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Letter to the Editor

Extended nonlinear guiding center theory of perpendicular diffusion

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ABSTRACT

Recently the nonlinear guiding center theory of cosmic ray perpendicular diffusion was proposed. In this paper it is demonstrated that at least the slab contribution to the perpendicular mean free path is calculated incorrectly within this nonlinear approach. An extended theory which includes an improved treatment of the slab contribution is presented in this paper. Contrary to the original theory, the extended nonlinear guiding center theory shows agreement with numerical simulations for slab and non-slab models.

Key words. cosmic rays – turbulence – diffusion

1. Introduction

The problem of cosmic ray transport perpendicular to a magnetic background field is well known in space and astrophysics and was described in many studies (e.g. Jokipii 1966; Urch 1977; Bieber & Matthaeus 1997; Giacalone & Jokipii 1999; Mace et al. 2000; Mazur et al. 2000; Kota & Jokipii 2000; Qin et al. 2002a,b; Matthaeus et al. 2003; Shalchi et al. 2004b; Shalchi 2005b). The first theoretical approach was the application of quasilinear theory (QLT, Jokipii 1966) where it was assumed that all diffusion coefficients can be calculated independently of each other by using a first order perturbation method. If the quasilinear *Ansatz* is combined with the assumption that guiding centers follow fieldlines and a magnetostatic slab model for the turbulence, a simple formula can be obtained well known as field line random walk limit (FLRW-limit). This quasilinear result provides us with a constant perpendiuclar diffusion coeofficient or mean free path which is usually interpreted as diffusive behaviour. By using test particle simulations (e.g. Qin et al. 2002a) it was demonstrated that the FLRW-limit is incorrect: perpendicular transport in a magnetostatic slab model behaves subdiffusively. In recent papers this conclusion was confirmed theoretically be using a compound diffusion model (Kota & Jokipii 2000) or by using more systematical calculations (Shalchi et al. 2005b).

It was also demonstrated by test particle simulations (e.g. Giacalone & Jokipii 1999; Qin et al. 2002b) that diffusion is recovered if the slab model is replaced by a slab/2D composite model. Some authors (e.g. Qin 2002b; Matthaeus et al. 2003) stated that for certain parameter regimes (e.g. high rigidities) the FLRW-limit agrees with these simulations. This statement is not valid because the FLRW-limit is a slab result and can therefore not be compared with simulations performed for composite geometry. The correct QLT result for the composite model is a superdiffusive behaviour (Shalchi & Schlickeiser 2004). Therefore, QLT is not appropriate for perpendicular transport, and a nonlinear description has to be applied. The first theoretical approach which seems to agree with the simulations for composite geometry was presented in Matthaeus et al. (2003) by applying the so called nonlinear guiding center (NLGC) theory. Another promising theory was obtained by combining the ideas of NLGC-theory with QLT. The resulting theory, the weakly nonlinear theory (WNLT, Shalchi et al. 2004b), is one theory for parallel and perpendicular transport and is also able to solve the recently discovered geometry problem of parallel transport (Shalchi et al. 2004b; Qin et al. 2006).

Although both theories successfully reproduced the simulations, there is one case in which both theories fail dramatically: they cannot describe subdiffusion in the slab model. One could argue, that this special case is only of academic interest and is therefore not important, but as demonstrated in this paper this statement is not correct: by using the methods proposed by Shalchi (2005b), it is demonstrated that the slab contribution behaves always subdiffusively even in the composite model. Therefore the NLGC-theory has to be modified.

2. The extended nonlinear guiding center theory

In this section we combine the methods proposed by Shalchi (2005b) with the NLGC-approach (Matthaeus et al. 2003). The starting point of our calculations is the assumption that guiding centers follow fieldlines:

$$v_x = v_z \frac{\delta B_x(x,y,z)}{B_0} = v_z \frac{\delta B_x^{\mathrm{slab}}(z)}{B_0} + v_z \frac{\delta B_x^{\mathrm{2D}}(x,y)}{B_0} \tag{1}$$

(in the magnetostatic model we neglect the time-dependence of the turbulent fields δB_x). Note that in the Matthaeus et al. (2003) paper (Eq. (1)) a parameter "a" was introduced. It can easily be demonstrated that this parameter has to be equal to 1 (Shalchi 2005b). By applying a fourier transformation, Eq. (1) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x &= \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 k \; \frac{\delta B_x^{\mathrm{slab}}(k_z)}{\mathrm{i} k_\| B_0} \mathrm{e}^{\mathrm{i} k_\| z} \\ &\quad + v_z \int \mathrm{d}^3 k \; \frac{\delta B_x^{\mathrm{2D}}(k_x, k_y)}{B_0} \mathrm{e}^{\mathrm{i} k_x x + \mathrm{i} k_y y}. \end{aligned} \tag{2}$$

By using Corrsin's independence hypothesis (Corrsin 1959), $\Delta x = x(t) - x(0)$, and by choosing $z(0) = 0$ we find

$$\kappa_{xx}(t) = \frac{1}{2tB_0^2} \int d^3k \, P_{xx}^{\rm slab}(\boldsymbol{k}) \frac{2 - 2\langle \cos[k_\parallel z(t)] \rangle}{k_\parallel^2} + \frac{1}{2tB_0^2} \int_0^t d\tau_1 \int_0^t d\tau_2 \langle v_z(\tau_1) v_z(\tau_2) \rangle \times \int d^3k \, P_{xx}^{\rm 2D}(\boldsymbol{k}) \left\langle e^{ik_x[x(\tau_1)-x(\tau_2)]+ik_y[y(\tau_1)-y(\tau_2)]} \right\rangle \quad (3)$$

where we defined a time dependent diffusion coefficient $\kappa_{xx}(t) = \langle (\Delta x)^2 \rangle / 2t$ and used the correlation tensor of the turbulent fields: $\left\langle \delta B_i(\boldsymbol{k}) \delta B_j^*(\boldsymbol{k}') \right\rangle = P_{ij}(\boldsymbol{k}) \delta(\boldsymbol{k} - \boldsymbol{k}')$. The slab contribution can be calculated in the same manner as proposed in Shalchi 2005b. We are mainly interested in the diffusion coefficients in the limit of large times, thus it seems to be appropriate to assume a Gaussian distribution of the guiding centers with a diffusive behaviour in the parallel direction. With

$$P_{xx}^{\rm slab}(\boldsymbol{k}) = \frac{C(\nu)}{2\pi} l_{\rm slab} \delta B_{\rm slab}^2 \left(1 + k_\parallel^2 l_{\rm slab}^2\right)^{-\nu} \frac{\delta(k_\perp)}{k_\perp} \quad (4)$$

we therefore obtain

$$\lim_{t\to\infty} \kappa_{xx}^{\rm slab}(t) \approx 2\sqrt{\pi} C(\nu) l_{\rm slab} \frac{\delta B_{\rm slab}^2}{B_0^2} \sqrt{\frac{\kappa_\parallel}{t}}. \quad (5)$$

In Eqs. (4) and (5) we used the function $C(\nu) = (2\sqrt{\pi})^{-1} \Gamma(\nu)/\Gamma(\nu - 1/2)$, the inertial range spectral index 2ν, the slab bendover scale $l_{\rm slab}$ and the strength of the slab fluctuations $\delta B_{\rm slab}^2 / B_0^2$. Equation (5) was already derived in Shalchi (2005b) (see Eqs. (40) and (41)) for pure slab geometry. Obviously, the slab contribution behaves also subdiffusively in the composite model.

The 2D contribution is much more difficult to calculate. The main problem is that the time-integrals reaches from small time-scales, where we expect a ballistic motion of the particle, up to large, diffusive scales. It is difficult to take these different time-scales into account if a nonlinear theory is formulated (see Shalchi 2005a). In the current paper we simply apply the models and approximations suggested by Matthaeus et al. (2003):

1) we assume a Gaussian distribution of the particles (or guiding centers) for all time-scales;
2) we assume that the mean square deviation of the particle (the width of the Gaussian function) behaves diffusively for all time-scales:

$$\left\langle e^{ik_x[x(\tau_1)-x(\tau_2)]+ik_y[y(\tau_1)-y(\tau_2)]} \right\rangle = e^{-(\kappa_{xx}k_x^2 + \kappa_{yy}k_y^2)|\tau_1 - \tau_2|}. \quad (6)$$

Furthermore we neglect the gyromotion of the particle;

3) we assume an isotropic and exponential form of the velocity correlation function

$$\langle v_z(\tau_1) v_z(\tau_2) \rangle = \frac{v^2}{3} e^{-v|\tau_1 - \tau_2|/\lambda_\parallel}. \quad (7)$$

By applying these three approximations we can determine the 2D contribution:

$$\kappa_{xx}^{\rm 2D}(t) = \frac{v^2}{6tB_0^2} \int d^3k \, P_{xx}^{\rm 2D}(\boldsymbol{k}) \times \int_0^t d\tau_1 \int_0^t d\tau_2 \, e^{-(\kappa_{xx}k_x^2 + \kappa_{yy}k_y^2 + v/\lambda_\parallel)|\tau_1 - \tau_2|}. \quad (8)$$

Because we are interested in large (diffusive) time scales we can calculate the two time integrals for the limit $t \to \infty$:

$$\lim_{t\to\infty} \kappa_{xx}^{\rm 2D}(t) = \frac{v^2}{3B_0^2} \int d^3k \, \frac{P_{xx}^{\rm 2D}(\boldsymbol{k})}{\kappa_{xx}k_x^2 + \kappa_{yy}k_y^2 + v/\lambda_\parallel}. \quad (9)$$

In the case of axisymmetric turbulence ($\kappa_\perp = \kappa_{xx} = \kappa_{yy}$) and with

$$P_{xx}^{\rm 2D}(\boldsymbol{k}) = \frac{2C(\nu)}{\pi} l_{\rm 2D} \delta B_{\rm 2D}^2 (1 + k_\perp^2 l_{\rm 2D}^2)^{-\nu} \delta(k_\parallel) \frac{k_y^2}{k_\perp^3} \quad (10)$$

we have

$$\lim_{t\to\infty} \kappa_{xx}^{\rm 2D}(t) = \frac{2v^2 C(\nu) l_{\rm 2D} \delta B_{\rm 2D}^2}{3B_0^2} \times \int_0^\infty dk_\perp \, \frac{(1 + k_\perp^2 l_{\rm 2D}^2)^{-\nu}}{\kappa_\perp k_\perp^2 + v/\lambda_\parallel}. \quad (11)$$

Which is the same 2D contribution that was already calculated by Matthaeus et al. (2003) and Shalchi et al. (2004a). In Eqs. (10) and (11) we used the 2D bendover scale $l_{\rm 2D}$ and the strength of the 2D fluctuations $\delta B_{\rm 2D}^2 / B_0^2$. The total coefficient consists of a subdiffusive slab (Eq. (5)) and a diffusive 2D contribution (Eq. (11)). According to these new considerations we can formulate the following statement: *the slab contribution behaves always subdiffusively and must therefore be neglected if the perpendicular mean free path is calculated for a magnetostatic composite model*. In the next section we test this statement by comparing the original and the improved (extended) NLGC-theory with test-particle simulations.

3. Comparison with test particle simulations

The considerations of the previous section lead to an important statement which must be considered in all applications of the NGLC-theory. Furthermore, it seems that there are three different time-regimes:

1) the small scales where the particle moves nearly unperturbed (ballistic regime);
2) the intermediate scales where the subdiffusive slab contribution is dominant. For such times the total mean free path behaves subdiffusively;
3) the large scales, where the time is large enough so that the slab contribution can be neglected and diffusion is recovered. This third regime doesn't exist in the case of pure slab geometry.

To test the ENGLC-theory we compare with test-particle simulations. We perform three runs to explore pure slab, strong slab and strong 2D turbulence. For all three runs a test-particle code, developed by Mace et al. (2000) and Qin et al. (2002a,b) was applied. The code itself is in detail described in these papers and in Qin (2002). Further informations can be obtained from the following webpage: `http://www.bartol.udel.edu/%7Ewhmgroup/Streamline/streamline.html`. To obtain the results presented in this section the trajectories of 2500 particles were calculated. For the turbulence parameters we assumed $l_{\rm slab} = 10 l_{\rm 2D} = 0.03$ AU, $2\nu = 5/3$ and $\delta B / B_0 = 1$ in the simulations and in the theoretical calculations. If the NLGC- or ENLGC-theory is applied, the parallel mean free path must be known as an input parameter. To replace the parallel mean free path we use the results from simulations. Both diffusion coefficient are calculated for a value $R = R_{\rm L}/l_{\rm slab} = 0.1$ of the dimensionless rigidity ($R_{\rm L}$ = Larmor radius).

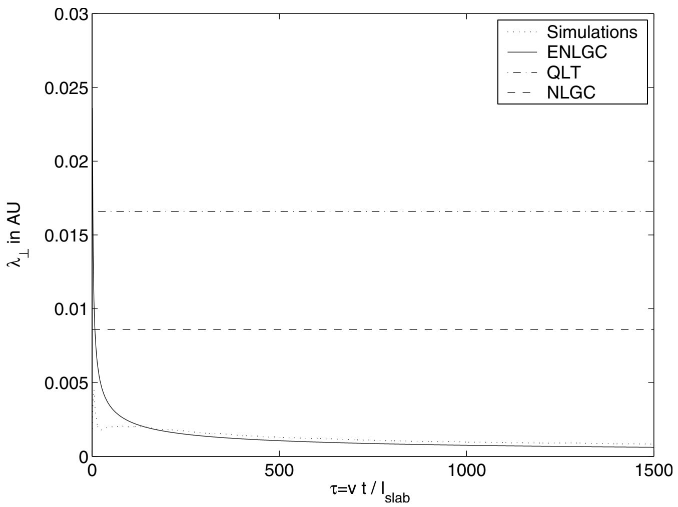

Fig. 1. The time-dependent perpendicular mean free path for pure slab geometry. Shown is the QLT-result (dash-dotted line), the NLGC-result (dashed line), the ENLGC-result (solid line) and the simulations (dotted line).

3.1. Run 1: pure slab geometry

In this first run we assume pure slab geometry. In this case the ENLGC-result is represented by Eq. (5) which can be written as

$$\lambda_\perp = 6C(\nu)\frac{\delta B_{\rm slab}^2}{B_0^2}\sqrt{\frac{\pi\lambda_\parallel l_{\rm slab}}{3\tau}} \tag{12}$$

where $\lambda_i = 3/\upsilon\kappa_i$ and the dimensionless time $\tau = \upsilon t/l_{\rm slab}$ were used. In Fig. 1 this formula is compared with the time-dependent result from the simulations. Both, ENLGC and the simulations provide a subdiffusive behaviour: $\lambda_\perp \sim \tau^{-0.5}$. The QLT and the NLGC results are constant in time and disagree with the simulations.

3.2. Run 2: strong slab geometry

In the second run we assume 90% slab and 10% 2D geometry. In this case diffusion is recovered (see Fig. 2). To obtain the ENLGC-result we neglect the slab contribution. In this case the original NLGC-theory fails to reproduce the simulations due to overestimating the slab contribution. The ENLGC-theory agrees very well with the simulations. The statement regarding the negligence of the slab contribution seems to be true.

3.3. Run 3: strong 2D geometry

Here we assume 20% slab/80% 2D composite geometry. The results are shown in Fig. 3. Here the NLGC and the ENLGC results are similar. The factor 2 between the extended theory and the simulations is expected to come due to the crude models which were applied to calculate the 2D contribution.

4. Analytic forms of the perpendicular mean free path

For applications it is useful to derive analytic forms of the perpendicular mean free path (Zank et al. 2004; Shalchi et al. 2004a). Because we must neglect the slab

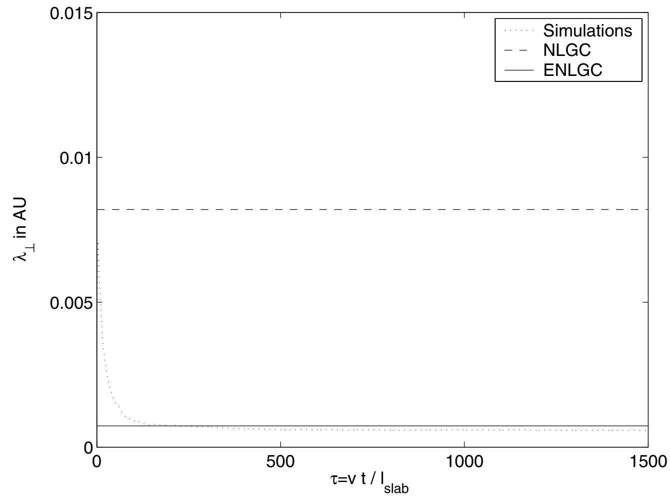

Fig. 2. The time-dependent perpendicular mean free path for strong slab geometry (90% slab/10% 2D). Shown is the NLGC-result (dashed line), the ENLGC-result (solid line) and the simulations (dotted line).

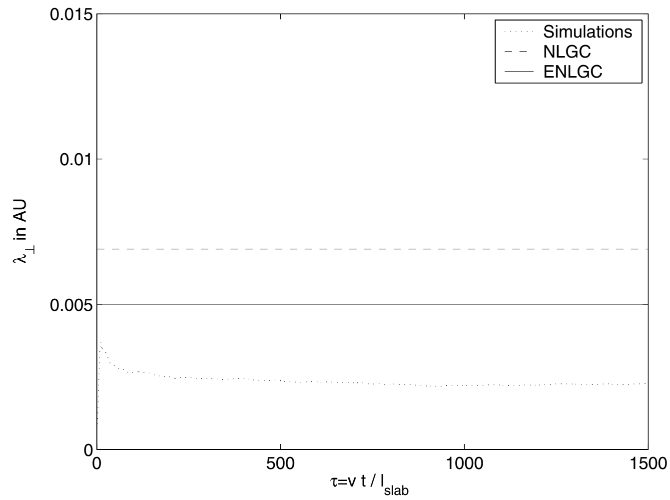

Fig. 3. The time-dependent perpendicular mean free path for strong 2D geometry (20% slab/80% 2D). Shown is the NLGC-result (dashed line), the ENLGC-result (solid line) and the simulations (dotted line).

contribution we can use the analytical results of Shalchi et al. (2004a) for pure 2D:

$$\lambda_\perp = \frac{1}{2}\frac{\delta B_{\rm 2D}^2}{B_0^2}\lambda_\parallel \tag{13}$$

if $\lambda_\parallel\lambda_\perp \ll 3l_{\rm 2D}^2$, and

$$\lambda_\perp = \left(\sqrt{3}\pi C(\nu) l_{\rm 2D}\frac{\delta B_{\rm 2D}^2}{B_0^2}\right)^{2/3}\lambda_\parallel^{1/3} \tag{14}$$

if $\lambda_\parallel\lambda_\perp \gg 3l_{\rm 2D}^2$. These two formulas must be used if the perpendicular mean free path is calculated for composite or pure 2D geometry. For pure slab we must apply Eq. (12). These three formulas (Eqs. (12)–(14)) are easy to apply and are therefore useful for applications in space and astrophysics.

5. Summary and conclusion

In this paper the NGLC-theory was improved by using a different treatment of the slab contribution to the perpendicular mean free path. The ENLGC-theory has the following advantages in comparison to the original NLGC-approach:

1) The ENLGC-theory can also be applied for slab and slab-like turbulence geometries. The agreement between the extended theory and test-particle simulations for such turbulence models is excelent (see Figs. 1 and 2).
2) The extended theory can explain the subdiffusive regime for intermediate time scales which can be seen in simulations (see Figs. 2 and 3).
3) For composite geometry and large enough times, we must neglect the slab contribution. Therefore the ENLGC-theory is much more easy to apply than the original NLGC-approach.

In the ENLGC-approach we calculated the slab and the 2D contribution in different ways. Whereas the slab contribution can be calculated in a systematic way the 2D contribution must be estimated by applying questionable models (exponential velocity correlation function, diffusive Gaussian distribution function). It must be subject of future work to find improved models for the velocity correlation function and the particle distribution function. The recently submitted work of Webb et al. (2006) will be valueable to achieve such an improvement.

Futhermore, it is unclear whether diffusion is recovered in non-static slab turbulence. In dynamical turbulence or plasmawave models for instance, a recovery of diffusion is possible. However, in this case we expect a very small slab coefficient and a very long time before diffusion is recovered. This statement agrees with test particle simulations performed for plasmawave turbulence (Michalek & Ostrowski 1996). As a final remark it should also be noted that the presented modifications must also be applied onto WNLT. Within WNLT we must set $D_\perp^{\mathrm{slab}} \to 0$, where $D_\perp^{\mathrm{slab}}$ is the slab contribution to the Fokker-Planck coefficient of perpendicular diffusion.

References

Bieber, J. W., & Matthaeus, W. H. 1997, ApJ, 485, 655
Corrsin, S. 1959, in Atmospheric Diffusion and Air Pollution, ed. F. Frenkiel, & P. Sheppard (New York: Academic), 161
Giacalone, J., & Jokipii, J. R. 1999, ApJ, 520, 204
Jokipii, J. R. 1966, ApJ, 146, 480
Kota, J., & Jokipii, J. R. 2000, ApJ, 531, 1067
Mace, J. W., Matthaeus, W. H., & Bieber, J. W. 2000, ApJ, 538, 192
Matthaeus, W. H., Qin, G., Bieber, J. W., et al. 2003, ApJ, 590, L53
Mazur, J. E., Mason, G. M., Dwyer, J. R., et al. 2000, ApJ, 532, L79
Michalek, G., & Ostrowski, M. 1996, Nonlinear Processes in Geophysics, 3, 66
Qin, G. 2002, Ph.D. Thesis, University of Delaware, USA
Qin, G., Matthaeus, W. H., & Bieber, J. W. 2002a, Geophys. Res. Lett., 29
Qin, G., Matthaeus, W. H., & Bieber, J. W. 2002b, ApJ, 578, L117
Qin, G., Matthaeus, W. H., & Bieber, J. W. 2006, ApJ, 640, L103
Shalchi, A., & Schlickeiser, R. 2004, A&A, 420, 821
Shalchi, A., Bieber, J. W., & Matthaeus, W. H. 2004a, ApJ, 604, 675
Shalchi, A., Bieber, J. W., Matthaeus, W. H., & Qin, G. 2004b, ApJ, 616, 617
Shalchi, A. 2005a, Phys. Plasmas, 12, 052324
Shalchi, A. 2005b, JGR, 110, A09103
Urch, I. H. 1977, Ap&SS, 46, 389
Webb, G. M., Zank, G. P., Kaghashvili, E. Kh., & leRoux, J. A., ApJ, submitted
Zank, G. P., et al. 2004, JGR, 109, A04107