A&A 463, 1211–1214 (2007)
DOI: 10.1051/0004-6361:20065271

Astronomy & Astrophysics

On cross-spectrum capabilities for detecting stellar oscillation modes *(Research Note)*

T. Appourchaux[1], J. Leibacher[1,2], and P. Boumier[1]

[1] Institut d'Astrophysique Spatiale, Université Paris XI–CNRS, Bâtiment 121, 91405 Orsay Cedex, France
e-mail: [Thierry.Appourchaux;Patrick.Boumier]@ias.u-psud.fr
[2] National Solar Observatory, Tucson, Arizona, USA
e-mail: John.Leibacher@ias.u-psud.fr

Received 24 March 2006 / Accepted 30 November 2006

ABSTRACT

Aims. Long-lived stellar oscillation modes are usually detected using Fourier transforms of time series of stellar radial velocities or brightness. It is commonly thought that one could use the cross spectrum of the two signals, or alternatively use the interleaved series of a single signal, to considerably improve the detection level by reducing the noise level.
Methods. We use a statistical analysis of the cross spectrum to compute its mean value and rms value, and use the associated signal-to-noise ratio for stochastically excited modes.
Results. Here, we demonstrate that the gain in the signal-to-noise ratio can indeed be improved for modes with a shorter lifetime than the observation time, but not those with a longer lifetime than the observation time.

Key words. methods: statistical – methods: analytical – Sun: helioseismology

1. Introduction

When measuring solar or stellar oscillation modes, the Fourier transform is commonly used for detecting long, as well as short-lived, p modes. It is sometimes useful to use independent measurements in order to increase the signal-to-noise ratio of the spectrum. For instance, measuring the solar radial velocity from two independent points would provide a $\sqrt{2}$ improvement in the signal-to-noise ratio in radial velocity measurements. Different parameters whose underlying signal is strongly correlated offer a similar, but reduced, gain in sensitivity for uncorrelated noise. It has been reported by García et al. (1999) that the use of the cross spectrum between interleaved data could significantly improve the signal-to-noise ratio, thereby decreasing the detection threshold of p modes. Such a technique has been applied to the data from the GOLF instrument aboard the SOHO spacecraft (Gabriel et al. 1995), providing an improvement as shown by García et al. (2004).

We show that the gain for the interleaved cross spectrum, although effective for short-lived modes (shorter than the observing time), is not effective for long-lived modes (longer than the observing time).

In the first section, we give mathematical preliminaries that are useful for understanding the statistics of *any* correlated signals. The second section uses what is known of solar p-mode spectra to explain how one gets improvements using the cross spectrum in the signal-to-noise ratio of short-lived modes.

2. Preliminaries

Here we show how one can derive the first two moments of the cross spectrum using a simple calculation based on the simple construction of two correlated random variables. In the formulation we have adopted, the interleaving of the time series has been left out since all the details are reduced to the coherence between two Fourier spectra. We assume that the Fourier spectra S_a, S_b can be described by the random variables

$$\begin{aligned} S_a &= X + \alpha Y, \\ S_b &= Y + \alpha X, \end{aligned} \tag{1}$$

where X and Y are given by

$$\begin{aligned} X &= (x_r, x_i), \\ Y &= (y_r, y_i), \end{aligned} \tag{2}$$

where x_r, y_r, x_i, and y_i are independent random variables, normally distributed, of mean 0 and rms value σ. Here any frequency dependence is explicitly left out. The rms values σ are the same for both the interleaved signals.

With this definition of the spectra, the coherence between the spectra is given by

$$\rho = \frac{E(S_a S_b^*)}{\sqrt{E(S_a S_a^*) E(S_b S_b^*)}} = \frac{2\alpha}{(1+\alpha^2)}, \tag{3}$$

where E is the expectation, and the * denotes the complex conjugate. We construct two Fourier spectra whose independence is specified by the parameter α. For $\alpha = 0$, the coherence is 0. For $\alpha = 1$, the coherence is 1.

3. Mean and rms value of the spectra

3.1. Cross spectrum

The cross spectrum is defined as

$$\mathrm{CS} = \mathrm{Real}(S_a S_b^*), \tag{4}$$

and thus the mean of the cross spectrum is

$$E(\mathrm{CS}) = 4\alpha\sigma^2, \tag{5}$$

and the rms value given by

$$\sigma_{\mathrm{CS}} = \sqrt{8\alpha^2 + 2(1+\alpha^2)^2}\sigma^2. \tag{6}$$

The appendix explains how this is derived.

When α is zero (independent spectra) the mean is zero, while the rms value is $\sqrt{2}\sigma^2$. If $\alpha = 1$, then the mean and the variance are the same, and in that case we have a power spectrum.

3.2. Mean spectrum

The mean spectrum is defined as

$$\mathrm{MS} = \frac{S_a S_a^* + S_b S_b^*}{2}. \tag{7}$$

With the assumptions given above, one can deduce the following for the mean and the rms value of the mean spectrum

$$E(\mathrm{MS}) = 2(1+\alpha^2)\sigma^2, \tag{8}$$

and the rms value is given by

$$\sigma_{\mathrm{MS}} = \sqrt{8\alpha^2 + 2(1+\alpha^2)^2}\sigma^2. \tag{9}$$

The appendix explains how this is derived.

When α is zero (independent spectra), the mean is $2\sigma^2$, while the rms value is $\sqrt{2}\sigma^2$. In that case, the mean spectrum has a χ^2 with 4 degrees-of-freedom (d.o.f.) statistics, which is to be compared with the spectrum of the original non-interleaved time series that has a χ^2 with 2 d.o.f. If $\alpha = 1$, then the mean and the variance are the same, and in that case we have a power spectrum.

3.3. Comparison

From the above

$$\frac{\sigma_{\mathrm{MS}}}{E(\mathrm{MS})} = \sqrt{\frac{1+\rho^2}{2}}. \tag{10}$$

As expected for two independent signals, the improvement is $\sqrt{2}$. We also have

$$\sigma_{\mathrm{CS}} = \sigma_{\mathrm{MS}}, \tag{11}$$

and

$$\frac{E(\mathrm{CS})}{E(\mathrm{MS})} = \rho. \tag{12}$$

From Eq. (12), we show that if the interleaved signals have a low coherence then indeed the *mean* of the cross spectrum will be lower than that of the power spectrum. On the other hand, since the statistical fluctuations over that *mean* are measured by σ_{CS}, Eq. (11) shows that the fluctuations have the same amplitude in either the cross spectrum or the power spectrum.

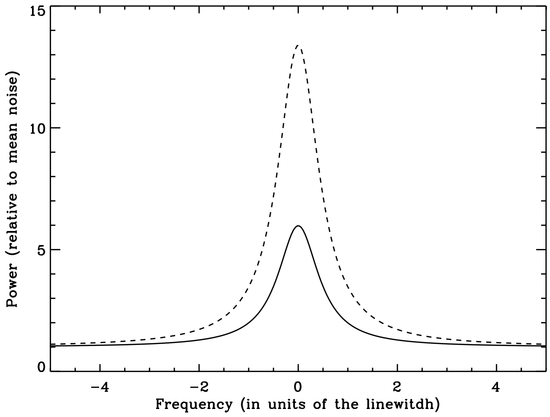

Fig. 1. Power spectrum normalized to the noise as a function of frequency for the original p-mode spectrum with a signal-to-noise ratio of 5 (solid line) and for the cross-spectrum with a coherence of the noise of 0.4 (dashed line). The mode profile is Lorentzian and symmetrical. The coherence and the rms noise are assumed to be constant.

4. Signal-to-noise ratio and coherence

Here we show that the improvement in signal-to-noise ratio as mentioned by García et al. (1999) is directly related to the variation in the coherence with frequency, e.g. across the profile of mode.

Using the work of García et al. (1999), we can make a simple mathematical model of the cross spectrum. We assume that given the coherence measured by García et al. (1999), the cross spectrum can be modeled as the sum of fully coherent part and of a partially coherent part

$$E(\mathrm{CS}) = A(\nu) + B(\nu)\rho_n(\nu), \tag{13}$$

where $A(\nu)$ represents the fully coherent part of the spectrum that includes the modes and the part of the noise correlated with the modes as introduced by Severino et al. (2001), and $B(\nu)$ represents the part of the noise partially correlated with a correlation of $\rho_n(\nu)$. The profile as demonstrated by Eq. (12) is similar to that of a regular mode profile, i.e. signal plus noise. With our description, the power spectrum can be modeled simply as

$$E(\mathrm{MS}) = A(\nu) + B(\nu). \tag{14}$$

The coherence of the noise with itself is one by definition. Figure 1 shows how the profile of the mode is improved (in a situation similar to that in Fig. 3 of García et al. 1999) for a case in which the signal-to-noise ratio is improved by a factor 2.5 (reciprocal of 0.4).

The impact of the signal-to-noise ratio ($SN = A(\nu)/B(\nu)$) on the coherence can be computed easily from our definition resulting in Eq. (12)

$$\rho = \frac{SN(\nu) + \rho_n(\nu)}{SN(\nu) + 1}. \tag{15}$$

Figure 2 shows how the coherence varies across a mode for a typical low signal-to-noise ratio. Equation (13) shows that the signal-to-noise ratio is improved as the inverse of the coherence of the noise $\rho_n(\nu)$; i.e., the lower the coherence the higher the signal-to-noise ratio.

5. Conclusion

There are two main conclusions. First as shown by García et al. (1999), the use of the cross spectrum for interleaved time-shifted

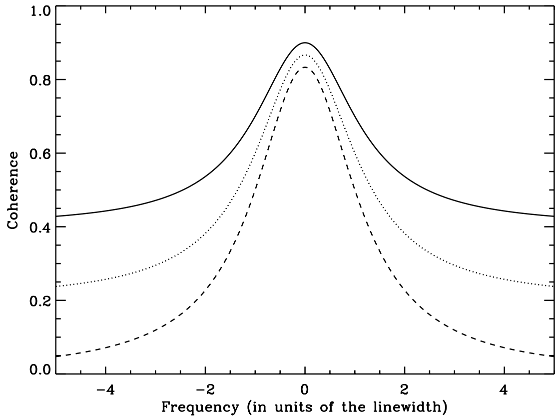

Fig. 2. Coherence, (as derived from Eq. (15)), as a function of frequency for a mode with a maximum signal-to-noise ratio of 5 for various values of the noise coherence: 0.4 (solid line) (same case as Fig. 1), 0.2 (dotted line), 0. (dashed line). The mode profile is Lorentzian and symmetrical. The coherence is assumed to be independent of frequency.

time series increases the signal-to-noise ratio for modes for which the lifetime of the mode is shorter than the observation time. The improvement in the mean signal-to-noise ratio is possible because an estimate of the mean mode profile is obtained when the mode is resolved by observations that are much longer than the mode lifetime. This improvement only applies to the mean spectrum.

Second, when the mode lifetime is longer than the observation time, the modes are unresolved and restricted to a single frequency bin. In that case, the average power in the mode is not available because, since the mode lifetime is not known, one single observation is linked to a single realization of the noise as well; there is no way to have several realizations of the noise. The decrease in the mean value of the cross spectrum does not improve the detection level when the mode lifetime is longer than the observation time, becasue this is the decrease in the rms value that is needed when one wants to detect faint signals above the mean value, i.e. $\sigma_{\rm CS}$. *The rms value of the cross spectrum and the mean spectrum being strictly identical, there is no gain in using the cross spectrum for detecting long-lived modes*.

Acknowledgements. This note was initiated by several discussions with Rafael García related to his work. We are grateful to the referee for comments that helped clarify the message of the note.

Appendix A: Derivation of Eqs. (6) and (9) of the Research note

A.1. Preliminaries

We assume that the Fourier spectra S_a, S_b can be described by the following random variables

$$S_a = X + \alpha Y$$
$$S_b = Y + \alpha X \quad \text{(A.1)}$$

where X and Y are given by:

$$X = (x_r, x_i)$$
$$Y = (y_r, y_i) \quad \text{(A.2)}$$

where x_r, y_r, x_i, and y_i are independent random variables, normally distributed with mean 0 and rms value σ. Here any frequency dependence is explicitly left out. The rms values σ are the same for both signals.

With this definition of the spectra, the coherence between them is given by

$$\rho = \frac{E(S_a S_b^*)}{\sqrt{E(S_a S_a^*)E(S_b S_b^*)}} = \frac{2\alpha}{(1+\alpha^2)}, \quad \text{(A.3)}$$

where E is the expectation, and the * denotes the complex conjugate. We construct two Fourier spectra whose independence is specified by the parameter α. For $\alpha = 0$, the coherence is 0. For $\alpha = 1$, the coherence is 1.

A.2. Mean spectrum

The mean spectrum is defined as follows

$$\mathrm{MS} = \frac{S_a S_a^* + S_b S_b^*}{2}. \quad \text{(A.4)}$$

A.2.1. Mean of the mean spectrum

This is defined as

$$E(\mathrm{MS}) = \frac{1}{2} E\left(S_a S_a^* + S_b S_b^*\right). \quad \text{(A.5)}$$

Replacing (A.1) in (A.5) we get

$$E(\mathrm{MS}) = \frac{1}{2} E\left((X+\alpha Y)(X+\alpha Y)^* + (\alpha X + Y)(\alpha X + Y)^*\right), \quad \text{(A.6)}$$

which we can rewrite as

$$E(\mathrm{MS}) = \frac{(1+\alpha^2)}{2}\left[(E(XX^*) + E(YY^*)\right] + \alpha\left[E(XY^*) + E(YX^*)\right]. \quad \text{(A.7)}$$

We only need to compute the first two terms because the latter are zero (independent variables). We then obtain

$$E(XX^*) = E(YY^*) = E(x_r^2) + E(x_i^2), \quad \text{(A.8)}$$

$$E(XX^*) = 2\sigma^2. \quad \text{(A.9)}$$

Replacing (A.8) and (A.9) in (A.7) we finally have

$$E(\mathrm{MS}) = 2\sigma^2(1+\alpha^2). \quad \text{(A.10)}$$

A.2.2. Rms value of the mean spectrum

The rms value is defined as

$$\sigma_{\rm MS}^2 = E(\mathrm{MS}^2) - E(\mathrm{MS})^2. \quad \text{(A.11)}$$

The latter term has already been calculated. We need to calculate the first term as

$$E(\mathrm{MS}^2) = \frac{1}{4}\left[E((S_a S_a^* + S_b S_b^*)^2)\right], \quad \text{(A.12)}$$

that can be rewritten as

$$E(\mathrm{MS}^2) = \frac{1}{4}\Big[E\Big(\left[(X+\alpha Y)(X^* + \alpha Y^*) + (Y+\alpha X)(Y^* + \alpha X^*)\right]^2\Big)\Big], \quad \text{(A.13)}$$

or simply

$$E(\mathrm{MS}^2) = \frac{1}{4}\Big[E\Big(\Big[(XX^* + YY^*)(1+\alpha^2) + 2\alpha(YX^* + XY^*)\Big]^2\Big)\Big]. \quad \text{(A.14)}$$

We can leave out all the cross terms having a single X or Y since their mean value is zero. We are then left with

$$E(\mathrm{MS}^2) = \frac{(1+\alpha^2)^2}{4}\left[E((XX^* + YY^*)^2)\right] + \alpha^2\left[E((XY^* + YX^*)^2)\right]. \quad (A.15)$$

The first term of Eq. (A.15) can be calculated as follows

$$E((XX^* + YY^*)^2) = E((XX^*)^2) + E((YY^*)^2) + 2E(XX^*)E(YY^*). \quad (A.16)$$

The first two terms of Eq. (A.16) can be derived as

$$E((XX^*)^2) = E((YY^*)^2) = E((x_r^2 + y_r^2)^2), \quad (A.17)$$

$$E((x_r^2 + y_r^2)^2) = E((x_r^4) + E((y_r^4) + 2E((x_r^2)E((x_r^2). \quad (A.18)$$

Here we recall that $E(x_r^4) = E(y_r^4) = 3\sigma^4$, then we have

$$E((x_r^2 + y_r^2)^2) = 3\sigma^4 + 3\sigma^4 + 2\sigma^4 = 8\sigma^4. \quad (A.19)$$

Using Eqs. (A.19) and (A.9) we obtain

$$E((XX^* + YY^*)^2) = 8\sigma^4 + 8\sigma^4 + 8\sigma^4 = 24\sigma^4. \quad (A.20)$$

The third term of Eq. (A.15) can the be calculated as follows

$$E((XY^* + YX^*)^2) = 4E(x_r y_r + x_i y_i)^2). \quad (A.21)$$

Ignoring again the cross terms leading to zero contribution, we have

$$E((XY^* + YX^*)^2) = 4E(x_r^2)E(y_r^2) + 4E(x_i^2)E(y_i^2), \quad (A.22)$$

finally

$$E((XY^* + YX^*)^2) = 8\sigma^4. \quad (A.23)$$

Now replacing Eqs. (A.20) and (A.23) in Eq. (A.15), we finally get

$$E(\mathrm{MS}^2) = 6(1+\alpha^2)^2\sigma^4 + 8\alpha^2\sigma^4. \quad (A.24)$$

Using Eqs. (A.10) and (A.24), we arrive at the solution of the RN

$$\sigma_{\mathrm{MS}} = \sqrt{2(1+\alpha^2)^2 + 8\alpha^2}\sigma^2. \quad (A.25)$$

A.3. Cross spectrum

The cross spectrum is defined as follows

$$\mathrm{CS} = \mathrm{Real}(S_a S_b^*). \quad (A.26)$$

A.3.1. Mean of the cross spectrum

This can easily be calculated as

$$E(\mathrm{CS}) = (1+\alpha)\mathrm{Real}(\mathrm{E}(XY^*)) + \alpha(E(XX^*) + E(YY^*)). \quad (A.27)$$

The first term is zero, we are then left using Eq. (A.9) with

$$E(\mathrm{CS}) = 4\alpha\sigma^2. \quad (A.28)$$

A.3.2. Rms value of the cross spectrum

Here we write directly, the following

$$\mathrm{CS} = (x_r + \alpha y_r)(y_r + \alpha x_r) + (x_i + \alpha y_i)(y_i + \alpha x_i). \quad (A.29)$$

The rms value is calculated from

$$E(\mathrm{CS}^2) = E([(x_r+\alpha y_r)(y_r+\alpha x_r) + (x_i+\alpha y_i)(y_i+\alpha x_i)]^2). \quad (A.30)$$

This can be expanded as follows

$$E(\mathrm{CS}^2) = E([(x_r + \alpha y_r)(y_r + \alpha x_r)]^2) + E([(x_i + \alpha y_i)(y_i + \alpha x_i)]^2) + 2E((x_r + \alpha y_r)(y_r + \alpha x_r)(x_i + \alpha y_i)(y_i + \alpha x_i)). \quad (A.31)$$

The first term and second terms are identical. Ignoring the cross terms which do not contribute, we have

$$E([(x_r+\alpha y_r)(y_r+\alpha x_r)]^2) = \alpha^2 E(x_r^4) + \alpha^2 E(y_r^4) + \left[2\alpha^2 + (1+\alpha^2)^2\right] E(x_r^2 y_r^2), \quad (A.32)$$

$$E([(x_r + \alpha y_r)(y_r + \alpha x_r)]^2) = 3\alpha^2\sigma^4 + 3\alpha^2\sigma^4 + \left[2\alpha^2 + (1+\alpha^2)^2\right]\sigma^4, \quad (A.33)$$

$$E([(x_r + \alpha y_r)(y_r + \alpha x_r)]^2) = (8\alpha^2 + (1+\alpha^2)^2)\sigma^4. \quad (A.34)$$

The last term of Eq. (A.30) can be calculated as follows

$$E((x_r + \alpha y_r)(y_r + \alpha x_r)(x_i + \alpha y_i)(y_i + \alpha x_i)) = E((x_r + \alpha y_r)(y_r + \alpha x_r))E((x_i + \alpha y_i)(y_i + \alpha x_i)). \quad (A.35)$$

The two terms of the product of the right hand side of Eq. (35) provide the same results, we have

$$E((x_r + \alpha y_r)(y_r + \alpha x_r)) = \alpha(E(x_r^2) + E(y_r^2)), \quad (A.36)$$

$$E((x_r + \alpha y_r)(y_r + \alpha x_r)) = 2\alpha\sigma^2. \quad (A.37)$$

Equation (A.35) can then be rewritten as

$$E((x_r + \alpha y_r)(y_r + \alpha x_r)(x_i + \alpha y_i)(y_i + \alpha x_i)) = 4\alpha^2\sigma^4. \quad (A.38)$$

Using Eqs. (A.34) and (A.38) in Eq. (A.31) we finally have

$$E(\mathrm{CS}^2) = 2(8\alpha^2 + (1+\alpha^2)^2)\sigma^4 + 8\alpha^2\sigma^4, \quad (A.39)$$

or simply

$$E(\mathrm{CS}^2) = (24\alpha^2 + 2(1+\alpha^2)^2)\sigma^4. \quad (A.40)$$

Using Eqs. (A.28) and (A.40) we arrive at

$$\sigma_{\mathrm{CS}} = \sqrt{2(1+\alpha^2)^2 + 8\alpha^2}\sigma^2. \quad (A.41)$$

A.4. Conclusion

As shown in the RN, we then have $\sigma_{\mathrm{MS}} = \sigma_{\mathrm{CS}}$.

References

Gabriel, A. H., Grec, G., Charra, J., et al. 1995, Sol. Phys., 162, 61
García, R. A., Jefferies, S. M., Toner, C. G., & Pallé, P. L. 1999, A&A, 346, L61
García, R. A., Jiménez-Reyes, S. J., Turck-Chièze, S., & Mathur, S. 2004, in SOHO 14 Helio- and Asteroseismology: Towards a Golden Future, ed. D. Danesy, ESA SP-559, 432
Severino, G., Magrì, M., Oliviero, M., Straus, T., & Jefferies, S. M. 2001, ApJ, 561, 444