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Astrophysical unipolar inductors powered by GW emission

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ABSTRACT

We consider the Unipolar Inductor Model (Goldreich & Lynden-Bell 1969, ApJ, 156, 59) applied to Double Degenerate Binaries (DDBs) with ultrashort periods (Wu et al. 2002, MNRAS, 331, 221).

In this model a magnetized primary white dwarf has a slight asynchronism between its spin and orbital motion, so that the (non-magnetic) secondary experiences a motional electric field when moving through the primary field lines. This induces a current flow between the two stars and provides an electric spin-orbit coupling mechanism for the primary.

We study the combined effect of Gravitational Wave emission and electric spin-orbit coupling on the evolution of the primary degree of asynchronism and the associated rate of electric current dissipation in such systems, assuming that the primary's spin is not affected by other mechanisms such as tidal interactions with the companion. In particular, we show that in ultrashort period binaries the emission of GW pumps energy in the electric circuit as to keep it steadily active. This happens despite the fact that spin-orbit coupling can rapidly synchronize the primary, because GW represent a slow desynchronizing mechanism that steadily substracts orbital angular momentum to the system. A slightly asynchronous steady-state is thus achieved, determined by the balance between these two competing effects. This can be shown to correspond to a condition where the total available electric energy is conserved, because of GW emission, while dissipation, synchronization and orbital shrinking continue.

Key words. gravitational waves – magnetic fields – stars: binaries: close – stars: dwarfs novae – X-rays: individuals: RX J0806.3+1527 – X-rays: individuals: RX J1914.4+2456

1. Introduction

The Unipolar Inductor Model (UIM from here on) has been originally proposed (Goldreich & Lynden-Bell 1969) for the Jupiter-Io system to explain the origin of bursts of decametric radiation received from the planet, whose occurrence and intensity were known to be strongly influenced by the orbital location of Io.

The model relies on the fact that Jupiter's spin differs from the system orbital period, Io's spin being locked to the orbit around the planet. Given the $\sim$10 G surface dipole magnetic field of Jupiter and the good electrical conductivity expected from the satellite, the system behaves as a remarkably simple electric circuit, that can be sketched as follows. A good conductor (the satellite) has a trasverse motion with respect to the field lines of the planet and this induces an e.m.f. across it[1]. The e.m.f. can accelerate free charges that are present in the ambient medium, giving rise to a flow of currents between the two objects. Currents are confined to a thin sheet along the sides of the flux tube connecting Jupiter and Io; hence, the cross section of the current layers at Jupiter's surface has the form of two arc-shaped strips. Currents flow along one side of the tube towards Jupiter and then vertically through its upper atmosphere and ionosphere. Jupiter's electrical conductivity (σ) in the ionosphere is expected to be almost isotropical (Goldreich & Lynden-Bell 1969, and references therein): hence, currents reaching it can propagate transverse to field lines, closing the circuit and returning back to Io along the opposite side of the flux tube. Charges along the flux tube are accelerated to mildly relativistic energies and lead to coherent cyclotron emission over a range of wavelenghts: this is the framework in which Jupiter's decametric radiation and its strong modulation by Io's position can be explained.

Among the main expectations of the model it is of interest here mentioning the effect of resistive dissipation of currents in the planetary atmosphere: this causes a local heating and an associated localized enhancement of the thermal emission.

Several confirmations to the model have been obtained over the years, the most spectacular being provided by the HST UV observations of Clarke et al. (1996). In particular, these revealed the emission from the resistive dissipation region on Jupiter's surface – Io's footprint – and its corotation

[1] It would be more appropriate to describe this in terms of the Lorenz force acting on the charge carriers within the conductor, but introducing the e.m.f. emphasises the analogy with an electrical circuit.

with the satellite, which means that the emission region drifts on the planet's surface just as expected in the UIM.

The same basic picture has been proposed by Li et al. (1998) for planetary companions to white dwarfs: if the UIM applied to such systems it would offer an alternative way of searching for extrasolar planets through the electromagnetic emission associated to the electrical circuit.

Wu et al. (2002) proposed a similar scenario for the case of two white dwarfs forming a close binary system. They consider a moderately magnetized primary, a non-magnetic secondary and a primary spin not perfectly synchronous with the orbital motion, while the secondary's is efficiently kept synchronous by tidal forces. The atmospheric electrical conductivity of white dwarfs is expected to be highly anisotropic, because of the WD magnetic field, up to at least ∼1 km depth (Li et al. 1998), so currents are forced to follow field lines in this situation: they can only cross them and close the circuit when σ becomes isotropical and cross-field diffusion can proceed efficiently.

This model was proposed to account for the observed properties of the two candidate ultrashort period binaries RX J0806+15 and RX J1914+24. These have significant soft X-ray emission ($\sim 10^{32}$ and 10^{34} erg s^{-1} respectivley) pulsed at periods ∼321 and 569 s: these periodicities are interpreted as likely due to orbital modulations, although the subject is still debated (see Cropper et al. 2003, for a review on this and several other aspects). Further, both sources have a measured orbital spin-up (Strohmayer 2004; Hakala et al. 2004; Israel et al. 2004), as if driven by GW with no matter transfer: this would thus require an alternative source for the X-ray emission, UIM representing an interesting possibility. In fact, given the very short orbital periods, possibly larger primary magnetic fields and the compactness of both components with respect to previous versions of the UIM, such systems may dissipate energy at a significant rate even for slight degrees of asynchronism, as shown by Wu et al. (2002). These authors showed that currents are resistively dissipated essentially in the primary atmosphere, the associated heating causing the observed soft X-ray emission: its source is ultimately represented by the energy of the relative motion between the primary spin and the orbit. Further, the process redistributes angular momentum – via the Lorentz torque on cross-field currents – between the primary spin and the orbit and thus acts to synchronize them.

In this application of the UIM, synchronization timescales ∼few 10^3 yrs were obtained for both RX J1914+24 and RX J0806+15, very short compared to the orbital evolutionary timescales, respectively $\sim 8 \times 10^6$ yrs and 3×10^5 yrs. Accordingly, a very low probability (significantly <1%) of detecting these systems during the asynchronous phase would be expected. This would in turn require a very large population of such systems in the Galaxy, much larger than predicted by population-synthesis models, since two of them have been detected in the short-lived, active phase.

Concerning this point, we focus in this work on a key aspect of the problem that has been overlooked in previous works, namely that in the framework of the UIM applied to DDBs perfect synchronization between the primary spin and the orbit is *never* reached: therefore, the current flow should not stop at any time. This happens even if the synchronization timescale (τ_α) is much shorter than the orbit evolutionary timescale ($\tau_o = \omega_o/\dot{\omega}_o$), because of the continuous loss of orbital angular momentum caused by GW. As tidal synchronization is expected to be efficient only for the lighter star, GW drive the primary out of synchronism on the longer timescale τ_o, thus continuosly feeding energy to the electric circuit.

Consider a perfectly synchronous system: the electric circuit would be turned off while GW would still cause orbital spin-up and, thus, a desynchronization of the primary's spin, which would in turn switch the circuit on: the system should thus evolve, over the timescale τ_α, towards a slightly asynchronous steady-state that is determined by the balance between the fast, synchronizing mechanism (UI) and the slow, desynchronizing one (GW).

In the following this issue will be addressed quantitatively: in particular we show that, in a binary system with UIM at work, the orbital period decreases because of GW emission and the primary spin is forced by the coupling to approach the orbital period, but perfect equality is not reached because of the energy fed by GWs to the circuit. Further, we derive a definition of τ_α as a function of system parameters, find conditions under which $\tau_\alpha \ll \tau_o$ is verified and discuss the salient evolutionary features of the UIM implied by this condition.

In a companion paper we apply these general considerations – and the related formulas that we derive here – to RX J0806+15 and RX J1914+24 and obtain interesting constraints on system parameters for the UIM to be applicable to these sources.

Detailed and systematic calculations and evolutionary implications are beyond the scope of these works and will be addressed elsewhere.

2. Asynchronous evolution in the unipolar inductor model

Following Wu et al. (2002), the primary asynchronism parameter is defined as $\alpha = \omega_1/\omega_o$, where ω_1 is the primary spin and ω_o the orbital motion.

Given an asynchronous system with orbital separation a, the secondary star will be moving across the primary magnetic field lines with the relative velocity $v = a(\omega_o - \omega_1) = [GM_1(1+q)]^{1/3}\,\omega_o^{1/3}(1-\alpha)$, where G is the gravitational constant, M_1 the primary mass, $q = M_2/M_1$ the system mass ratio. The electric field induced through the secondary is $\boldsymbol{E} = \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}$, with an associated e.m.f. $\Phi = 2R_2E$, R_2 being the secondary's radius. Because of the induced e.m.f. a current flows between the two component stars, whose resistive dissipation – which takes place essentially in the primary atmosphere – has two effects: first, it causes significant heating of the dissipation region, thus powering its soft X-ray emission. Second, the Lorentz torque associated to currents crossing field lines in the primary atmosphere (and in the conducting secondary) redistributes angular momentum between the primary spin and the orbital motion. In presence of significant GW-emission, as expected for two white dwarfs orbiting each other at ultrashort periods, there is an additional effect: orbital angular momentum is continuously lost from the system, causing the orbital period to steadily decrease (and the orbit to shrink). Hence, as long as the primary is not efficiently kept synchronous by tidal forces, its spin will lag behind the orbital motion on the orbital evolutionary timescale; electric coupling will thus be active and exchanging angular momentum between the orbit and the primary spin.

We stress that the absence of any mechanism other than the Unipolar Inductor able to affect the primary's spin over the orbit evolutionary timescale is an essential assumption of our analysis: in Appendix A we show that tidal synchronization of the primary is indeed not expected to be efficient on this timescale, while it may well be effective in rapidly synchronizing a low-mass companion. The latter effect is discussed in Appendix A as well, being however of much lower relevance.

In this work we consider binary systems consisting of two degenerate white dwarfs, with the following mass-radius relation (Nauenberg 1972; Marsh & Steeghs 2002):

$$\frac{R}{R_\odot} = 0.0112\left[\left(\frac{M}{1.433}\right)^{-\frac{2}{3}} - \left(\frac{M}{1.433}\right)^{\frac{2}{3}}\right]^{\frac{1}{2}} \tag{1}$$

where M is expressed in solar masses.

In order to make all above features more quantitative we need proper expressions for the physical quantities of interest. Let us start recalling Eq. (E4) from Appendix E of Wu et al. (2002), that describes the evolution of the orbital period of a binary system with the coupled effects of UIM and GW:

$$\frac{\dot\omega_{\rm o}}{\omega_{\rm o}} = \frac{1}{g(\omega_{\rm o})}\left(\dot E_g - \frac{W}{1-\alpha}\right). \tag{2}$$

Here $\dot E_g = -(32/5)G/c^5[q/(1+q)]^2 M_1^2 a^4 \omega_{\rm o}^6$ is the energy loss rate through GW emission (Landau & Lifshitz 1951), the second term in parenthesis represents the contribution of electric coupling between the primary spin and the orbit and W is the electric current dissipation rate (source luminosity). The function $g(\omega_{\rm o}) < 0$ is:

$$g(\omega_{\rm o}) = -\frac{1}{3}\left[\frac{q^3 G^2 M_1^5 \omega_{\rm o}^2}{(1+q)}\right]^{\frac{1}{3}}\left[1 - \frac{6}{5}(1+q)\left(\frac{R_2}{a}\right)^2\right]. \tag{3}$$

The first term (including the coefficient −1/3) corresponds to the ratio between $\dot E_g$ and $\dot\omega_{\rm o}/\omega_{\rm o}$ for two point masses with no spin-orbit coupling of any kind. It represents two thirds of the total orbital (gravitational plus kinetic) energy of the binary system (E_g). The second term in square brackets – call it $h(a)$ – is of order unity for most plausible system parameters: in particular, $0.6 < h(a) < 1$ for orbital periods longer than 200 s, primary mass $M_1 > 0.4\ M_\odot$ and secondary mass $M_2 > 0.08\ M_\odot$ but, unless extreme values of all parameters at the same time are assumed, it is >0.85 in most cases. This term accounts for the secondary spin being tidally locked to the orbit; as the system shrinks due to GW emission, an additional tiny amount of orbital angular momentum is lost to spin up the secondary and the resulting $\dot\omega_{\rm o}/\omega_{\rm o}$ will be just slightly higher, given M_1, q and $\omega_{\rm o}$. Equation (3) can thus be written concisely as $g(\omega_{\rm o}) = (2/3)E_g h(a)$, a physically much clearer expression which will be frequently used especially in Sect. 3. As both $\dot E_g$ and $g(\omega_{\rm o})$ are negative, GW always give a positive contribution to $\dot\omega_{\rm o}$ while electric coupling ($W > 0$ by definition) may either favour or oppose the orbital spin-up depending on the sign of $(1-\alpha)$.

Since the two candidate ultrashort period binaries have a measured orbital spin-up, in the following we will be interested in systems where the orbit is shrinking: if $\alpha < 1$ this is warranted, but when $\alpha > 1$ spin-orbit coupling transfers angular momentum *to* the orbit and, if sufficiently strong, it may even overcome the effect of GW.

From a general point of view, however, there is no a priori reason not to consider systems subject to such a phase of orbital spin-down, due to a strong spin-orbit coupling. In Sect. 3.1 and Appendix B we briefly comment on this situation. For the moment, however, we focus only on systems where $\dot\omega_{\rm o} > 0$, a condition that, if $\alpha > 1$, must be implicitly expressed as $|\dot E_g| > |W/(1-\alpha)|$ and will be made more explicit in the next sections.

The quantity W can be expressed as:

$$W = \frac{\Phi^2}{\mathfrak{R}} = \left(\frac{\mu_1 R_2}{c}\right)^2 \frac{[GM_1(1+q)]^{-\frac{4}{3}}}{\mathfrak{R}}\ \omega_{\rm o}^{14/3}(1-\alpha)^2 \tag{4}$$

where μ_1 is the primary magnetic moment and the system's effective resistance $\mathfrak{R}$ is (see Wu et al. 2002):

$$\mathfrak{R} = \frac{1}{2\overline{\sigma}R_2}\left(\frac{H}{\Delta d}\right)\jmath(e)\left(\frac{a}{R_1}\right)^{3/2} = N\omega_{\rm o}^{-1} \tag{5}$$

where N includes G, M_1 and q after writing a according to Kepler's third law. In the above formula $\overline{\sigma}$ is the height averaged WD atmospheric conductivity, H the atmospheric depth at which currents cross magnetic field lines and return back to the secondary and Δd the thickness of the arc-like cross section of the current layer at the primary atmosphere. Finally, the geometric factor $\jmath(e) \sim 1$ when the orbital period is less than 1 h or so (see Wu et al. 2002). Combining the two above relations we obtain:

$$\begin{aligned} W &= \left(\frac{\mu_1}{c}\right)^2 \frac{2\overline{\sigma}R_1^{3/2}R_2^3}{[GM_1(1+q)]^{11/6}}\frac{\omega_{\rm o}^{17/3}(1-\alpha)^2}{(H/\Delta d)\jmath(e)} \\ &= k\,\omega_{\rm o}^{\frac{17}{3}}(1-\alpha)^2 \end{aligned} \tag{6}$$

where the last equality defines k.

Finally, combining Eqs. (E4) and (E5) from Appendix E of Wu et al. (2002), the following expression for the evolution of α is obtained:

$$\begin{aligned} \frac{\dot\alpha}{\alpha} &= \frac{\dot\omega_1}{\omega_1} - \frac{\dot\omega_{\rm o}}{\omega_{\rm o}} = -\frac{\dot\omega_{\rm o}}{\omega_{\rm o}} + \frac{W}{\alpha(1-\alpha)I_1\omega_{\rm o}^2} \Rightarrow \\ \dot\alpha &+ \left(\frac{\dot\omega_{\rm o}}{\omega_{\rm o}} + \frac{k}{I_1}\omega_{\rm o}^{\frac{11}{3}}\right)\alpha - \frac{k}{I_1}\omega_{\rm o}^{\frac{11}{3}} = 0 \end{aligned} \tag{7}$$

where I_1 is the moment of inertia of the primary star.

2.1. General considerations and approximations

Equation (7) in its general form is a non linear, first order differential equation for α, with time-dependent coefficients, the non-linear term being due to the coupling between UI and GW in the orbital evolution, i.e. by the torque that exchanges angular momentum between the primary spin and the orbit.

Independent of the exact solution, a general remark can be made that provides a better understanding of the problem and allows finding approximations that simplify its mathematical treatment without affecting its very nature.

The key point is that the orbital period changes continuously over time: hence, synchronization would be maintained only if the primary spin changed continuously as well. If UI was the only mechanism affecting the primary spin – as we are assuming – the electrical circuit would have to remain always active in order to allow ω_1 to track ω_o.

Note that this conclusion holds in general, whatever the functional form of $\dot{\omega}_o$ is. On the other hand, the details of how the system evolves and the state to which it is led depend on the functional form of $\dot{\omega}_o$, i.e. on the model adopted.

In the following we obtain an analytic solution to the evolutionary equation in a specific approximation of the UIM and show that it implies the existence of a slightly asynchronous, asymptotic state[2]. The physical conditions under which our particular solution applies are described in detail, but we stress that the conclusion concerning the existence of a slightly asynchronous, asymptotic state has a more general validity.

The most natural choice to begin with is to focus on the simplified problem in which the effects of UI and GW on the evolution of the asynchronism can be decoupled, thus neglecting the non-linear term in the evolutionary equation. The primary spin evolution is clearly independent of GW and is driven by UI alone; so let us specialize to the case where GW alone drive the orbital evolution, UI providing only a negligible contribution to it.

2.1.1. Validity of the approximation

We check the relative magnitudes of the terms[3] in Eq. (7) to determine the conditions that allow this approximation to be introduced consistently. Using Eq. (2) to re-express $\dot{\omega}_o/\omega_o$ in (7) it obtains:

$$\frac{\dot{\alpha}}{\alpha} = -\underbrace{\frac{\dot{E}_g}{g(\omega_o)}}_{A} + \underbrace{\frac{W}{g(\omega_o)(1-\alpha)}}_{B} + \underbrace{\frac{W}{\alpha(1-\alpha)I_1\omega_o^2}}_{C} \cdot \qquad (8)$$

Let us first compare (B), the contribution of UI to the orbital evolution to (C), the contribution of UI to the primary spin evolution. The condition $B \ll C$ (say, $B < 10^{-1}C$) will be met if:

$$\alpha < \frac{g(\omega_o)}{10\, I_1\omega_o^2} \propto \omega_o^{-\frac{4}{3}} \qquad (9)$$

where the right-hand term is a decreasing function of ω_o and, for component masses in the range ($M_1 \sim 0.5 \div 1\ M_\odot$), gives $\alpha < 2$ even at an orbital period $P_o \sim 300$ s. Therefore, if $P_{orb} > 300$ s the condition $B \ll C$ is met for most plausible values of the asynchronism.

[2] The meaning of this will be extensively discussed in the rest of the paper.

[3] More precisely, we compare their absolute values.

We still need to determine the conditions under which $B \ll A$ is true as well and, then, that GW mostly determine the orbital evolution. By requiring $B < 10^{-1}A$ we obtain:

$$\omega_o < \omega_{GW} = 8.5\times10^{-27} q^{6/7}(1+q)^{1/2}\frac{M_1^{31/14}}{(2\sigma R_2^3 R_1^{3/2})^{3/7}} \times\frac{(H/\Delta d)^{3/7}}{(1-\alpha)^{3/7}\,\mu_1^{6/7}} \quad \text{or}$$

$$P_o > P_{GW} = 410\frac{1}{q^{\frac{6}{7}}(1+q)^{\frac{1}{2}}}\frac{\left(\frac{R_2}{1.7\times10^9}\right)^{\frac{9}{7}}\left(\frac{R_1}{6\times10^8}\right)^{\frac{9}{14}}}{\left(\frac{M_1}{0.9\ M_\odot}\right)^{\frac{31}{14}}} \times\left(\frac{\mu_1}{2\times10^{30}}\right)^{\frac{6}{7}}\left[\frac{H}{\Delta d}J(e)\right]^{-\frac{3}{7}}(1-\alpha)^{\frac{3}{7}} \qquad (10)$$

where c.g.s. units are omitted in the normalizations and a height-averaged conductivity $\overline{\sigma} = 3\times10^{13}$ e.s.u. has been assumed (Wu et al. 2002). Here μ_1 has been normalized to a low value because this gives more easily shrinking orbits. Indeed, it can be directly checked from the above equation that, if $\mu_1 \sim 10^{32}$ G cm^3 as suggested originally by Wu et al. (2002), $P_{GW} \geq 10^4$ s even for $(1-\alpha) \sim 0.1$. This can be stated as follows: relatively highly magnetized systems with $\alpha > 1$ do not shrink unless their degree of asynchonism is very low, because spin-orbit coupling dominates over GW emission. They can shrink only if either they are formed with α very close to one (which would require some spin-orbit coupling even during the common envelope phase that likely leads to their formation) or begin their evolution with $\alpha < 1$. What is more likely is that at an orbital period of, say, 5 h the primary spin is faster than that (say, $\alpha \sim 5{-}10$). In this case, a 10^{32} G cm^3 magnetic moment would cause the orbit to widen until $(1-\alpha)\omega_o^{17/3}$ is small enough that GW dominate over spin-orbit coupling in $\dot{\omega}_o$.

Let us restrict attention to weakly magnetized systems ($\mu_1 < 10^{31}$ G cm^3): even in this case, condition (10) is more constraining than (9): it can be met by low mass binaries, for $\mu_1 \sim$ a few 10^{30} G cm^3 and $P \leq 2000$ s, only if $(1-\alpha)$ is quite small ($<10^{-1}$). For higher mass systems ($M_1 \geq 0.8\ M_\odot$), on the other hand, this condition is met at short periods even with $(1-\alpha) = (0.1 \div 1)$.

The choice of low magnetic moments is completely arbitrary here and justified only because it gives ultrashort period systems where $\dot{\omega}_o$ is determined by GW alone. In Paper II we apply the model to RX J0806+15 and RX J1914+24, showing that such low magnetic moments are not simply convenient but seem to be required for the model to work, which it does quite well indeed.

In summary, B can be neglected even for highly asynchronous systems at orbital periods longer than a few thousands seconds and $\mu_1 < 10^{31}$ G cm^3, while at shorter periods $(100 \div 2000)$ s this approximation holds only if they are almost synchronous ($1-\alpha < 10^{-1}$) or have sufficiently high-mass primaries. In Fig. 1 we show an example of the numerical integration of the evolutionary Eq. (7) for two different values of the primary magnetic moment. Representative, although arbitrary, system parameters and initial conditions have been chosen (see captions). The early phase of orbital spin-down (the period

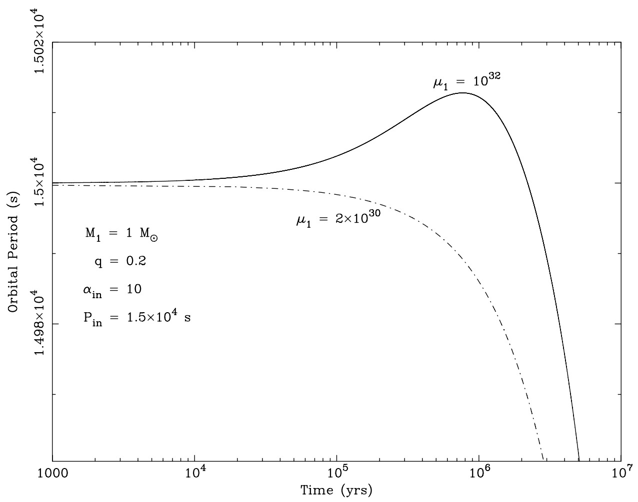

Fig. 1. Illustrative example of the early orbital evolution of a DDB subject to the UI mechanism. The plots are obtained through a numerical integration of the coupled evolutionary equations for α and ω_o. Initial conditions are chosen quite arbitrarily, for illustrative purposes only: the initial orbital period is $P_{in} = 1.5 \times 10^4$ s ($\sim$4 h) and the primary initial spin $\sim$25 min ($\alpha_{in} = 10$). Other system parameters are indicated in the figure. Two primary magnetic moments were tried and their values, in G cm^3, label the corresponding curves.

actually changes just slightly) for the highly magnetized system is seen clearly in the upper curve of Fig. 1.

2.2. A steady-state solution

At this point we introduce a major simplification of the problem by assuming that the timescale over which α changes is considerably shorter than the evolutionary timescale of ω_o.

On one hand, this restricts the physical regime of interest to conditions that will be carefully explored in the next section. On the other hand it affords a very simple and straightforward solution, providing physical insight on the problem at hand.

With the above approximations, Eq. (7) becomes a first order differential equation for α with constant coefficients whose solution is given by:

$$\alpha(t) = (\alpha_0 - \alpha_\infty)\mathrm{e}^{-t/\tau_\alpha} + \alpha_\infty \tag{11}$$

where α_0 is the (arbitrary) initial value and τ_α and α_∞ are defined as follows:

$$\begin{aligned} \tau_\alpha &= \left(\frac{\dot{\omega}_o}{\omega_o} + \frac{k}{I_1}\omega_o^{11/3}\right)^{-1} \\ \alpha_\infty &= \frac{k\omega_o^{11/3}}{I_1}\tau_\alpha \quad \text{hence} \quad \lim_{t\to\infty} \alpha = \alpha_\infty < 1. \end{aligned} \tag{12}$$

This solution requires $\dot{\omega}_o$ to be driven by GW alone and both $(\dot{\omega}_o/\omega_o)$ and ω_o to be strictly constant: the latter requires $\tau_\alpha \ll \tau_o$, as both the above quantities evolve on a timescale $\sim \tau_o$.

The timescale τ_α and the parameter $(1 - \alpha_\infty)$ themselves are determined by the orbital period and its derivative, so they will be subject to secular evolution as well: in particular, they both decrease as the system shrinks. Hence, once the system has reached the asymptotic asynchronous state – starting from an arbitrarily asynchronous configuration – α will be locked to its "instantaneous" steady-state value during the subsequent evolution.

Expressions for τ_α and W can now be rewritten in terms of α_∞, I_1 and the measured quantities ω_o and $\dot{\omega}_o$.

$$\begin{aligned} \tau_\alpha &= \frac{\omega_o}{\dot{\omega}_o}(1 - \alpha_\infty) = \tau_o(1 - \alpha_\infty) \\ W &= \alpha_\infty I_1 \omega_o \dot{\omega}_o \frac{(1-\alpha)^2}{1 - \alpha_\infty}\cdot \end{aligned} \tag{13}$$

2.3. Validity of the exponential solution

The picture introduced in the previous section holds under well defined assumptions that will be addressed here. The condition that α changes rapidly with respect to the timescale over which $\omega_o^{11/3}$ evolves corresponds to the following statement: after a time τ_α, $\omega_o^{11/3}$ must have changed by a small amount, say less than 10%, in order for its approximation to a constant coefficient to be acceptable. Equation (13) then implies:

$$1 - \alpha_\infty < \frac{3}{110} \tag{14}$$

that can be translated to the following condition on the orbital period:

$$\begin{aligned} \omega_o > \omega_{\rm fast} &= \frac{(36 I_1)^{\frac{3}{11}} \; [(H/\Delta d)\jmath(e)]^{\frac{3}{11}} \; [G M_1 (1+q)]^{\frac{1}{2}}}{(2\sigma R_2^3 R_1^{3/2})^{\frac{3}{11}}} \\ &\quad \times \left(\frac{c}{\mu_1}\right)^{\frac{6}{11}} \left(\frac{\dot{\omega}_o}{\omega_o}\right)^{\frac{3}{11}}\cdot \end{aligned} \tag{15}$$

We insert in this equation the pure GW expression for $\dot{\omega}_o/\omega_o$ and, for the sake of simplicity, write it as $B\omega_o^{8/3}$, where the constant B incorporates all physical constants and system constant parameters. It readily obtains:

$$\omega_o > \omega_{\rm fast} = 36\frac{B I_1}{k} \tag{16}$$

which translates to the following limiting period:

$$\begin{aligned} P < P_{\rm fast} &= 1.2 \times 10^3 \left(\frac{M_1}{0.9\,M_\odot}\right)^{-\frac{7}{2}} \left(\frac{\mu_1}{2.5 \times 10^{30}}\right)^2 \left(\frac{R_1}{6 \times 10^8}\right)^{\frac{3}{2}} \\ &\quad \times \left(\frac{R_2}{1.7 \times 10^9}\right)^3 \frac{\left(\frac{I_1}{2.8 \times 10^{50}}\right)^{-1}}{q(1+q)^{\frac{3}{2}}\left(\frac{H}{\Delta d}\right)} \end{aligned} \tag{17}$$

where c.g.s. units in the normalizations and $\jmath(e)$ in the denominator have been omitted. From the above one obtains fundamental indications concerning the evolutionary scenario implied by the UIM: there exists a critical period (in the time-dependent case this will essentially correspond to a range of periods) shortwards of which the synchronization becomes fast compared to the system orbital evolution. Double degenerate binaries are thought to be born with orbital periods of a few hours: hence, if asynchronous at birth, their asynchronism will not change much faster than ω_o until they shrink to a sufficiently small orbital separation. All that happens *before* a system meets the fast synchronization requirement depends on its

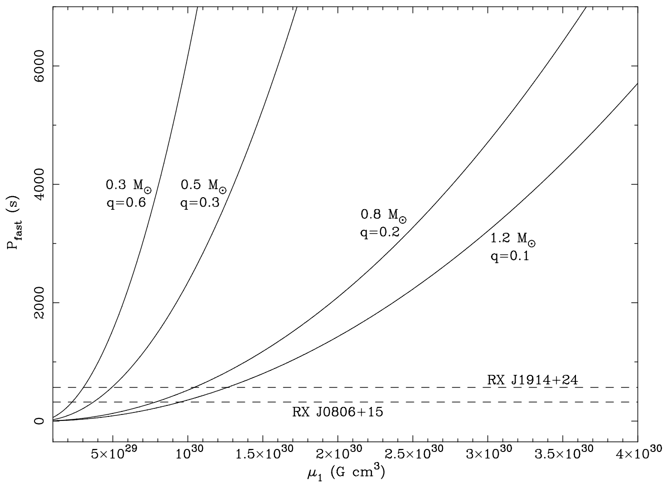

Fig. 2. The orbital period for which $\tau_\alpha \ll \tau_o$ (P_{fast}) as a function of the primary magnetic moment (μ_1) for four different values of M_1. The measured $\dot{\omega}_o$ and ω_o of both candidate ultrashort period binaries – RX J0806+15 and RX J1914+24 – constrain secondary masses in the range (0.1÷0.35) $M_\odot$ approximately, increasing for decreasing M_1 (see Paper II): we chose here q in order to give an approximately constant $M_2 \sim 0.2\ M_\odot$, for illustrative purposes. The rightmost curve has a significantly lower secondary mass, reflecting a similar result of Paper II, although it would have changed very little if we had allowed for a somewhat larger M_2. A general trend is clear: if $\mu_1 > 4 \times 10^{30}$ G cm^3, the condition for fast synchronization is reached for fairly long orbital periods, whatever the system masses. Further, the value of P_{fast} is sensitive to the primary mass, low-mass systems reaching fast synchronization at longer periods than high mass ones for this particular choice (and most plausible choices) of the mass ratios. Finally, the measured orbital periods of the candidate ultrashort period binaries are indicated by the dashed lines: if their measured orbital spin-up is due to GW alone, they should both be in the fast synchronization regime (or close to it), unless $\mu_1 < 10^{30}$ G cm^3. This is discussed in Paper II and is reported here just for the sake of illustration.

essentially unknown initial conditions (initial value of α) and on the particular evolutionary path it follows. However, once the orbital period is short enough, the UI mechanism becomes so efficient as to cause fast synchronization of the primary spin, on a timescale $\tau_\alpha \ll \tau_o$: in this regime, the value of α is brought to the corresponding α_∞ while ω_o remains essentially unchanged.

The state of a system *after* it has gone through fast synchronization is completely independent of initial conditions and its previous evolution; it becomes a function of the orbital period and the fundamental parameters M_1, q and μ_1 only.

In particular, higher mass systems reach the fast synchronization regime at shorter periods than lower mass ones (see Fig. 2), because in the latter the current dissipation rate is stronger, for a given orbital period and magnetic moment and, at the same time, GW are weaker. Hence, high mass sytems can reach shorter periods still maintaining a relatively high degree of asynchronism, whose exact value depends on the initial one.

Upon inverting relation (17) (or directly from the example of Fig. 2) it is also found that, if $\mu_1 > 10^{31}$ G cm^3, the fast synchronization regime is reached at periods longer than $10^3 \div 10^4$ s, whatever the component masses. Given the strong efficiency of UI in such highly magnetic systems, they will be characterized by very low steady state values of the asynchronism and very short timescales to reach it. Therefore, somewhat contrary to intuition, the luminosity of highly magnetized systems at short orbital periods will be quite low because of the tiny degree of asynchronism they can sustain.

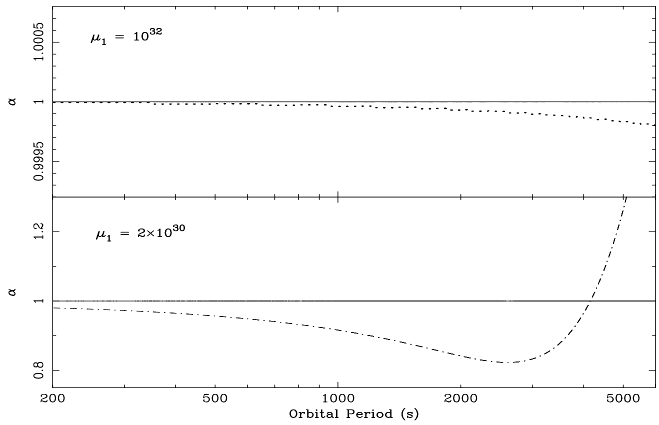

Fig. 3. Late evolution of the asynchronism parameter α, obtained from the numerical integration of Eq. (7), for a Double Degenerate Binary with $M_1 = 1\ M_\odot$, $q = 0.2$, and two different values of μ_1. These are indicated in the figure and expressed in G cm^3. The initial spin period is ~4 h and $\alpha_{in} = 10$ but, once $\alpha = \alpha_\infty$, its value is independent of the previous history of the system: it is determined only by system parameters, given the orbital period. The dependence of α_∞ on ω_o can be understood here only at a qualitative level; in Sect. 3 we obtain it in full generality.

Finally, recall that we have found in Sect. 2.2 that, at periods less than ~2000 s, UI is negligible with respect to GW in low-mass sytems only if their asynchronism is quite low. High-mass sytems, on the other hand, may fulfil that requirement even with a higher degree of asynchronism. However, from Eq. (17) it follows that low-mass systems reach steady-state at orbital periods significantly longer than 10^3 s, so they will certainly have a low value of $(1 - \alpha)$ shortwards of that. On the contrary, systems with sufficiently high-mass primaries (and low μ_1) can reach periods ~10^3 s or less without experiencing fast synchronization and are ultimately more likely not to fulfil the requirement UI $\ll$ GW at short orbital periods. In Fig. 3 we show the late evolution of the asynchronism parameter as obtained through a numerical integration of Eq. (7). The same system parameters of Fig. 1 have been used. The behaviour of α in the plot confirms that the system never becomes exactly synchronous as it shrinks; the value of α_∞ has a marked dependence on μ_1, in the same way expressed by Eq. (12).

3. Energy budget in the UIM: powering the circuit through GW emission

There is a general line of reasoning, based on energetic arguments, that leads to the definition of a system's steady-state without solving the evolutionary equation and without simplifying assumptions[4].

In the UIM a simple electric circuit is devised, with the secondary star acting as a generator and the primary as a resistance. The e.m.f. driving the current is due to the asynchonism between the primary spin and the orbital motion: hence, we can write the total energy available to the circuit simply as:

$$E_{\rm UIM} = \frac{1}{2}I_1(\omega_1^2 - \omega_{\rm o}^2) = E_1^{\rm sync}(\alpha^2 - 1) \tag{18}$$

where $E_1^{\rm sync} = (1/2)I_1\omega_{\rm o}^2$ is the primary spin energy for perfectly synchronous rotation.

With this definition $E_{\rm UIM} > 0$ if $\omega_1 > \omega_{\rm o}$ while it is negative in the opposite situation. However, what counts is the absolute value of $E_{\rm UIM}$, so that a more negative value corresponds to a larger energy reservoir.

From the above equation, the rate of change of the total available energy is readily obtained:

$$\dot{E}_{\rm UIM} = I_1\omega_{\rm o}(\dot{\omega}_1\alpha - \dot{\omega}_{\rm o}) \tag{19}$$

and we see that when $\alpha = 1$ the available electric energy goes to zero but has a nonzero, negative first derivative; indeed, in this situation $\dot{\omega}_1 = 0$ because the circuit is switched off, while GW still contribute to $\dot{\omega}_{\rm o}$. Because of this, α becomes smaller than 1 and the negative $\dot{E}_{\rm UIM}$ – that with $\alpha > 1$ implied a decreasing energy reservoir – now causes the circuit energy to increase again, in absolute value: GW start feeding energy to the circuit from this point on.

In the UIM, dissipation of the electric current works against the e.m.f. and acts to synchronize the primary spin with the orbital motion. This is done through the torque $N_1 = I_1\dot{\omega}_1$, that transfers angular momentum between the primary and the orbit. The torque is readily obtained remembering that the primary spin derivative is (see Eq. (7)):

$$\frac{\dot{\omega}_1}{\omega_1} = \frac{W}{\alpha(1-\alpha)I_1\omega_{\rm o}^2}\cdot \tag{20}$$

The rate of work done by the torque on the primary – the rate of spin energy change $\dot{E}^{(1)}_{\rm spin}$ – is just $N_1\omega_1$ so that, finally, we have:

$$\dot{E}^{(1)}_{\rm spin} = \frac{\alpha}{1-\alpha}W. \tag{21}$$

Since no net angular momentum loss arises from the coupling, the same torque with an opposite sign acts on the orbit as well, its rate of work being now $\dot{E}^{\rm (UIM)}_{\rm orb} = -N_1\omega_{\rm o} = -\dot{E}^{(1)}_{\rm spin}/\alpha = -W/(1-\alpha)$.

Equations (20) and (2) can now be subsituted into Eq. (19) and, remembering Eq. (21), the following is obtained:

$$\dot{E}_{\rm UIM} = \dot{E}^{(1)}_{\rm spin} - 3\frac{E_1^{\rm sync}}{E_g}\dot{E}_g - 3\dot{E}^{\rm (UIM)}_{\rm orb}\frac{E_1^{\rm sync}}{E_g}\cdot \tag{22}$$

[4] In the following we neglect the small term $h(a)$ in $g(\omega_{\rm o})$ (Sect. 2). This term accounts for the secondary spin being tidally locked to the orbit and can be easily added to all formulas. It is neglected only for the sake of clarity, as is it does not alter the physical content of the discussion, leaving all emphasis on physically more relevant features and somewhat simplifying the notation.

This last expression shows that the rate of change of $E_{\rm UIM}$ receives three different contibutions:

- the first term, $\dot{E}^{(1)}_{\rm spin}$, represents the rate of change of the primary spin because of the work done by the Lorentz torque. Obviously, this term always causes the absolute value of $E_{\rm UIM}$ to decrease, since $E_{\rm UIM}$ and $\dot{E}^{(1)}_{\rm spin}$ always have opposite signs;
- the second term is always *negative* and represents the effect of GW emission. Having a negative sign it causes the absolute value of $E_{\rm UIM}$ to decrease if $\omega_1 > \omega_{\rm o}$, but in the opposite case it causes $|E_{\rm UIM}|$ to increase and, thus, represents a mechanism for injecting energy in the circuit.
  This is the only term that would be present if there was no spin-orbit coupling at all. Its effect is better understood in this framework, indeed. Suppose ω_1 is a constant, greater than $\omega_{\rm o}$: since GW cause $\omega_{\rm o}$ to increase, the ratio α will decrease, thus consuming the available electric energy. From the point when $\alpha = 1$ on, the further increase of $\omega_{\rm o}$ makes α to become increasingly smaller than 1, thus powering the circuit battery once again: while the loss of orbital energy consumes the electric energy when $\alpha > 1$, it powers the electric circuit when $\alpha < 1$, thus showing that the sign of $E_{\rm UIM}$ in our definition (Eq. (18)) has a direct physical meaning.
- the last term describes spin-orbit coupling itself. It represents the rate of work done on the orbit by the torque N_1, times the ratio between the primary and orbital moments of inertia, where the latter is defined as $I_{\rm o} = M_1a^2q/(1+q)$ and $E_g = -(1/2)I_{\rm o}\omega_{\rm o}^2$.
  This term always acts to increase $E_{\rm UIM}$ *in absolute value*, as it can be checked (remember that $E_g < 0$ by definition).

3.1. Conserving the total electric energy

The above considerations lead naturally to the following question: given that only one term out of three in Eq. (22) always dissipates electric energy, can a condition be reached where dissipation is balanced by the terms feeding energy to the circuit? If this was the case, one would expect the flow of currents never to stop and the X-ray emission associated to their dissipation never to fade away.

To answer this point consider again $\dot{E}_{\rm UIM}$ in the form of Eq. (19): we see that the energy first derivative is zero if $\dot{\omega}_1\alpha = \dot{\omega}_{\rm o}$. Further, we can write the second derivative:

$$\ddot{E}_{\rm UIM} = I_1\dot{\omega}_{\rm o}(\dot{\omega}_1\alpha - \dot{\omega}_{\rm o}) + I_1\omega_{\rm o}(\ddot{\omega}_1\alpha + \dot{\omega}_1\dot{\alpha} - \ddot{\omega}_{\rm o}) \tag{23}$$

and see that this is zero as well, if $\dot{\omega}_1\alpha = \dot{\omega}_{\rm o}$. Hence, if this equality is verified, it implies that the available energy in the generator stays constant following the system evolution: the dissipation of electric current is exactly balanced by the spin-orbit coupling and GW emission. We call this the "energy steady-state".

We remember here that in the present work we are essentially concerned with the case when $\dot{\omega}_{\rm o} > 0$. In this regime, it must be $\dot{\omega}_1\alpha > 0$ in order for $\alpha\dot{\omega}_1 = \dot{\omega}_{\rm o}$ to be possible. When the primary spin has the same verse as the orbital motion ($\alpha > 0$) this condition exists only for $\alpha < 1$, because the primary spins down ($\dot{\omega}_1 < 0$) if $\alpha > 1$. When the primary spin is antialigned ($\alpha < 0$), Eq. (20) shows that $\dot{\omega}_1$ is always positive and again the condition $\dot{E}_{\rm UIM} = 0$ cannot be met.

Therefore, the energy steady-state *in the presence of orbital spin-up* exists only with the primary spin aligned with and somewhat slower than the orbital motion. This is not surprising, since we have seen that GW feed energy to the circuit only when $\alpha < 1$.

When $\alpha > 1$ and $\dot{\omega}_{\rm o} > 0$, the energy first derivative receives a negative contribution from both terms in Eq. (19) and $E_{\rm UIM}$ is an ever decreasing quantity. A steady-state solution with $\alpha > 1$ can in principle exist only if $\dot{\omega}_{\rm o} < 0$, i.e. when spin-orbit coupling dominates over GW emission. If sufficiently strong, this effect could overcome all others and allow $E_{\rm UIM}$ to stay constant or even increase while the orbit widens.

However, $\alpha_\infty^{\rm en} = |\dot{\omega}_{\rm o}|/|\dot{\omega}_1| > 1$ must be compared to the definition of the two derivatives (Eqs. (2) and (20)). Consider the largest possible value of $\dot{\omega}_{\rm o}$, the one obtained if GWs are completely negligible; even in this case, in order for $\dot{\omega}_{\rm o} > \dot{\omega}_1$ to hold we must have $\alpha < 3(I_1/I_{\rm o})$. The right-hand side of this disequality is very hardly larger than 1 (one needs quite small component masses – $M_1 \leq 0.5 M_\odot, q \leq 0.2$ – and an orbital period shorter than 400 s). We conclude that steady-state during spin-down is very unlikley to exist in real systems. Furthermore, this case is of no relevance for the two ultrashort period binaries RX J0806+15 and RX J1914+24 and we will neglect it from here on. For the sake of completeness, in Appendix B we show that orbital spin-down in the UIM is necessarily a transient phase, after which every system must eventually start shrinking.

3.2. The "energy steady-state" with orbital spin-up

We can now write $\dot{\omega}_1$ and $\dot{\omega}_{\rm o}$ explicitly in Eq. (19) taking their expressions given, respectively, in Eqs. (7) and (2). From this we find a simple expression for the value of α at which $E_{\rm UIM}$ stays constant, that we indicate as $\alpha_\infty^{\rm en}$ to distinguish it from the steady-state α_∞ of the approximate solution (Sect. 2.2).

By imposing $\dot{E}_{\rm UIM} = 0$ it obtains, after a little algebra[5]

$$\alpha_\infty^{\rm en}(1 - \alpha_\infty^{\rm en}) = \frac{I_1\ (\dot{\omega}_{\rm o}/\omega_{\rm o})}{k\omega_{\rm o}^{11/3}} = A \tag{24}$$

a second order equation with the following roots:

$$\alpha_\infty^{\rm en} = \frac{1 \pm \sqrt{1-4A}}{2}\cdot \tag{25}$$

From the above we see that, in order for a steady-state solution to exist, it must be $A \leq 1/4$. If A is greater than that, the discriminant of the above equation is negative, which ultimately implies $\dot{E}_{\rm UIM} < 0$. We address the meaning of the requirement $A < 1/4$ starting from the definition:

$$\frac{\dot{\omega}_{\rm o}}{\omega_{\rm o}} \leq \frac{1}{4}\frac{k\omega_{\rm o}^{11/3}}{I_1} \tag{26}$$

that, remembering the general expression (2) for $\dot{\omega}_{\rm o}/\omega_{\rm o}$ and writing the pure GW contribution to it as $B\omega_{\rm o}^{8/3}$, leads to the following:

$$B\omega_{\rm o}^{\frac{8}{3}} - \frac{3k\omega_{\rm o}^{\frac{17}{3}}}{2E_g}(1-\alpha) \leq \frac{1}{4}\frac{k\omega_{\rm o}^{\frac{11}{3}}}{I_1} \quad \text{or}$$
$$1 - \alpha \leq \frac{2E_g B}{3k\omega_{\rm o}^3} - \frac{1}{6}\frac{E_g}{I_1\omega_{\rm o}^2} = n(\omega_{\rm o}) \tag{27}$$

which assures that $A < 1/4$.

This condition can in principle be met for any value of α, greater or smaller than 1, depending on system parameters and initial conditions, because it concerns only the existence of a steady-state configuration but not the fact that the system has actually reached it. However, the condition must clearly hold when the system actually reaches steady-state, and thus has $\alpha = \alpha_\infty^{\rm en} < 1$: this implies $n(\omega_{\rm o}) > 0$, since $1 - \alpha_\infty^{\rm en}$ is positive, which translates to:

$$\omega_{\rm o} > 4\frac{BI_1}{k} = \frac{\omega_{\rm fast}}{9}\cdot \tag{28}$$

This corresponds to a limiting period $P_{\rm steady} = 9P_{\rm fast}$ (Eq. (17)) longwards of which the condition $A < 1/4$ for the existence of an "energy steady-state" is incompatible with $\alpha_\infty^{\rm en} < 1$. Stated differently, this result implies than no steady state exists at orbital periods longer than $P_{\rm steady}$ and $\dot{E}_{\rm UIM}$ can only be negative in these cases.

3.3. Asynchronous steady-state: the general expression and its approximation

We comment here on the relation between the "energy steady-state", obtained in full generality, and the one derived in the previous section with neglect of the spin-orbit coupling and temporal variations of $\omega_{\rm o}$.

First of all, the "energy steady-state" has two different roots while in Sect. 2.2 we only found one. This is due to the non-linearity of the problem: the complete evolutionary equation for α has a second order term, which leads to the two roots of Eq. (25). On the other hand, in Sect. 2.2 we explicitly reduced the equation to the linear form, thus excluding one of the two solutions. Concerning this point we can note that, once $A < 1/4$, one solution tends to be close to 1 (although always somewhat smaller) and the other tends to zero (although always somewhat higher). Therefore, it seems natural to neglect the latter if the system evolves from $\alpha > 1$ towards steady-state. If the system started from an almost zero initial spin, on the other hand, the solution with the smallest $\alpha_\infty^{\rm en}$ would be met first and should be taken into account.

It can be easily realized that the approximate expression for α_∞ (Eq. (12)) differs from the general one for $\alpha_\infty^{\rm en}$ (Eq. (25)) just by a small quantity. Indeed, from Eq. (12) we can write:

$$1 - \alpha_\infty = \frac{A}{1+A} \tag{29}$$

[5] We leave $(\dot{\omega}_{\rm o}/\omega_{\rm o})$ because this is a measured quantity, independent of assumed system parameters. Writing $\dot{\omega}_{\rm o}$ explicitly, α_∞ is obtained as a function of system parameters only. Put $B = (3/2E_g)I_1\omega_{\rm o}^2$ and $C = (3/2)(\dot{E}_g/E_g)I_1/(k\omega_{\rm o}^{11/3})$.
Then $\alpha_\infty = (1/2)\left[(1-B) \pm \sqrt{(1+B)^2 - 4C}\right]$. Requiring the expression under square root to be positive, a limit very similar to Eq. (28) is obtained, coincident with it if $I_1/I_{\rm o} \to 0$, an appropriate limit indeed.

from which the fast synchronization condition (Eq. (14)) becomes:

$$\frac{A}{1+A} \leq \frac{3}{110} \quad \text{or}$$
$$A \leq \frac{3}{107}. \tag{30}$$

In this case, since from Eq. (12) $\alpha_\infty = (1+A)^{-1}$ and A must be this small, we can re-write it, at first order in A, as $\alpha_\infty \simeq 1 - A$. Expanding Eq. (25) at first order in A as well, the same expression is obtained, thus proving that $\alpha_\infty^{\rm en}$ and α_∞ are coincident, at lowest order in A.

Further, Eq. (30) shows that an upper limit on A is not different from a constraint on the synchronization timescale, so that condition $A < 1/4$ can be given this more intuitive meaning, resulting simply in a less restrictive statement than the short synchronization condition (Eq. (14)). It happens to be less restrictive than that just because it describes a general result, based on no approximation. $A < 3/107$ corresponds indeed to a very specific case, where temporal variations of $\omega_{\rm o}$ are so slow that can be neglected in the evolution of α (although they ultimately determine the very existence of an asynchronous steady-state). The result of this section does not rely on that assumption: it adds to the previous analysis the conclusion that a well defined steady-state exists even when temporal variations of $\omega_{\rm o}$ are non-negligible. Nevertheless, the details of if and how the system evolves towards it cannot be addressed in the time-independent, linear approximation, in the general case. Hence, steady-state is already defined for systems meeting the requirement (28), but these are evolving towards it on a timescale that is not much shorter than $\tau_{\rm o}$: the fast synchronization regime ensues only at the even shorter orbital period $P_{\rm fast}$.

The discussion of this section shows that *the steady-state degree of asynchronism towards which the system evolves corresponds to a stationary state for the available electric energy*: the system adjusts its parameters as to dissipate only the energy that is fed by GW emission and spin-orbit coupling, maintaining a somewhat negative, constant electric energy reservoir.

4. Conclusions

In the present work we have discussed some important implications of the Unipolar Inductor Model applied to ultrashort period DDBs, that had been overlooked in previous works. In particular, we have focussed attention on systems with orbital spin-up. The main result of our study can be summarized as follows: *in the framework of the UIM, and in systems whose orbit is shrinking, the dissipation of the e.m.f. and associated currents can be balanced by spin-orbit coupling and mainly, when the primary spin is slower than the orbital motion, by the emission of gravitational waves.*

When the asynchronism parameter α is smaller than unity GW feed the the circuit battery – at the rate at which the orbit shrinks – by driving the primary spin out of synchronism and through the orbital spin up itself. This is a remarkable circuit, in which gravitational energy is "converted" to electric energy, powering a continuous flow of currents. An equilibrium with the energy dissipation process can thus be reached, such that the electric circuit is not expected to switch off.

In the early evolutionary stages – long orbital periods, say a few hours – it is likely that the primary spin be faster than the very slow orbital motion: the degree of asynchronism decreases at a rate comparable to that at which $\omega_{\rm o}$ itself evolves, because the Lorentz torque is initially much weaker than the GW torque.

When the orbital period becomes sufficiently short ($P < P_{\rm steady}$), on the other hand, a steady-state solution exists independent of the current value of α and such systems start evolving towards it. When they reach the even shorter period $P_{\rm fast}$ they enter the fast synchronization regime, where steady-state is achieved in a very short time compared to $\tau_{\rm o}$. The existence of this regime is due to the fact that the rate of dissipation of currents is a stronger function of $\omega_{\rm o}$ than GW are: hence, a point is reached where most of the residual electric energy is consumed over a short time – during which GW affect it only very slightly – at the expense of the primary spin. From this point on the circuit is forced to work in an *almost* synchronous state, in which the associated dissipation rate is balanced by the energy being fed by GW and spin-orbit coupling.

Therefore, one may say that the electric energy reservoir of systems born with $\alpha > 1$ (and sufficiently weak primary magnetic moment to allow the orbit to shrink) must initially decrease. They dissipate this energy at a rate determined by the orbital parameters, magnetic moment *and* degree of asynchronism. Their lifetime is thus a dependent variable, fixed by the ratio between the initial $E_{\rm UIM}$ and the dissipation rate $\dot{E}_{\rm UIM}$. When they reach $\alpha = 1$ the initial reservoir is completely consumed.

At this point a change in the nature of the circuit occurs: GW revive it by substracting further orbital energy (GW "inject" negative energy in the circuit) and $\omega_{\rm o}$ continues increasing, thus becoming larger than ω_1. As this happens, the primary spin starts tracking the orbital spin-up because of the spin-orbit coupling, although remaining somewhat lower than $\omega_{\rm o}$. Since $\alpha < 1$, the steady-state condition $\dot{\omega}_1 = \dot{\omega}_{\rm o}/\alpha$ can be met, *if* the orbital period has become sufficiently short (Eq. (28)). Once steady-state is reached, ω_1 continues increasing somewhat more rapidly than $\omega_{\rm o}$: α becomes a slowly incresing parameter, approaching 1 from below over a timescale much longer than $\tau_{\rm o}$ itself. However, it cannot reach exactly unity since the total energy in the circuit remains constant and cannot vanish.

The system lifetime is now virtually infinite (apart from the possible onset of mass transfer) while the degree of asynchronism becomes a dependent variable, being fixed by the condition that the dissipation rate always match the rate of work done by GW and spin-orbit coupling.

Several details of the UIM deserve further investigation in order to better assess its predictions; a number of observational constraints that were not considered here should still be taken into account. In particular, the simple UIM we have appealed to has been debated since its early proposal for the Jupiter-Io system. Assuming a perfectly field-aligned current system implies complete neglect of plasma inertia effects. In fact, it has been argued by several authors (Drell et al. 1965;

Neubauer 1980; Neubauer 1998; Russell & Huddleston 2000; Saur et al. 2004; Lopes & Williams 2005) that "field-aligned" currents could be associated to standing (in Io's frame) Alfvén disturbances; these distort significantly the dipole (unperturbed) field lines. The very presence of magnetic interaction between Jupiter and its satellite, and the presence of a large-scale current flow between them, is not questioned. Therefore, overall system energetics, the intensity of the current flow and exchange of angular momentum through the currents themselves remain essentially valid. On the other hand, the way the coupling works may be more complicated than assumed here. Plasma inertia effects could, even significantly in principle, affect the efficiency of the coupling and the geometry of the current flow, as well as its stability.

The main goal of the present work was that of demonstrating the viability of the model, by showing that the UI phase is not short-lived because the emission of GW can keep the circuit working at any time. To this aim, we have referred to the basic UIM since this appeared to give the best physical insight into this problem. Account for MHD effects, which may well be of relevance to a more detailed description of system properties, is delayed to future work.

Appendix A

Here we check that the primary tidal synchronization timescale is expected to be longer than the GW evolutionary timescale of the binary system, in the weak viscosity approximation first introduced by Darwin (1879). In this approximation it is assumed that the star shape has a quadrupole distortion (to leading order) induced by the tidal influence of the companion. In a frame corotating with the star under study, the asynchronism between spin and orbit reflects in an apparent rotation of the companion at the beat frequency $\omega_b = \omega_1 - \omega_o$. If stellar matter had no viscosity at all, its shape would instantaneously adjust to the present position of the companion and the tidal bulge would be aligned with the line of the centres. A small but finite viscosity introduces a (small) lag in the reaction of the tidal bulge to the changing position of the companion, so the tides will lag behind the line of the centres by a small time lapse τ which measures the viscosity itself. The small viscosity approximation translates in the requirement that τ be sufficiently small, so that the stellar shape at time t will be adjusted as to align the axis of its tidal bulge with the position the companion had at time $t - \tau$.

The expression for the evolution of a component spin due to the tidal interaction[6] with a companion in this framework was derived by Hut (1981) in Appendix A and is:

$$\frac{1}{T_t} = \left(\frac{\dot{\omega}_1}{\omega_1}\right)_t = 3\frac{k}{T}\frac{q^2}{r_g^2}\left(\frac{R}{a}\right)^6 \frac{1-\alpha}{\alpha} \tag{31}$$

where in that equation we have assumed a circular orbit (e = 0) with orbital and spin axes aligned (i = 0). Here k is the apsidal motion constant of the star (~0.12 for a white dwarf, Verbunt & Hut 1983), r_g is the star gyration radius, defined as $I = r_g MR^2$ and T is a characteristic timescale of the tide, related to τ simply by (Hut 1981):

$$T = \frac{R^3}{GM\tau}. \tag{32}$$

Note the strong dependence of T_t on the ratio (R/a) that measures how small is the star with respect to its Roche lobe, an intuitive result indeed. Significantly different values of T_t are thus expected for components with significantly different masses.

The lag time τ can be related to the mean viscosity of the star through (Alexander 1973):

$$\overline{\mu} = \frac{75}{224\pi}\frac{GM_1^2}{R^4}k\tau. \tag{33}$$

In order to neglect tidal interactions in the evolution of the primary spin we must require that T_t be considerably longer than the orbital evolutionary timescale (driven by GW). Hence:

$$3\frac{k}{T}\frac{q^2}{r_g^2}\left(\frac{R}{a}\right)^6 \frac{|1-\alpha|}{\alpha} < \frac{96}{50}\frac{(GM_1)^{5/3}}{c^5}\frac{q\omega_o^{8/3}}{(1+q)^{1/3}} \tag{34}$$

where the right-hand expression corresponds to one tenth of $\dot{\omega}_o/\omega_o$ expected from GW emission.

With the above expression we finally obtain (rescaling the orbital period to the shortest known, that of RX J0806+15):

$$\overline{\mu} < 10^{14}\frac{(1+q)^{5/3}}{q}\left(\frac{M}{M_\odot}\right)^{\frac{14}{3}} R_9^{-7}\left(\frac{P}{321}\right)^{4/3}\frac{\alpha}{|1-\alpha|} \tag{35}$$

where P is expressed in seconds and R_9 is the star radius in units of 10^9 cm. We need to introduce numbers in order to check the meaning of this condition: we consider the case of a relatively massive (say $M_1 > 0.8\ M_\odot$) primary star and a significantly less massive secondary (say, $M_2 < 0.25\ M_\odot$), in order to describe the more likley situation for the two candidate ultrashort period binaries (see Paper II). From Eq. (35) it is seen that the most favourable case for tidal synchronization to be efficient is that of short period systems, so we consider only them here.

With a primary mass $M_1 = 0.8\ M_\odot$ it obtains, at $P \sim 1000$ s and $\alpha = 2$ (an extremely asynchronous system at such a short period), $\overline{\mu} > 10^{17}$ g cm^{-1} s^{-1} in order for $T_t < 10\tau_o$, or $\overline{\mu} = 10^{18}$ for the two times to be comparable. In a more likely situation, such as the 321.5 orbital period source RX J0806+15, with $M_1 = 0.8\ M_\odot$ and a smaller asynchronism, $(1 - \alpha) \sim 10^{-2}$, an even larger value for the mean viscosity obtains, $\overline{\mu} > 10^{18}$ g cm^{-1} s^{-1}, in order for tidal effects to act on a timescale shorter than 10 τ_o. Note that even stronger constraints obtain when considering systems with a larger orbital separation than RX J0806+15.

Hence, tides are likely to always have long a synchronization timescale with respect to the orbital evolutionary timescale, unless $\overline{\mu} \geq (10^{18} \div 10^{19})$ g cm^{-1} s^{-1}, a hardly plausible value.

Indeed, according to Kopal (1968), plasma (or radiative) viscosity in non-degenerate stars is at most $\sim(10^3 \div 10^4)$ g cm^{-1} s^{-1}. In a degenerate star, however, it could well be of the same order of magnitude or somewhat stronger than turbulent viscosity in normal stars with convective

[6] we consider only the equilibrium tide, whose effects are expected in general to be much stronger than the dynamical tide (Zahn 1977).

envelopes, $\geq 10^{10} \div 10^{11}$ g cm^{-1} s^{-1} (see also Alexander 1973). Essentially the same conclusions were reached by Iben et al. (1998). They found that a mean viscosity much larger than 10^{13} g cm^{-1} s^{-1} would be required in order for tidal synchronization to be efficient over the GW timescale: they referred to the suggestion of Smarr & Blandford (1976) in order for a relevant viscosity to be obtained, namely that a magnetic field $>10^4$ G with a proper orientation may strongly enhance a WD viscosity. In particular, they speculate an extreme value $\sim 10^{18}$ g cm^{-1} s^{-1} can be reached, assuming an electrical conductivity $\sim 10^{19}$ e.s.u. and a sufficiently deep layer where dissipation of tidal energy takes place. We stress that this highly speculative suggestion is the only one made in the literature for such a high value white dwarf internal viscosity.

We focus now on the secondary component: consider the representative case of a secondary with $M_2 \sim 0.2\ M_{\odot}$: hence, a plausible value $\overline{\mu} \sim (10^{10} \div 10^{12})$ g cm^{-1} s^{-1} suffices to make tidal synchronization at least as fast as orbital evolution even for long orbital periods ($\sim 10^4$ s). Overall, then, it seems at least plausible that secondary stars are efficiently synchronized by tidal forces in these systems. However, given that some room for doubt may be left on this subject, we want to stress here that the condition of tidal synchronization of the secondary enters our model mainly through the function $h(a)$, of the order unity, defined in Sect. 2.

Hence, as long as the spin of the secondary does not signficantly alter the e.m.f. calculated in Sect. 2, the major consequence of a free secondary's spin in our model would be setting $h(a) = 1$, not a significant change indeed. The conclusion of this section can only be that tidal effects are at least extremely unlikely to be important in the evolution of the primary spin in the systems considered here. Althoguh large uncertainties still exist in the determination of white dwarf viscosities, all estimates point to it being not sufficiently strong to affect the primary spin over a timescale at least comparable to $\tau_{\rm o}$.

We note however that, even if tides acted on the same timescale as GW emission (they would be faster than GW only if $\overline{\mu} > 10^{18}$, even in the extremely favourable case of highly asynchronous systems at a period ≤ 1000 s) this would in any case make UIM the strongest spin-orbit coupling mechanism once the fast synchronization regime is reached. Hence, even in the regime where tides may possibly become as fast as the orbital evolution ($P \leq 10^3$ s), UIM would be much more efficient in affecting the primary's spin.

Finally, low-mass secondaries are much more easily synchronized by tidal interactions and it seems that they are fully consistent with having a tidal synchronization timescale shorter than $\tau_{\rm o}$ even at relatively long orbital periods.

Appendix B

We have shown in the text that the state with $\alpha^{\rm en}_{\infty} > 1$ cannot exist for orbital periods longer than a few hundred seconds or for $M_1 + M_2 > 0.6\ M_{\odot}$ or so. Further, we have briefly discussed systems undergoing orbital spin-down, assuming that they must eventually stop and reverse their orbital evolution. However, this latter conclusion has not been demonstrated and we address it here. Namely, we show that a maximum orbital separation (period) exists in this case, beyond which GWs take over spin-orbit coupling and the two stars can only spiral-in. We begin writing the condition that the orbit keeps widening, that is $\dot{\omega}_{\rm o}/\omega_{\rm o} < 0$ or $g^{-1}(\omega_{\rm o})[\dot{E}_g - W/(1-\alpha)] < 0$ at any orbital period. Put $g(\omega_{\rm o}) = -\overline{g}\omega_{\rm o}^{2/3}$, $\dot{E}_g = B_g \omega_{\rm o}^{10/3}$ and thus obtain:

$$\frac{B_g}{\overline{g}}\omega_{\rm o}^{\frac{8}{3}} + \frac{k}{\overline{g}}\omega_{\rm o}^{5}(1-\alpha) \leq 0 \tag{36}$$

from which:

$$\alpha \geq 1 + \frac{B_g}{k}\omega_{\rm o}^{\frac{7}{3}}. \tag{37}$$

By adding or subtracting 1 from Eq. (37) we eventually obtain:

$$(\alpha^2 - 1) \geq \frac{2B_g}{k\omega_{\rm o}^{7/3}}\left(1 + \frac{1}{2}\frac{B_g}{k\omega_{\rm o}^{7/3}}\right) \tag{38}$$

that would require an ever increasing $E_{\rm UIM}$, in order for spin-down to continue.

Concerning this point note that, *during orbital spin-down*, this condition can be met if spin-orbit coupling transfers more energy to the orbit than that lost by dissipation of currents and by the decrease of $\omega_{\rm o}$. Indeed, when $\alpha > 1$ ($E_{\rm UIM} > 0$) spin-orbit coupling is the only term with a positive contribution to $\dot{E}_{\rm UIM}$ (Eq. (22)). Hence, in order for $E_{\rm UIM}$ to increase, remembering Eq. (22) and the relation $(\dot{E}_s^{(1)}/\alpha) = -\dot{E}_{\rm orb}^{\rm (UIM)}$ (cf. Sect. 3), we obtain:

$$\alpha \leq 3\frac{E_1^{\rm sync}}{|E_g|}\left(1 - 3\frac{|\dot{E}_g|}{\dot{E}_{\rm orb}^{\rm (UIM)}}\right) \tag{39}$$

where all terms have been written as to be positive. In our hypothesis spin-orbit coupling ($\dot{E}_{\rm orb}^{\rm (UIM)}$) is stronger than GWs ($\dot{E}_g$): in particular, in the extreme case that GWs were completely negligible, one would obtain from Eq. (39) the largest upper limit on α. However, even the largest upper limit is too constraing: indeed, neglecting the second term on the righ-hand side of Eq. (39):

$$\alpha \leq 3\frac{I_1}{I_{\rm o}} \tag{40}$$

the orbital momentum of inertia $I_{\rm o}$ being defined in Sect. 3.

As already stated in the text, the right-hand side of this equation is very hardly greater than 1, especially for periods longer than a few hundred seconds, while $\alpha > 1$ is a necessary condition here. So, during orbital spin-down, $E_{\rm UIM}$ is not expected to increase as the system evolves. Thus write:

$$\alpha^2 - 1 \propto \omega_{\rm o}^{-(2-\delta)} \tag{41}$$

where δ is positive and can be smaller or larger than 2.

The evolution of the asynchronism is thus not fast enough for $\dot{\omega}_{\rm o}$ to remain negative. In fact, conditions (41) and (38) are not compatible in general: their right-hand sides define two functions, $y(\omega_{\rm o})$ and $w(\omega_{\rm o})$, with w more strongly dependent on $\omega_{\rm o}$ than y. It drops to zero more quickly for large $\omega_{\rm o}$ and diverges more quickly as $\omega_{\rm o}$ becomes small.

Then, suppose to have a system at a sufficiently short orbital period and high $\alpha > 1$ that spin-orbit coupling drives the two component stars apart. As the orbit widens, $(\alpha^2 - 1) \propto \omega_{\rm o}^{-(2-\delta)}$, while the condition for orbital spin-down to continue requires a stronger, *negative*, dependance. A point is reached, at which the degree of asynchronism is not sufficient anymore to sustain spin-orbit coupling against GWs and the orbit must start shrinking. Therefore, binary systems can only spin down via this mechanism for a finite time, after which they must shrink and eventually reach a state with $\alpha_{\infty}^{en} < 1$.

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