Nonradial and nonpolytropic astrophysical outflows

VIII. A GRMHD generalization for relativistic jets[★]

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Received 26 July 2005 / Accepted 10 October 2005

ABSTRACT

Steady axisymmetric outflows originating at the hot coronal magnetosphere of a Schwarzschild black hole and surrounding accretion disk are studied in the framework of general relativistic magnetohydrodynamics (GRMHD). The assumption of meridional self-similarity is adopted for the construction of semi-analytical solutions of the GRMHD equations describing outflows close to the polar axis. In addition, it is assumed that relativistic effects related to the rotation of the black hole and the plasma are negligible compared to the gravitational and other energetic terms. The constructed model allows us to extend previous MHD studies for coronal winds from young stars to spine jets from Active Galactic Nuclei surrounded by disk-driven outflows. The outflows are thermally driven and magnetically or thermally collimated. The collimation depends critically on an energetic integral measuring the efficiency of the magnetic rotator, similarly to the non relativistic case. It is also shown that relativistic effects quantitatively affect the depth of the gravitational well and the coronal temperature distribution in the launching region of the outflow. Similarly to previous analytical and numerical studies, relativistic effects tend to increase the efficiency of the thermal driving but reduce the effect of magnetic self-collimation.

Key words. stars: winds, outflows – acceleration of particles – stars: mass-loss – galaxies: jets – magnetohydrodynamics (MHD) – relativity

1. Introduction

The formation of relativistic jets around compact objects and Active Galactic Nuclei (AGNs) is one of the most intriguing and yet not fully understood astrophysical phenomena (Ferrari 1998; Mirabel & Rodríguez 1998; Mirabel 2003). In those jets, velocities reach a fraction of the speed of light with the corresponding Lorentz factor ranging from values $\gamma \sim 2$ –10 in Seyfert Galaxies and radio loud AGNs (Piner et al. 2003; Urry & Padovani 1995) up to the inferred values $\gamma = 10^3$ in GRBs; AGN jets are also characterized by the rather narrow opening angles of a few degrees (Biretta et al. 2002; Tsinganos & Bogovalov 2005).

MHD models for coronal or disk-jets rely on the basic idea that the gravitational energy of the central object is transferred

to the accreting plasma which via a collimation mechanism then produces the jet. This energy released by accretion increases with the mass of the central object, a fact which may explain the wide variety of the powerful jets observed. Several analytical and numerical efforts have been invested to investigate the mechanisms of jet acceleration and collimation. The formation of collimated jets seems to be closely related to the presence of large scale magnetic fields (e.g., Gabuzda 2003) and the existence of a gaseous disk and/or a hot corona around the central object (Königl & Pudritz 2000; Livio 2002).

For the *energy source* of jets, it is usually assumed that at their base they are powered either by a spinning black hole (Blandford & Znajek 1977; Rees et al. 1982; Begelman et al. 1984), or, by the surrounding accretion disk (Miller & Stone 2000). Furthermore, they are plausibly Poynting flux dominated (Sikora et al. 2005) with their central spine hydrodynamically dominated (Meliani et al. 2004) and their plasma composed by protons-electrons or by electron-positron pairs.

[★] All appendices are only available in electronic form at

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The *collimation* of the outflow, this is likely to be due mainly to the hoop stress resulting from the toroidal magnetic field generated by the rotation of the source (Bogovalov 1995). Recent VLBI observations suggest that the direction of the magnetic field vectors is transverse to the jet axis. This is the case in BL Lac objects, e.g., in Mrk 501 (Gabuzda 2003), or/in quasars where the central faster part of the jet is characterized by toroidal magnetic fields (Asada et al. 2002). Magnetic self-collimation has been shown to be efficient in the non relativistic context (Heyvaerts & Norman 1989, 2003; Livio 2002; Honda & Honda 2002; Tsinganos & Bogovalov 2002). In the relativistic limit however it is slower due to the decollimating effect of the electric force and the higher inertia of the flow, but still possible (Vlahakis & Königl 2003, 2004). Alternatively, collimation in relativistic jets may be due to the external pressure of a surrounding slower and easily collimated disk wind, Bogovalov & Tsinganos (2005).

Magnetized no relativistic jets from extended accretion disks were first modelled analytically by Blandford & Payne (1982), wherein the plasma acceleration relies on the magnetic extraction of angular momentum and rotational energy from the underlying cold Keplerian disk. This energy is channeled along the large-scale open magnetic fieldlines anchored in the corona or the rotating disk. The ionized fluid is forced to follow the fieldlines and to rotate with them while it is magnetocentrifugally accelerated if the angle between the poloidal magnetic field and the disk is less than 60° . Cao & Spruit (1994) showed that in the relativistic case this condition is less severe and that close to the black hole the magnetocentrifugal acceleration may be efficient even at higher angles. Analytical, radially self-similar disk-wind solutions were extended to special relativistic cold winds in Li et al. (1992), and Contopoulos (1994), by neglecting gravity to allow the separation of the variables. Thermal effects were introduced into these relativistic models by Vlahakis & Königl (2003) to analyze the formation of a relativistic flow from hot magnetized plasmas, showing that such solutions could be applied to Gamma Ray Bursts wherein the flow is thermally driven at the base. However, most of the acceleration is of magnetic origin and there is an efficient conversion of Poynting to kinetic flux of the order of 50%. They also applied this disk wind solution to AGN jets (Vlahakis & Königl 2004) showing that they could trace the observed parsec scale expansion of the wind. Another approach to solve the relativistic MHD equations for outflows around black holes is to solve numerically the transfield equation in the force-free limit (Camenzind 1986a), a study further developed in GRMHD by using first a Schwarzschild metric and then extending it to a Kerr metric (Fendt 1997).

Radial self-similarity is usually used in disk-wind models due to the complexity of the non linear system of MHD equations. However, such solutions cannot describe the flow close to the rotational axis where they become singular. On the other hand, meridional self-similarity provides a better alternative to study the outflow close to the symmetry and rotation axis of the central corona. In the central part where the thermal energy is rather high, the wind may be thermally driven. Spherically symmetric relativistic hydrodynamical models have been proposed to study the formation of such outflows (Michel 1972;

Das 1999; Meliani et al. 2004). Those models are restricted to the case where the magnetic effects in the acceleration are negligible. In these models, a wind forms in the hot corona because of the internal shock maintained by the centrifugal barrier (Chakrabarti 1989; Das 2001), or by the pressure induced via a first order Fermi mechanism (Das 1999).

An important alternative to analytical models are numerical simulations. In the special relativistic domain, simulations have been presented for coronal winds (Bogovalov & Tsinganos 1999; Tsinganos & Bogovalov 2002), and in the general relativistic domain for disk winds (e.g. Koide et al. 1999, 2001) to model the formation and collimation of relativistic outflows in the vicinity of black holes. The difficulty for relativistic outflows in a single-component model to be collimated led Bogovalov & Tsinganos (2005) to propose a two-component model wherein a relativistic central wind is collimated by a surrounding non relativistic disk-wind. Shocks may also develop as the disk-wind collides and collimates the inner relativistic wind. Note that all such simulations are performed by using time-dependent codes. Analytical models conversely have presented more sophisticated steady solutions of outflows to be used as initial conditions in more complex simulations, albeit sacrificing freedom on the chosen boundary conditions.

In this article we present an extension of the non relativistic meridionally self-similar solutions (Sauty & Tsinganos 1994, hereafter ST94) to the case of relativistic jets emerging from a spherical corona surrounding the central part of a Schwarzschild black hole and its inner accretion disk.

We will not discuss here the origin of the plasma, assuming that it can come from, e.g., the accretion disk or pair creation. Furthermore, only the outflow process is considered, with the base of the corona placed at a few Schwarzschild radii, just above the so called separating surface (Takahashi et al. 1990).

Attention is also given to the contribution of the different mechanisms, hydrodynamic and magnetic, to the acceleration and collimation of the outflow, as in previous papers of this series (Sauty et al. 1999, 2002, 2004, hereafter STT99, STT02 and STT04). This is also a way to extend to 2D outflows, thermally driven, spherically symmetric (1D) wind models (Price et al. 2003; Meliani et al. 2004).

In the following two sections, the basic steady axisymmetric GRMHD equations are presented using a 3+1 formalism (Sect. 2), together with their integrals. The assumptions leading to the self-similar model are presented in Sect. 3. In Sect. 4, the analytical expressions of the model together with the derivation of an extra free integral controlling the efficiency of the magnetic rotator, as in STT99, are given. In Sect. 5, an asymptotic analysis of the solutions is performed as well as the link to the boundary conditions in the source. Section 6 is devoted to a parametric study of various solutions to emphasize the main difference obtained with relativistic flows. We discuss the acceleration and collimation of these new solutions (Sect. 7) and compare them with the non relativistic model in Sect. 8. In the last Sect. 9 we summarize our results and shortly outline their main astrophysical implications. The confrontation of the present model with observed jets from radio loud extragalactic jets, such as those associated with FRI and FRII sources, is postponed to a following paper, as it involves special

techniques for constraining the parameters by using observational data and a specific iterative scheme to use the model.

2. Basic equations

In this section we briefly present, in order to establish notation, the governing equations for magnetized fluids in the background spacetime of a Schwarzschild black hole and also the corresponding MHD integrals.

2.1. The 3+1 formalism for steady flows

2.1.1. Schwarzschild metric

The gravitational potential due to the matter outside the black hole is assumed to be negligible. In Schwarzschild coordinates (ct, r, θ, φ) the background metric is written as,

$$ds^2 = -h^2 c^2 dt^2 + \frac{1}{h^2} dr^2 + r^2 d^2\theta + r^2 \sin^2\theta d\varphi^2, \quad (1)$$

where

$$h = \sqrt{1 - \frac{2\mathcal{G}M_\bullet}{c^2 r}} = \sqrt{1 - \frac{r_G}{r}}, \quad (2)$$

is the redshift factor induced by gravity at a distance r from the central black hole of mass $M_\bullet$, expressed in terms of the Schwarzschild radius $r_G = 2\mathcal{G}M_\bullet/c^2$. Note that the time line element or the lapse function is usually denoted by h_0 or α . In this paper, for further convenience and simplification we have used the symbol h .

In the following we find convenient to use a 3+1 split of space-time, following the usual approach of MHD flow treatment in general relativity (Thorne & McDonald 1982; Thorne et al. 1986; Mobarrry & Lovelace 1986; Camenzind 1986a). The 3+1 approach allows to obtain equations similar to the familiar classical equations. We write all quantities in the FIDucial Observer frame of reference, known as FIDO, which corresponds to observers in free fall around the Schwarzschild black hole. For the FIDO, space time is locally flat.

2.1.2. Particle conservation

The equation of conservation of particles $(nu^a)_{;a} = 0$ in the 3+1 formalism is

$$\nabla \cdot (h\gamma n \mathbf{V}) = 0. \quad (3)$$

Here n is the proper number density of particles, $\mathbf{V}$ is the fluid three-velocity as measured by FIDOs, $u^a = (\gamma c, \gamma \mathbf{V})$ and

$$\gamma = (1 - V^2/c^2)^{-1/2}, \quad (4)$$

is the Lorentz factor.

2.1.3. Maxwell's equations and Ohm's law

Maxwell's equations written in the 3+1 formulation (Thorne & MacDonald 1982; Breitmoser & Camenzind 2000) are¹

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e, \quad (5)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (6)$$

$$\nabla \times (h\mathbf{E}) = 0, \quad (7)$$

$$\nabla \times (h\mathbf{B}) = \frac{4\pi h}{c} \mathbf{J}_e, \quad (8)$$

where $(\mathbf{E}, \mathbf{B})$ is the electromagnetic field and $(\rho_e, \mathbf{J}_e)$ the associated charge and current densities. Ohm's law for a plasma of high conductivity is

$$\mathbf{E} = -\frac{\mathbf{V} \times \mathbf{B}}{c}. \quad (9)$$

2.1.4. Euler equation

Euler's equation is obtained by projecting the conservation of the energy-momentum tensor, $T_{;b}^{ab} = 0$, onto the spatial coordinates ($a = 1, 2, 3$) and combined with Maxwell's equations (see Breitmoser & Camenzind 2000; or, for an expression closer to ours, albeit restricted to special relativity Goldreich & Julian 1970; Appl & Camenzind 1993; Heyvaerts & Norman 2003),

$$\gamma n (\mathbf{V} \cdot \nabla) \left(\frac{\gamma w \mathbf{V}}{c^2} \right) = -\gamma^2 n w \nabla \ln h - \nabla P + \rho_e \mathbf{E} + \frac{\mathbf{J}_e \times \mathbf{B}}{c}, \quad (10)$$

where P and w are the pressure and enthalpy per particle, respectively. The form of Eq. (10) is equivalent to that in Mobarrry & Lovelace (1986).

2.1.5. Thermodynamics

The first law of thermodynamics is obtained by projecting the conservation of the energy-momentum tensor along the fluid four-velocity, $u_a T_{;b}^{ab} = 0$. In fact, for ideal MHD fluid the corresponding contribution of the electromagnetic field is null due to the assumed infinite conductivity. Thus, only the thermal energy affects the variation of the proper enthalpy of the fluid,

$$n \mathbf{V} \cdot \nabla w = \mathbf{V} \cdot \nabla P. \quad (11)$$

2.2. Integrals of axisymmetric MHD outflows

Assuming axisymmetry of the plasma flow allows us to reduce the number of differential equations by integrating some of them and thus obtaining conserved quantities along the streamlines. We follow the notations of Tsinganos (1982).

¹ Apparently, Eqs. (5)–(8) have some differences with their equivalent forms in a globally flat spacetime. The h appearing in Eqs. (7) and (8) results from formulating the laws of differentiation on a “per-unit-t-base” while the vectors $\mathbf{E}$ and $\mathbf{B}$ are defined on a “per-unit-FIDO-time” (Thorne et al. 1986).

From Eqs. (6) and (3) we can introduce a magnetic flux function A ,

$$\mathbf{B}_p = \nabla \times \left(\frac{A}{r \sin \theta} \mathbf{e}_\varphi \right) = \frac{\nabla A}{\varpi} \times \mathbf{e}_\varphi, \quad (12)$$

and a stream function Ψ which gives the particle flux,

$$4\pi h \gamma n \mathbf{V}_p = \nabla \times \left(\frac{\Psi}{r \sin \theta} \mathbf{e}_\varphi \right) = \frac{\nabla \Psi}{\varpi} \times \mathbf{e}_\varphi, \quad (13)$$

where the subscript p denotes the poloidal components and $\varpi = r \sin \theta$.

From Eq. (7), we can define an electric potential associated to the electric field, $\mathbf{E} = (\nabla \Phi)/h$. Thus, the previous equation and axisymmetry imply $E_\varphi = 0$. In addition from the flux freezing condition (Eq. (9)) and Eqs. (12) and (13) we get that Ψ is constant on surfaces of constant A on which the corresponding streamlines and fieldlines are roped, $\mathbf{V}_p \parallel \mathbf{B}_p$. It follows that $d\Psi/dA = \Psi_A$ is a function of A and we can write

$$\mathbf{V}_p = \frac{\Psi_A}{4\pi h \gamma n} \mathbf{B}_p. \quad (14)$$

Since $\mathbf{B}_p \cdot \nabla \Phi/h = \mathbf{B}_p \cdot \mathbf{E} = \mathbf{B} \cdot \mathbf{E} = 0$, the surfaces of constant electric potential are also surfaces of constant magnetic flux, so $\Phi = \Phi(A)$. Thus $\Omega = -c d\Phi/dA$ is also a function of A . From Eq. (9) the toroidal components V_φ and B_φ are related

$$V_\varphi = \frac{\Psi_A}{4\pi h \gamma n} B_\varphi + \frac{\varpi \Omega}{h}. \quad (15)$$

This is called the isorotation law because each stream/fieldline rotates rigidly with an angular speed Ω corresponding to the angular speed Ω of the footpoints of this poloidal stream/fieldline.

The azimuthal component of the momentum equation yields the conservation of the total specific angular momentum,

$$L = \gamma \frac{w}{c^2} \varpi V_\varphi - h \frac{\varpi B_\varphi}{\Psi_A}. \quad (16)$$

The generalized Bernoulli integral (including rest mass)

$$\mathcal{E} = h \gamma w - h \frac{\varpi \Omega}{\Psi_A} B_\varphi, \quad (17)$$

may be obtained by integrating the equation of motion along each streamline. The first part of the r.h.s. represents the hydrodynamical energy flux transported by the fluid while the second part corresponds to the Poynting flux.

We have obtained the usual four integrals of motion Ψ_A , Ω , $\mathcal{E}$, L (Heyvaerts & Norman 2003) that are constant along a fieldline for a stationary and axisymmetric plasma. They can be used to find algebraic relations between the Lorentz factor, the toroidal velocity and the toroidal magnetic field. Defining the poloidal ‘‘Alfvénic’’ number M (Michel 1969; Camenzind 1986b; Breitmoser & Camenzind 2000)

$$M^2 = \frac{4\pi h^2 n w \gamma^2 V_p^2}{B_p^2 c^2} = \frac{w \Psi_A^2}{4\pi n c^2}, \quad (18)$$

and the cylindrical distance in units of the light cylinder distance (although in our case it is not a cylinder),

$$x = \frac{\varpi \Omega}{ch}, \quad (19)$$

we find,

$$V_\varphi = \frac{c}{x} \left[\frac{M^2 x_A^2 - x^2 h^2 (1 - x_A^2)}{M^2 - h^2 (1 - x_A^2)} \right], \quad (20)$$

$$B_\varphi = -\frac{\mathcal{E} \Psi_A}{cx} \left[\frac{x^2 - x_A^2}{M^2 - h^2 (1 - x_A^2)} \right], \quad (21)$$

$$h \gamma w = \mathcal{E} \left[\frac{M^2 - h^2 (1 - x_A^2)}{M^2 - h^2 (1 - x_A^2)} \right]. \quad (22)$$

The quantity x_A^2 ,

$$x_A^2 = \frac{\Omega L}{\mathcal{E}} \quad (23)$$

is a measure for the amount of energy carried by the electromagnetic field. It is a measure of the energy flux of the magnetic rotator in units of the total energy flux (Breitmoser & Camenzind 2000).

The MHD equations possess a well known singularity at the Alfvén surface where the denominator of Eqs. (20)–(22) vanishes. Then, the numerators should vanish simultaneously to ensure a regular behavior, implying at the Alfvén point,

$$x^2|_{\text{Alfvén}} = \left(\frac{\varpi_A \Omega}{h_\star c} \right)^2 = x_A^2, \quad (24)$$

$$M^2|_{\text{Alfvén}} = h_\star^2 (1 - x_A^2) = M_A^2. \quad (25)$$

Using Eq. (18) we find

$$V_\star^2 = \frac{B_\star^2 c^2}{4\pi \gamma_\star^2 n_\star w_\star}, \quad (26)$$

where the subscript $\star$ denotes quantities evaluated at the Alfvén point along the polar axis. Note that the position of the Alfvén surface is shifted with respect to the classical case because of the lapse function h and the existence of the light cylinder $x = 1$.

In the Newtonian limit, Eqs. (23) and (24) give $L = m \varpi_A^2 \Omega$ where $m = w/c^2$ is the particle mass.

3. Construction of the meridionally self-similar model

Our goal in this section is to find semi-analytical solutions of the r – and θ – components of the Euler Eq. (10), by means of separating the variables r and θ . In order to facilitate the analysis it is convenient to use dimensionless quantities normalizing at the Alfvén radius along the polar axis. Using the notations introduced in ST94 we define a dimensionless radius R and magnetic flux function α ,

$$R = \frac{r}{r_\star}, \quad A = \frac{r_\star^2 B_\star}{2} \alpha. \quad (27)$$

We introduce two dimensionless parameters to describe the gravitational potential. The *first* is ν which represents the escape speed at the Alfvén point along the polar axis in units of $V_\star$,

$$\nu = \frac{V_{\text{esc},\star}}{V_\star} = \sqrt{\frac{2\mathcal{G}M_\bullet}{r_\star V_\star^2}}. \quad (28)$$

The *second* parameter² is the ratio of the Schwarzschild radius over the Alfvén radius r_* ,

$$\mu = \frac{r_G}{r_*} = \frac{V_{\text{esc},*}^2}{c^2} \quad (29)$$

which is also the escape speed in units of the speed of light.

Combining Eqs. (28) and (29) we get a condition that restrict the parametric space to

$$\frac{\mu}{\nu^2} = \frac{V_*^2}{c^2} < 1. \quad (30)$$

3.1. Separation of the variables

3.1.1. Magnetic flux and “Alfvénic” number

As $\alpha = 0$ on the rotational axis and we are interested on the central component of the jet, we assume that the cross section area of a given magnetic flux tube can be expanded to first order in α ,

$$S(R, \alpha) = \pi \varpi^2 = \pi r_*^2 G^2(R) \alpha. \quad (31)$$

Normalizing at the Alfvén surface, we choose $G(R = 1) = 1$ such that G is the cylindrical radius in units of the Alfvénic cylindrical radius. Thus $G(R) = \varpi / \varpi_A$ with $\varpi_A = r_* \sqrt{\alpha}$.

This is equivalent to assume, as in ST94, that the magnetic flux function $\alpha(R, \theta)$ has a dipolar latitudinal dependence,

$$\alpha = \frac{R^2}{G^2(R)} \sin^2 \theta. \quad (32)$$

We also introduce the expansion factor F

$$F = \left. \frac{\partial \ln \alpha}{\partial \ln R} \right|_{\theta} = 2 - \frac{d \ln G^2}{d \ln R}. \quad (33)$$

In addition, we assume that the surfaces of constant “Alfvénic” number are spheres, such that,

$$M^2(R, \alpha) = M^2(R), \quad (34)$$

with $M^2(R = 1) = h_*^2$.

3.1.2. Pressure

The pressure dependence is obtained by making a first order expansion in α

$$P = P_0 + \frac{1}{2} \gamma_*^2 n_* \frac{w_*}{c^2} V_*^2 \Pi(R) (1 + \kappa \alpha), \quad (35)$$

with P_0, κ constants and $\Pi(R)$ a dimensionless function.

3.1.3. Free integrals

Combining Eqs. (18) and (34) we deduce that

$$\frac{4\pi n c^2 M^2}{w} = \Psi_A^2. \quad (36)$$

² μ corresponds to the parameter m used by Daigne & Drenkhahn (2002).

Expanding the r.h.s. to first order in α , we find

$$\Psi_A^2 = 4\pi c^2 \frac{n_* h_*^2}{w_*} (1 + \delta \alpha), \quad (37)$$

where δ is a free parameter describing the deviations from spherical symmetry of the ratio number density/enthalpy and not of the density itself as in ST94.

Similarly we expand $L\Psi_A$

$$L\Psi_A = h_* \lambda B_* r_* \alpha, \quad (38)$$

where λ is a constant measuring rotation.

Finally we choose for Ω a form similar to the one in ST94

$$\Omega = \lambda h_* \frac{V_*/r_*}{\sqrt{1 + \delta \alpha}}. \quad (39)$$

From Eq. (19) we can now express x^2 in terms of α and $G(R)$,

$$x^2 = \lambda^2 \frac{V_*^2}{c^2} \frac{h_*^2}{h^2} G^2 \frac{\alpha}{1 + \delta \alpha} = \frac{\mu \lambda^2}{\nu^2} \frac{h_*^2}{h^2} G^2 \frac{\alpha}{1 + \delta \alpha}. \quad (40)$$

Similarly, from Eq. (24) we find

$$x_A^2 = \lambda^2 \frac{V_*^2}{c^2} \frac{\alpha}{1 + \delta \alpha} = \frac{\mu \lambda^2}{\nu^2} \frac{\alpha}{1 + \delta \alpha}, \quad (41)$$

which gives from Eq. (23) the form of the Bernoulli integral,

$$\mathcal{E} = h_* \gamma_* w_*. \quad (42)$$

Note that the parameter λ in Eq. (39) is the same constant as in Eq. (38) because the energy $\mathcal{E}$ must be equal to its hydrodynamic part $h_* \gamma_* w_*$ on the rotational axis where the Poynting flux vanishes.

3.2. Electric force

Conversely to the non relativistic limit, we cannot neglect the charge separation and the presence of the electric field. From the previous assumptions, we can calculate the electric force $\rho_e \mathbf{E}$ which has the following two components

$$\begin{aligned} \rho_e E_r = & \frac{B_*^2}{4\pi r_* G^4} \left\{ \frac{h_*^2}{h} \frac{F}{2} x_A^2 G^2 \sin^2 \theta \right. \\ & \times \left[\frac{h^2}{4} \left(2 \frac{dF}{dR} + \frac{F^2}{R} + 2F - \frac{8}{h^2 R} \right) \right. \\ & \left. \left. + \frac{1}{R(1 + \delta \alpha)} \left(h^2 \frac{F^2}{4} - 1 \right) \right] \right. \\ & \left. + \frac{h_{0*}^2}{h} \frac{F}{2} \lambda^2 \frac{V_*^2}{c^2} R \sin^2 \theta \frac{2 + \delta \alpha}{(1 + \delta \alpha)^2} \right\}, \end{aligned} \quad (43)$$

$$\begin{aligned} \rho_e E_\theta = & \frac{B_*^2}{4\pi r_* G^4} \left\{ \left(\frac{h_*}{h} \right)^2 x_A^2 G^2 \sin \theta \cos \theta \right. \\ & \times \left[\frac{h^2}{4} \left(2 \frac{dF}{dR} + \frac{F^2}{R} + 2F - \frac{8}{h^2 R} \right) \right. \\ & \left. \left. + \frac{1}{R(1 + \delta \alpha)} \left(h^2 \frac{F^2}{4} - 1 \right) \right] \right. \\ & \left. + \left(\frac{h_*}{h} \right)^2 \lambda^2 \frac{V_*^2}{c^2} R \sin \theta \cos \theta \frac{2 + \delta \alpha}{(1 + \delta \alpha)^2} \right\}. \end{aligned} \quad (44)$$

Note that in this form of the expressions of the forces the variables are not separable.

3.3. Non relativistic rotation

Contrary to the non relativistic case and in order to separate the variables (R, θ) in the r - and θ - components of the Euler Eq. (10) we need some further assumptions. Basically we expand these equations with respect to θ .

We suppose that the rotational speed of the fluid remains always subrelativistic. In other words, we focus on streamlines that never cross the light cylinder such that the later does not affect the dynamics ($x \ll 1$, which implies $x_A \ll 1$). Of course this reduces the domain of validity of the solutions to the vicinity of the rotational axis because x_A^2 as it is given by Eq. (41) is sufficiently small only for relatively small α . The region of validity of our model depends on how small the parameter $\lambda^2 V_\star^2 / c^2 = \lambda^2 \mu / v^2$ is, though. The requirement that the light cylinder lies further away from the Alfvénic surface ($x_A < 1$) constrains the parametric space to $\delta > \lambda^2 \mu / v^2$. On the other hand, there is a possibility that all streamlines never cross the light cylinder. This happens when the equation

$$x = 1 \Leftrightarrow \frac{1}{\alpha} = \frac{\lambda^2 \mu}{v^2} \frac{G^2 h_\star^2}{h^2} - \delta \quad (45)$$

cannot be satisfied for any R , or equivalently $\delta > (\lambda^2 \mu / v^2) [G^2 h_\star^2 / h^2]_{\max}$. Weak rotation (small λ) or significant deviation of the particle flux from spherical symmetry (high δ which results in a fast decrease of Ω as we move away from the rotation axis) contribute to the validity of the above inequality.

A consequence of this is that the jet is thermally driven. Indeed the ratio between the Poynting flux and the matter energy flux is,

$$\frac{-h\varpi\Omega B_\varphi / \Psi_A}{h\gamma w} = h^2 \frac{x^2 - x_A^2}{M^2 - h^2(1 - x_A^2)}. \quad (46)$$

Thus, in the approximation $x \ll 1$ the contribution of the Poynting flux is negligible in accelerating the flow.

More generally, after expanding with respect to x_A^2 , we neglect terms of the order $x_A^2 \sin \theta$ or higher. This is also consistent with our previous assumption of keeping terms only up to α in the integrals. For example, the expression of the azimuthal magnetic field, Eq. (21), becomes

$$B_\varphi \approx -\frac{\lambda B_\star}{G^2} \frac{h_\star}{h} \frac{G^2 h_\star^2 - h^2}{M^2 - h^2} R \sin \theta. \quad (47)$$

Equivalently, after expanding all terms with respect to θ we neglect terms of the order of $\sin^3 \theta$ or higher.

As another example, the approximate form of the electric force is

$$\rho_e E_r \approx \frac{B_\star^2}{4\pi r_\star G^4} \frac{h_{0\star}^2}{h} F \lambda^2 \frac{V_\star^2}{c^2} R \sin^2 \theta, \quad (48)$$

$$\rho_e E_\theta \approx 2 \frac{B_\star^2}{4\pi r_\star G^4} \left(\frac{h_\star}{h}\right)^2 \lambda^2 \frac{V_\star^2}{c^2} R \sin \theta \cos \theta, \quad (49)$$

where we have further approximated $\frac{2+\delta\alpha}{(1+\delta\alpha)^2} \approx 2$, since this factor is multiplied with $x_A^2 / \sin \theta \propto \sin \theta$ and $\alpha \propto \sin^2 \theta$. Though this approximation is consistent with the fact that we neglect any terms of the order of $\sin^3 \theta$ or higher, it gives an extra restriction. Thus, the model applies only to the region near the

rotational axis where $\alpha \ll 1/\delta$. We shall calculate both inequalities ($x_A^2 \ll 1$ and $\alpha \ll 1/\delta$) *a posteriori* in order to determine the regime where the solution is valid.

Note that in the very vicinity of the black hole $h \rightarrow 0$ and since the factor hx does not vanish (see Eq. (19)), x becomes larger than unity, as expected by the presence of the second light cylinder close to the horizon (e.g., Takahashi et al. 1990). However, it is enough that the coronal base is at a distance of a few gravitational radii ($R \gtrsim 2\mu$) to guarantee that $x \ll 1$ at the base of the outflow (this condition is fulfilled in our solutions).

4. Equations of the model

4.1. Expressions of the fields and the enthalpy

The velocity and magnetic fields can now be written exclusively in terms of unknown functions of R . For later convenience, as in ST94, we shall denote by N_B , N_V and D the following quantities that appear in various components of the fields,

$$N_B = \frac{h^2}{h_\star^2} - G^2, \quad (50)$$

$$N_V = \frac{M^2}{h_\star^2} - G^2, \quad (51)$$

$$D = \frac{h^2}{h_\star^2} - \frac{M^2}{h_\star^2}. \quad (52)$$

Thus,

$$B_r = \frac{B_\star}{G^2} \cos \theta, \quad (53)$$

$$B_\theta = -\frac{B_\star}{G^2} \frac{hF}{2} \sin \theta, \quad (54)$$

$$B_\varphi = -\frac{\lambda B_\star}{G^2} \frac{h_\star}{h} \frac{N_B}{D} R \sin \theta, \quad (55)$$

$$V_r = \frac{V_\star M^2}{h_\star^2 G^2} \frac{1}{\sqrt{1+\delta\alpha}} \left(\cos \theta + \frac{\mu \lambda^2}{v^2} \frac{N_B}{D} \alpha \right), \quad (56)$$

$$V_\theta = -\frac{V_\star M^2}{h_\star^2 G^2} \frac{hF}{2} \frac{1}{\sqrt{1+\delta\alpha}} \sin \theta, \quad (57)$$

$$V_\varphi = -\frac{h}{h_\star} \frac{\lambda V_\star}{G^2} \frac{N_V}{D} \frac{R \sin \theta}{\sqrt{1+\delta\alpha}}. \quad (58)$$

The electric field can be deduced from the flux freezing condition and the above equations (see Eqs. (48), (49)).

Similarly, the enthalpy and the particle number density are given by

$$h\gamma w = h_\star \gamma_\star w_\star \left(1 - \frac{\mu \lambda^2}{v^2} \frac{N_B}{D} \alpha \right), \quad (59)$$

$$h\gamma n = h_\star \gamma_\star n_\star \frac{h_\star^2}{M^2} \left(1 + \delta\alpha - \frac{\mu \lambda^2}{v^2} \frac{N_B}{D} \alpha \right), \quad (60)$$

where we used Eq. (37) to deduce Eq. (60), while the pressure is given by Eq. (35).

4.2. Ordinary differential equations and numerical technique

There are three equations given in Appendix A that determine the three unknown functions $\Pi(R)$, $F(R)$ and $M^2(R)$. We recall that G^2 is related to F through Eq. (33). Before discussing in detail the results of the parametric study we outline the method for the numerical integration of Eq. (33) and Eqs. (A.1)–(A.3). We start integrating the equations from the Alfvén critical surface. In order to calculate the toroidal components of the fields, i.e. N_B/D and $N_V/D = N_B/D - 1$, we apply L'Hôpital's rule,

$$\left. \frac{N_B}{D} \right|_{\star} = \frac{h_{\star}^2(2 - F_{\star}) - \mu}{h_{\star}^2 p}, \quad p = \left. \frac{1}{h_{\star}^2} \frac{dM^2}{dR} \right|_{\star} - \frac{\mu}{h_{\star}^2}. \quad (61)$$

To avoid kinks in the fieldline shape, we need to satisfy a regularity condition (Heyvaerts & Norman 1989). This means that Eq. (A.3) should be regular at $R = 1$. As in ST94 this extra requirement is equivalent to $N_F, D = 0$ which eventually gives a third order polynomial equation for p

$$C_3 p^3 + C_2 p^2 + C_1 p + C_0 = 0, \quad (62)$$

$$C_0 = -\frac{\lambda^2}{2} \left(F_{\star} - 2 + \frac{\mu}{h_{0\star}^2} \right)^2, \quad (63)$$

$$C_1 = \lambda^2 \left(F_{\star} - 2 + \frac{\mu}{h_{0\star}^2} \right), \quad (64)$$

$$C_2 = \frac{1}{2} \lambda^2 - \frac{h_{\star}^2}{8} F_{\star}^2 + \frac{1}{2} (1 - \kappa \Pi_{\star}) + \lambda^2 \frac{\mu}{v^2}, \quad (65)$$

$$C_3 = h_{\star}^2 \frac{F_{\star}}{4}. \quad (66)$$

Once we have determined the regularity conditions at the Alfvén point, we integrate downwind and upwind and cross all the other existing critical points as in the non relativistic case.

It is worth noticing that the shape of the streamlines $F_{\star}$ at $R = 1$ is determined by the regular crossing of the slow magnetosonic surface. We point out further that, besides the free parameters listed at the beginning of the section, solutions depend also on $\Pi_{\star}$, i.e. the pressure at the Alfvén surface. As in the classical case its value has been chosen such that the gas pressure is always positive. More details on the numerical technique can be found in ST94 and STT02.

4.3. The integral ϵ

As in ST94, it is possible to find a constant ϵ for all fieldlines. This parameter ϵ has been used in ST94 and in the following papers to classify the various solutions. We shall use a similar technique to construct such a constant in the present model.

Equation (11), after substituting n from Eq. (18) and using $\mathbf{V} \cdot \nabla \propto \partial/\partial R|_{\alpha}$ (derivative keeping α constant), can be re-written as

$$\begin{aligned} -8\pi M^2 \left. \frac{\partial P}{\partial R} \right|_{\alpha} &= -\frac{\partial}{\partial R} \left(\frac{\Psi_A^2 w^2}{c^2} \right) \Big|_{\alpha} \\ &= \frac{\partial}{\partial R} \left[\frac{\Psi_A^2 (\mathcal{E}^2 - w^2)}{c^2} \right] \Big|_{\alpha}, \end{aligned} \quad (67)$$

where $\Psi_A^2 w^2$ is proportional to the energy *per unit volume* of the fluid in the comoving frame, i.e. reduced to the thermal content. Thus $\Psi_A^2 (\mathcal{E}^2 - w^2)$ in essence measures the variation between the total energy and the thermal energy of the fluid.

By writing the term

$$\begin{aligned} \frac{\Psi_A^2 w^2}{c^2} &= \frac{\Psi_A^2 w^2 \gamma^2 (1 - V_{\varphi}^2/c^2 - V_p^2/c^2)}{c^2} \\ &= \frac{\Psi_A^2}{c^2 h^2} (h\gamma w)^2 \left(1 - \frac{V_{\varphi}^2}{c^2} \right) - \frac{M^4 B_p^2}{h^2}, \end{aligned} \quad (68)$$

and using Eqs. (20) and (22) we find

$$\begin{aligned} \frac{\Psi_A^2 w^2}{c^2} &= \frac{\Psi_A^2 \mathcal{E}^2}{c^2 h^2} \left[\frac{M^2 - h^2(1 - x_A^2)}{M^2 - h^2(1 - x^2)} \right]^2 \\ &\quad - \frac{\Psi_A^2 \mathcal{E}^2}{c^2 h^2} \left[\frac{M^2 x_A^2/x - x h^2(1 - x_A^2)}{M^2 - h^2(1 - x^2)} \right]^2 - \frac{M^4 B_p^2}{h^2}. \end{aligned} \quad (69)$$

In the particular model that we examine, the form of the pressure is $P = f_1(R)(1 + \kappa\alpha)/8\pi$. We also know the θ dependence in all quantities in the expression for $\Psi_A^2 w^2/c^2$, and after expanding with respect to $\sin^2 \theta$ we find $\Psi_A^2 (\mathcal{E}^2 - w^2)/c^2 = f_2(R) + f_3(R)\alpha$. Then Eq. (67) gives

$$\begin{aligned} -M^2 \frac{df_1}{dR} (1 + \kappa\alpha) &= \frac{df_2}{dR} + \frac{df_3}{dR} \alpha \\ &\Leftrightarrow \begin{cases} -M^2 df_1 = df_2 \\ -M^2 \kappa df_1 = df_3. \end{cases} \end{aligned} \quad (70)$$

Eliminating df_1 we get the integral $f_3(R) - \kappa f_2(R) = \epsilon = \text{const.}$ After substituting the expressions for $f_2(R)$ and $f_3(R)$, we arrive at

$$\begin{aligned} \epsilon &= \frac{M^4}{h_{\star}^4 R^2 G^2} \left(\frac{F^2}{4} - \frac{1}{h^2} - \kappa \frac{R^2}{h^2 G^2} \right) - \frac{(\delta - \kappa) v^2}{h^2 R} \\ &\quad + \frac{\lambda^2}{G^2 h_{\star}^2} \left(\frac{N_V}{D} \right)^2 + \frac{2\lambda^2 N_B}{h^2 D}, \end{aligned} \quad (71)$$

where ϵ is a constant, the same for all fieldlines.

Similarly to what was done in ST99, we can calculate this constant at the base of the flow R_o assuming the poloidal velocity is negligible there [$M(R_o) \approx 0$]. Let's express $\epsilon/2\lambda^2$ in terms of the conditions at the source boundary,

$$\frac{\epsilon}{2\lambda^2} = \frac{\mathcal{E}_{R,o} + \mathcal{E}_{\text{Poynt},o} + \Delta \mathcal{E}_G^*}{\mathcal{E}_{\text{MR}}}, \quad (72)$$

where $\mathcal{E}_{\text{MR}} = h^2 L \Omega$ is the energy of the magnetic rotator, $\mathcal{E}_{\text{Poynt}} = -h \varpi \Omega B_{\varphi} / \Psi_A$ is the Poynting flux and

$$\mathcal{E}_{R,o} = \frac{\mathcal{E}}{c^2} \frac{V_{\varphi,o}^2}{2}, \quad (73)$$

is the rotational energy per particle. It is proportional to the specific rotational energy $V_{\varphi,o}^2/2$, with the factor $\mathcal{E}/c^2$ having the dimensions of a mass. Finally we have

$$\Delta \mathcal{E}_G^* = -\frac{\mathcal{E}}{c^2} \frac{\mu c^2 (\delta - \kappa) \alpha}{2 R_o (1 + \delta \alpha)}, \quad (74)$$

a term similar to the nonrelativistic case where it measures the excess or the deficit on a non polar streamline, compared to the

polar one, of the gravitational energy per unit mass which is not compensated by the thermal driving (STT99). As in the classical case, ϵ measures the efficiency of the magnetic rotator to collimate the flow. Thus if $\epsilon > 0$ we have an Efficient Magnetic Rotator (EMR) where magnetic collimation may dominate, while if $\epsilon < 0$ we have an Inefficient Magnetic Rotator (IMR) where collimation cannot be but of thermal origin.

5. Asymptotic behaviour

In the region far from the base where the jet attains its asymptotic velocity, assuming it becomes cylindrical, the forces in the radial direction become negligible, since the jet is no longer accelerated. In the transverse direction, the following four forces balance each other: the transverse pressure gradient, f_P , the total magnetic stress (magnetic pinching plus magnetic pressure gradient) of the toroidal magnetic field component, f_B , the centrifugal force, f_C and the charge separation electric force, f_E ,

$$f_C + f_B + f_P + f_E = 0. \quad (75)$$

The full expressions of these forces are given in Appendix B. In the asymptotic region, $\theta \sim 0$, they can be written as follows in cylindrical coordinates,

$$\begin{aligned} f_C &= \gamma^2 n \frac{w}{c^2} \frac{V_\varphi^2}{\varpi} \\ &= \frac{B_\star^2}{4\pi G_\infty^4} \frac{h_\star^2 \lambda^2}{M_\infty^2} \varpi_\infty \left(\frac{N_{V\infty}}{D_\infty} \right)^2, \end{aligned} \quad (76)$$

$$\begin{aligned} f_B &= -\frac{1}{4\pi\varpi} \left(B_\varphi^2 + \frac{1}{2} \frac{dB_\varphi^2}{d\varpi} \varpi \right) \\ &= -\frac{B_\star^2}{2\pi G_\infty^4} h_\star^2 \lambda^2 \varpi_\infty \left(\frac{N_{B\infty}}{D_\infty} \right)^2, \end{aligned} \quad (77)$$

$$f_P = -\frac{dP}{d\varpi} = -\frac{B_\star^2}{4\pi G_\infty^2} \Pi_\infty \kappa \varpi_\infty, \quad (78)$$

$$f_E = \rho_e E_\varpi = \frac{B_\star^2}{2\pi G_\infty^4} h_\star^2 \frac{\lambda^2 \mu}{v^2} \varpi_\infty. \quad (79)$$

The centrifugal and electric forces have a decollimating effect on the jet, while the pinching magnetic force collimates it because of our choice on the current distribution. For asymptotically underpressured jets where $\kappa > 0$ and $\Pi_\infty > 0$ (or $\kappa < 0$ and $\Pi_\infty < 0$), the pressure increases away from the polar axis which helps collimation. The opposite holds for overpressured jets where $\kappa < 0$ and $\Pi_\infty > 0$ (or $\kappa > 0$ and $\Pi_\infty < 0$).

Combining Eq. (75) with Eqs. (76)–(79) we obtain,

$$\frac{\kappa}{2\lambda^2} \Pi_\infty = \frac{h_\star^2}{G_\infty^2} \left[\frac{1}{2M_\infty^2} \left(\frac{N_{V\infty}}{D_\infty} \right)^2 + \frac{\mu}{v^2} - \left(\frac{N_{B\infty}}{D_\infty} \right)^2 \right]. \quad (80)$$

The second equation controlling the asymptotic transverse force balance is given by ϵ , Eq. (71). In the asymptotic region, for a cylindrically collimated jet $F_\infty \rightarrow 2$ and ϵ becomes,

$$\frac{\epsilon}{2\lambda^2} = -\frac{\kappa}{2\lambda^2} \frac{M_\infty^4}{h_\star^4 G_\infty^4} + \frac{1}{2h_\star^2 G_\infty^2} \left(\frac{N_{V\infty}}{D_\infty} \right)^2 + \frac{N_{B\infty}}{D_\infty}. \quad (81)$$

We notice that the asymptotic behaviour of the jet is described by the asymptotic pressure Π_∞ and the two parameters $\epsilon/2\lambda^2$

and $\kappa/2\lambda^2$. These equations are similar to the classical model except for the decollimating effect of the electric field and charge separation. Besides that note also that in D , N_V and N_B the space curvature at the Alfvén critical surface also appears.

At the base of the wind the Alfvén number vanishes, $M_o \rightarrow 0$, while the opening of the jet is weak, $G_o^2 \ll 1$. We can use this criterion to define the distance R_o where the outflow starts. Namely from the expression of ϵ , Eq. (71), we get,

$$R_o = \frac{(\delta - \kappa)v^2 - \mu\epsilon}{2\lambda^2 - \epsilon}. \quad (82)$$

We see that in order to have acceleration ($R_o > \mu$) we must have approximately $(\delta - \kappa)/2 > \mu\lambda^2/v^2$ for $2\lambda^2 - \epsilon > 0$, extending the criterion found in the classical regime ($\delta > \kappa$; the above expression reduces to Eq. (14) of STT02 for $\mu \rightarrow 0$). In particular we note that for $\epsilon > 0$ or $\epsilon < 0$ relativistic effects enlarge or reduce the size of the sub-Alfvénic region, respectively.

Note that the definition of R_o coincide with the so called separating surface (see e.g. Takahashi et al. 1990, for a cold plasma in a Kerr metric). Above this surface the plasma is outflowing. Below this surface other critical surfaces exist (e.g., Beskin & Kuznetsova 2000) but remain out of our consideration. If the plasma is created via pair production this may constrain the boundary conditions at the base of the flow. However we do not discuss the origin of the coronal plasma in this paper.

6. Parametric analysis

As a first step of the numerical analysis we have performed a study of the effects on the solution of the free parameters of the model μ , v , λ , δ and κ . With respect to the classical case, the relativistic effects are ruled by the new parameter μ .

6.1. Effect of space curvature and gravity (μ , v)

The parameters μ and v denote the escape speed in units of the light and Alfvén speed, respectively. However similar they look, they have opposite effects on the initial acceleration and the terminal speed. In the super-Alfvénic region the acceleration is not strongly affected by different values of μ for $\mu \leq 0.1$; in fact, the effect of relativistic gravity is negligible after $10r_G$ (Fig. 1). So the effects we are discussing now refer to the base of the flow, in the sub-Alfvénic regime.

The parameter μ , the ratio between the Schwarzschild and the Alfvénic radius, representing also the escape speed at the Alfvén radius in units of c , is related to the relativistic effects of gravity in this model. Basically μ controls the extension of the corona and the acceleration of the flow in the sub-Alfvénic region. Increasing μ increases also the asymptotic velocity, as it can be seen in Fig. 1a. A simple physical interpretation may be given to this behaviour. When the Alfvénic surface approaches the Schwarzschild surface, gravity in the sub-Alfvénic region, and thus in the corona, increases. Consequently, to support gravity the thermal energy increases too. This larger amount of thermal energy in the corona will be transformed in turn largely into kinetic energy along the flow. In other words, the increase of μ implies a stronger density gradient of the flow in the

Fig. 1. Variation of the velocity vs. r/r_* for different values of μ in **a)** and ν in **b)**. The other parameters are $\nu = 1.185$, $\delta = 2.113$, $\kappa = 0.5$, $\lambda = 1.9995$ in **a)**, and $\mu = 0.1$, $\delta = 1.0$, $\kappa = 0.5$, $\lambda = 1.0$ in **b)**.

sub-Alfvénic region, increasing the radial pressure gradient $d\Pi/dR$ and leading to a stronger expansion and acceleration.

This behaviour can be also understood considering that larger values of μ imply a larger space curvature, increasing also the expansion of the streamlines and thence the efficiency of the acceleration, as it has been shown in the study of the relativistic Parker wind (see Meliani et al. 2004).

Conversely, increasing ν decreases the asymptotic velocity as well, since it reduces the size of the corona, keeping μ constant, that is the distance of the Alfvén surface to the Black Hole (see Fig. 1b). This figure shows that the base of the flow gets closer to the Alfvén radius and farther from the Schwarzschild radius. As in the non relativistic case the parameter ν^2 is the ratio of gravitational to kinetic energy at the Alfvén surface, Eq. (28). An increase of ν reduces the fluid velocity at the Alfvén radius with respect to escape speed needed to get out of the black hole’s attraction. The reduction of the size of the corona is also consistent with the reduction of the velocity. The behaviour is as expected from the solutions in the classical regime (ST94, STT02, STT04): the higher is the value of ν the lower is the asymptotic velocity, although we didn’t show it explicitly, as in the present Fig. 1b. There is also a minimum value of ν to have mass ejection, below that value the thermal energy cannot support gravity (see STT02).

6.2. Effect of rotation (λ)

The parameter λ is related to the rotation of the flow and to the axial electric current (Fig. 2b). As for non relativistic outflows (ST94, STT02) it rules the jet dynamics through the Lorentz force, collimating asymptotically the jet via the toroidal magnetic field, while the centrifugal force has instead a decollimating effect. We have checked that the behaviour of the solutions is similar to the classical case. The increase of λ leads to more collimated and slower jets (Fig. 2a). This can be understood as follows. Increasing λ increases the axial current (Fig. 2b) which increases the toroidal magnetic pinching. In order to preserve equilibrium the flows reacts by increasing the centrifugal force and thus its rotational speed by reducing its cross section. The reduction of the expansion factor reduces as usual the pressure

Fig. 2. Comparison between two jet solutions with $\lambda = 1.0$ ($\epsilon = 0.089$) and 1.2 ($\epsilon = 0.824$). In **a)** we plot the velocity along the polar axis and in **b)** the dimensionless electric current density along the polar axis. The other parameters are $\mu = 0.1$, $\nu = 0.65$, $\delta = 1.4$, $\kappa = 0.2$ and $\Pi_* = 0.75$.

gradient and the thermal driving efficiency thus reducing the asymptotic speed.

In addition we must take into account that λ is related to the electric potential Φ . Consequently it controls the charge separation and the corresponding electric force ($\rho_e E \propto x^2 \propto \lambda^2$). This force becomes dominant where the jet rotation speed becomes relativistic ($\varpi\Omega \sim c$). In other words the higher is λ the larger the effects of the light cylinder (Fig. 3).

Fig. 3. Morphology in the poloidal plane of the streamlines of the two solutions of the previous figure and their light cylinders for $\lambda = 1.0$ ($\epsilon = 0.089$) in **a**) and $\lambda = 1.2$ ($\epsilon = 0.824$) in **b**). As in Fig. 2, the other parameters are $\mu = 0.1$, $\nu = 0.65$, $\delta = 1.4$, $\kappa = 0.2$ and $\Pi_* = 0.75$. The light cylinders are represented by the two thick solid lines which surround the jets, and the different regions of validity of our solutions are shown: solid, dashed and dashed-dotted lines correspond to streamlines where the two quantities (defined in Sect. 2.2) $x_A^2 G^2$ and $(2 + \delta\alpha)(1 + \delta\alpha)^2 - 2$ are <0.01 , <0.1 and >0.1 , respectively.

6.3. Effect of pressure and density anisotropies (κ , δ)

The physical meaning of κ remains the same as in non relativistic flows. For κ positive or negative the gas pressure increases or decreases with colatitude, respectively. Then in the first case the gas contributes to the thermal confinement of the flow (underpressured jets), while in the second to its thermal support (overpressured jets). We have limited ourselves to present here the study of underpressured flows. The behaviour of the relativistic solutions with κ is analogous to that of classical flows. For higher κ both the asymptotic velocity and jet cross sections decrease (see STT99; and STT02 for details).

In the present model the parameter δ controls the variation of the ratio n/w with colatitude, or equivalently in the direction perpendicular to the rotational axis. This is a relativistic generalization of the classical solutions where it governs the transverse profile of the mass density. As a result, the effects are similar to those found for non relativistic solutions. A larger δ means a larger gravitational potential of the external streamlines with respect to the axis, where the acceleration is more efficient. Then the asymptotic velocity increases with δ (ST94; STT02).

7. Jet dynamics, acceleration and collimation

We will address now the question of the process of acceleration and collimation of the jet in the case of EMRs and IMRs. Keeping fixed $\mu = 0.1$ we have displayed the results for an IMR solution with $\nu = 0.781$, $\delta = 2.613$, $\kappa = 0.490$, $\lambda = 0.880$ and $\Pi_* = 1.4$ ($\epsilon = -1.747$) in Fig. 4a (see also Figs. 5a and 6a), and an EMR with $\nu = 0.541$, $\delta = 3.253$, $\kappa = 0.39$, $\lambda = 1.401$ and $\Pi_* = 1.26$ ($\epsilon = 1.128$) in Fig. 4b (see also Figs. 5b and 6b).

We see that the shape of the streamlines in the two cases (Figs. 4) looks similar to the corresponding non relativistic case (STT02; see later Figs. 7). IMRs show a fast expansion in an intermediate region, while far from the base the collimated streamlines show strong oscillations. EMRs show conversely a continuous expansion with relatively mild oscillations, or even no oscillations at all when pressures are lower. We also display for the two solutions the energies along a given streamline (Fig. 5) and the forces perpendicular to the flow (Fig. 6).

7.1. Acceleration

By construction of this model, the wind is basically thermally driven. At the lowest order we have $\mathcal{E} \approx h\gamma w$, while the first order term corresponds to the Poynting flux which remains however of the same order than the thermal terms in the transverse direction. This supposes that there is a high temperature corona around the black hole as proposed by Chakrabarti (1989) and Das (1999, 2000). The latest has shown that the stronger is the thermal energy, the more stable is the corona.

We can study the acceleration of the jet analysing the contribution of the different energies and their conversion from one to the other along the streamlines. The dominating energy at the base of the outflow is the enthalpy. Part of it is used to balance gravity and the remaining part is converted into kinetic energy in the region of expansion of the jet and stops when the streamlines collimate (compare Figs. 4 and 5). In fact, during the expansion of the jet, the plasma density decreases which also induces a decrease of the enthalpy. In turn, it creates a strong pressure gradient ($\nabla P = n\nabla w$) that accelerates the jet. For the parameters we have chosen, we see that the EMR solution has a larger expansion factor than the IMR one (Figs. 4),

Fig. 4. Morphology of the poloidal streamlines for two solutions corresponding to an Inefficient Magnetic Rotator (IMR, $\epsilon = -1.747$) in **a**), and an Efficient Magnetic Rotator (EMR, $\epsilon = 1.128$) in **b**). We chose $\mu = 0.1$ in both cases, while the other parameters are $\nu = 0.781$, $\kappa = 0.49$, $\delta = 2.613$, $\lambda = 0.880$ and $\Pi_\star = 1.4$ in **a**), and $\nu = 0.541$, $\kappa = 0.39$, $\delta = 3.253$, $\lambda = 1.401$ and $\Pi_\star = 1.26$ in **b**). The various regions of validity of our solutions are shown as in Fig. 3.

Fig. 5. Variation of the energy flux normalized to the mass energy, along the external streamline for the IMR solution in **a**) and the EMR solution in **b**) of the previous figure. The Poynting energy is $\leq 1\%$ of the enthalpy at the base of the flow and is not plotted. The parameters are the same as in Fig. 4.

Fig. 6. Plot of the transverse forces for the relativistic IMR **a**) and EMR **b**). Forces are normalized by their maximum value, usually reached at the base of the flow. The parameters are the same as in Fig. 4.

because of the thermal driving. This is correlated to the larger Lorentz factor of the EMR solution ($\gamma \sim 2.8$) as compared to the IMR one ($\gamma \sim 2.4$). This result is not related to the nature of the magnetic rotator. For other parameters, we would get different results but the asymptotic Lorentz factor always increases with the increase of the expansion factor because of the thermal

driving. We verified that the Poynting flux in the two solutions is negligible, representing at maximum only 1% of the enthalpy at the basis of the flow.

The IMR solution undergoes a strong expansion in this region until a distance of $100r_\star$ and then recollimates and consequently decelerates because of the compression. On the other

Fig. 7. Morphology of the poloidal streamlines for non relativistic IMR **a)** and EMR **b)**: the parameters are the same as in Fig. 4, with $\mu = 10^{-5}$.

hand, the EMR solution collimates already at a distance of approximately $50r_*$ and accelerates all the way downwind. In other words, the nature of the collimation affects obviously the velocity profile.

7.2. Collimation

The collimation of the flow is controlled by different types of forces that depend on the morphology of the streamlines in the jet. We have plotted for the two solutions the forces perpendicular to the streamlines in Fig. 6. Asymptotically the centrifugal force is the dominant term which supports the wind against either the magnetic confinement in EMR or the pressure gradient in IMR.

The behaviour of the other forces depends on the shape of the streamlines (Fig. 6) and they play a relevant role in the intermediate region before collimation is fully achieved. In particular, the stress tensor from the poloidal magnetic field and the gravity favor deviation from radial expansion while the thermal pressure gradient initially tends to maintain the radial expansion. In the region of formation of the jet, the strong gravity along the polar axis generates a strong pressure gradient. As density and pressure increase with colatitude ($\delta, \kappa > 0$), it also generates a total force $f_{\nabla_\perp P} + f_{\nabla_\perp \ln h} \propto v^2(\delta - \kappa)$ which further out in the jet provides the thermal confinement.

In an IMR, neither the pressure gradient nor the pinching force from the toroidal magnetic field can brake the expansion of the flow, and the recollimation occurs where the curvature of the poloidal streamlines becomes relevant. In the asymptotic region, the collimation is mainly provided by the transverse pressure gradient, $\kappa\Pi$, which balances the centrifugal force. The pressure confined jets undergo strong oscillations similarly to the non relativistic case (STT94; STT99).

In an EMR, conversely, the pinching force of the toroidal magnetic field provides collimation all along the flow. The pressure gradient may help this collimation as for the present solution or be completely negligible for other sets of

parameters. The magnetic pinching force is balanced asymptotically mainly by the centrifugal force which tends to decollimate the jet. We must notice also that, as expected in the relativistic case, the electric force is always positive and its effect in decollimating the jet may be comparable with the centrifugal force, differently from IMRs (see Fig. 6).

8. Relativistic vs. non relativistic outflows

We analyse here in more detail the main differences between the relativistic and non relativistic solutions already discussed in previous Sect. 6. By increasing the value of μ we increase the depth of the potential well. In our calculations we have assumed $\mu = 0.1$ for the relativistic solutions and this can be justified as follows.

We supposed that the Alfvén surface is roughly at a distance of $10r_o$ from the central object. This typical distance is usually chosen because it corresponds to the case where the wind carries away all the accreted angular momentum, provided about 10% of the accreted mass goes to the jet (Livio 1999). This is of course arbitrary but allows us to compare our solutions to other models. In the case of young stellar jets, the star has a mass of the order of $\sim 1 M_\odot$. The corresponding Schwarzschild radius is approximately $r_G \approx 3$ km. Therefore, space curvature at the Alfvén surface corresponds to a value of $\mu \approx 10^{-5}$. Conversely, for AGN jets, with a central super massive black hole of $\sim 10^9 M_\odot$, we have $r_G \approx 10^4 R_\odot$ which corresponds to the value we have chosen $\mu \approx 0.1$.

We have drawn the corresponding morphologies of two non relativistic solutions associated with an IMR and an EMR in Figs. 7, keeping the other parameters as in Fig. 4. We also plotted the corresponding transverse forces for the non relativistic solutions in Figs. 8.

8.1. Jets from Inefficient Magnetic Rotators (IMR)

Let's first turn our attention to the solution from an IMR. The morphologies of classical and relativistic jets show

Fig. 8. Plot of the transverse forces for the non relativistic IMR **a)** and EMR **b)**. We assumed $\mu = 10^{-5}$ while the other parameters are identical to the corresponding relativistic solutions displayed in Figs. 4.

indeed small differences. The relativistic jet, though, undergoes an expansion slightly more important in the intermediate region (Fig. 4a) than in the classical case (Fig. 7a). This expansion induces a slight relative increase of the curvature forces (inertial and magnetic) compared to other forces. In the asymptotic region, the relativistic jet recollimation is comparable to the non relativistic one (Fig. 9a).

Note also that in the asymptotic region, the relativistic jet pinching by the toroidal magnetic field is almost null, while in the non relativistic solution this force is of the order of the pressure gradient (Figs. 6a and 8a). This behaviour is a consequence of the decrease of the collimation efficiency in relativistic jets.

Last, we know that in the relativistic solutions, there is an extra electric force, $\rho_e E$, due to the non negligible charge separation, which also decollimates. In IMRs, its influence remains however weak because of the low magnetic field.

8.2. Jets from Efficient Magnetic Rotators (EMR)

In the jet solution associated with an EMR, the solution is effectively magnetically collimated because of the toroidal magnetic field pinching force. This solution clearly shows the role of the magnetic field in collimating the jet but also the contribution of the charge separation to the electric field and then to the decollimation.

The morphology of EMR jets is more affected by relativistic effects than the one of IMR jets. This can be seen by comparing Figs. 4b and 7b. In particular, the jet radius, or equivalently the expansion factor, asymptotically becomes more important in relativistic jets because of the increase of the centrifugal and electric forces (Figs. 6b and 8b). Simultaneously, the magnitude of the Lorentz force decreases and the thermal acceleration increases.

We can give a simple explanation to this relativistic effect. An increase of gravity at the base of the jet induces a decrease of the normalized electric current in the jet because $\bar{I} \propto h_*$ where $\bar{I} = I/(c/2r_*B_*)$. We have plotted the dimensionless electric current in Fig. 10 for the two EMR solutions.

Fig. 9. Plot of the asymptotic dimensionless cross section of the jet G_s^2 in **a)**, and the asymptotic velocity in **b)** as a function of the parameter μ on a logarithmic scale. The other parameters are those given for the previous relativistic IMR and EMR solutions.

The electric current $\bar{I}$ flows through a given cross sectional area $S = \pi\omega^2$ of the classical and relativistic solutions. We use the normalized electric current because of the different scaling of the classical and the relativistic solutions. The decrease of the current goes with a decrease of the toroidal magnetic field and with an increase of the expansion of the jet as explained in Sect. 6.2 and, consequently, with an increase of the poloidal velocity. Note that this increase of the velocity corresponds to the increase of the relativistic gravity as we already discussed. The new point is that, conversely to hydrodynamical models, it also decollimates the flow. On the other hand, the magnetocentrifugal driving of the Poynting flux becomes weaker in

Fig. 10. Density of the electric current normalized $\bar{I}_z$, in the classical and the relativistic jets for EMR shown in the previous figures.

relativistic thermally driven winds, as expected. Thus, as the rest mass increases, the Poynting flux is getting weaker relatively to the other energies, $\mathcal{E}_{\text{Poynt.}} \ll mc^2$, while, the thermal energy becomes relativistic and comparable to the rest mass $(w - mc^2) \approx mc^2$.

The relativistic effects on the jet acceleration become remarkable only for a distance between the Alfvén surface and the Schwarzschild surface smaller than 100 (i.e. $\mu > 0.01$, see Fig. 11a). For $\mu < 0.01$, using this model, we see that outflows from a star with mass $1 M_\odot$ and starting at $100r_G$ are simply scaling with μ (Figs. 11b and 11c). Similarly the ratio between the energetics of the two jets are simply proportional to μ . The scaling just reflects the linear growth of the flow formation region with gravity, i.e. with μ .

Conversely, for $\mu > 0.01$, the jet is formed at a distance smaller than $100r_G$, this linear scaling with μ of the dynamics and the energetics does not hold any longer because of non linear relativistic effects. The thermal energy converts more efficiently into kinetic energy (Fig. 11a) as in the spherical case (Meliani et al. 2004). It increases even more because of the stronger expansion of the relativistic jet in the super-Alfvénic region. In fact, in the relativistic solution displayed in Fig. 11a, collimation starts at 50 Alfvén radii, while, in the non relativistic solution, Fig. 11c, collimation occurs only at 10 Alfvén radii.

In summary let us just point out that the mass of the central object and the properties of the jet (acceleration, morphology and energy) are not simply proportional to each other in the context of strong gravitational fields.

8.3. The effect of charge separation

As we have seen, solutions obtained with this model are essentially thermally driven winds but collimation is either thermal (pressure confinement) or magnetic (toroidal magnetic pinching). However, conversely to the non relativistic case an extra decollimating force exists which is the electric force, despite that we neglect the light cylinder effects.

The strength of the electric field results from the induction term $V_p B_\phi / c$ and it is higher for higher magnetic fields. As a matter of fact it gets more important when magnetic effects are important and when the light cylinder is closer to the streamlines. In relativistic flows where the poloidal velocity is of the

Fig. 11. Plot of the energies normalized by the parameter μ . In **a**) is shown the relativistic solution for an EMR, previously displayed in Fig. 5b with $\mu = 10^{-1}$. In **c**) is shown the corresponding non relativistic solution with $\mu = 10^{-5}$. In **b**) we plot an intermediate solution with $\mu = 10^{-2}$. Plots **b**) and **c**) are identical which shows that for such small values of μ , the energies vary linearly with gravity, while this is not true in **a**). In **d**) we plot the ratio between the asymptotic velocity and the escape velocity at the surface of the corona as a function of $\log(1/\mu)$.

order of the light velocity, the contribution in the transverse direction of this force increases in the super-Alfvénic domain.

Therefore, the contribution of this force, in relativistic jets from EMRs, is of the order of the pinching force and the

centrifugal force. Conversely, in the non relativistic limit, the poloidal velocity remains largely subrelativistic, $V_p \ll c$, and the electric field remains weak ($E \sim V_p^2 B_\phi^2 / c^2 / \varpi \rightarrow 0$; see for comparison Figs. 6b and 8b). The electric field does not affect the collimation of the non relativistic jet and the charge separation can be neglected.

Conversely, the effects of the electric force are negligible in pressure confined jets from IMRs as the magnetic effects themselves are very weak or completely negligible, $B_\phi^2 / \varpi \ll P / \varpi \Rightarrow \rho E \ll P / \varpi$.

Parallely, the effects of the electric force on the acceleration of the jet are very weak even for EMRs. The correction introduced on the asymptotic speed is negligible. This is because the electric force is perpendicular to the streamlines and, so, it affects mainly the morphology of the jet.

9. Conclusion

In this paper, we have investigated the problem of the formation and collimation of relativistic jets. We have explored these problems by means of a semi-analytical model, which is the first meridional self-similar model of relativistic jets, including general relativistic effects. We constructed it on the basis of the classical model developed in ST94 to study jets from young stars. We have made an extension of this model for a black hole characterized by weak angular momentum, $a = JG/(M_\bullet^2/c) \ll 1$. Therefore, we treated the problem of GRMHD outflows in a Schwarzschild metric. Moreover, we concentrated our efforts on modelling the jet close to its polar axis. In the construction of this model, we were limited to describing jets possessing a weak rotation velocity compared to the speed of light and we neglected the effects of the light cylinder. Thus, we restricted our study to thermally accelerated jets with a weak contribution of the Poynting flux. However, in the collimation of the jet, both electric and magnetic terms are comparable to the inertial and pressure gradient ones. We have also studied the collimation effects by magnetic and thermal forces and the decollimation effect of centrifugal and electric forces. Our model is restricted to outflow solutions only, and we do not address to the problem of the origin of the hot coronal plasma.

We found that the influence of the electric force and the charge separation in the jet depends on the collimation regime. In the case of EMRs where jets are magnetically collimated, the electric decollimating force plays an important role. This force is of the order of the magnitude of the centrifugal force and of the magnetic pinching. Therefore, relativistic jets from EMRs are less collimated than their non relativistic counterparts. Conversely, jets from IMRs, where collimation is mainly of thermal origin, are not very sensitive to the electric decollimation. In this type of jets, the contribution of $\rho_e E$ is balanced by the increase of the external pressure.

We have also used the model to compare classical jets to relativistic jets. We undertook this comparative study by changing the free parameter μ . In fact, this new parameter gives in the relativistic model the space curvature which is induced by gravity near the central object. We used typical values of μ from $\mu \sim 10^{-5}$ for Jets from Young Stellar Objects to $\mu \sim 0.1$ for jets

from compact sources. We found that the difference between these two types of jets is only a scaling effect on μ for $\mu < 10^{-2}$. In this case, the spatial dimensions and energies are linear functions of μ . For $\mu > 10^{-2}$, the relativistic effects increase together with the thermal acceleration of the jet. Simultaneously, strongly relativistic effects tend to decollimate the jets because of the decrease of the electric current density.

To conclude we have seen that relativistic effects and particularly relativistic gravity tend to enhance the thermal acceleration (as in Meliani et al. 2004) and reduce the magnetic collimation of the jet (as in Bogovalov & Tsinganos 2005). Still collimation can be obtained either by thermal or magnetic means but relativistic effects lower the efficiency of the magnetic rotator. This means that despite quantitative changes, we can use this generalized model to verify, for the classification of AGN jets, the conjectures given in Sauty et al. (2001) using the non relativistic model. Mainly we see that the observed types of jets from radio loud galaxies can be connected to the efficiency of their magnetic rotator and to the environment of the host galaxy. By using the present model, a more precise and quantitative analysis of the observed jets will be presented in a following paper.

Acknowledgements. We acknowledge financial support for our visits from the French Foreign Office and the Greek General Secretariat of Research and Technology (Program PLATON), from the European RTN JETSET (MRTN-CT-2004-005592) and the Observatoire de Paris. E.T. acknowledges financial support by the Italian Ministry for Education, University and Research (MIUR) under the grant Cofin 2003/027534-002. Part of this work has been supported by the European RTN ENIGMA (HPRN-CY-2002-00231).

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Online Material

Appendix A: Ordinary differential equations

$$\frac{d\Pi}{dR} = -\frac{2}{h^2} \frac{1}{G^4} \left(\frac{dM^2}{dR} + \frac{F-2}{R} M^2 \right) - \frac{1}{h^4 R^2 M^2} \left(v^2 h_\star^4 - \mu \frac{M^4}{G^4} \right), \quad (\text{A.1})$$

$$\frac{dM^2}{dR} = \frac{\mathcal{N}_M}{\mathcal{D}}, \quad (\text{A.2})$$

$$\frac{dF}{dR} = \frac{\mathcal{N}_F}{\mathcal{D}}, \quad (\text{A.3})$$

$$\mathcal{D} = -\left(1 + \kappa \frac{R^2}{G^2}\right) D + \lambda^2 R^2 \frac{N_B^2}{D^2} + \frac{h^4 F^2}{4h_\star^2}, \quad (\text{A.4})$$

$$\begin{aligned} \mathcal{N}_M = & \frac{1}{4h_\star^2} \frac{M^4}{R} \left(8 + 8\kappa \frac{R^2}{G^2} - 2F - 4F\kappa \frac{R^2}{G^2} - h^2 F^2 \right) + \frac{h^2}{h_\star^2} \frac{M^2}{R} \left[-2 - 2\kappa \frac{R^2}{G^2} - F \frac{\lambda^2 \mu}{v^2} \frac{h_\star^2}{h^2} R^2 + F\kappa \frac{R^2}{G^2} \right. \\ & \left. + \frac{h^2}{4} \left(1 + \frac{\mu}{Rh^2} \right) F^2 + \frac{h^2}{8} F^3 \right] - D \frac{v^2 h_\star^4}{2h^2} \frac{G^2}{M^2} (\delta - \kappa) + \frac{h^2}{h_\star^2} \frac{1}{2} \kappa \Pi R G^2 F M^2 \\ & - \frac{M^2}{2h^2} \frac{\mu}{R^2} \left(1 + \kappa \frac{R^2}{G^2} \right) D + \lambda^2 \frac{N_B N_V}{D^2} \mu + \lambda^2 R \left(\frac{N_B^2}{D^2} - \frac{h^2}{2M^2} \frac{N_V^2}{D^2} \right) [2M^2 + h^2(F-2)] \\ & - \lambda^2 R h^2 (F-2) \frac{N_B}{D} + \mu \lambda^2 \frac{G^2 h_\star^4 N_B}{h^2 M^2}, \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \mathcal{N}_F = & -\frac{M^2 F}{R h_\star^2} \left[\left(1 - \frac{F}{2} \right) \left(1 + \kappa \frac{R^2}{G^2} + \lambda^2 R^2 \frac{N_B^2}{D^3} \right) + \frac{F}{2} \left(1 - h_0^2 \frac{F}{2} \right) \right] \\ & + \frac{1}{R} \left(\frac{h}{h_\star} \right)^2 \left(1 + \kappa \frac{R^2}{G^2} - \lambda^2 R^2 \frac{N_B^2}{D^3} - \frac{h^2}{4} F^2 \right) \left(F^2 - F - \frac{2}{h^2} - \frac{h_\star^2}{h^4} \frac{\lambda^2 \mu}{v^2} 4R^2 \right) \\ & + \left(\frac{2R\Pi G^2 \kappa}{h_\star^2} + \frac{\mu F}{h_\star^2 R^2} \right) \left(1 + \kappa \frac{R^2}{G^2} - \lambda^2 R^2 \frac{N_B^2}{D^3} - \frac{h^2}{4} F^2 \right) \\ & - \frac{v^2 G^2 h_\star^2}{2M^2 h^2} F (\delta - \kappa) - \frac{\mu F M^2}{2R^2 h^2 h_\star^2} \left(1 + \kappa \frac{R^2}{G^2} \right) + \frac{\lambda^2}{h_\star^2} \left(\frac{N_B^2}{D^2} - \frac{N_B}{D} \right) \frac{F}{D} \mu - \frac{\lambda^2 R h^2}{h_\star^2} F (F-2) \frac{N_B}{D^2} \\ & + \frac{4\lambda^2 R}{h^2} \left(\frac{N_B^2}{D^2} - \frac{1}{2M^2} h^2 \frac{N_V^2}{D^2} \right) \left(1 + \kappa \frac{R^2}{G^2} - \lambda^2 R^2 \frac{N_B^2}{D^3} - \frac{h^2}{2} F \right) + \mu \lambda^2 \frac{G^2 h_\star^2 F N_B}{h^2 M^2} \frac{N_B}{D}. \end{aligned} \quad (\text{A.6})$$

Appendix B: The forces on the plasma

The momentum equation can be written as

$$-\frac{n}{\Psi_A^2 w/c^2} (\mathbf{U} \cdot \nabla) \mathbf{U} - \nabla P + \frac{(\nabla \cdot \mathbf{E})}{4\pi} \mathbf{E} + \frac{[\nabla \times (h\mathbf{B})]}{4\pi h} \times \mathbf{B} - \gamma^2 n w \nabla \ln h = 0, \quad (\text{B.1})$$

where the generalized velocity $\mathbf{U}$ is given by

$$\mathbf{U}_p = \Psi_A \gamma \frac{w}{c^2} \mathbf{V}_p = \frac{M^2}{h} \mathbf{B}_p, \quad (\text{B.2})$$

$$U_\varphi = \Psi_A \gamma \frac{w}{c^2} V_\varphi = \lambda h_\star B_\star \frac{R}{G^2} \left[\frac{M^2 - h_\star^2 G^2 (1 - x_A^2)}{M^2 - h^2 + G^2 h_\star^2 x_A^2} \right] \sin \theta. \quad (\text{B.3})$$

In the following we give the expressions of the various terms.

B.1. Advection force

$$-\frac{n}{\Psi_A^2 w/c^2} [(U \cdot \nabla) U] \cdot \mathbf{e}_R = -\frac{B_\star^2}{4\pi r_\star G^4} \left\{ \frac{1}{h^2} \frac{dM^2}{dR} + \frac{M^2}{hR} \left(F - 2 - \frac{\mu}{2Rh^2} \right) \right. \\ \left. + \sin^2 \theta \left[-\frac{1}{h} \frac{dM^2}{dR} - \frac{M^2}{hR} \left(h^2 \frac{F^2}{4} + \frac{F}{2} - 2 + \frac{\mu}{2Rh^2} \right) - \lambda^2 h h_\star^2 \frac{R}{M^2} \frac{N_V^2}{D^2} \right] \right\}, \quad (\text{B.4})$$

$$-\frac{n}{\Psi_A^2 w/c^2} [(U \cdot \nabla) U] \cdot \mathbf{e}_\theta = \frac{B_\star^2}{4\pi r_\star G^4} \sin \theta \cos \theta \\ \times \left[\frac{F}{2} \frac{dM^2}{dR} + \frac{M^2}{R} \frac{F}{2} \left(\frac{F}{2} - 1 \right) + \frac{M^2}{2} \frac{dF}{dR} + \frac{\lambda^2 h_\star^2 R}{M^2} \frac{N_V^2}{D^2} \right] \quad (\text{B.5})$$

B.2. Pressure force

$$\mathbf{f}_{\text{Pr}}^R = -\frac{h}{r_\star} \frac{\partial P}{\partial R} = -\frac{h}{8} \frac{B_\star^2}{\pi r_\star G^4} \left[\frac{d\Pi}{dR} G^4 \left(1 + \kappa \frac{R^2}{G^2} \sin^2 \theta \right) + \Pi F \kappa R G^2 \sin^2 \theta \right] \mathbf{e}_R, \quad (\text{B.6})$$

$$\mathbf{f}_{\text{Pr}}^\theta = -\frac{1}{r_\star R} \frac{\partial P}{\partial \theta} = -\frac{B_\star^2}{4\pi r_\star G^4} R G^2 \Pi \kappa \sin \theta \cos \theta \mathbf{e}_\theta. \quad (\text{B.7})$$

B.3. Electric force

$$\mathbf{f}_E^R = \frac{B_\star^2}{4\pi r_\star G^4} \frac{h_{0\star}^2}{h} F \lambda^2 \frac{V_\star^2}{c^2} R \sin^2 \theta \mathbf{e}_R, \quad (\text{B.8})$$

$$\mathbf{f}_E^\theta = \frac{B_\star^2}{2\pi r_\star G^4} \left(\frac{h_\star}{h} \right)^2 \lambda^2 \frac{V_\star^2}{c^2} R \sin \theta \cos \theta \mathbf{e}_\theta, \quad (\text{B.9})$$

B.4. Magnetic force

B.4.1. Magnetic hoop stress

$$\mathbf{f}_{S,B\varphi}^R = -\frac{1}{4\pi r_\star} B_\varphi^2 \left(\frac{h}{R} + \frac{dh}{dR} \right) \mathbf{e}_R = -\frac{B_\star^2}{4\pi r_\star G^4} \left(\frac{h}{R} + \frac{dh}{dR} \right) \frac{\lambda^2 h_\star^2}{h^2} \left[\frac{\left(\frac{h}{h_\star} \right)^2 - G^2}{\frac{M^2}{h_\star^2} - \left(\frac{h}{h_\star} \right)^2} \right]^2 R^2 \sin^2 \theta \mathbf{e}_R \quad (\text{B.10})$$

$$\mathbf{f}_{S,B\varphi}^\theta = -\frac{1}{4\pi r_\star R} B_\varphi^2 \cot \theta \mathbf{e}_\theta = -\frac{\lambda^2 B_\star^2}{4\pi r_\star G^4} \frac{N_B^2}{D^2} \frac{h_\star^2}{h^2} R \sin \theta \cos \theta \mathbf{e}_\theta. \quad (\text{B.11})$$

$$\mathbf{f}_{S,B\rho}^R = \left[-\frac{1}{4\pi r_\star} B_\theta^2 \left(\frac{h}{R} + \frac{dh}{dR} \right) + \frac{B_\theta}{4\pi r_\star R} \frac{\partial B_r}{\partial \theta} \right] \mathbf{e}_R \\ = \left[-\frac{B_\star^2}{16\pi r_\star G^4} h^2 F^2 \left(\frac{h}{R} + \frac{dh}{dR} \right) \sin^2 \theta + h \frac{B_\star^2}{8\pi r_\star G^4} \frac{F}{R} \sin^2 \theta \right] \mathbf{e}_R, \quad (\text{B.12})$$

$$\mathbf{f}_{S,B\rho}^\theta = \left[\frac{1}{4\pi r_\star} B_r B_\theta \left(\frac{dh}{dR} + \frac{h}{R} \right) + \frac{h B_r}{4\pi r_\star} \frac{\partial B_\theta}{\partial R} \right] \mathbf{e}_\theta \\ = -\frac{B_\star^2}{8\pi r_\star G^4} \frac{h}{h^2} \left[h \frac{dF}{dR} + h \frac{F^2}{R} - h \frac{F}{R} + F \frac{\mu}{h R^2} \right] \sin \theta \cos \theta \mathbf{e}_\theta. \quad (\text{B.13})$$

B.4.2. Magnetic gradient pressure

$$\begin{aligned}
f_{Pr,B\varphi}^R &= -\frac{h}{8\pi r_\star} \frac{\partial B_\varphi^2}{\partial R} \mathbf{e}_R = -\frac{h_\star^2}{h^2} \frac{\lambda^2 B_\star^2}{4\pi r_\star G^4} \left[\left(\frac{h}{h_\star} \right)^2 - G^2 \right]^2 R^2 \sin^2 \theta \\
&\quad \times \left[\frac{h}{R} - \frac{\mu}{2R^2 h} + h \frac{F-2}{R} + h \frac{\frac{\mu}{R^2 h_\star^2} + \frac{G^2}{R} (F-2)}{\left(\frac{h}{h_\star} \right)^2 - G^2} - h \frac{\frac{\mu}{R^2 h_\star^2} - \frac{dM^2}{h_\star^2 dR}}{\left(\frac{h}{h_\star} \right)^2 - \frac{M^2}{h_\star^2}} \right] \mathbf{e}_R,
\end{aligned} \tag{B.14}$$

$$f_{Pr,B\varphi}^\theta = -\frac{1}{8\pi r_\star R} \frac{\partial B_\varphi^2}{\partial \theta} \mathbf{e}_\theta = -\lambda^2 \frac{B_\star^2}{4\pi r_\star G^4} \frac{N_B^2}{D^2} \left(\frac{h_\star}{h} \right)^2 R \sin \theta \cos \theta \mathbf{e}_\theta. \tag{B.15}$$

$$\begin{aligned}
f_{Pr,Bp}^R &= -\frac{h}{8\pi r_\star} \frac{\partial B_\theta^2}{\partial R} \mathbf{e}_R \\
&= -h^2 \frac{B_\star^2}{16\pi r_\star G^4} F^2 \left[\frac{\mu}{2R^2 h} + h \left(\frac{1}{F} \frac{dF}{dR} + \frac{F-2}{R} \right) \right] \sin^2 \theta \mathbf{e}_R,
\end{aligned} \tag{B.16}$$

$$f_{Pr,Bp}^\theta = -\frac{1}{8\pi r_\star R} \frac{\partial B_\theta^2}{\partial \theta} \mathbf{e}_\theta = \frac{B_\star^2}{4\pi r_\star G^4} \frac{1}{R} \sin \theta \cos \theta \mathbf{e}_\theta. \tag{B.17}$$

B.5. Gravity force

$$-\gamma^2 n w \nabla \ln h = -\frac{\gamma^2 n w}{r_\star} \frac{dh}{dR} \mathbf{e}_R = -\frac{B_\star^2}{8\pi r_\star} \frac{h_\star^4}{h^3} \frac{\nu^2}{R^2 M^2} \left(1 + \delta\alpha - 2 \frac{\mu \lambda^2}{\nu^2} \frac{N_B}{D} \alpha \right) \mathbf{e}_R, \tag{B.18}$$