A&A 459, 981–985 (2006)
DOI: 10.1051/0004-6361:20065897

Astronomy & Astrophysics

Precession-nutation procedures consistent with IAU 2006 resolutions*

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Received 24 June 2006 / Accepted 28 August 2006

ABSTRACT

Context. The 2006 IAU General Assembly has adopted the P03 model of Capitaine et al. (2003a) recommended by the WG on precession and the ecliptic (Hilton et al. 2006) to replace the IAU 2000 model, which comprised the Lieske et al. (1977) model with adjusted rates. Practical implementations of this new "IAU 2006" model are therefore required, involving choices of procedures and algorithms.

Aims. The purpose of this paper is to recommend IAU 2006 based precession-nutation computing procedures, suitable for different classes of application and achieving high standards of consistency.

Methods. We discuss IAU 2006 based procedures and algorithms for generating the rotation matrices that transform celestial to terrestrial coordinates, taking into account frame bias (B), P03 precession (P), P03-adjusted IAU 2000A nutation (N) and Earth rotation. The NPB portion can refer either to the equinox or to the celestial intermediate origin (CIO), requiring either the Greenwich sidereal time (GST) or the Earth rotation angle (ERA) as the measure of Earth rotation. Where GST is used, it is derived from ERA and the equation of the origins (EO) rather than through an explicit formula as in the past, and the EO itself is derived from the CIO locator.

Results. We provide precession-nutation procedures for two different classes of full-accuracy application, namely (i) the construction of algorithm collections such as the Standards Of Fundamental Astronomy (SOFA) library and (ii) IERS Conventions, and in addition some concise procedures for applications where the highest accuracy is not a requirement. The appendix contains a fully worked numerical example, to aid implementors and to illustrate the consistency of the two full-accuracy procedures which, for the test date, agree to better than 1 μas.

Conclusions. The paper recommends, for case (i), procedures based on angles to represent the PB and N components and, for case (ii), procedures based on series for the CIP X, Y. The two methods are of similar efficiency, and both support equinox based as well as CIO based applications.

Key words. astrometry – reference systems – ephemerides – celestial mechanics – time

1. Introduction

Resolution 1 of the XXVIth General Assembly of the International Astronomical Union (Prague 2006) adopted the recommendation of the Working Group on precession and the ecliptic (Hilton et al. 2006) to introduce from 2009 the "P03" models of Capitaine et al. (2003a). P03, that we will here refer to as "IAU 2006 precession", supersedes the IAU 2000 precession. The latter comprised the precession of Lieske et al. (1977) with simple rate corrections (Mathews et al. 2002); the IAU 2006 models provide a replacement that is consistent with dynamical theory.

In this paper we provide detailed and efficient procedures, including numerical examples, for implementing the IAU 2006 precession in applications that, as is normally the case, also involve nutation and Earth rotation. The recommended procedures offer different blends of canonical rigor, internal consistency, flexibility and ease of use. The choices address a range of performance needs, from cases where low accuracy (~1 arcsec) is sufficient to applications requiring sub-microarcsecond precision, and support the use of both the equinox and the celestial intermediate origin (CIO).

The IAU resolution adopting the P03 precession does not stipulate a specific parameterization, expressly stating that the user makes this choice. The starting point for the procedures recommended by the present paper was the wide range of options discussed in Capitaine & Wallace (2006), denoted C06 in the following. That paper identifies a variety of different approaches, including six ways of forming the bias-precession-nutation (NPB) matrix, three ways of generating the position of the celestial intermediate pole (CIP) and eight ways of locating the celestial intermediate origin (CIO). By examining the needs of two important but representative applications, namely the SOFA software (Wallace 1998) and the IERS Conventions (2003), we have developed two procedures that can be recommended for general use and that achieve high standards of consistency, both internal and mutual.

In this paper we use the basic precession angles only for numerical comparison purposes, and the two procedures recommended for practical use start with (different) derived quantities.

* Appendix is only available in electronic form at http://www.aanda.org

The two procedures aim in particular to provide the 3 × 3 matrices $\mathbf{M}_{\rm CIO}$ and $\mathbf{R}$ in the expressions:

$$\begin{aligned} \boldsymbol{v}_{\rm TIRS} &= \mathbf{R}(\rm TT, UT) \cdot \boldsymbol{v}_{\rm GCRS}, \\ \boldsymbol{v}_{\rm CIRS} &= \mathbf{M}_{\rm CIO}(\rm TT) \cdot \boldsymbol{v}_{\rm GCRS}, \end{aligned} \quad (1)$$

where the 3-vector $\boldsymbol{v}_{\rm GCRS}$ is a direction in the GCRS and $\boldsymbol{v}_{\rm CIRS}$ and $\boldsymbol{v}_{\rm TIRS}$ are the same direction with respect to the celestial intermediate and terrestrial intermediate reference systems.

Also of interest is $\mathbf{M}_{\rm class}$, the classical (i.e. equinox based) counterpart to $\mathbf{M}_{\rm CIO}$. These matrices are related to each other and to the GCRS-to-TIRS matrix $\mathbf{R}$ by the expressions (discussed in C06, Sect. 4.2):

$$\begin{aligned} \mathbf{R} &= R_3(\rm ERA) \cdot \mathbf{M}_{\rm CIO} \\ &= R_3(\rm ERA) \cdot R_3(-\rm EO) \cdot \mathbf{M}_{\rm class} \\ &= R_3(\rm GST) \cdot \mathbf{M}_{\rm class}, \end{aligned} \quad (2)$$

where ERA is the Earth rotation angle, GST is the Greenwich sidereal time and EO = ERA−GST is the equation of the origins.

The recommendations made in this paper differ from past procedures in the sense that in both approaches (equinox or CIO based) the fundamental expression for measuring Earth rotation is the ERA and the quantity s is used to locate the CIO on the equator of the celestial intermediate pole.

The Earth rotation angle has a conventional linear relationship with UT1 (see Capitaine et al. 2000):

$$\begin{aligned} \theta(T_u) = 2\pi\,(& 0.7790572732640 \\ & + 1.00273781191135448\, T_u), \end{aligned} \quad (3)$$

where T_u = (Julian UT1 date − 2451545.0), and UT1 = UTC + dUT1. UTC is Coordinated Universal Time and dUT1, provided by the IERS, is UT1 − UTC[1].

IERS Conventions (2003) obtains s by evaluating a series for $s + XY/2$ (which is too large to be reproduced here but is available electronically[2]), and this probably remains the best method overall: the series for $s + XY/2 + D$ (cf. C06, Sect. 4.1) is more concise, but the real performance gains are arguably outweighed by the extra complications. Once the CIP coordinates X, Y are available, s can be obtained.

The two full-accuracy procedures to be described start, respectively, with angles to represent the bias, precession and nutation components (Sect. 2) and with series that generate the CIP X, Y vector components directly (Sect. 3). Table 1 provides the number of terms of the development series used in the procedures described in Sects. 2 and 3 with a cut-off of 0.1 μas after a century. The X, Y series based procedure includes for the first time the ability to generate equinox based as well as CIO based products. This is done by introducing, as an additional basis, the same ecliptic as that used in the angles based procedure, thus ensuring a high degree of consistency and symmetry between these two hitherto quite distinct schemes. Finally, for applications where a different trade-off between speed and accuracy is required, Sect. 4 provides a selection of simplified algorithms.

[1] As with any rapidly changing angle, for high-precision purposes it is important to arrange the calculation of θ in such a way that rounding errors are minimized. This involves eliminating integer multiples of 2π at as early a stage as possible.

[2] Tables for the P03 series for X, Y and $s + XY/2$ are available at the CDS via anonymous ftp to
`cdsarc.u-strasbg.fr (130.79.128.5)` or via
`http://cdsweb.u-strasbg.fr/cgi-bin/qcat?J/A+A/459/981`,
and at `http://syrte.obspm.fr/iau2006`.

Table 1. Comparison between various series for (i) the precession-nutation angles (first four lines; see Sect. 2) and (ii) the GCRS CIP X, Y coordinates (final two lines; see Sect. 3); 0.1 μas resolution.

quantity	number of terms in					
	t^0	t	t^2	t^3	t^4	t^5
$\bar{\gamma}$	1	1	1	1	1	1
$\bar{\phi}$	1	1	1	1	1	1
$\bar{\psi}$	1321	38	1	1	1	1
ϵ_A	1038	20	1	1	1	1
X	1307	254	37	5	2	1
Y	963	278	31	6	2	1
$s + XY/2$	34	4	26	5	2	1

The appendix contains a fully worked numerical example, in sufficient detail to overcome any implementation difficulties. The material also illustrates the consistency of the two full-accuracy procedures. For the test date in 2006, the agreement is better than 1 μas. Other trials showed that even after 2 centuries the largest differences are under 10 μas (cf. C06 Fig. 6).

In all cases it must be understood that the algorithms are optimized for the current era, say 1800−2200. For more remote dates they must be used with circumspection. This is especially true of the X, Y series based procedure (Sect. 3).

2. Procedure based on bias-precession and nutation angles

For many applications, rather than developing a specialized solution it is better to make use of existing tools, in the form of a general-purpose suite of algorithms that can be combined in various ways. For such a suite, canonical rigor is an important consideration, but convenience and efficiency must also be taken into account. To achieve a balance, some use of derived quantities is acceptable, as long as they are clearly traceable to published models. Internal consistency is of paramount importance, and overall the result must be concise, modular and easily understood, as well as versatile and comprehensive. These requirements lie behind the design of general-purpose software libraries, such as that provided by the IAU Standards Of Fundamental Astronomy initiative (SOFA, Wallace 1998).

One aspect of the need for versatility is that such a library must support both the traditional equinox based methods and the newer CIO based methods. The obvious approach is simply to provide separate facilities for the two methods. However, not only would this be at odds with the need for conciseness (for a start, duplication of the nutation series, which are large, must be avoided) but numerical consistency would be hard to guarantee.

A better plan is first to identify a complete but non-redundant set of canonical models, and then to use them in different ways to achieve the full range of quantities and transformations. For example, a comprehensive suite of high-precision precession-nutation and matching Earth rotation functions can be derived from the following basic components:

1. A precession model, in this case IAU 2006.
2. A nutation model, in this case IAU 2000A with slight adjustments to make it compatible with the IAU 2006 precession.
3. A model for the quantity s, to locate the CIO on the CIP equator, that is consistent with the adopted precession-nutation model.
4. The expression for Earth rotation angle as a function of UT1.

The products that can be derived from these components include:

- Precession-nutation matrices, that rotate GCRS coordinates into either CIP/CIO coordinates or CIP/equinox coordinates.
- The equation of the origins, EO = ERA−GST, which is the distance between the CIO and equinox and provides the link between the two approaches.
- Greenwich (apparent) sidereal time, GST.

The non-redundancy in the set of canonical models naturally delivers a concise result but, more importantly, ensures consistency. Where a given application – the computation of hour angles is a prime example – can be implemented by more than one route, the answers will agree rigorously, limited only by different computational rounding errors.

In the above proposal, it is important to note what is *not* included in the list of canonical models. There is no need for a formula giving Greenwich mean sidereal time, something that has in the past always accompanied a new precession model. Nor are there series for either the equation of the equinoxes or the equation of the origins. All of these quantities could, if necessary to support older applications, be derived from the canonical bases listed. The non-redundant design not only leads to numerical consistency, with no examples of multiple models competing with each other, but also allows the canonical components to be replaced individually if and when required, including new precession models, use of simplified nutation series etc.

2.1. Computation of the basis quantities

For the IAU 2006 precession model, the recommended parameterization is the 4-rotation Fukushima-Williams method (Fukushima 2003). This is concise and versatile, and can be referred directly to the GCRS pole and origin without requiring the frame bias to be applied separately. The Fukushima-Williams angles with respect to the GCRS are obtained using the following series (from Table 1 of Hilton et al. 2006):

$$\begin{aligned}
\bar{\gamma} &= -0''.052928 + 10''.556378\,t + 0''.4932044\,t^2 \\
&\quad -0''.00031238\,t^3 - 0''.000002788\,t^4 \\
&\quad +0''.0000000260\,t^5 \\
\bar{\phi} &= +84381''.412819 - 46''.811016\,t + 0''.0511268\,t^2 \\
&\quad +0''.00053289\,t^3 - 0''.000000440\,t^4 \\
&\quad -0''.0000000176\,t^5 \\
\bar{\psi} &= -0''.041775 + 5038''.481484\,t + 1''.5584175\,t^2 \\
&\quad -0''.00018522\,t^3 - 0''.000026452\,t^4 \\
&\quad -0''.0000000148\,t^5 \\
\epsilon_A &= +84381''.406 - 46''.836769\,t - 0''.0001831\,t^2 \\
&\quad +0''.00200340\,t^3 - 0''.000000576\,t^4 \\
&\quad -0''.0000000434\,t^5.
\end{aligned} \tag{4}$$

The IAU 2000A nutation components $\Delta\psi_{2000A}$ and $\Delta\epsilon_{2000A}$ are from the luni-solar and planetary terms of the Mathews et al. (2002) series, as set out in the MHB_2000 code, which is available electronically from the IERS. The small corrections that make the IAU 2000A nutation consistent with the IAU 2006 precession (Capitaine et al. 2005, Sect. 3.6) are made as follows:

$$\begin{aligned}
\Delta\psi &= \Delta\psi_{2000\mathrm{A}} + (0.4697\times 10^{-6} + f)\,\Delta\psi_{2000\mathrm{A}} \\
\Delta\epsilon &= \Delta\epsilon_{2000\mathrm{A}} + f\,\Delta\epsilon_{2000\mathrm{A}}
\end{aligned} \tag{5}$$

where $f \equiv (\dot{J}_2/J_2)t = -2.7774\times 10^{-6}\,t$, and t is the time interval since J2000 in Julian centuries (TT).

The remaining basis quantities are (i) the quantity s, which can be obtained from the series for $s + XY/2$ (Sect. 1) once X and Y are available (Sect. 2.2) and (ii) the Earth rotation angle, which is obtained from Eq. (3).

2.2. Computation of the NPB matrices

The IAU-2006-compatible nutations given by Eq. (5) can be used to transform the PB angles $\bar{\gamma}, \bar{\phi}, \bar{\psi}, \epsilon_A$ given by Eq. (4) into NPB angles $\bar{\gamma}, \bar{\phi}, \psi, \epsilon$ simply by adding the nutations to the corresponding two angles:

$$\begin{aligned}
\psi &= \bar{\psi} + \Delta\psi, \\
\epsilon &= \epsilon_A + \Delta\epsilon.
\end{aligned} \tag{6}$$

These angles[3] can be used to form the equinox based NPB matrix if required:

$$\mathbf{M}_{\mathrm{class}} = R_1(-\epsilon)\cdot R_3(-\psi)\cdot R_1(\bar{\phi})\cdot R_3(\bar{\gamma}). \tag{7}$$

However, in order to obtain the CIO based NPB matrix $\mathbf{M}_{\mathrm{CIO}}$, we do not require the whole of $\mathbf{M}_{\mathrm{class}}$. The bottom row is the GCRS unit vector to the CIP, and so the CIP X, Y coordinates can be efficiently computed using the expressions for elements [3,1] and [3,2] only:

$$\begin{aligned}
X &= \sin\epsilon\sin\psi\cos\bar{\gamma} - (\sin\epsilon\cos\psi\cos\bar{\phi} \\
&\quad - \cos\epsilon\sin\bar{\phi})\sin\bar{\gamma}, \\
Y &= \sin\epsilon\sin\psi\sin\bar{\gamma} + (\sin\epsilon\cos\psi\cos\bar{\phi} \\
&\quad - \cos\epsilon\sin\bar{\phi})\cos\bar{\gamma}.
\end{aligned} \tag{8}$$

In high-accuracy applications, these X, Y predictions can be supplemented by adding IERS corrections dX, dY, to take account of VLBI observations and in particular to compensate for the free core nutation (∼0.3 mas).

With X, Y and $s + XY/2$ now available, the quantity s can be calculated.

The remaining seven elements of the CIO based NPB matrix $\mathbf{M}_{\mathrm{CIO}}$ are functions of X, Y and s (cf. C06, Eq. (25)):

$$\begin{aligned}
\mathbf{M}_{\mathrm{CIO}}[1,1] &= \cos s + aX(Y\sin s - X\cos s) \\
\mathbf{M}_{\mathrm{CIO}}[1,2] &= -\sin s + aY(Y\sin s - X\cos s) \\
\mathbf{M}_{\mathrm{CIO}}[1,3] &= -(X\cos s - Y\sin s) \\
\mathbf{M}_{\mathrm{CIO}}[2,1] &= \sin s - aX(Y\cos s + X\sin s) \\
\mathbf{M}_{\mathrm{CIO}}[2,2] &= \cos s - aY(Y\cos s + X\sin s) \\
\mathbf{M}_{\mathrm{CIO}}[2,3] &= -(Y\cos s + X\sin s) \\
\mathbf{M}_{\mathrm{CIO}}[3,1] &= X \\
\mathbf{M}_{\mathrm{CIO}}[3,2] &= Y \\
\mathbf{M}_{\mathrm{CIO}}[3,3] &= Z
\end{aligned} \tag{9}$$

where $Z = (1 - X^2 - Y^2)^{1/2}$ and $a = 1/(1 + Z)$.

It should be noted that, although Eqs. (9) are rigorous, s is so small in the present era that writing s (in radians) for $\sin s$ and 1 for $\cos s$ would be acceptable, introducing errors much smaller than come from the resolution of the series used to obtain s.

The matrix $\mathbf{M}_{\mathrm{CIO}}$ can be used to transform GCRS vectors into the celestial intermediate reference system (CIRS), with onward transformation into the terrestrial intermediate reference

[3] The same formulation provides convenient access to two other matrices. Omitting nutation, i.e. using $\bar{\psi}$ and ϵ_A instead of ψ and ϵ, will produce the PB matrix. Evaluating $\bar{\gamma}$, $\bar{\phi}$, $\bar{\psi}$ and ϵ_A for the date J2000.0 will produce the B matrix, i.e. frame bias on its own.

system (TIRS) requiring only the Earth rotation angle. Similarly, the matrix $\mathbf{M}_{\rm class}$ can be used in an equinox based application to express GCRS vectors with respect to the true equator and equinox of date, but in this case the transformation into terrestrial coordinates requires Greenwich sidereal time (GST). In the past this has always been done using an explicit GST(UT1) formula, but that approach introduces a number of complications for high-precision applications: the formulas are rather complicated, involve both UT1 and dynamical time, and have to match the adopted precession model.

The approach advocated here avoids the need for such a GST(UT1) formula by instead expressing GST in terms of the Earth rotation angle (cf. Capitaine et al. 2003b, Eqs. (40), (41)). The difference ERA−GST, namely the equation of the origins, is a function of $\mathbf{M}_{\rm class}$ and s, and can be obtained as follows (see C06, Sect. 4.5).

2.3. Computation of EO and GST

From C06 Eq. (24) we obtain:

$$\mathbf{M}_{\rm class} = R_3(-[-{\rm EO} + s]\,) \cdot \mathbf{M}_\Sigma \tag{10}$$

and hence

$$R_3({\rm EO} - s) \;=\; \mathbf{M}_{\rm class} \cdot \mathbf{M}_\Sigma^T \tag{11}$$

where the matrix $\mathbf{M}_\Sigma$ is given by C06 Eq. (4). Writing out elements (1,1) and (2,1) of the 3×3 matrix $R_3({\rm EO} - s)$, we obtain

$$\begin{aligned} \cos({\rm EO} - s) \;&= \boldsymbol{\Upsilon} \cdot \boldsymbol{\Sigma} \\ -\sin({\rm EO} - s) \;&= \mathbf{y} \cdot \boldsymbol{\Sigma} \end{aligned} \tag{12}$$

where the unit vectors $\boldsymbol{\Upsilon}$ and $\mathbf{y}$ are the top and middle rows of $\mathbf{M}_{\rm class}$ (i.e. they are the x and y axes of the equinox based equatorial triad) and $\boldsymbol{\Sigma}$ is the top row of $\mathbf{M}_\Sigma$ (i.e. the unit vector directed to the point Σ in C06 Fig. 1):

$$\begin{aligned} \boldsymbol{\Upsilon}[1] \;&=\; \cos\psi\cos\bar\gamma + \sin\psi\cos\bar\phi\sin\bar\gamma \\ \boldsymbol{\Upsilon}[2] \;&=\; \cos\psi\sin\bar\gamma - \sin\psi\cos\bar\phi\cos\bar\gamma \\ \boldsymbol{\Upsilon}[3] \;&=\; -\sin\psi\sin\bar\phi \end{aligned} \tag{13}$$

$$\begin{aligned} \mathbf{y}[1] \;&=\; \cos\epsilon\sin\psi\cos\bar\gamma \\ &\quad -(\cos\epsilon\cos\psi\cos\bar\phi + \sin\epsilon\sin\bar\phi)\sin\bar\gamma \\ \mathbf{y}[2] \;&=\; \cos\epsilon\sin\psi\sin\bar\gamma \\ &\quad +(\cos\epsilon\cos\psi\cos\bar\phi + \sin\epsilon\sin\bar\phi)\cos\bar\gamma \\ \mathbf{y}[3] \;&=\; \cos\epsilon\cos\psi\sin\bar\phi - \sin\epsilon\cos\bar\phi \end{aligned} \tag{14}$$

$$\begin{aligned} \boldsymbol{\Sigma}[1] \;&=\; 1 - X^2/(1+Z) \\ \boldsymbol{\Sigma}[2] \;&=\; -XY/(1+Z) \\ \boldsymbol{\Sigma}[3] \;&=\; -X. \end{aligned} \tag{15}$$

Using Eqs. (12), and substituting the values of s and the vector components (13)–(15), we obtain the angle EO by evaluating:

$${\rm EO} \;=\; s - \tan^{-1}\frac{\mathbf{y}\cdot\boldsymbol{\Sigma}}{\boldsymbol{\Upsilon}\cdot\boldsymbol{\Sigma}}. \tag{16}$$

Greenwich sidereal time is simply:

$${\rm GST} \;=\; {\rm ERA} - {\rm EO}, \tag{17}$$

while the EO can also be used to transform intermediate (i.e. CIO based) right ascensions to classical apparent right ascensions:

$$\alpha_{\rm eqx} \;=\; \alpha_{\rm CIO} - {\rm EO}. \tag{18}$$

3. Procedure based on CIP *X,Y* coordinates

For highly focused applications that have specific accuracy requirements, practical issues take precedence, with adaptability to different applications a less important consideration. The chosen procedure must be straightforward, self-contained and difficult to mis-apply.

In such cases, direct series for the CIP X, Y (cf. C06, Sect. 3.1) and for computing the CIO locator s (cf. C06, Sect. 4.1) are appropriate, and this is the method set out in IERS Conventions (2003). The scheme is straightforward, with little to go wrong as long as the series are correctly implemented. In particular, there are no opportunities for applying matrix rotations in the wrong order, always a danger when the frame bias, precession and nutation are implemented as a chain of individual rotations.

The procedure based on the X, Y series is optimized for the case where transformation between celestial and terrestrial coordinates is the goal. No ecliptic or equinox is needed, and the Earth rotation angle is used directly. However, as we shall show in Sect. 3.3, by introducing an ecliptic as an additional basis, it is possible to generate all the equinox based products efficiently and accurately, should this be required.

3.1. Computation of the basis quantities

As for the angles based procedure (Sect. 2.1), two elements of the canonical basis exist as trigonometric series that are too large to be reproduced here but are available electronically:

- The X, Y components of the IAU 2006 CIP unit vector; see C06, Sect. 3.1. These series encapsulate the complete chain of rotations, comprising frame bias, P03 precession and the IAU 2000A nutation with P03 adjustments.
- The series for $s + XY/2$.

The first series is evaluated, giving X, Y; the quantity s can then be obtained using the second series. Once ERA has been calculated, using Eq. (3), everything needed to form the matrices $\mathbf{M}_{\rm CIO}$ and $\mathbf{R}$ is at hand.

3.2. Computation of the M_{CIO} and R matrices

Given X, Y, s and the ERA θ, the elements of (i) the CIO based NPB matrix $\mathbf{M}_{\rm CIO}$ and (ii) the GCRS-to-TIRS matrix $\mathbf{R}$ can be obtained from the expressions:

$$\begin{aligned} \mathbf{A}[1,1] \;&=\; \cos\beta + aX(Y\sin\beta - X\cos\beta) \\ \mathbf{A}[1,2] \;&=\; -\sin\beta + aY(Y\sin\beta - X\cos\beta) \\ \mathbf{A}[1,3] \;&=\; -(X\cos\beta - Y\sin\beta) \\ \mathbf{A}[2,1] \;&=\; \sin\beta - aX(Y\cos\beta + X\sin\beta) \\ \mathbf{A}[2,2] \;&=\; \cos\beta - aY(Y\cos\beta + X\sin\beta) \\ \mathbf{A}[2,3] \;&=\; -(Y\cos\beta + X\sin\beta) \\ \mathbf{A}[3,1] \;&=\; X \\ \mathbf{A}[3,2] \;&=\; Y \\ \mathbf{A}[3,3] \;&=\; Z \end{aligned} \tag{19}$$

where $Z = (1 - X^2 - Y^2)^{1/2}$ and $a = 1/(1+Z)$; for $\mathbf{A} \equiv \mathbf{M}_{\rm CIO}$, $\beta = s$ (cf. Eq. (9)); for $\mathbf{A} \equiv \mathbf{R}$, $\beta = \theta + s$.

3.3. Computation of M_{class}, EO and GST

Should equinox based products be required, these can be obtained by introducing an ecliptic. The first two

Table 2. Various formulations of the $\mathbf{M}_{\rm CIO}$ matrix elements corresponding to different accuracies over 2 centuries.

matrix element	expressions				
$R(1,1)$	$1 - X^2(1 + X^2/4)/2$	$1 - X^2/2$	$1 - X^2/2$	$1 - X^2/2$	1
$R(1,2)$	$-s - XY(1 + X^2/4)/2$	$-s - XY/2$	$-s - XY/2$	0	0
$R(1,3)$	$-X + sY$	$-X$	$-X$	$-X$	$-X$
$R(2,1)$	$s(1 - X^2/2) - XY(1 + X^2/4)/2$	$s - XY/2$	$s - XY/2$	0	0
$R(2,2)$	$1 - Y^2/2$	$1 - Y^2/2$	$1 - Y^2/2$	1	1
$R(2,3)$	$-Y - sX$	$-Y - sX$	$-Y$	$-Y$	$-Y$
$R(3,1)$	X	X	X	X	X
$R(3,2)$	Y	Y	Y	Y	Y
$R(3,3)$	$1 - (X^2(1 + X^2/4) + Y^2)/2$	$1 - (X^2 + Y^2)/2$	$1 - (X^2 + Y^2)/2$	$1 - X^2/2$	1
accuracy over ± 2 cy better than	0.1 μas	165 μas	3.7 mas	0.38 arcsec	0.85 arcsec
≤ during 21st cy	0.001 μas	8 μas	0.4 mas	0.08 arcsec	0.12 arcsec

Fukushima-Williams angles (Eq. (4)) provide a convenient method. The rotations $R_3(\bar{\gamma})$ followed by $R_1(\bar{\phi})$ (cf. Eq. (7)) produce a matrix the z-axis of which is the GCRS unit vector for the ecliptic pole of date, $\boldsymbol{k}$:

$$\boldsymbol{k} = [\sin\bar{\phi}\sin\bar{\gamma}, \quad -\sin\bar{\phi}\cos\bar{\gamma}, \quad \cos\bar{\phi}] \tag{20}$$

Writing $\boldsymbol{n}$ for the CIP vector (X, Y, Z), the equinox based NPB matrix is:

$$\mathbf{M}_{\rm class} = [\langle \boldsymbol{n}\times\boldsymbol{k}\rangle, \quad \boldsymbol{n} \times\langle \boldsymbol{n}\times\boldsymbol{k}\rangle, \quad \boldsymbol{n}] \tag{21}$$

(cf. Murray 1983 Eq. (5.4.12) and C06 Eq. (22)). The EO and GST can then be obtained using the methods of Sect. 2.3.

4. Approximate procedures

When transforming between celestial and terrestrial coordinates, the accuracy requirements will depend on the application, and in many cases a trade-off between accuracy, conciseness and computing costs can be considered. The CIO based NPB matrix $\mathbf{M}_{\rm CIO}$ is particularly well-suited to approximation as its elements are simple functions of the CIP coordinates X, Y and the quantity s, the latter a very small angle in the present era. To add to these advantages, the required Earth rotation measure is ERA, a straightforward linear transformation of UT1. Various approximate formulations of the matrix elements of $\mathbf{M}_{\rm CIO}$ and their accuracy limitations are summarized in Table 2.

The comparisons in the table assume full-accuracy X, Y and (where present at all) s. However, in cases where an approximate formulation is advantageous it will also be usual to introduce other simplifications, such as truncating the series. To take an extreme example, if the simplest of the formulations is used:

$$\mathbf{M}_{\rm CIO} \simeq \begin{pmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \\ X & Y & 1 \end{pmatrix} \tag{22}$$

and X, Y are computed as follows:

$$X = 2.6603\times10^{-7}\tau - 33.2\times10^{-6}\sin\Omega$$
$$Y = -8.14\times10^{-14}\tau^2 + 44.6\times10^{-6}\cos\Omega \tag{23}$$

where τ is the number of days since J2000.0, and:

$$\Omega = 2.182 - 9.242\times10^{-4}\tau \quad \text{(radians)}, \tag{24}$$

the maximum error in the 21st century is less than 0.″9, still adequate for many purposes such as predicting source visibility or pointing a small telescope.

5. Summary

In this paper we have provided two full-accuracy procedures for implementing the IAU 2006 precession. The two are of similar efficiency, and each supports the use of equinox based as well as CIO based applications. In addition, we have listed a selection of simplified methods for applications where the highest accuracy is not a requirement.

The appendix, below, contains a fully worked numerical example, demonstrating both of the full-accuracy procedures and the most basic of the simplified procedures.

Acknowledgements. We are indebted to the anonymous referee, who made useful suggestions one of which led to a clearer and rigorous reformulation of Eq. (16). We also wish to thank John Irwin for his extremely detailed checking of the numerical values in the paper and associated computer code.

References

Capitaine, N., & Wallace, P. T. 2006, A&A, 450, 855 (C06)
Capitaine, N., Guinot, B., & McCarthy, D. D. 2000, A&A, 355, 398
Capitaine, N., Wallace, P. T., & Chapront, J. 2003a, A&A, 412, 567
Capitaine, N., Wallace, P. T., & McCarthy, D. D. 2003b, A&A, 406, 1135
Capitaine, N., Wallace, P. T., & Chapront, J. 2005, A&A, 432, 355
Fukushima, T. 2003, AJ, 126, 494
Hilton, J., Capitaine, N., Chapront, J., et al. 2006, Celest. Mech. Dyn. Astron., 94, 351
IERS Conventions 2003, IERS Technical Note 32, ed. D. D. McCarthy, & G. Petit, Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie, 2004
Lieske, J. H., Lederle, T., Fricke, W., & Morando, B. 1977, A&A, 58, 1
Mathews, P. M., Herring, T. A., & Buffett, B. A. 2002, J. Geophys. Res., 107, B4, 10.1029/2001JB000390
Murray, C. A. 1983, Vectorial Astrometry, (Bristol: Adam Hilger Ltd)
Wallace, P. T. 1998, Highlights of Astronomy Vol. 11A, ed. J. Andersen (Kluwer Academic Publishers), 11, 191

Online Material

Appendix: Numerical example

Here we present a worked numerical example where the procedures described in Sects. 1–3 are used to generate (i) the CIO based NPB matrix $\mathbf{M}_{\rm CIO}$, (ii) the equinox based NPB matrix $\mathbf{M}_{\rm class}$ and (iii) the GCRS-to-TIRS matrix $\mathbf{R}$, in several different ways. A calculation precision of about of 33 decimal digits was used, and to facilitate comparisons the results in most cases are reported to several more digits than the underlying models justify.

The date selected for the tests is:

$$\mathrm{UTC} = 2006 \text{ January } 15,\ \ 21^{\rm h}\,24^{\rm m}\,37^{\rm s}\!.5 \tag{A.1}$$

corresponding to:

$$\mathrm{TT} = 2400000.5 + 53750.892855138888889\ \mathrm{(JD)} \tag{A.2}$$

$$t = +0.06040774415164651\ \mathrm{Jcy} \tag{A.3}$$

Adopting for the purposes of this example the value:

$$\mathrm{UT1} - \mathrm{UTC} = +0^{\rm s}\!.3341 \tag{A.4}$$

we obtain:

$$\mathrm{UT1} = 2400000.5 + 53750.892104561342593\ \mathrm{(JD)} \tag{A.5}$$

A.1. Reference method

The basis of the reference method is the matrix $\mathbf{M}_{\rm class}$ calculated as the product of the three individual rotation matrices $\mathbf{B}$ (bias) followed by $\mathbf{P}$ (precession) and then $\mathbf{N}$ (nutation), using basic rather than derived quantities:

$$\begin{aligned}\mathbf{M}_{\rm class} &= \mathbf{N}\cdot\mathbf{P}\cdot\mathbf{B}\\ &= R_1(-[\epsilon_A+\Delta\epsilon])\cdot R_3(-\Delta\psi)\cdot R_1(\epsilon_A)\cdot R_3(\chi_A)\cdot R_1(-\omega_A)\cdot R_3(-\psi_A)\cdot R_1(\epsilon_0-\eta_0)\cdot R_2(\xi_0)\cdot R_3(d\alpha_0)\end{aligned} \tag{A.6}$$

The various angles are defined in C06, Sect. 2.2.1. The IAU 2000A frame bias angles are (Capitaine et al. 2003a):

$$d\alpha_0 = -0''\!.014600000,\quad \xi_0 = -0''\!.041775000\cdot\sin(84381''\!.448000000),\quad \eta_0 = -0''\!.006819200 \tag{A.7}$$

and the P03 J2000 obliquity is:

$$\epsilon_0 = 84381''\!.406000000 \tag{A.8}$$

The P03 precession angles are ψ_A, ω_A, χ_A and ϵ_A, and the nutation components $\Delta\psi$ and $\Delta\epsilon$ are the IAU 2000A values adjusted to match the P03 precession. Using the P03 series (Capitaine et al. 2005, Tables 3 and 4) we can obtain the precession angles for the test date:

$$\psi_A = +304''\!.359364139,\quad \omega_A = +84381''\!.404629617,\quad \chi_A = +0''\!.628998164,\quad \epsilon_A = +84378''\!.576696215 \tag{A.9}$$

Evaluating the IAU 2000A nutation series, using the SOFA subroutine iau_NUT00A, for the test date gives:

$$\Delta\psi_{\rm 2000A} = -1''\!.071332645,\qquad \Delta\epsilon_{\rm 2000A} = +8''\!.656842472 \tag{A.10}$$

and we use Eq. (5) to obtain the corrections needed to match the IAU 2006 precession:

$$d\Delta\psi = -0''\!.000000323,\qquad d\Delta\epsilon = -0''\!.000001452 \tag{A.11}$$

The nutation components are therefore:

$$\Delta\psi = -1''\!.071332969,\qquad \Delta\epsilon = +8''\!.656841020 \tag{A.12}$$

We now have everything needed to evaluate (A.6) and form the classical NPB matrix $\mathbf{M}_{\rm class}$:

$$\mathbf{M}_{\rm class} = \begin{pmatrix} +0.99999892304984813 & -0.00134606989019584 & -0.00058480338117601\\ +0.00134604536886632 & +0.99999909318492607 & -0.00004232245847787\\ +0.00058485981985452 & +0.00004153524101576 & +0.99999982810689266\end{pmatrix} \tag{A.13}$$

The bottom row of this matrix is the CIP unit vector (X, Y, Z). We evaluate the series for $s + XY/2$ and subtract $XY/2$ to obtain the quantity s. Then, by substituting X, Y, Z and s into Eqs. (9), we obtain the CIO based NPB matrix:

$$\mathbf{M}_{\rm CIO} = \begin{pmatrix} +0.99999982896948063 & +0.00000000032319161 & -0.00058485982037244\\ -0.00000002461548515 & +0.99999999913741188 & -0.00004153523372294\\ +0.00058485981985452 & +0.00004153524101576 & +0.99999982810689266\end{pmatrix} \tag{A.14}$$

All that remains is to use Eq. (3) to obtain:

$$\mathrm{ERA} = 76^\circ\!.265431053522 \qquad \equiv 5^{\rm h}\,05^{\rm m}\,03^{\rm s}\!.703452845 \tag{A.15}$$

and with the first of Eqs. (2) to obtain the GCRS-to-TIRS matrix:

$$\mathbf{R} = \begin{pmatrix} +0.23742421473054043 & +0.97140604802742436 & -0.00017920749858993\\ -0.97140588849284692 & +0.23742427873021975 & +0.00055827489427310\\ +0.00058485981985452 & +0.00004153524101576 & +0.99999982810689266\end{pmatrix} \tag{A.16}$$

A.2. Procedure based on bias-precession and nutation angles

Evaluating Eqs. (4) for the test date gives the following Fukushima-Williams angles:

$$\bar{\gamma} = +0''.586558662, \quad \bar{\phi} = +84378''.585257806, \quad \bar{\psi} = +304''.327212171, \quad \epsilon_A = +84378''.576696215 \tag{A.17}$$

The last of these is the same as (A.9), obtained earlier. Adding the nutations (Eq. (6)), we obtain:

$$\psi = +303''.255879203, \qquad \epsilon = +84387''.233537235 \tag{A.18}$$

If required, we can use these angles in Eq. (7) to obtain the whole of the equinox based NPB matrix:

$$\mathbf{M}_{\rm class} = \begin{pmatrix} +0.99999892304984688 & -0.00134606989112466 & -0.00058480338117619 \\ +0.00134604536979454 & +0.99999909318492478 & -0.00004232245950000 \\ +0.00058485981985612 & +0.00004153524203735 & +0.99999982810689262 \end{pmatrix} \tag{A.19}$$

The total rotational difference with respect to (A.13) is 0.28 μas. This is consistent with the rounding precision used for the polynomial coefficients of the respective parameterizations. In the normal case, where only the bottom row is wanted, Eqs. (8) can be used to compute X and Y alone. Then, by evaluating the series for $s + XY/2$ and subtracting $XY/2$, we obtain the quantity s:

$$s = -0''.002571986 \tag{A.20}$$

Substituting X, Y, Z and s into Eqs. (9), we obtain the CIO based NPB matrix:

$$\mathbf{M}_{\rm CIO} = \begin{pmatrix} +0.99999982896948063 & +0.00000000032319161 & -0.00058485982037403 \\ -0.00000002461548575 & +0.99999999913741183 & -0.00004153523474454 \\ +0.00058485981985612 & +0.00004153524203735 & +0.99999982810689262 \end{pmatrix} \tag{A.21}$$

With the first of Eqs. (2), we apply ERA to obtain the GCRS-to-TIRS matrix:

$$\mathbf{R} = \begin{pmatrix} +0.23742421473053985 & +0.97140604802742432 & -0.00017920749958268 \\ -0.97140588849284706 & +0.23742427873021974 & +0.00055827489403210 \\ +0.00058485981985612 & +0.00004153524203735 & +0.99999982810689262 \end{pmatrix} \tag{A.22}$$

For this and the previous result, the agreement with (A.14) and (A.16) respectively is 0.23 μas. To obtain the equation of the origins, we evaluate Eqs. ((13)–(15)) to produce the three vectors:

$$\boldsymbol{\Upsilon} = (+0.99999892304984688 \;\; -0.00134606989112466 \;\; -0.00058480338117619) \tag{A.23}$$
$$\mathbf{y} = (+0.00134604536979454 \;\; +0.99999909318492478 \;\; -0.00004232245950000) \tag{A.24}$$
$$\boldsymbol{\Sigma} = (+0.99999982896948086 \;\; -0.00000001214614813 \;\; -0.00058485981985612) \tag{A.25}$$

and then use Eq. (16), which gives:

$$\mathrm{EO} = -277''.646996035 \tag{A.26}$$

Combining EO and ERA with Eq. (17) gives us Greenwich (apparent) sidereal time:

$$\mathrm{GST} = 05^{\rm h}\,05^{\rm m}\,{}^{\rm s}\!.22213252581 \tag{A.27}$$

and with the third of Eqs. (2) we can form the GCRS-to-TIRS matrix $\mathbf{R}$ using GST instead of ERA. The results are identical to (A.22).

A.3. Procedure based on CIP X,Y coordinates

The procedures given in Sect. 3 start with the direct evaluation of the CIP coordinates X, Y from series. Evaluating the series for the test date using the fundamental arguments given in IERS Conventions (2003), we obtain:

$$X = +120''.635997299064, \qquad Y = +8''.567258740044 \tag{A.28}$$

Converting into radians, we obtain the CIP vector components:

$$X = +0.000584859819249, \qquad Y = +0.000041535242468 \tag{A.29}$$

These agree with the reference values from (A.1) to 0.12 μas in X and 0.30 μas in Y. The corresponding Z value is:

$$Z = +0.999999828106893 \tag{A.30}$$

Evaluating the series for $s + XY/2$ and subtracting $XY/2$ we obtain:

$$s = -0''.002571986 \tag{A.31}$$

Substituting X, Y, Z and s into Eqs. (9), we obtain the CIO based NPB matrix:

$$\mathbf{M}_{\rm CIO} = \begin{pmatrix} +0.99999982896948099 & +0.00000000032319161 & -0.00058485981976671 \\ -0.00000002461548598 & +0.99999999913741182 & -0.00004153523517497 \\ +0.00058485981924879 & +0.00004153524246778 & +0.99999982810689296 \end{pmatrix} \qquad \text{(A.32)}$$

and with the first of Eqs. (2) we can form the GCRS-to-TIRS matrix:

$$\mathbf{R} = \begin{pmatrix} +0.23742421473053972 & +0.97140604802742430 & -0.00017920749985661 \\ -0.97140588849284746 & +0.23742427873021973 & +0.00055827489333995 \\ +0.00058485981924879 & +0.00004153524246778 & +0.99999982810689296 \end{pmatrix} \qquad \text{(A.33)}$$

The agreement with (A.16) is $0.32\,\mu$as. Should we wish to generate the equinox based products, we would take the first two Fukushima-Williams angles, $\bar{\gamma}$ and $\bar{\phi}$ (see (A.17), above), and use Eq. (20) to obtain the ecliptic pole GCRS vector:

$$\mathbf{k} = (+0.00000113112930755 \;\; -0.39776442218982286 \;\; +0.91748758271636401) \qquad \text{(A.34)}$$

This and the CIP vector (X, Y, Z) give us, via Eq. (21), the equinox based NPB matrix:

$$\mathbf{M}_{\rm class} = \begin{pmatrix} +0.99999892304984912 & -0.00134606988972260 & -0.00058480338056834 \\ +0.00134604536839225 & +0.99999909318492665 & -0.00004232245992880 \\ +0.00058485981924879 & +0.00004153524246778 & +0.99999982810689296 \end{pmatrix} \qquad \text{(A.35)}$$

The rotational disagreement with respect to the reference version is $0.34\,\mu$as. The slight increase relative to the (A.33) result is due to the propagation of the CIP disagreement into the position of the equinox. Using Eqs. ((13)–(17)), we obtain the equation of the equinoxes and the Greenwich sidereal time:

$$\mathrm{EO} = -277\overset{''}{.}646995746, \qquad \mathrm{GST} = 05^{\rm h}\,05^{\rm m}\,22\overset{\rm s}{.}213252562 \qquad \text{(A.36)}$$

These differ from the values obtained in (A.26) and (A.27) by $0.29\,\mu$as. With the third of Eqs. (2) we can form the GCRS-to-TIRS matrix **R** using GST instead of ERA. The results are identical to (A.33).

A.4. Approximate procedure

Using Eqs. (22)–(24), for our test date we obtain $\tau = +2206.392855139$ days, $\Omega = 4\overset{\circ}{.}092400420$, $X = +0.000582240$, $Y = +0.000043749$ and hence a crude approximation to the GCRS to CIRS matrix (cf. (A.32)):

$$\mathbf{M}_{\rm CIO} \simeq \begin{pmatrix} +1.00000000000000000 & +0.00000000000000000 & -0.00058224012792061 \\ +0.00000000000000000 & +1.00000000000000000 & -0.00004374943683668 \\ +0.00058224012792061 & +0.00004374943683668 & +1.00000000000000000 \end{pmatrix} \qquad \text{(A.37)}$$

The rotational error is 0.7 arcsec.