\Phi'(r)$ in the corresponding regions can be determined in the same way as for a star with a radiative envelope.

Table 2. The orders in ε of the divergence, the radial component, and the transverse component of the Lagrangian displacement for different regions in a star composed of a convective core in adiabatic equilibrium, a radiative zone, and a convective envelope.

	Conv. core	Rad. zone	Boundary	Conv. env.	Surface
$\alpha(r)$	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-(n_e+3/2)}$
$\xi(r)$	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-1/6}$	ε^{0}	$\varepsilon^{-(n_e-1/2)}$
$\eta(r)$	$\varepsilon^{-1/6}$	ε^{-1}	$\varepsilon^{-5/6}$	ε^{-1}	$\varepsilon^{-(n_e+3/2)}$

4.6. Identification of the radial order of a g^+-mode

From the uniformly valid asymptotic solutions given by Eqs. (94) and (158), it results that the nodes of the asymptotic solutions for $\alpha(r)$ and $\xi(r)$ coincide in the intermediate radiative zone, so that these solutions have the same number of nodes in the zone considered. The approximate positions of the nodes are still given by Eq. (101).

In order to determine the approximate position of the last node, we transform asymptotic solution $\xi^{(o)}(r;\varepsilon)$ by means of Eqs. (145). The solution then becomes

$$\xi^{(o)}(r;\varepsilon) = B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\,\frac{c^2(r)}{g(r)}\,K_5(r)$$
$$\times\left[K^{1/6}\,\sin\left(\tau_b+\frac{\pi}{12}\right)+K^{-1/6}\,\cos\left(\tau_b-\frac{\pi}{12}\right)\right]\cdot \quad (161)$$

When the slope of N^2 is continuous at the upper boundary of the intermediate radiative zone, $K = 1$, and the asymptotic solution reduces to

$$\xi^{(o)}(r;\varepsilon) = B_{0,c}\,\frac{\sqrt{\pi}}{K_{1,b}^{1/4}\,K_{5,b}}\,\frac{c^2(r)}{g(r)}\,K_5(r)$$
$$\times 2\,\cos\frac{\pi}{6}\,\cos\left(\tau_b-\frac{\pi}{4}\right) \quad (162)$$

and has nodes at

$$\tau_b^0 = \left(2\,k-\frac{1}{2}\right)\frac{\pi}{2},\quad k = 1,2,3,\ldots \quad (163)$$

The node associated with $k = 1$ is located at $\tau_b^0 = 3\,\pi/4$ and is related to the first node of the uniformly valid solutions $\alpha^{(b,u)}(r;\varepsilon)$ and $\xi^{(b,u)}(r;\varepsilon)$ counted from the boundary between the radiative zone and the convective envelope. The position of the last node of the asymptotic solutions for $\alpha(r)$ and $\xi(r)$ counted from the boundary between the convective core and the radiative zone is then, in terms of the fast variable $\tau(r)$,

$$\tau^0(\text{last}) = \tau_{\text{Rad}} - \tau_b^0(\text{first}) = \left[2(n-1)-\frac{5}{6}\right]\frac{\pi}{2}.$$

In accordance with Eq. (101), the last node is associated with $j = n-1$. Consequently, both the divergence and the radial component of the Lagrangian displacement display $n-1$ nodes in the intermediate radiative zone.

The fact that the divergence and the radial component of the Lagrangian displacement have opposite signs at $r = R$ indicates that at least one additional node appears in one of the asymptotic solutions valid in the convective envelope. From the asymptotic solutions $\alpha^{(s,u)}(r;\varepsilon)$ and $\xi^{(s,u)}(r;\varepsilon)$, which are uniformly valid from the surface to a distance sufficiently large from the boundary between the radiative zone and the convective envelope, it

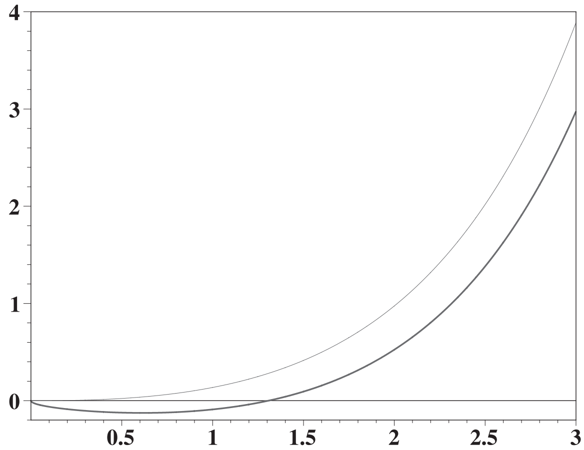

Fig. 3. The functions $H_4(\tau_s)$ (thin line) and $H_5(\tau_s)$ (thick line) for $n_e = 1$ and $\Gamma_1 = 5/3$.

results that the radial component of the Lagrangian displacement displays an additional node in the convective envelope at the point at which

$$1+2\,\frac{N_s^2}{g_s}\,\frac{1}{\tau_s^2}\left[\frac{\mathrm{d}\ln I_{n_e+1}(\tau_s)}{\mathrm{d}\ln\tau_s}+(n_e+1)\right] = 0.$$

In illustration, the parts of the uniformly valid asymptotic solutions $\alpha^{(s,u)}(r;\varepsilon)$ and $\xi^{(s,u)}(r;\varepsilon)$ that depend on the fast variable τ_s, respectively

$$H_4(\tau_s) = \tau_s^{1/2}\,I_2(\tau_s)$$

and

$$H_5(\tau_s) = H_4(\tau_s)\left\{1+2\,\frac{N_s^2}{g_s}\,\frac{1}{\tau_s^2}\left[\frac{\mathrm{d}\ln I_2(\tau_s)}{\mathrm{d}\ln\tau_s}+2\right]\right\},$$

are represented in Fig. 3 for $n_e = 1$ and $\Gamma_1 = 5/3$.

5. Concluding remarks

We have developed a first-order asymptotic theory for low-degree, higher-order g^+-modes in stars with a convective core. We considered stars containing a radiative envelope as well as stars containing an intermediate radiative zone and a convective envelope. In both cases, we regarded the convective core to be in adiabatic equilibrium.

We started from the fourth-order system of differential equations in the divergence $\alpha(r)$ and the radial component $\xi(r)$ of the Lagrangian displacement that stems from Pekeris (1938). To this system, we applied two-variable expansion procedures in the central parts of the radiative and the convective regions, and boundary-layer theory near the boundary and the turning points. The two methods are commonly presented for single second-order differential equations, but we extended their application to the fourth-order system of differential equations considered. In contrast with Willems et al. (1997), we no longer adopted boundary-layer variables that correspond to the fast variable used at larger distances from the boundary or singular point.

At the boundaries between a radiative and a convective region, we imposed the continuity of the divergence, the radial component, and the first derivative of the radial component of the Lagrangian displacement, in order to ensure that the Lagrangian displacement and the Lagrangian perturbation of pressure be continuous there.

The system of differential equations in the divergence $\alpha(r)$ and the radial component $\xi(r)$ of the Lagrangian displacement is particularly appropriate for the development of the asymptotic theory, since, in the various regions of a star, the lowest-order asymptotic approximation of $\alpha(r)$ is solution of a *homogeneous* second-order differential equation. The lowest-order asymptotic approximation of $\xi(r)$, is subsequently obtained as solution of an *inhomogeneous* second-order differential equation, which involves the lowest-order asymptotic approximation of $\alpha(r)$ in its inhomogeneous part. Consequently, the divergence of the Lagrangian displacement plays a basic role in the development of the asymptotic theory, as it does in the asymptotic theory for higher-order p-modes.

In the central parts of the radiative zones, differential Eq. (3) is formally comparable with a second-order differential equation that governs a linear oscillator of constant frequency and small damping. Consequently, in these zones, the eigenfunctions $\alpha(r)$ and $\xi(r)$ are given by trigonometric functions of a fast radial variable, whose amplitudes are slowly modulated in terms of the radial coordinate r.

Our asymptotic approach shows that different eigenfunctions may be of different orders in the small expansion parameter ε in a given region of a star, and that the order of an eigenfunction may vary from one region to another.

The eigenfrequency equations are given by Eqs. (87) and (150). We have verified in two cases that the number n corresponds to the radial order of the g^+-mode: in Eq. (87) when $n_e = 3$, and in Eq. (150) when the slope of N^2 is continuous at the boundary between the intermediate radiative zone and the convective envelope.

In accordance with Cowling's classification of non-radial oscillations of stars, the asymptotic eigenfunction for $\xi(r)$ of a g^+-mode of radial order n displays n nodes between $r = 0$ and $r = R$. On its side, the asymptotic eigenfunction $\alpha(r)$ displays one node less. In stars with a radiative envelope, the nodes are situated in the radiative envelope, while in stars with a convective envelope, they are situated in the intermediate radiative zone, apart from the last node of $\xi(r)$, which is situated in the convective envelope.

For a star containing a convective envelope, the number κ that appears in Eq. (A.13) of Tassoul (1980) corresponds to the number of nodes displayed by the asymptotic eigenfunction $\xi(r)$ in the intermediate radiative zone, and not to the total number of nodes displayed by that function between $r = 0$ and $r = R$. Therefore, the number κ corresponds to the radial order of the g^+-mode minus one.

In the whole star, with the exception of the small boundary layer near the surface, the asymptotic eigenfunction $\xi(r)$ is related to the asymptotic eigenfunction $\alpha(r)$ as $\xi(r) = \left[c^2(r)/g(r)\right]\alpha(r)$. This relation implies that the Eulerian perturbation of the pressure is identically zero, that the asymptotic eigenfunctions $\alpha(r)$ and $\xi(r)$ have the same sign, and that the first $n-1$ nodes of the asymptotic eigenfunction $\xi(r)$ coincide with the $n-1$ nodes of the asymptotic eigenfunction $\alpha(r)$. Since the two eigenfunctions have the same sign, the mass elements are subject to a dilation $[\alpha(r) > 0]$ in regions in which $\xi(r) > 0$, and to a contraction $[\alpha(r) < 0]$ in regions in which $\xi(r) < 0$.

Near the surface, the eigenfunctions $\alpha(r)$ and $\xi(r)$ have opposite signs, as follows from Eqs. (89) and (152). Hence, there the ratio of the Eulerian perturbation of pressure to the mass density is different from zero, and mass elements are subject to a dilation when $\xi(r) < 0$, and to a contraction when $\xi(r) > 0$. The dilations and the contractions of the mass elements are now dominated by the transverse components of the Lagrangian displacement. In this connection, it may be observed that, near the surface, the ratio $\eta(r)/\xi(r)$ is of the order of ε^{-2}, as it appears from Tables 1 and 2.

Acknowledgements. One of the authors, A.M., acknowledges the support of the Direccion General de Investigacion under project ESP2004-03855-C03-01. He also acknowledges a recent stay of two years at the Observatoire de Paris-Meudon, and especially the hospitality offered to him during this stay. The authors thank an anonymous referee for his careful reading of the manuscript and for his helpful comments.

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