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Influence of turbulence on the shape of a spectral line

The analytical approach

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ABSTRACT

We consider the propagation of a spectral-line radiation in a correlated turbulent atmosphere. The ensembles of realizations of turbulent velocities $\boldsymbol{u}(\boldsymbol{r}, t)$ and optical depth τ_ν are assumed to be Gaussian. We investigate the explicit analytical solution of the stochastic radiative transfer equation for the intensity I_ν of radiation. The scattering term is not taken into account. It is shown that, in addition to the usual Doppler broadening of the spectral line, correlated turbulent motions of atoms and molecules give rise to considerable changes in the shape of a spectral line. It was found for the first time that the mean intensity $I_\nu^{(0)}$ ($I_\nu = I_\nu^{(0)} + I'_\nu$, $\langle I'_\nu \rangle = 0$) obeys the usual radiative transfer equation with renormalized extinction factor $\alpha_\nu^{\rm eff}$ if the correlation length R_0 of the turbulence is small as compared to a photon free path. A simple analytical expression for $\alpha_\nu^{\rm eff}$ is given. This expression integrally depends on the two-point correlation function of the turbulent velocity field. We also discuss the problem how to obtain the main turbulence parameters from the analysis of the shape of a spectral line.

Key words. line: formation – turbulence – masers

1. Introduction

In turbulent atmospheres of stars and in interstellar clouds the optical depth τ_ν ($d\tau_\nu = \alpha_\nu ds$) of the radiation path is a stochastic quantity. As a result, the intensity $I_\nu(s)$ of the radiation, its polarization and the shape of the spectral line acquire a stochastic component. A typical example is the effect of star light variation due to its propagation through the turbulent Earth's atmosphere.

As for the shape of a spectral line, it was found that, apart from thermal broadening, there exists additional Doppler broadening due to small-scale chaotic turbulent motions of atoms and molecules. The total line broadening $\Delta\nu_{\rm D}$ is determined by the expression:

$$(\Delta\nu_{\rm D})^2 = (\Delta\nu_{\rm Th})^2 + (\Delta\nu_k)^2 = \frac{2}{3}\cdot\frac{\nu_0^2}{c^2}\left(u_{\rm Th}^2 + u_k^2\right), \quad (1)$$

where $u_{\rm Th}^2 = 2k_{\rm B}T/m$ and $u_k^2 = \langle u^2(\boldsymbol{r}, t)\rangle$ are rms values of thermal and turbulent velocities respectively, ν_0 is the central frequency of the spectral line, c is the speed of light, $k_{\rm B}$ is the Boltzmann constant, T is the temperature of the atmosphere. The brackets $\langle I\rangle$ denote the ensemble average of the stochastic quantity I.

Formula (1) has been derived under the assumption that both the thermal velocity of atoms or molecules and the velocity of turbulent motions are, so to say, "white noise", i.e., these velocities are not correlated even in the nearest space-time vicinity.

The real turbulent velocity $\boldsymbol{u}(\boldsymbol{r}, t)$ is always correlated. There exist spatial interval $|\boldsymbol{r} - \boldsymbol{r}'| \equiv R \simeq R_0$ and time interval $|t - t'| \equiv \tau \simeq \tau_0$ where the velocity values are correlated. The case of uncorrelated velocities formally corresponds to the limits $R_0 \to 0$ and $\tau_0 \to 0$.

Turbulence gives rise to stochasticity of physical parameters of the atmosphere such as magnetic field, number density of scattering or absorbing particles, etc. As a result, the extinction coefficient α_ν is also a stochastic quantity. The radiative transfer equation (see Rybicki & Lightman 1979)

$$(\boldsymbol{n}\nabla)I_\nu(\boldsymbol{n}, s) \equiv \frac{dI_\nu(\boldsymbol{n}, s)}{ds} = -\alpha_\nu(\boldsymbol{n}, s)I_\nu(\boldsymbol{n}, s) + j_\nu(\boldsymbol{n}, s) \quad (2)$$

in a turbulent atmosphere acquires a stochastic meaning. Here $\boldsymbol{n}$ is the direction of the light propagation, s is the path along $\boldsymbol{n}$, j_ν is the source of radiation, α_ν is the extinction factor.

Usually, thermal Doppler broadening $\Delta\nu_{\rm Th}$ is much greater than the natural linewidth, and the extinction factor can be taken as

$$\alpha_\nu = \alpha_0 \exp\left(-\frac{\nu - \nu_0 - \boldsymbol{U}_0\boldsymbol{n}\dfrac{\nu_0}{c} - \boldsymbol{u}\boldsymbol{n}\dfrac{\nu_0}{c}}{\Delta\nu_{\rm Th}}\right)^2. \quad (3)$$

Here $\boldsymbol{U}_0$ is the regular velocity of the medium, $\boldsymbol{u}$ is the stochastic (turbulent) velocity ($\langle\boldsymbol{u}\rangle = 0$), $\alpha_0 = N_0\sigma_0\sqrt{\pi}/\Delta\nu_{\rm Th}$ is the extinction factor at the line center, N_0 is the number density of atoms or molecules at the resonant level, σ_0 is the quantum-mechanical cross section at $\nu = \nu_0$. Strictly speaking, factor α_0 is also a stochastic quantity, because it is proportional to the number density N_0. Obviously, N_0, as a certain passive admixture, in the turbulent medium is a stochastic value. Studying the shape of a spectral line, we neglect the stochasticity of N_0, because it gives rise to effects analogous to those in the continuum radiation. As shown by Silant'ev (2005), the effect of turbulence in non-magnetized atmospheres results in a small decrease of the optical depth as compared to its direct averaged value.

The relation of the solutions of stochastic equation (2) to the observed radiation intensity depends on the particular conditions of the observation: exposure time, angular resolution of the telescope, parameters R_0, τ_0, u_k, etc. In this paper we study the shape of a spectral line only for the average intensity. The conditions when the observed intensity coincides with the ensemble averaged value $I_\nu^{(0)}$ ($I_\nu = I_\nu^{(0)} + I'_\nu$, $\langle I'_\nu \rangle = 0$) were discussed in detail by Levshakov & Kegel (1997). Usually, the exposure time is less than the lifetime τ_0 of turbulent motions and the coincidence of $I_\nu^{(0)}$ with the observed intensity requires the existence of multiple turbulent cells in the observed region. In observations of stars this condition is implemented nearly always; it takes place more seldom in the observation of light propagation in turbulent interstellar clouds. In the latter case it is necessary to observe a very large part of a cloud.

As will be shown, the effect of correlated turbulence on the line shape increases with increasing optical depth of the light path. It seems that this effect can be observed most readily in the propagation of maser emission. Usually the regions with inverted populations of molecular levels are observed as small "spots" inside the large molecular clouds. Intense maser emission can be observed even after the passage of an optically thick layer. In this case the radiation penetrates many turbulent cells and the observed emission coincides with the average intensity $I_\nu^{(0)}$.

Of course, the analysis of the fluctuations of the observed intensity is also very informative for estimation of turbulence parameters in stellar atmospheres, interstellar clouds and circumstellar envelopes.

The effect of turbulence on the propagation of continuum radiation is rather small (see Silant'ev 2005). It consists mostly in a decrease of the effective optical depth

$$\tau_{\rm eff} \cong \tau_0 - \tau_1 \frac{\langle \alpha'^2 \rangle}{\alpha_0^2}, \tag{4}$$

where $\tau_1 = \alpha_0 R_0$ is the mean optical depth of the region of turbulent correlations, α_0 and α' are the mean and fluctuation parts of the extinction factor respectively. Note that $\tau_{\rm eff}$ describes the propagation of the mean intensity $I_\nu^{(0)}$.

This gives rise to the coincidence of the angular distribution and polarization of radiation outgoing from a turbulent semi-infinite atmosphere with those for a non-turbulent one. However, in a magnetized plasma atmosphere (due to the Faraday rotation effect) the magnetic field fluctuations, frozen into the turbulent conducting medium, decrease very strongly the linear polarization degree $p(\lambda)$. The spectra of $p(\lambda)$ and the position angle $\chi(\lambda)$ of the outgoing radiation change very strongly compared to those for a non-turbulent plasma atmosphere (Silant'ev 2005).

The effect of turbulent correlations on the spectral line shape was investigated in a number of papers (Hundt 1973; Auvergne et al. 1973; Gail et al. 1974, 1975; Schatzman & Magnan 1975; Traving 1975; Magnan 1976; Frisch & Frisch 1976; Mitskevich et al. 1993; Levshakov & Kegel 1994, 1996, 1997; Böger et al. 2003; etc.). The influence of turbulence on the shape of X-ray lines in clusters of galaxies was investigated by Inogamov & Sunyaev (2003).

Schatzman & Magnan (1975) have taken for the first time into account the correlation between the velocity and temperature in the large-scale turbulence (convective bubbles). This allowed them to explain the shift and asymmetry of lines formed in thermally driven turbulence.

Magnan (1976) has elaborated the "effective cells" approach: "cells" qualitatively represent regions of the atmosphere with correlated turbulent velocities. Inside an effective cell the turbulent velocity is uniform in every realization, and the one-dimensional distribution of turbulent velocities is described by the Gaussian ensemble of realizations. The velocities in neighbouring cells are statistically independent. Considering the cells as plane-parallel layers of the atmosphere and introducing the coefficients of reflection and transmission of radiation for an individual sell, Magnan has constructed an efficient numerical procedure to find these coefficients for atmospheres consisting of n "effective cells". The case $n = 1$ corresponds to a macroturbulent atmosphere and $n \gg 1$ represents a usual turbulent atmosphere in the absence of a finite correlation length. Seemingly Magnan has found for the first time that statistically the turbulent medium with a finite length of velocity correlation (the thickness of an "effective layer" in his theory) is more transparent than the turbulent atmosphere with a vanishing correlation length ("short-correlated" turbulence). In spite of its crude assumptions, Magnan's theory qualitatively truly describes the influence of the turbulence on the shape of a spectral line.

Gail et al. (1974) and Traving (1975) have developed a more realistic approach to solve the problem. They approximated the stochastic radiative transfer Eq. (2), depending on the turbulent velocities $\boldsymbol{u}(\boldsymbol{r}, t)$, by a stochastic Fokker-Planck equation. The radiative transfer process is assumed to be Markovian. The derived Fokker-Planck equation was numerically solved in a number of the above-mentioned papers. As in Magnan's paper, it was found that a finite turbulent correlation length changes very strongly the spectral line shape, both for absorption and emission lines.

Especially impressive results were obtained numerically by Böger et al. (2003) for a maser emission line in a turbulent cloud with inverted populations of molecular levels. Böger et al. calculated the intensity and line shape in the unsaturated mode. The calculated maser spectra have a very complicated shape, which looks like really observed maser spectra. Usually, one considers such spectra as a superposition of emission from separate maser spots. Böger et al. mostly present non-averaged spectra. They correspond to various particular realizations of the turbulent velocity field, and many of them have very small linewidths. It seems that these small linewidths are the consequence of very large values of negative optical depths ($\tau_0 = -110, -400$) at the line center. In the averaged spectra these features disappear.

Levshakov & Kegel (1997), using the Fokker-Planck equation, explain the line asymmetry by introducing an additional parameter – hydrodynamic velocity u_0 at the edge of the turbulent cloud. To some extent this looks like introducing a surface cell in Loucif & Magnan (1982).

The numerical calculations in the above-mentioned papers have demonstrated that, in contrast to the usually used short-correlated model of turbulence, turbulence with a finite length of correlation changes significantly the line shape of both absorption and emission spectral lines.

All the features of the influence of turbulence on the spectral line shape exist in the presented below simple analytical theory of radiative transfer in turbulent atmospheres. Here we first derive the usual radiative transfer equation for the averaged intensity in turbulent atmospheres with renormalized (effective) absorption coefficient $\alpha_\nu^{\rm eff}$. The simple analytical expression for $\alpha_\nu^{\rm eff}$ obtained in our paper depends on the explicit form of the two-point turbulent velocity correlation function. In addition, we give a qualitative explanation of these effects based on explicit formulae of our analytical approach, which essentially complements the analogous discussion of Magnan (1976).

Our discussion stresses more explicitly the statistical nature of the effects and maintains also the discussion about the physical realization of the "effective cells" model.

In this paper we investigate the effect of finite correlation length R_0 of turbulence on the spectral line shape using the assumptions that stochastic optical length τ_ν ($\mathrm{d}\tau_\nu = \alpha_\nu(z)\mathrm{d}s$) is a Gaussian stochastic quantity and the turbulence is statistically uniform and isotropic. As known (see Van Kampen 1981; Gardiner 1985), Gaussian stochastic fields very often occur in the nature. Most theoretical papers studying turbulence use this assumption. This approach is simpler than the Fokker-Planck equation approach, because we use the exact explicit solution of the stochastic radiative transfer Eq. (2) without the source term:

$$I_\nu(s) = I_\nu(0)\mathrm{e}^{-\tau_\nu(s)}, \tag{5}$$

$$\tau_\nu(s) \equiv \tau_\nu^{(0)}(s) + \tau'_\nu(s) = \int_0^s \mathrm{d}s'\alpha_\nu(s'). \tag{6}$$

Below we study mainly the shape of the absorption line, which corresponds to the mean intensity $I_\nu^{(0)}(s)$ from Eq. (5). The analysis of this expression shows that for the case $s \gg R_0$ (s is the thickness of a turbulent layer) the averaged intensity $I_\nu^{(0)}(s)$ obeys the usual radiative transfer Eq. (2) with renormalized extinction factor $\alpha_\nu^{\mathrm{eff}}$. This factor depends linearly on the mean optical thickness of the correlation length $\tau_1 = \langle\alpha_\nu(\nu = \nu_0)\rangle R_0$ at the line center:

$$\alpha_\nu^{\mathrm{eff}} = \langle\alpha_\nu\rangle[\, 1 - \tau_1\, f_\nu(\xi)\,], \tag{7}$$

where function $f_\nu(\xi)$ integrally depends on the correlation function of the turbulent velocities, and $\xi = \Delta\nu_{\mathrm{Th}}/\Delta\nu_k = u_{\mathrm{Th}}/u_k$ is the ratio of the thermal Doppler width to the turbulent one. With decreasing τ_1 and increasing ξ, the second term in Eq. (7) vanishes. In this case correlations of turbulent velocities do not affect the line shape and only the usual turbulent Doppler width $\Delta\nu_k$ is present.

A simple model with a constant source function $S_\nu = j_\nu/\alpha_\nu$ is used very often (see Rybicki & Lightman 1979; Böger et al. 2003). In this case the analytical solution of the radiative transfer Eq. (2) also has a very simple form:

$$I_\nu(s) = I_\nu(0)\mathrm{e}^{-\tau_\nu(s)} + S_\nu\left(1 - \mathrm{e}^{-\tau_\nu(s)}\right). \tag{8}$$

The mean intensity $I_\nu^{(0)}$ in this model is also determined by the term $\langle\exp(-\tau_\nu(s))\rangle$, which, in the case of Gaussian processes, has a very simple expression:

$$\langle\mathrm{e}^{-\tau_\nu(s)}\rangle = \mathrm{e}^{-\tau_\nu^{(0)}(s)}\,\mathrm{e}^{\frac{1}{2}\langle\tau_\nu'^2(s)\rangle}. \tag{9}$$

The mean value of the square of fluctuations $\langle\tau_\nu'^2(s)\rangle$ is related to the turbulent correlation function in a rather simple way (see below).

Thus, averaging Eq. (5) yields the following expression for the mean intensity:

$$I_\nu^{(0)}(s) = I_\nu(0)\mathrm{e}^{-\tau_\nu^{(0)}(s)}\,\mathrm{e}^{\frac{1}{2}\langle\tau_\nu'^2(s)\rangle} \equiv I_\nu(0)\mathrm{e}^{-\tau_\nu^{\mathrm{eff}}(s)}. \tag{10}$$

The intensity fluctuations can be estimated as

$$\langle I_\nu'^2(s)\rangle = \left(I_\nu^{(0)}(s)\right)^2\left[\mathrm{e}^{\langle\tau_\nu'^2(s)\rangle} - 1\right]. \tag{11}$$

From Eq. (10) follows the definition of the effective optical depth $\tau_\nu^{\mathrm{eff}}(s)$:

$$\tau_\nu^{\mathrm{eff}}(s) = \tau_\nu^{(0)}(s) - \frac{1}{2}\langle\tau_\nu'^2(s)\rangle.$$

According to this formula, $\tau_\nu^{\mathrm{eff}} \le \tau_\nu^{(0)}$, i.e., the turbulent atmosphere is effectively more transparent than the non-turbulent one for all the frequencies. This is a purely statistical effect. Remember that $\tau_\nu^{\mathrm{eff}}(s)$ describes the propagation of the mean intensity $I_\nu^{(0)}(s)$ in the atmosphere. The mean intensity has only a statistical meaning. Earlier we have briefly discussed the condition when this statistical value coincides with the observed intensity. In an atmosphere with inverted populations of resonant atoms or molecules $\tau_\nu^{(0)} < 0$, and the absolute value of τ_ν^{eff} is larger than $|\tau_\nu^{(0)}|$, i.e., a turbulent medium increases the intensity of resonant radiation more efficiently than a non-turbulent one.

Our explicit formulae enable us to estimate the correlation length R_0 of the turbulence, its characteristic velocity u_k and the ratio of thermal and turbulent velocities u_{Th}/u_k. It is easier to analyze the usual radiative transfer equation for the mean intensity with the effective absorption factor, derived here for the first time, than the more complicated Fokker-Planck equation.

2. Basic formulae

To obtain the explicit relationship for the mean intensity $I_\nu^{(0)}(s)$ (see Eq. (10)), we calculate first $\langle\tau_\nu'^2(s)\rangle$; it is

$$\langle\tau_\nu'^2(s)\rangle = \langle\tau_\nu^2(s)\rangle - \langle\tau_\nu(s)\rangle^2. \tag{12}$$

When deriving terms in Eq. (12), we must know the mean values $\langle\alpha_\nu(s)\rangle$ and $\langle\alpha_\nu(s)\alpha_\nu(s')\rangle$, which can be readily obtained using formulae:

$$\alpha_\nu(s) = \frac{\alpha_0\Delta\nu_{\mathrm{Th}}}{\sqrt{\pi}} \times \int_0^\infty \mathrm{d}t\cos\left[\left(\nu - \nu_0 - \frac{\boldsymbol{U}_0\boldsymbol{n}}{c}\nu_0 - \frac{\boldsymbol{un}}{c}\nu_0\right)t\right]\mathrm{e}^{-\Delta\nu_{\mathrm{Th}}^2t^2/4}, \tag{13}$$

$$\langle\cos\left(\nu - \nu_0 - \frac{\boldsymbol{U}_0\boldsymbol{n}}{c}\nu_0 - \frac{\boldsymbol{un}}{c}\nu_0\right)t\rangle = \cos\left[\left(\nu - \nu_0 - \frac{\boldsymbol{U}_0\boldsymbol{n}}{c}\nu_0\right)t\right]\mathrm{e}^{-\Delta\nu_k^2t^2/4}. \tag{14}$$

Recall that $\Delta\nu_k^2 = 2u_k^2\nu_0^2/3c^2$, and $u_k^2 = \langle u^2\rangle$. In the derivation of expression (14) we take into account that for a Gaussian ensemble of turbulent velocities $\boldsymbol{u}$ the mean value of an odd number of velocity components is zero and the mean value of an even number of components is equal to the sum of all possible pair correlations. For example,

$$\langle u_i(1)u_j(2)u_p(3)u_q(4)\rangle = \langle u_i(1)u_j(2)\rangle\langle u_p(3)u_q(4)\rangle + \langle u_i(1)u_p(3)\rangle\langle u_j(2)u_q(4)\rangle + \langle u_i(1)u_q(4)\rangle\langle u_j(2)u_p(3)\rangle. \tag{15}$$

For brevity we denote here the space-time coordinates by numbers. Expression (9) was also obtained using these simple rules.

Using Eqs. (13) and (14), we can average Eq. (6):

$$\langle\tau_\nu(s)\rangle \equiv \tau_\nu^{(0)}(s) = \langle\alpha_\nu\rangle s, \tag{16}$$

$$\langle\alpha_\nu\rangle = \alpha_0\frac{\Delta\nu_{\mathrm{Th}}}{\Delta\nu_{\mathrm{D}}}\mathrm{e}^{-(x-x_0)^2}, \tag{17}$$

where the total Doppler linewidth $\Delta\nu_{\mathrm{D}}$ can be found from the relationship $(\Delta\nu_{\mathrm{D}})^2 = (\Delta\nu_{\mathrm{Th}})^2 + (\Delta\nu_k)^2$. Thus, it consists of the contributions of thermal and turbulent chaotic velocities. We also introduce here dimensionless frequencies:

$$x = \frac{\nu - \nu_0}{\Delta\nu_{\rm D}}, \qquad x_0 = \frac{\boldsymbol{U}_0\boldsymbol{n}}{c} \cdot \frac{\nu_0}{\Delta\nu_{\rm D}}. \tag{18}$$

Remember that the velocity of the regular motion of atoms and molecules $\boldsymbol{U}_0$ is assumed to be constant. Due to homogeneity of turbulence, the mean extinction factor $\langle\alpha_\nu(s)\rangle \equiv \langle\alpha_\nu\rangle$ is independent of the distance s. It is visible from Eq. (17) that $\langle\alpha_\nu\rangle$ has a Gaussian shape with total linewidth $\Delta\nu_{\rm D}$.

The expression $\langle\alpha_\nu(s)\alpha_\nu(s')\rangle$ can be obtained in a similar way:

$$\langle\alpha_\nu(s)\alpha_\nu(s')\rangle = \frac{\alpha_0^2\,\Delta\nu_{\rm Th}^2}{2\pi}\int_0^\infty {\rm d}t\int_0^\infty {\rm d}t'{\rm e}^{-\Delta\nu_{\rm Th}^2(t^2+t'^2)/4}$$
$$\times\left\{{\rm e}^{-\Delta\nu_{\rm Th}^2(t^2+t'^2+2b_{||}(R)tt')/4}\cos\left[\left(\nu-\nu_0-\frac{\boldsymbol{U}_0\boldsymbol{n}}{c}\nu_0\right)(t+t')\right]\right.$$
$$\left.+{\rm e}^{-\Delta\nu_{\rm Th}^2(t^2+t'^2-2b_{||}(R)tt')/4}\cos\left[\left(\nu-\nu_0-\frac{\boldsymbol{U}_0\boldsymbol{n}}{c}\nu_0\right)(t-t')\right]\right\}. \tag{19}$$

Here $R = |s - s'|$ is the distance along the line of sight $\boldsymbol{n}$, $b_{||}(R)$ is the correlation function of the uniform and isotropic turbulence. It is determined by the expression:

$$\langle u_{||}(s)u_{||}(s')\rangle = \frac{u_k^2}{3}b_{||}(R), \qquad u_{||}(s) = \boldsymbol{un}. \tag{20}$$

Function $b_{||}(R)$ at $R = 0$ is equal to unity and it rapidly vanishes at $R > R_0$. Remember that R_0 is the characteristic length of turbulent eddies (cells).

The following models of a turbulence are frequently used:

$$(1) \qquad b_{||}(R) = 1 - \left(\frac{R}{R_0}\right)^2 \text{ for } R \le R_0,$$
$$b_{||}(R) = 0 \qquad \text{for } R > R_0, \tag{21}$$

$$(2) \qquad b_{||}(R) = \exp\left(-\frac{R}{R_0}\right). \tag{22}$$

A special case is the so called acoustic turbulence, i.e., chaotic superposition of acoustic waves. For acoustic turbulence one frequently takes

$$(3) \qquad b_{||}(R) = \frac{\sin(R/R_0)}{R/R_0}. \tag{23}$$

Introducing new variables $T = t + t'$, $\tau = t - t'$, expression (19) can be calculated in the analytical form:

$$\langle\alpha_\nu(s)\alpha_\nu(s')\rangle \equiv C(R)$$
$$= \langle\alpha_\nu\rangle\frac{\alpha_0\Delta\nu_{\rm Th}}{\Delta\nu_{\rm D}}\frac{\exp\left[-(x-x_0)^2\,\dfrac{1-\eta b_{||}(R)}{1+\eta b_{||}(R)}\right]}{\sqrt{1-\eta^2 b_{||}^2(R)}}, \tag{24}$$

where we introduce dimensionless parameters:

$$\eta = \frac{\Delta\nu_k^2}{\Delta\nu_{\rm D}^2} = \frac{\Delta\nu_k^2}{\Delta\nu_{\rm Th}^2 + \Delta\nu_k^2} \equiv \frac{1}{1+\xi^2},$$
$$\xi = \frac{\Delta\nu_{\rm Th}}{\Delta\nu_k} = \frac{u_{\rm Th}}{u_k}. \tag{25}$$

At $R \gg R_0$ ($b_{||}(R) \simeq 0$) the values of $\alpha_\nu(s)$ and $\alpha_\nu(s')$ are virtually independent and the expression (19) tends to $\langle\alpha_\nu(s)\alpha_\nu(s')\rangle \to \langle\alpha_\nu\rangle^2$.

Using Eq. (24), we can write the formula for $\langle\tau_\nu^2\rangle$:

$$\langle\tau_\nu^2\rangle = \int_0^s {\rm d}s'\int_0^s {\rm d}s''\langle\alpha_\nu(s')\alpha_\nu(s'')\rangle$$
$$= \int_0^s {\rm d}s'\int_0^s {\rm d}s''\,C(|s'-s''|)$$
$$= \int_0^s {\rm d}R\,C(R)\int_R^{2s-R}{\rm d}r = 2\int_0^s {\rm d}R\,(s-R)\,C(R). \tag{26}$$

Here we have used substitutions $r = s' + s''$ and $R = s' - s''$ and taken into account that $C(R)$ is an even function of R.

Finally, expression (12) for $\langle\tau_\nu'^2(s)\rangle$ becomes:

$$\langle\tau_\nu'^2(s)\rangle = 2\tau_\nu^{(0)}(s)\frac{\alpha_0\Delta\nu_{\rm Th}}{s\Delta\nu_{\rm D}}\left\{-\frac{s^2}{2}{\rm e}^{-(x-x_0)^2}\right.$$
$$\left.+\int_0^s {\rm d}R\left[(s-R)\frac{\exp\left[-(x-x_0)^2\,\dfrac{1-\eta b_{||}(R)}{1+\eta b_{||}(R)}\right]}{\sqrt{1-\eta^2 b_{||}^2(R)}}\right]\right\}. \tag{27}$$

Note that $\alpha_0 \propto \Delta\nu_{\rm Th}$ and Eq. (27) depends only on $\Delta\nu_{\rm D}$ and $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k$.

Now mean intensity (10) can be written as

$$I_\nu^{(0)}(s) = I_\nu(0)A(s, x-x_0), \tag{28}$$

where transmission coefficient A_ν is

$$A(s, x-x_0) = {\rm e}^{-\tau_\nu^{\rm eff}(s)}, \tag{29}$$

$$\tau_\nu^{\rm eff}(s) = \tau_\nu^{(0)}(s)\left\{1 + \frac{\tau_1 s}{2R_0}{\rm e}^{-(x-x_0)^2}\right.$$
$$\left.-\tau_1\int_0^{s/R_0}{\rm d}y\left(\left(1-y\frac{R_0}{s}\right)\frac{\exp\left[-(x-x_0)^2\,\dfrac{1-\eta b_{||}(y)}{1+\eta b_{||}(y)}\right]}{\sqrt{1-\eta^2 b_{||}^2(y)}}\right)\right\}. \tag{30}$$

Here we have used the following dimensionless quantities:

$$y = \frac{R}{R_0}, \qquad \tau_1 = \alpha_0\frac{\Delta\nu_{\rm Th}}{\Delta\nu_{\rm D}}R_0, \qquad \tau_0 = \tau_1 s/R_0. \tag{31}$$

Parameters τ_1 and τ_0 are the optical depths of the correlation length R_0 and of the layer s at the line center respectively; $\tau_\nu^{(0)}(s)$ is the mean optical depth of the distance s (see Eq. (16)). The terms with τ_1 in Eq. (30) describes the contribution of turbulent motions to the effective optical depth $\tau_\nu^{\rm eff}(s)$ of the layer s. When the atmosphere is non-turbulent ($\boldsymbol{u} = 0$, $\Delta\nu_k = 0$, $\eta = 0$) this term is zero for any τ_1. In this case $\tau_\nu^{(0)}(s)$ is determined by Eqs. (16) and (17), where $\Delta\nu_{\rm D} = \Delta\nu_{\rm Th}$. Of course, the choice of parameters must preserve $\tau_\nu^{\rm eff}(s) > 0$ for the usual medium with non-inverted population of atomic levels.

If the correlation length $R_0 \to 0$, i.e., $\tau_1 \to 0$ (but $\langle \boldsymbol{u}^2 \rangle \neq 0$), then the terms with τ_1 vanishes for any correlation functions $b_{\|}(R)$ and any values of η. This limit corresponds to the case of uncorrelated turbulence when the effect of turbulence is reduced to an additional line broadening only. If τ_1 does not tend to zero, but parameter $\eta \ll 1$ ($\xi \gg 1$), the term with the correlation function $\eta b_{\|}(R) \ll 1$ is insignificant. In these limits the transmission coefficient is

$$A_0(s, x - x_0) = \mathrm{e}^{-\tau_\nu^{(0)}(s)}. \tag{32}$$

Below we will compare coefficients $A(s, x-x_0)$ and $A_0(s, x-x_0)$.

For estimations we can use the explicit expression of $\tau_\nu^{\mathrm{eff}}(s)$ for the crude correlation function $b_{\|}(R) = 1$ at $R \leq R_0$ and equal to zero at $R > R_0$:

$$\tau_\nu^{\mathrm{eff}}(s) = \tau_\nu^{(0)}(s)$$
$$\times \left\{ 1 - \tau_1 \left(1 - \frac{R_0}{2s}\right) \left[\frac{\exp\left(-(x-x_0)^2 \dfrac{1-\eta}{1+\eta}\right)}{\sqrt{1-\eta^2}} - \mathrm{e}^{(x-x_0)^2} \right] \right\}. \tag{33}$$

The largest deviation of τ_ν^{eff} from the short-correlated value $\tau_\nu^{(0)}$ occurs at the line center $x = x_0$. In this case Eq. (33) acquires a simple form:

$$\tau_0^{\mathrm{eff}} = \tau_0 \cdot \left\{ 1 - \tau_1 \left(1 - \frac{R_0}{2s}\right) \left[\frac{1+\xi^2}{\xi\sqrt{2+\xi^2}} - 1 \right] \right\}. \tag{34}$$

Here we have used a physically clearer parameter $\xi = u_{\mathrm{Th}}/u_k$ (see Eq. (25)).

Expression (34) can be used for the estimation of the contribution of finite-correlation turbulence to the effective optical depth. It is clear that the case $s = R_0$, ($\tau_1 = \tau_0$) is the limiting case when we can at all speak about the turbulence in a layer of material. It is the limiting case of macroturbulence. Thus, the maximum contribution is $\tau_0[(1+\xi^2)/(\xi\sqrt{2+\xi^2}-1)]/2$. The term in square brackets for $\xi = 0.1$, 0.2, 0.5, 1 and 2 is equal to 6.14, 2.64, 0.67, 0.15 and 0.02, respectively. It should be noted that the exponential dependence of transmission coefficients $A(s, x - x_0)$ and $A_0(s, x - x_0)$ increases the relative contribution of finite-correlation turbulence to the spectral line shape for larger optical depths τ_0:

$$\frac{A(s,0)}{A_0(s,0)} = \exp\left\{ \tau_1\tau_0 \left[\frac{1+\xi^2}{\xi\sqrt{2+\xi^2}} - 1 \right] \left(1 - \frac{R_0}{2s}\right) \right\}. \tag{35}$$

Let us take for example $\xi = 0.5$, $R_0/s = 0.1$. Formulae (34) and (35) yield $(\tau_0^{\mathrm{eff}} - \tau_0^{(0)}) \equiv \Delta\tau = 0.063\tau_0^2$, $\Delta\tau/\tau_0 = 6.3\tau_0\%$, $\Delta A/A_0 = (\exp(\Delta\tau) - 1)$. For $\tau_0 = 1, 2, 3, 4$ the quantities $\Delta\tau/\tau_0$ and $\Delta A/A_0$ take the values 6.3%, 12.6%, 18.9%, 25.2% and 6.5%, 13.4%, 76.3%, 174%, respectively. Thus, relatively small differences between τ_ν^{eff} and $\tau_\nu^{(0)}$ result in very large differences in the transmission coefficients for $\tau_0 \geq 2$.

It is interesting to compare our results with those obtained by Magnan (1976), who developed the "effective cells" approximation to describe turbulence with a finite correlation length. To use the method of addition of layers, Magnan considered plane-parallel layers with thickness R_0 as statistically independent "effective cells" with a uniform distribution of turbulent velocities inside every cell. The one-dimensional distribution of velocities of each cell is taken as a Gaussian one. Thus, a crude correlation function $b_{\|}(|z-z'|) = 1$ at $|z-z'| \leq R_0$, and $b_{\|} = 0$ at $|z-z'| > R_0$ was used. Here z and z' are the coordinates along the normal to the surface of the layer.

Firstly, we compare the values of $A(s, 0) = \exp(-\tau_0^{\mathrm{eff}})$ corresponding to our three-dimensional crude correlation function (see Eq. (34)) with Magnan's $A(s, 0)$ at $\tau_0 = 2.3025$ ($A_0(s, 0) = \exp(-\tau_0) = 0.1$) and $\xi = u_{\mathrm{Th}}/u_k = 0.2$. For $s/R_0 \equiv n = 16$ we have, instead of Magnan's value 0.15, the value 0.23. For $n = 8$ the corresponding values are 0.21 and 0.52. The difference in the results is rather large.

The analogous comparison of the results for the correlation functions (21), (22) and (23) yields, instead of Magnan's value 0.15 (for $n = 16$), the values 0.13, 0.116 and 0.19, respectively. For $n = 8$, instead of Magnan's value 0.21, we have obtained the values 0.166, 0.133 and 0.35, respectively.

Thus, the comparison of the results demonstrates that the "effective cells" model can be considered only as a qualitative model. However, qualitatively this model describes all the features of the influence of finite-correlation turbulence on the shape of a spectral line.

3. The radiative transfer equation for the mean intensity $I_\nu^{(0)}(s)$

The terms with τ_1 in Eq. (30) depends on the distance s, i.e., $\tau_\nu^{\mathrm{eff}}(s)$ is a nonlinear function of s, and the expression (29) is not interpreted as a solution of radiative transfer Eq. (2) with any effective extinction coefficient $\alpha_\nu^{\mathrm{eff}}$. However, in the case $R_0 \ll s$ this is possible. Indeed, in this case the term with τ_1 virtually does not depend on s, because correlation function $b_{\|}(R)$ rapidly vanishes at $R \geq R_0$. In this approximation ($s \gg R_0$) expression (30) becomes

$$\tau_\nu^{\mathrm{eff}}(s) = \tau_\nu^{(0)}(s) \left\{ 1 + \tau_1 \left(1 - \frac{R_0}{2s}\right) \mathrm{e}^{-(x-x_0)^2} \right.$$
$$\left. -\tau_1 \int_0^1 \mathrm{d}y \left(\left(1 - y\frac{R_0}{s}\right) \frac{\exp\left[-(x-x_0)^2 \dfrac{1-\eta b_{\|}(y)}{1+\eta b_{\|}(y)}\right]}{\sqrt{1-\eta^2 b_{\|}^2(y)}} \right) \right\} \cdot \tag{36}$$

For the crude correlation function $b_{\|}(r) = 1$ at $R \leq R_0$ this expression transforms to expression (33).

Neglecting small terms $\simeq R_0/s \ll 1$, we obtain that the term with τ_1 in Eq. (30) does not depend on s. In this case formula (29) can be considered as the solution of the radiative transfer equation for the mean intensity $I_\nu^{(0)}(s)$ with renormalized effective extinction factor:

$$\frac{\mathrm{d}I_\nu^{(0)}(s)}{\mathrm{d}s} = -\alpha_\nu^{\mathrm{eff}} I_\nu^{(0)}(s) + \langle j_\nu(s) \rangle, \tag{37}$$

$$\alpha_\nu^{\mathrm{eff}} = \langle \alpha_\nu \rangle \left\{ 1 + \tau_1 \mathrm{e}^{-(x-x_0)^2} \right.$$
$$\left. -\tau_1 \int_0^1 \mathrm{d}y \frac{\exp\left[-(x-x_0)^2 \dfrac{1-\eta b_{\|}(y)}{1+\eta b_{\|}(y)}\right]}{\sqrt{1-\eta^2 b_{\|}^2(y)}} \right\}. \tag{38}$$

When deriving Eq. (38), we assume that R_0 corresponds to the distance at which the correlation of turbulent velocities almost disappears, i.e., $b_{||}(R_0) \simeq 0$. For this reason, in Eq. (38) the upper integration limit can be taken to be ∞; this makes the calculations simpler. In the radiative transfer theory the characteristic length is the unit optical depth. Therefore, the validity condition of the usual radiative transfer Eq. (37) is $\tau_1 \ll 1$, i.e., the optical depth at the center of the line of the correlation region is to be small.

Simple formulae (33)–(35) demonstrate that at $\tau_1 \ll 1$ the difference between $\tau_\nu^{\rm eff}$ and $\tau_\nu^{(0)}$ can be sufficiently large if parameter $\xi \leq 1$. Just at these conditions the new radiative transfer Eq. (37) describes the sufficiently large influence of the correlated turbulence on the shape of spectral lines.

The solution methods for the usual radiative transfer Eq. (37) are known. It is easier to solve Eq. (37) for the calculation of the mean intensity $I_\nu^{(0)}(s)$ than to use the more complicated Fokker-Planck equation, derived by Gail et al. (1974) and Traving (1975). Particularly, the method of addition of layers, successfully applied by Magnan (1976, 1985, 1993) to the solution of radiative transfer problems in turbulent atmospheres, can be used for solving the exact Eq. (37). In this case one does not need to employ the rather crude model of effective turbulent cells, introduced in these papers.

4. Qualitative discussion of the results

It is visible from Eqs. (30) and (38) that the correlations of turbulent velocities give rise to a decrease in $\alpha_\nu^{\rm eff}$ and $\tau_\nu^{\rm eff}(s)$ as compared to the mean values $\langle\alpha_\nu\rangle$ and $\tau_\nu^{(0)}(s)$ at $\alpha_0 > 0$, and, on the contrary, these effective coefficients are greater than the mean values at $\alpha_0 < 0$ (the case of an atmosphere with inverted populations of atomic levels). This is a purely statistical effect. Below we present two qualitative models which help us to understand this effect.

First we consider non-statistically the simple model of correlated turbulence and demonstrate that this effect ($A(x) > A_0(x)$) arises even in this model for $x \approx 0$. For simplicity we accept that the regular motion of atoms and molecules is absent, i.e., $\boldsymbol{U}_0 = 0$. The condition $\langle\boldsymbol{u}\rangle = 0$ means that inside the correlation region R_0 there are two subregions with mutually opposite turbulent velocities $\boldsymbol{u}$ and $-\boldsymbol{u}$, so the mean value $\langle\boldsymbol{u}\rangle = 0$. Let the optical depths of these subregions at the line center be $\tau_1/2$. The total optical depth of these regions is $\tau_\nu(x) = (\tau_1/2)\exp[-(x-x')^2] + (\tau_1/2)\exp[-(x+x')^2]$. This yields for the transmission coefficient the expression

$$A(x) = A_1(x-x')A_2(x+x') = \exp\left[-\frac{\tau_1}{2}\left(\mathrm{e}^{-(x-x')^2} + \mathrm{e}^{-(x+x')^2}\right)\right], \tag{39}$$

where $x' = (u_{||}/c)(\nu_0/\Delta\nu_D)$. We consider here the case $x'^2 \leq 0.5$ when expression (39) has only one maximum at $x = 0$. Note that Eq. (39) depends on two characteristic velocities, corresponding to x' and $-x'$.

We compare this expression with the transmission coefficient $A_0(x)$ (see Eq. (32)), which does not take into account the finite length of the turbulent correlation. In our model the $A_0(x)$-coefficient takes the form:

$$A_0(x) = \exp\left[-\tau_1\mathrm{e}^{-x^2}\right]. \tag{40}$$

The ratio of these transmission coefficients is:

$$\frac{A(x)}{A_0(x)} = \exp\left[\tau_1\mathrm{e}^{-x^2}\left(1 - \mathrm{e}^{-x'^2}\cosh(2xx')\right)\right]. \tag{41}$$

The expression (41) at $\tau_1 > 0$ takes the maximum value at $x = 0$:

$$\frac{A(0)}{A_0(0)} = \exp\left[\tau_1\left(1 - \mathrm{e}^{-x'^2}\right)\right] > 1. \tag{42}$$

The inequality $A(x) > A_0(x)$ follows from Eq. (41) for $x \leq 0.7$ and then there exists the little opposite inequality $A(x) < A_0(x)$. Finally, at $x \to \infty$ one has $A(x) \to A_0(x)$. Thus, even this non-statistical consideration of simple correlated turbulence demonstrates that a usual, non-inverted medium becomes more transparent as compared to the case when correlations are not taken into account.

On the contrary, for an atmosphere with inve rted populations of atomic levels ($\tau_1 < 0$) we have $A(x)/A_0(x) < 1$, i.e., taking into account the finite correlation length R_0 gives rise to an effective obscuration of the medium.

The existence of inequality $A(x) > A_0(x)$ in correlated turbulence is a pure statistical effect. It can be demonstrated in a general form for all x in simple model of two realizations. Let we have two realizations of the turbulence in a layer. The first realization corresponds to optical depth $\tau_\nu = \tau_\nu^{(0)} + \tau'_\nu$, and the second one to $\tau_\nu = \tau_\nu^{(0)} - \tau'_\nu$ with the mean optical depth of the layer equal to $\tau_\nu^{(0)}$. The mean value of the intensity is

$$\langle I_\nu(s)\rangle = \frac{1}{2}\left(I(0)\mathrm{e}^{-(\tau_\nu^{(0)}+\tau'_\nu)} + I(0)\mathrm{e}^{-(\tau_\nu^{(0)}-\tau'_\nu)}\right) = I(0)\mathrm{e}^{-\tau_\nu^{(0)}}\cosh(\tau'_\nu) \geq I(0)\mathrm{e}^{-\tau_\nu^{(0)}}, \tag{43}$$

i.e., the effective optical depth $\tau_\nu^{\rm eff}$ is smaller for every frequency than the mean value $\tau_\nu^{(0)}$. For small τ'_ν we have $\cosh(\tau'_\nu) \approx 1 + \tau'^2_\nu/2 \approx \exp(\tau'^2_\nu/2)$, which coincides with the formula below Eq. (11). It means that $A(x) \equiv \exp(-\tau_\nu^{\rm eff}) \geq A_0(x)$ for any values of dimensionless frequencies x. The numerical calculations with Eq. (29) for various models of turbulence confirm this qualitative analysis. Integration of $A(x)$ and $A_0(x)$ from $(-\infty)$ to $(+\infty)$ evidently conserves the inequality. Remember that this integration is connected with the equivalent width of an absorption line.

This contradicts the statement of Magnan (1976): "the macroturbulence does not change the equivalent width". What is the cause of this discrepancy? Magnan has checked this statement only for one "effective cell". In this case there exists only one characteristic turbulent velocity u_k, and, introducing a new variable $y = x - x_k$ ($x_k = (u_k\nu_0/c)/\Delta\nu_{\rm D}$), we obtain that integration of $A(x)$ and $A_0(x)$ yields the same value. Real turbulence consists of many "effective cells", each with its own independent characteristic velocity u_k: $A(x) = A_1(x, u_k^{(1)})A_2(x, u_k^{(2)})\cdots A_n(x, u_k^{(n)})$. Clearly, in this case a common new variable y, which transforms all the $A_n(x, u^{(n)})$ to the form $A_n(y)$, does not exist.

The limiting case of macroturbulence in Magnan (1976) corresponds to $n = 1$, which means that there exists only one "effective cell", which corresponds physically to a plane-parallel layer with a uniform velocity of material in each realization. Such a realization does not describe any turbulence at all. A realistic implementation of turbulent motion must keep the position of the center of masses at rest, i.e., every physically feasible realization must have positive as well as negative velocities of material. Of course, one may consider an ensemble with positive and negative velocity directions, as it was made by Magnan, but physical implementation of such an ensemble in the nature is questionable: too artificial are the realizations of such an ensemble! It seems that the case of many "effective cells" indeed represents to some extent the real turbulent atmosphere. As a minimum, using the concept of "effective cells", we are to consider the case of two cells ($n = 2$), where Magnan's statement is already invalid. To some extent our simple model of turbulence (see Eq. (39)) corresponds to two "effective layers".

Thus, the true statement is: "the finite length turbulence diminishes the equivalent width".

Our above-considered qualitative model has taken into account only the dependence on the finite correlation length R_0. Formula (30) for $\tau_\nu^{\rm eff}(s)$ also depends on parameters s/R_0 and $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k$ (remember that $\eta = 1/(1+\xi^2)$). In a more interesting case $s \gg R_0$ the s/R_0-dependence nearly disappears. On the contrary, the dependence of $\tau_\nu^{\rm eff}$ on the parameter ξ is very strong. The reason for this dependence has a purely kinematic character.

Remember that in the rest atom frame of an atom the extinction factor α_ν (see Eq. (3)) has linewidth $\Delta\nu_{\rm Th}$. Since an atom participates in two statistically independent motions-thermal and chaotic, uncorrelated turbulent motions, the absorption of a propagating radiation beam occurs in a broad frequency interval $\Delta\nu_{\rm D}$ ($\Delta\nu_{\rm D}^2 = \Delta\nu_{\rm Th}^2 + \Delta\nu_k^2$). The propagation of radiation through the layer $\tau_1/2$ decreases the intensity due to the transparation coefficient $A_1(x) = \exp(-\tau_1/2\exp(-x^2))$. The second layer with the same mean optical depth $\tau_1/2$ in the uncorrelated turbulence does not differ physically from the first one and gives rise to the same absorption of radiation, $A_2(x) = A_1(x)$. The total absorption of radiation in these two layers is described by $A(x) = A_1(x)A_2(x) \equiv A_0(x)$.

In a turbulent atmosphere with the finite size R_0 of the correlation length the situation is different: more probably, the second gas volume has velocity $-\boldsymbol{u}$, as it was accepted earlier in our simple model of turbulence. If the velocity difference $\sim 2u_k$ lies beyond the linewidth $\sim\Delta\nu_{\rm Th}$, this second absorption practically does not occur. Thus, correlated turbulence is more transparent (if $\alpha_\nu > 0$) that uncorrelated one.

It follows from this qualitative analysis that the transparation effect is more profound for a relatively large optical depth of the medium when the probability of the second absorption of light is sufficiently large. This was earlier mentioned in Levshakov & Kegel (1994) as a consequence of numerical Fokker-Planck equation calculations. Our calculations and illustrative model with crude correlation function $b_{\|}(R) = 1$ at $R \leq R_0$ (see Eqs. (33)–(35)) confirm this distinctly.

If the velocity difference $\sim 2u_k$ is about the characteristic thermal velocity $u_{\rm Th}$ (the parameter $\xi \cong 1$), the absorption in the second layer occurs practically with the same intensity as in the first layer and the effect of transparation is small. In this case the extinction coefficients in the first and second layer overlap widely. When inequality $u_k > u_{\rm Th}$ takes place, overlap of the extinction coefficients is small, only at the wings, and the absorption of radiation in the second layer virtually does not occur. This corresponds to the large transparation effect. It is clear from these qualitative considerations that for $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k = u_{\rm Th}/u_k > 1$ resonant absorption occurs more frequently than for $\xi < 1$, and the transparation effect is larger in the latter case. Thus, for $\tau_0 = \langle\alpha_\nu(\nu = \nu_0)\rangle s = 2$ and $s = 10R_0$ turbulent model (21) yields $A(0) = 0.221$ at $\xi = 0.1$ and $A(0) = 0.152$ at $\xi = 0.5$. For uncorrelated (or, better to say, short-correlated) turbulence we have $A(0) = 0.135$. For turbulent model (22) the corresponding values are 0.168 and 147, i.e., are closer to the case of short-correlated turbulence.

The transmission coefficients $A(x)$ and $A_0(x)$ for a number of values of τ_0 and ξ are presented in Figs. 1–7. The first three figures correspond to the layer thickness of $10R_0$, and Figs. 4–6

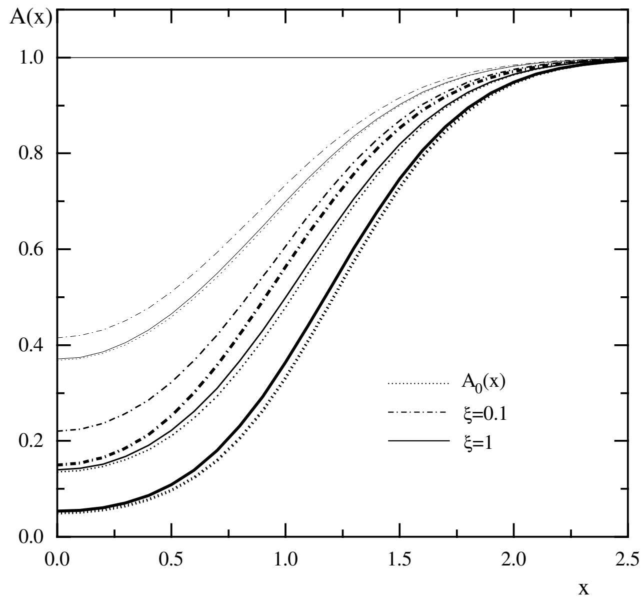

Fig. 1. Transmission functions $A(x)$ and $A_0(x)$ for turbulence model (21) for optical depths of the layer at the line center $\tau_0 = 1$, 2 and 3 (from top to bottom, respectively). The dotted line corresponds to $A_0(x)$ (short-correlated turbulence). The dash-dotted curves show $A(x)$ for $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k = 0.1$. The solid lines represent the case $\xi = 1$. The layer thickness is $10R_0$.

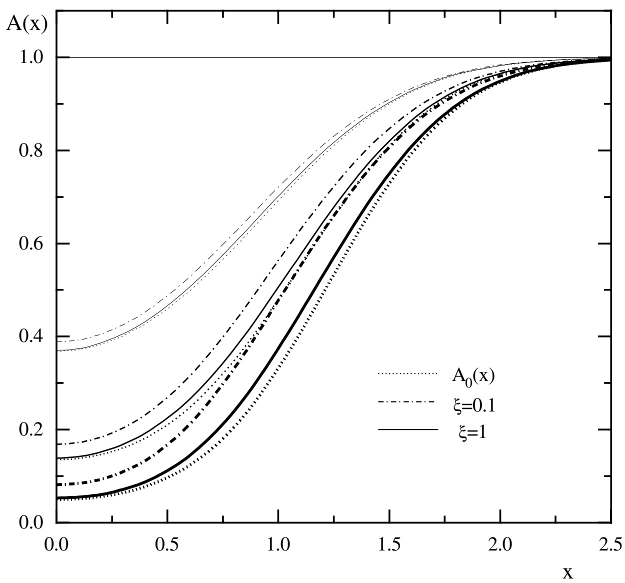

Fig. 2. Same as in Fig. 1, but for turbulence model (22) ($b_{\|}(R) = \exp(-R/R_0)$).

represent the case $\tau_0/\tau_1 = 5$. The optical depth takes the values $\tau_0 = 1$, 2 and 3 (curves from top to bottom). Figures 1 and 4 correspond to the model of turbulence (21), Figs. 2 and 5 represent the exponential model (22), and Figs. 3 and 6 correspond to acoustic turbulence (23). Firstly, we see that the larger is τ_0 the greater is the contribution of the finite correlation length R_0. Model (22) with the exponential correlation function gives a lesser transparency effect as compared to models (21) and (23). The most efficient one for transparency is acoustic turbulence. This is due to the slow decrease of correlation function (23) as compared to functions (21) and (22). The figures demonstrate that the transparency effect is very sensitive to the value of $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k$. Above we have discussed the cause of this.

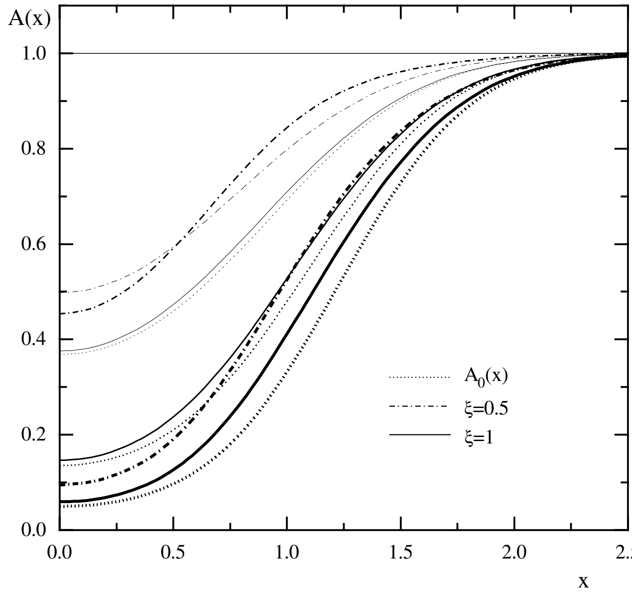

Fig. 3. Same as in Fig. 1, but for turbulence model (23 (acoustic turbulence with $b_{\|}(R) = \sin(R/R_0)/(R/R_0)$). The dash-dotted lines correspond to $\xi = 0.5$.

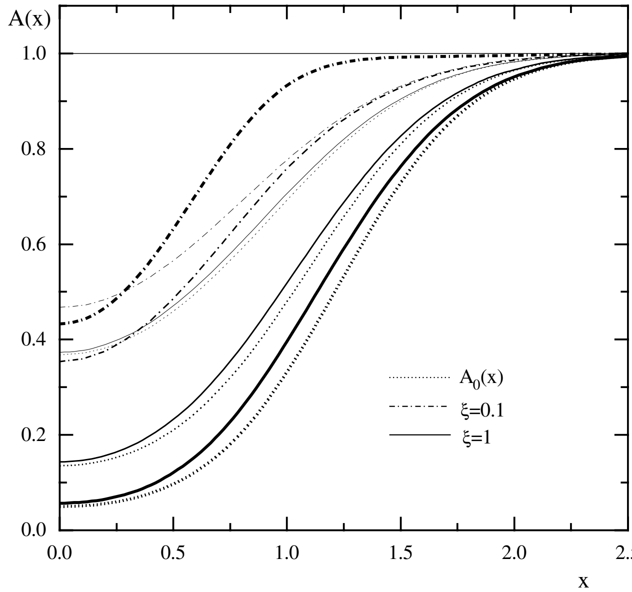

Fig. 4. Transmission functions $A(x)$ and $A_0(x)$ for turbulence model (21) with optical depths of the layer at the line center $\tau_0 = 1$, 2 and 3 (*from top to bottom*, respectively). The dotted line corresponds to $A_0(x)$ (short-correlated turbulence). The dash-dotted curves show $A(x)$ for $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k = 0.1$. The solid lines represent the case $\xi = 1$. The layer thickness is 5 R_0.

From the figures it follows that at $\xi = 1$ the transparency effect is very small, but for $\xi = 0.1$ it is large.

The comparison of Figs. 1, 2 with 4, 5, presented in Fig. 7, shows that turbulence with a larger correlation length is more transparent than that with a smaller correlation length. We see also that the $A(x)$-coefficient acquires the characteristic half-value $(1 - A(x)) = 0.5(1 - A(0))$ at lesser values of x than the $A_0(x)$-function. It means that turbulence with a finite correlation length gives rise to a smaller linewidth as compared to that for the short-correlated turbulence. The forms of $A(x)$ in all the considered cases look as the $A_0(x)$ curve, but they lie in a region

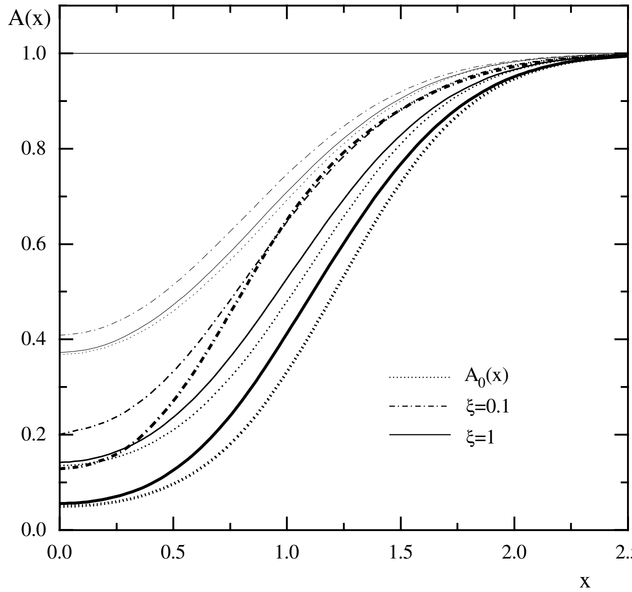

Fig. 5. Same as in Fig. 4, but for turbulence model (22) ($b_{\|}(R) = \exp(-R/R_0)$).

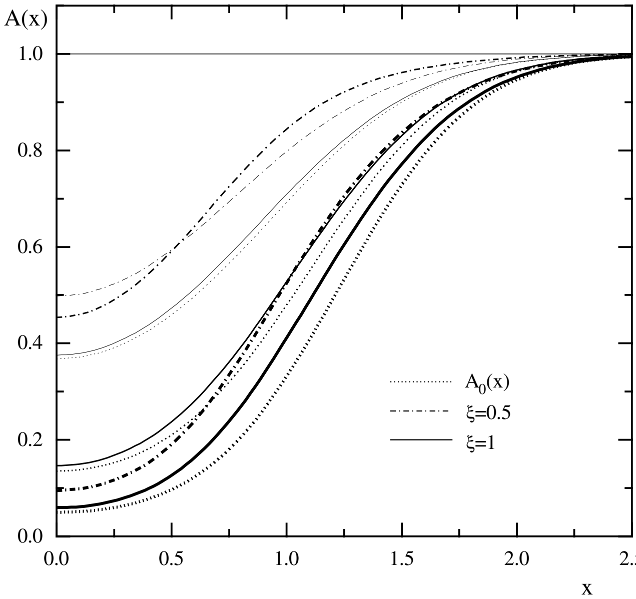

Fig. 6. Same as in Fig. 4, but for turbulence model (23) (acoustic turbulence with $b_{\|}(R) = \sin(R/R_0)/(R/R_0)$). The dash-dotted lines correspond to $\xi = 0.5$.

above the $A_0(x)$ values. Purely geometrically, it is rather evident that the the halfwidth values of x for $A(x)$ have to be smaller than the halfwidth value for $A_0(x)$. The curve a is an especially impressive example. The reason for this effect lies in our previous qualitative considerations. The sharpness of the line profile is the integral characteristic of a line. It is difficult to explain it in a purely qualitative manner, so the following qualitative consideration cannot be considered as some "proof" of the effect. The statement that the halfwidth of a spectral line in a turbulent medium is smaller than that in a non-turbulent case also has a statistical nonlinear character as the effect of transparency (see the term $\langle\tau_\nu'^2\rangle/2$ in the expression for $\tau_\nu^{\rm eff}$).

The existence of a finite correlation length gives rise to an incomplete overlap of the extinction factors in the first and second layers of the atmosphere. In each realization of turbulent motion

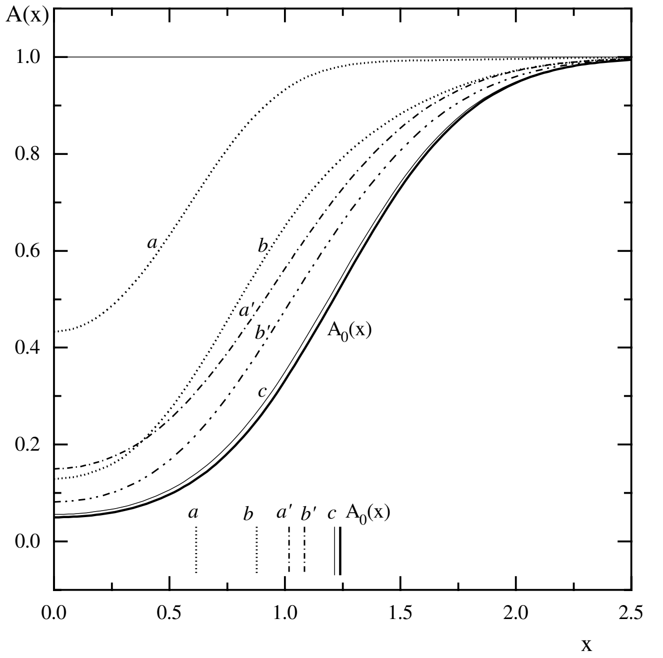

Fig. 7. Comparison of the transmission functions $A(x)$ for turbulence models (21) (a–curves) and (22) (b–curves). Parameter $\xi = \Delta\nu_{\rm Th}/\Delta\nu_k = u_{\rm Th}/u_k = 0.1$ and the optical depth of the layer s at the line center $\tau_0 = 3$. The dotted lines correspond to $s = 5\,R_0$, and the dash-dotted represent the case $s = 10\,R_0$. At $s = 100\,R_0$ (small-scale turbulence) both models yield virtually the same $A(x)$ and are represented by c)–curve. At the bottom the x-positions of the halfwidth level are shown. It is visible that finite-correlated turbulence produces narrower absorption lines than short–correlated turbulence (the curve $A_0(x)$).

the overlap pattern is, evidently, not symmetric: overlap in one of the absorption profile wings is lesser (denote it as a left wing) than in the opposite (right) wing. The radiation with a frequency of the left wing will be absorbed less during its propagation in the second layer and the left-side profile of the absorption line will be steeper than the right-side one. Thus, one realization of turbulence gives rise to the asymmetry of a spectral line. In usual isotropic turbulence the velocities $\boldsymbol{u}$ and $-\boldsymbol{u}$ occur with equal probability. Thus, in a realization with opposite velocities the right wing of the spectral line will be steeper than the left one. Due to nonlinear dependence of the effective extinction factor on $\langle\tau_\nu'^2\rangle$, the resulting steepness of these opposite realizations preserves.

From Fig. 7 we see that a turbulent layer with many (>100) turbulent cells practically does not differ from a layer with short-correlated turbulence. Moreover, in this limit different forms of the correlation function $b_{\|}(R)$ yield the same shape of the spectral line $A(x)$. This result is rather evident. In this case every turbulent cell of size R_0 is not correlated with the neighbouring cell (we remind that the length of correlation is R_0). These chaotically moving cells are similar to a cloud of chaotically moving absorbing atoms. This effect was discussed for the first time by Magnan (1976).

There exists another way to obtain the limit of short-correlated turbulence even if the number of turbulent cells is not large. Remember that $A(x)$ depends on the parameter $\xi = u_{\rm Th}/u_k$. If this parameter is large, $\xi \geq 1$, i.e., the characteristic turbulent velocity u_k is smaller than the thermal velocity of atoms inside the turbulent cells, the existence of finite correlation length R_0 does not affect the shape of the spectral line. This effect is very large. Thus, in the case when the turbulent layer has thickness $s = R_0$ (the limiting case of large-scale turbulence) the difference of $\tau_\nu^{\rm eff}$ at the line center from τ_0 is ≈30% and 4% for $\xi = 1$ and 2, respectively. This effect is also evident. The case $u_k \ll u_{\rm Th}$ corresponds to a very small shift of the center of the α_ν-profile (see Eq. (3)), for either correlated or uncorrelated turbulence. Large values of ξ correspond to small values of parameter $\eta = 1/(1+\xi^2) \leq 1$. This parameter is always related to correlation function $b_{\|}(R/R_0) \leq 1$ (see Eq. (30)) and characterizes the effectiveness of the finite-length turbulent velocity correlations.

Let us summarize the discussion. A very large number of turbulent cells ($n \geq 100$) within the layer is a sufficient condition for the existence of the short-correlated limit. If this condition is not valid, the short-correlated limit occurs when parameter $\xi = u_{\rm Th}/u_k \geq 1$. In the intermediate case ($n < 100$ and $\xi \leq 1$) the effects of finite correlation length of turbulent velocities on the line shape can be large, especially if the optical depth $\tau_0 > 2$.

5. Estimation of characteristic turbulent parameters

We discuss now the problem of how to obtain the main turbulence parameters u_k, R_0 and $U_{0\|} = \boldsymbol{U}_0\boldsymbol{n}$ from the analysis of the shape of a spectral line. The effective optical depth $\tau_\nu^{\rm eff}(s)$ (see Eq. (30)) depends also on thermal velocity $u_{\rm Th}$ and thickness s of the layer. Dimensionless frequency $x = (\nu-\nu_0)/\Delta\nu_{\rm D}$ depends on linewidth $\Delta\nu_{\rm D}$, which is related to unknown characteristic turbulent velocity $u_k = (\langle\boldsymbol{u}^2(\boldsymbol{r},t)\rangle)^{1/2}$ and thermal velocity $u_{\rm Th}$ (see Eq. (3)). Thus, the unknown quantities are u_k, $u_{\rm Th}$, R_0, s and $U_{0\|}$. Of course, $\tau_\nu^{\rm eff}(s)$-coefficient depends also on the particular form of the correlation function $b_{\|}(R/R_0)$, but this dependence is integral and not very sensitive to the choice of a particular model (see Figs 1–6). To estimate five unknown quantities, we can use one of the models (21) or (22). It seems that the case of acoustic turbulence (23) is not so plausible in the nature as the former ones. The observed spectra can be presented as functions of arbitrary dimensionless frequency $x = (\nu-\nu_0)/\Delta\nu_*$, where $\Delta\nu_*$ is some arbitrary linewidth used in the reduction of the data. Usually one takes $\Delta\nu_*$ corresponding to a velocity of 1 km s^{-1}. To compare our theoretical formulae with the observational data, we have to write them as functions of $x = (\nu-\nu_0)/\Delta\nu_*$. It means that in all the formulae we substitute, instead of old x and x_0, the values ax and ax_0, where new parameter a is

$$a = \frac{\Delta\nu_*}{\Delta\nu_{\rm D}}. \tag{44}$$

Thus, the expression for $\tau_\nu^{\rm eff}(s)$ becomes:

$$\tau_\nu^{\rm eff}(s) = \tau_\nu^{(0)}(s)\left\{1 + \frac{\tau_1 s}{2R_0}{\rm e}^{-a^2(x-x_0)^2} - \tau_1\int_0^{s/R_0}{\rm d}y \right.$$
$$\left.\times\left(\left(1-y\frac{R_0}{s}\right)\frac{\exp\left[-a^2(x-x_0)^2\,\dfrac{1-\eta b_{\|}(y)}{1+\eta b_{\|}(y)}\right]}{\sqrt{1-\eta^2 b_{\|}^2(y)}}\right)\right\}. \tag{45}$$

$$\tau_\nu^{(0)}(s) = \tau_0{\rm e}^{-a^2(x-x_0)^2}. \tag{46}$$

Here x is the dimensionless frequency used in the reduction of observational data.

We consider two observational situations corresponding to propagation of continuum radiation and spectral line (maser) through the layer s, respectively.

5.1. Absorption line in turbulent medium

Considering propagation of continuum radiation in a resonant turbulent medium, we can take the intensity of incident radiation equal to a constant value inside the frequency interval of the absorption line. For simplicity we take this constant equal to unity. Evidently, the shape of the absorption line is independent of the frequency shift x_0, and we take $x_0 = 0$. Usually the intensity of the absorption line is taken from the local level of the continuum radiation intensity. Thus, the shape of an absorption line is described by the expression:

$$H(x) = 1 - A(x) \equiv 1 - \mathrm{e}^{-\tau_\nu^{\mathrm{eff}}(s)}. \tag{47}$$

The maximum value of $H(x)$ corresponds to $x = 0$:

$$H(0) \equiv H_0 = 1 - \mathrm{e}^{-\tau_0^{\mathrm{eff}}}, \tag{48}$$

$$\tau_0^{\mathrm{eff}} = \tau_0 \left\{ 1 + \frac{\tau_1 s}{2R_0} - \tau_1 \int_0^{s/R_0} \mathrm{d}y \left(\left(1 - y\frac{R_0}{s}\right) \frac{1}{\sqrt{1-\eta^2 b_\parallel^2(y)}} \right) \right\}. \tag{49}$$

Here τ_0^{eff} is the effective optical depth of the layer s at the center of a spectral line. Remember that $\tau_0^{\mathrm{eff}} < \tau_0$.

Thus, from Eq. (48) we can obtain the value of τ_0^{eff}, which depends on parameters τ_0, τ_1, ξ ($s/R_0 \equiv \tau_0/\tau_1$ and $\eta = 1/(1+\xi^2)$):

$$\tau_0^{\mathrm{eff}} = -\ln(1 - H_0). \tag{50}$$

If $\tau_0^{\mathrm{eff}} > 1$ then one also has $\tau_0 > 1$, and the influence of turbulence (if it exists!) can be really observable. The case $\tau_0^{\mathrm{eff}} < 1$ does not warrant that $\tau_0 < 1$ also takes place. Note that, according to Eq. (50), the value of H_0 determines just the τ_0^{eff}-value, not the τ_0-coefficient. If we approximate the profile of the spectral line by the formula in short-correlated turbulence limit

$$H(x) = 1 - A_0(x) \equiv 1 - \exp(-\tau_0 \mathrm{e}^{-a^2x^2}), \tag{51}$$

we obtain that the parameter τ_0 is equal to τ_0^{eff}. Thus, the approximation (51) indeed means "mixed" approximation with the transmission coefficient

$$A_{\mathrm{mixed}}(x) = \exp\left(-\tau_0^{\mathrm{eff}} \mathrm{e}^{-a^2x^2}\right). \tag{52}$$

Turbulent transmission coefficient $A(x)$ and function $A_{\mathrm{mixed}}(x)$ are equal at $x = 0$. For this reason the "mixed" approximation $A_{\mathrm{mixed}}(x)$ lies nearer to the real $A(x)$-function than the $A_0(x)$-coefficient. The largest discrepancy between these functions occurs at $x \approx 0.8$. Turbulence model (21) demonstrates that $A_{\mathrm{mixed}}(0.8)$ is less by 1.5%, 6% and 13% than the real value of $A(0.8)$ at $\xi = 0.1$ and $\tau_0 = 1$, 2 and 3, respectively. The non-turbulent $A_0(x)$-function differs from $A(x)$ larger: it is less by 7.7%, 27% and 51%, respectively. Note that there exist inequalities $A(x) > A_{\mathrm{mixed}}(x) > A_0(x)$. Nevertheless, the "mixed" approximation of the absorption line

$$H_{\mathrm{mixed}}(x) = 1 - \exp\left(-\tau_0^{\mathrm{eff}} \mathrm{e}^{-a^2x^2}\right) \tag{53}$$

can be useful. To obtain parameter a^2 in this model, we can use the formula:

$$a^2 \equiv \left(\frac{\Delta\nu_*}{\Delta\nu_{\mathrm{D}}}\right)^2 = -\frac{1}{x^2} \ln \frac{\ln(1 - H(x))}{\ln(1 - H_0)}. \tag{54}$$

It is interesting that formally we can choose any x – the right side of Eq. (54) does not depend on x for $H_{\mathrm{mixed}}(x)$. This property can be used to examine whether finite-correlation turbulence exists in the atmosphere or not. The statistical confidence of this statement can be obtained by using many values of x from the observed line shape.

If the $H_{\mathrm{mixed}}(x)$-curve differs sufficiently (outside the observational errors) from the observed profile then we make sure that finite correlation turbulence indeed exists in the layer s. In this case we have to construct the profile using a more sophisticated model (46), which depends on four parameters, τ_0, τ_1, ξ, and a^2. To obtain these parameters, we can use relationships (46) at four x-values from the observed line shape. This yields four nonlinear equations for the above-mentioned parameters. It should be stressed that a good coincidence of model (53) with the observed line shape does not guarantee that turbulent effects are insufficient. This is due to the fact that the "mixed" approximation uses the most important parameter τ_0^{eff} that is indeed a function of τ_0, τ_1 and ξ. Strictly speaking, we must always use model (46). If, as a result, we obtain that τ_1 is very small or $\xi \geq 1$, then we are sure that $\tau_0^{\mathrm{eff}} \simeq \tau_0$ and turbulence effects are really insignificant.

5.2. Propagation of maser radiation

As we have seen, the influence of finite-length turbulent correlations is most profound in the propagation of radiation in optically thick layers. These effects can be observed most readily in the propagation of a maser emission. Intense maser emission can be observed even after the passage of an optically thick layer. In this case radiation penetrates many turbulent cells, and the observed emission coincides with average intensity $I_\nu^{(0)}$. It seems that maser emission from a region with inverted populations of molecular levels is statistically independent of conditions in the turbulent layer with the normal distribution of the level populations. Very frequently a maser emission line has a Gaussian shape with the Doppler width $\Delta\nu_{\mathrm{S}}$. First, using our general formulae, we demonstrate the influence of turbulence on the shape of maser radiation. We obtain the following expression for the mean intensity:

$$I_\nu^{(0)}(x) = I_0 \mathrm{e}^{-b^2x^2} A(x - x_0) \equiv I_0 \mathrm{e}^{-(b^2x^2 + \tau_\nu^{\mathrm{eff}}(s))}. \tag{55}$$

The term $I_0 \exp(-b^2x^2)$ is the incident maser intensity from the source with the Doppler width $\Delta\nu_{\mathrm{S}}$. Here x and x_0 are dimensionless frequencies (18) with the Doppler width $\Delta\nu_{\mathrm{D}}$ in the turbulent layer s, $b = \Delta\nu_{\mathrm{D}}/\Delta\nu_{\mathrm{S}}$.

In Fig. 8 we present normalized intensities $I_\nu^{(0)}(x)/I_{\nu\max}^{(0)} \equiv H(x)$ for values $b = 0.5$, 1, 2 and mean optical depths at the center of a line $\tau_0 = 1$, 2 and 3. Figures (an), (bn) and (cn) correspond to the values $b = 0.5$, 1 and 2, respectively. We have considered the simplest case with $x_0 = 0$, i.e., absence of a regular motion in the layer. Solid lines correspond to turbulence with $\xi \equiv u_{\mathrm{Th}}/u_k = 0.1$ and $s = 5\,R_0$. Dotted lines represent the transformation of the line shape when turbulence is short-correlated ($\tau_1 = 0$):

$$I_\nu^{(0)}(x) = I_0 \mathrm{e}^{-b^2x^2} A_0(x - x_0) \equiv I_0 \mathrm{e}^{-(b^2x^2 + \tau_\nu^{(0)}(s))}. \tag{56}$$

First of all, we see that turbulence with a finite correlation size makes lines narrower than short-correlated turbulence. This is due to the fact that transmission coefficient $A(x)$ is narrower than the $A_0(x)$-coefficient (see Figs. 1–6). The case $b = 0.5$ corresponds to broadband incident emission as compared to the characteristic width of the transmission coefficients $A(x)$ and $A_0(x)$.

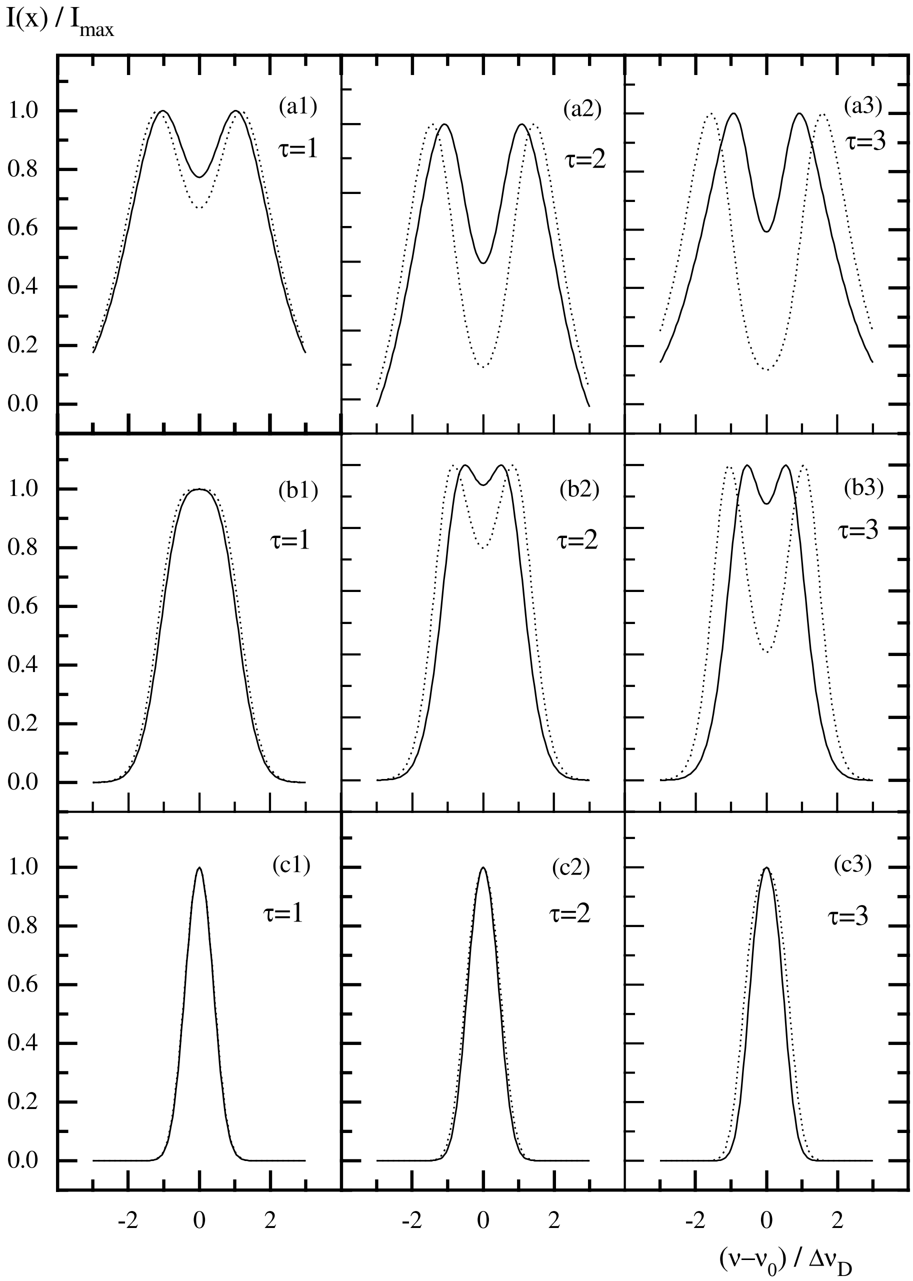

Fig. 8. Normalized shape $I_\nu(x)/I_{\nu\max}$ of the maser line with a Gaussian initial profile after the passage of a turbulent cloud (solid lines). The dotted lines correspond to the usual model of short-correlated turbulence. The cases (an), (bn) and (cn) represent $\Delta\nu_{\rm D}/\Delta\nu_{\rm S} = 0.5, 1$ and 2, respectively; $\Delta\nu_{\rm D}$ and $\Delta\nu_{\rm S}$ are the Doppler widths in the cloud and maser source. The optical depth of the cloud τ at the line center takes the values 1, 2 and 3. Turbulence correlation length R_0 is taken to be one-fifth of the layer's depth, and parameter $\xi = u_{\rm Th}/u_k = 0.1$; $u_{\rm Th}$ and u_k-values are characteristic thermal and turbulent velocities in the cloud.

This produces central minima in the line shape. Naturally, layers with a large optical depth give rise to more profound minima than layers with lesser thickness. The case $b = 3$ corresponds to narrowband incident emission. In this case the line shape virtually does not change. The intensity decreases only by its value proportionally to $A(0)$ and $A_0(0)$.

Thus, the presented figures demonstrate that the shape of a maser line changes quite substantially after the passage of an optically thick turbulent layer with a finite length of correlation as compared to the model of short-correlated turbulence.

To approximate the observed lines, we must substitute to our theoretical formulae the values of x and x_0 with quantities ax and ax_0, as it was explained in detail in the previous subsection (see Eq. (44)). Note that after this substitution x is the dimensionless frequency used by reduction of the observational data. The observed intensity at $x = 0$ is related to the $\tau_0^{\rm eff}$-coefficient:

$$I_\nu^{(0)}(0) = I_0 {\rm e}^{-\tau_0^{\rm eff}}. \tag{57}$$

This yields

$$\tau_0^{\rm eff} = -\ln \frac{I_\nu^{(0)}(0)}{I_0}. \tag{58}$$

As in the case of an absorption line, using approximation (56) instead of τ_0 gives rise to $\tau_0^{\rm eff}$. Thus, this approximation indeed denotes the "mixed" approximation:

$$I_\nu^{(0)}(x) = I_0 \exp\left(-a^2 b^2 x^2 - \tau_0^{\rm eff} {\rm e}^{-a^2 x^2}\right). \tag{59}$$

The normalized form of function (59) is

$$H_{\rm mixed}(x) \equiv \frac{I_\nu^{(0)}(x)}{I_{\nu\max}^{(0)}}$$

$$= \exp\left[-a^2 b^2 (x^2 - x_{\rm m}^2) - \tau_0^{\rm eff}\left({\rm e}^{-a^2 x^2} - {\rm e}^{-a^2 x_{\rm m}^2}\right)\right], \tag{60}$$

where x_m corresponds to the maximum value of the observed intensity. The derivative of $H_{\text{mixed}}(x)$ is equal to zero at $x = 0$ and at x_m:

$$x_m^2 = \frac{1}{a^2} \ln \frac{\tau_0^{\text{eff}}}{b^2}. \tag{61}$$

If $\tau_0^{\text{eff}}/b^2 < 1$, there exists a unique maximum at $x = 0$. To obtain the formulae for τ_0^{eff}, a^2 and b^2 from Eq. (60), we introduce, for brevity, notation

$$H_0 \equiv H_{\text{mixed}}(0), \qquad H_n \equiv H_{\text{mixed}}(x_n), \qquad \mathrm{e}^{-x_1^2 a^2} \equiv y. \tag{62}$$

Taking the natural logarithm of Eq. (60), we obtain:

$$-\ln \frac{H_{\text{mixed}}(x)}{H_0} = a^2 b^2 x^2 + \tau^{\text{eff}} \left(\mathrm{e}^{-x^2 a^2} - 1\right). \tag{63}$$

Taking this equation at $x = x_1$, $x_2 = \sqrt{2}x_1$ and $x_3 = \sqrt{3}x_1$ we obtain a set of equations for y, y^2 and y^3, which can be easily solved:

$$y = \frac{\ln (H_2^2/H_1 H_3)}{\ln (H_1^2/H_0 H_2)}. \tag{64}$$

This solution yields for parameters a^2, b^2 and τ_0^{eff} the following expressions:

$$a^2 = -\frac{1}{x_1^2} \ln y,$$

$$\tau_0^{\text{eff}} = \frac{\ln (H_1^2/H_0 H_2)}{(y-1)^2},$$

$$b^2 = \frac{1}{x_1^2 a^2} \left(\ln \frac{H_0}{H_1} - (y-1)\tau_0^{\text{eff}}\right). \tag{65}$$

We emphasize that x_1 is some arbitrary value of x. Thus, varying the position of x_1 at the observed curve $H(x)$ we can check whether parameters a^2, b^2 and τ_0^{eff} change their values or not. If they take various values it is clear that there exists the finite-turbulence influence on the shape of a spectral line. In this case it is necessary to use general formula (55). Note that, to obtain three parameters (a^2, b^2 and τ_0^{eff}) analytically, we have used the values of the observational shape curve at four points: $x = 0$, x_1, $x_2 = \sqrt{2}x_1$ and $x_3 = \sqrt{3}x_1$.

The second proof of the validity of the "mixed" approximation is substituting the obtained parameters a^2, b^2 and τ_0^{eff} to the exact relationship (61), where x_m is known from observational line shape.

The asymmetric maser lines are observed more frequently. In these cases we have to use the formulae with $x_0 \neq 0$. In this case the "mixed" approximation has the form:

$$H_{\text{mixed}}(x) = \mathrm{e}^{-a^2 b^2 (x^2 - x_m^2)} \times \mathrm{e}^{-\tau_0^{\text{eff}}\left(\mathrm{e}^{-(x-x_0)^2 a^2} - \mathrm{e}^{-(x_m - x_0)^2}\right)}. \tag{66}$$

Instead of expression (61) for the position of the maximum of the line shape, we have a more complicated formula:

$$(x_m - x_0)^2 a^2 = \ln\left[\frac{x_m \tau_0^{\text{eff}}}{(x_m - x_0) b^2}\right]. \tag{67}$$

In this case we have not obtained analytical formulae for four parameters a^2, b^2, x_0 and τ_0^{eff}. So, they have to be obtained by numerical methods from set of Eqs. (66) taken at various points x_n. The substitution of the obtained parameters to the exact formula (67) can be used to check whether the finite-correlation turbulence influences the line shape or not.

The obtained parameters $a = \Delta\nu_*/\Delta\nu_D$, $b = \Delta\nu_D/\Delta\nu_S$, x_0 and τ_0^{eff} allow us to construct the "mixed" approximation formulae (57) or (66). If this approximation lies beyond the observed line shape $H(x)$, then it is clear that the effects of the finite-correlation turbulence play an important role and one must use the more complex approximation (55). It seems that parameters a, b, x_0 and τ_0^{eff} obtained from "mixed" approximation can be used as a starting point to construct approximation (55). Note once more that the good coincidence of $H_{\text{mixed}}(x)$ with the observed $H(x)$ does not guarantee that the influence of turbulence is small (remember that the obtained value τ_0^{eff} is a function of τ_0, τ_1 and ξ). In the general case approximation (55) depends on six parameters τ_0, τ_1, ξ, a, b and x_0. To obtain these parameters we must use additional values of the observed line profile $H(x)$. A detailed investigation of this complicated procedure is beyond the scope of this paper.

It should be emphasized that the obtained parameters a and ξ allow us to estimate separately characteristic thermal velocity u_{Th} and turbulent velocity u_k. This is impossible if one uses the short-correlated model of turbulence.

5.3. Estimation of the parameters of the H_2O maser source in S252A

To illustrate our theory, we have chosen a flare of the H_2O maser in the source S252A at $V_{\text{LSR}} = 10$ km s^{-1}, not a single two-peak profile, because this one may be produced by a simple overlap of two physically unassociated components with similar velocities V_{LSR}. In addition, it is important to stress that during strong flares the maser operation mode may change, resulting in a variation in linewidths of individual features. However, in such cases maximum linewidth variations are a few times less than the linewidth itself. For the flare in S252A at $V = 10$ km s^{-1} the minimum separation between the components was 0.5 km s^{-1}, which is close to the linewidth. The maximum separation reached 0.8 km s^{-1}. Hence it is clear that the changes in the line structure are not related to the maser operation mode. Finally, important evidence favouring our choice of the flare were time-correlated flux variations of both components.

Thus, the two-peak line profile at 10 km s^{-1} in S252A, in which the components have similar radial velocities and their fluxes change in phase can be a proof of physical association of the components, i.e., the source of emission at 10 km s^{-1} is a single maser feature.

The observed convergence of the components during the flare and their subsequent mutual recession (see Fig. 9b) took place also in other maser sources. For this phenomenon in W75S and W31A we have proposed a mechanism based on coherent properties of the radiation (see, e.g., Lekht et al. 1996). Here we propose another possible mechanism for the explanation of this phenomenon.

The H_2O maser source in the S252A region was observed at Pushchino Radio Astronomical Observatory (Russia) during 1987–1988. As all the masers in star-forming regions H_2O maser in S252A is located inside a dense molecular cloud. The fragments of the observed spectra are presented in Fig. 9a (Berulis et al. 1996). In the reduction of these spectra they were represented as a sum of two Gaussian profiles (Lekht et al. 1996).

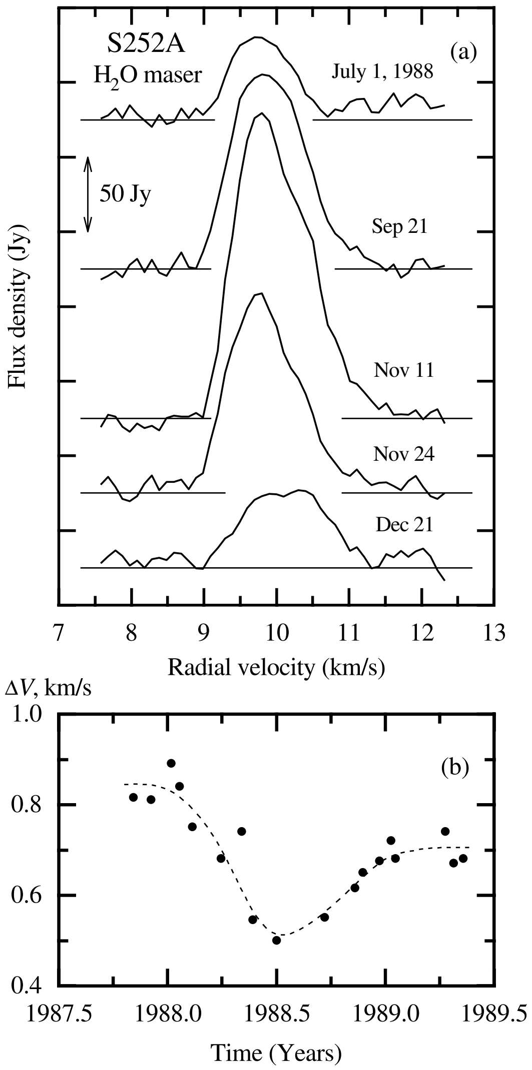

Fig. 9. a) Fragments of the H_2O maser spectra of S252A in 1988; **b)** variations in the difference between the centers of the approximating Gaussians.

It was found that after a burst of the emission at the beginning of 1988 the distance between these two Gaussian profiles decreased systematically up to May 1988, and then the distance increased to a new level, lower than the initial one. This is clearly seen in Fig. 9b. What is the cause of this effect? We have seen from Fig. 8 that propagation of maser emission through a turbulent layer with a finite correlation length decreases the width of the spectral line. Thus, it is natural to explain the effect by this mechanism.

Here we suppose that after the burst the existing turbulent layer has acquired an additional regular-velocity motion and, possibly, other parameters have also changed. Here we present only a preliminary solution of this problem. Our main goal will be to demonstrate that the simple analytical theory developed in this paper allows us to find very quickly the basic parameters of turbulence by the "hit and fit" method. For this purpose we use general formulae (55) and (66) with unknown parameters τ_0, x_0, a^2 and b^2. We take the two initial Gaussian curves with $x_0 = 0$ and $x_0 = 0.8$ and seek at what parameters the distance between the centers of these curves takes the value $x_0 = 0.5$. The calculation of $\tau_\nu^{\rm eff}$ was done for correlation function (21). Of course, in such a simple statement of the problem the solution is not unique. But we restrict our solution by the condition that the final linewidth is about the observed value, 1–1.1 km s^{-1}. It seems that this restriction does not make the solution unique, but anyway it makes the range of possible values of the parameters narrower.

As a result, we have found that the parameters characterizing the layer are: $\tau_0 = 3$, optical depth in center of a line, regular velocity along the line of sight $U_0 \simeq 1$, mean turbulent velocity $u_k \simeq 0.5$, and the thermal velocity $u_{\rm Th} \simeq 0.1$ km s^{-1}. It is interesting that the "mixed" approximation yields a shift of 0.46 instead of 0.5 km s^{-1} for the general formula (55). The difference is rather large and the solution for the layer with a finite correlation length is more preferable. The obtained preliminary results can be used in the construction of a physical model of the maser source in S252A.

Thus, this illustrative example shows that the simple analytical formulae obtained in our paper can be used to estimate possible parameters of turbulence in interstellar clouds. Of course, any additional information about the conditions in molecular cloud, which usually exists, can help to obtain the unique solution.

6. Conclusion

Let us summarize the results of the paper. First of all, we have developed the analytical theory of radiative transfer of a resonant line in turbulent atmospheres with a finite length of correlation. We have found for the first time that mean intensity $I_\nu^{(0)} \equiv \langle I_\nu \rangle$ obeys the usual radiative transfer equation with renormalized extinction factor $\alpha_\nu^{\rm eff}$ if correlation length R_0 is small as compared to the photon free path. Effective absorption coefficient $\alpha_\nu^{\rm eff}$ does not coincide with mean absorption factor $\alpha_\nu^{(0)} \equiv \langle \alpha_\nu \rangle$. The transition $\alpha_\nu^{\rm eff} \to \alpha_\nu^{(0)}$ takes place if the optical depth at the line center τ_1 of correlation length R_0 tends to zero (the case of short-correlated turbulence), or characteristic thermal velocity $u_{\rm Th}$ is equal to or greater than characteristic turbulent velocity u_k (parameter $\xi = u_{\rm Th}/u_k \geq 1$).

Assuming that the ensemble of turbulent velocities is Gaussian, we have obtained an explicit analytical formula for $\alpha_\nu^{\rm eff}$, which integrally depends on the correlation function of turbulent velocities. It is found that $\alpha_\nu^{\rm eff} \leq \alpha_\nu^{(0)}$. It means that statistically a turbulent layer with a finite correlation length is more transparent then a layer of short-correlated turbulence.

It is shown that, in addition to the usual Doppler broadening of the spectral line, correlated turbulent motions of atoms and molecules produce considerable changes in the shape of a spectral line. Averaged spectral line intensity $I_\nu^{(0)}$ is narrower than that described by radiative transfer equation with averaged extinction factor $\alpha_\nu^{(0)}$. This distortion of the shape increases with increasing optical depth τ_0 of the turbulent layer at the line center. An increase in the parameter τ_1 and decrease in ξ also give rise to an increase in the distortion of the shape. If the turbulent layer contains many ($\tau_0/\tau_1 \geq 100$) turbulent cells, the effects of finite correlation length turbulence disappear. If the number of turbulent cells is not large, but $u_{\rm Th} \geq u_k$, the finite correlation length effects also tend to disappear.

Explicit analytical formulae allow us to obtain physically obvious qualitative explanations of all the effects. They are very useful for a quick interpretation of the observed spectral lines.

These effects were mentioned earlier in various papers as the results of purely numerical calculations, without a simple physical explanation. The simple analytical formulae derived in this paper allow us to understand the expected effects of turbulence without complicated numerical calculations. In particular, we have found that turbulence with finite correlation length diminishes the equivalent width of an absorption line, which is opposite to the statement of Magnan (1976).

Finally, we briefly discuss the problem how to obtain the main turbulent parameters from the analysis of the shape of an absorption line or maser line propagating through a turbulent interstellar cloud. We present tests whether the shape of a spectral line is distorted by finite correlation length turbulence or not. This analysis basically allows us to estimate separately the cloud temperature and characteristic turbulent velocity u_k; this is not possible when using the usual short-correlated model of turbulence. As an illustrative example, we have estimated the possible parameters of the H_2O maser source in S252A. We hope that the obtained analytical formulae will be useful for a detailed explanation of various observational data.

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