A&A 443, 777–780 (2005)
DOI: 10.1051/0004-6361:20054119

Astronomy & Astrophysics

A toy model for coupling accretion disk oscillations to the neutron star spin *(Research Note)*

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Received 1 July 2005 / Accepted 2 September 2005

ABSTRACT

Lee et al. (2004, ApJ, 603, L93) demonstrated numerically that rotation of neutron star couples with oscillations of its accretion disk to excite resonances. No specific coupling was assumed, but a magnetic field was suggested as the most likely one. Following this idea, we showed (Pétri 2005a, A&A, 439, L27, Paper I) that, if the neutron star is non-axially symmetric and rotating, its gravity might provide the coupling and excite resonances. Here, we return to the original suggestion that the coupling has a magnetic origin, and demonstrate how it works in terms of a simple analytic toy model.

Key words. accretion, accretion disks – magnetohydrodynamics (MHD) – relativity – stars: neutron – X-rays: binaries

1. Introduction

In some neutron star sources, the observed QPO frequencies obviously depend on the neutron star spin. For example, the difference in frequencies of the double-peaked QPO in the millisecond pulsar SAX J1808.4-3658 is clearly equal to half of the pulsar spin (Wijnands et al. 2003). This led Kluźniak et al. (2004) to suggest that a resonance is excited by coupling accretion disk oscillation modes to the neutron star spin. The suggestion was fully confirmed by numerical simulations of the coupling (Lee et al. 2004). It was found that a resonant response occurs when the difference between the frequencies of the two modes equals one-half of the spin frequency as observed in SAX J1808.4-3658 and other "fast rotators", and when it equals the spin frequency, as observed in "slow rotators" like XTE J1807-294.

Lee et al. (2004) suggested that the coupling is most likely a magnetic one, but did not specify any concrete mechanism in their numerical simulations and introduced the coupling by a purely formal ansatz. Following their idea, we went on to discuss in terms of a simple analytic toy-model (Pétri 2005a, Paper I) that also a non-axially symmetric gravitational field of a rotating neutron star excite resonances in accretion disk oscillations.

Another point of view to account for the dichotomy between slow and fast rotators is given by Lamb & Miller (2003), who explain it in the framework of the sonic-point beat frequency model.

In this research note, we discuss yet another toy model that provides the coupling by the neutron star magnetic field. We use the same notation as in Paper I.

2. The model

In this section, we describe the main features of the model, starting with a simple treatment of the accretion disk, assumed to be made of non-interacting charged single particles orbiting in the equatorial plane of the star. Magnetohydrodynamical aspects of the disk, such as pressure and current, are therefore neglected. Particles evolve in a perfectly spherically symmetric gravitational potential. The asymmetry arises from an eccentric, misaligned dipolar magnetic field corotating with the neutron star.

2.1. Eccentric stellar magnetic field

The periodically varying epicyclic frequencies are introduced by adding a rotating asymmetric dipolar magnetic field to the background gravity. Generally, when dealing with an oblique rotator, the location of the magnetic moment $\boldsymbol{\mu}$ generating the dipolar magnetic field coincides with the centre of the neutron star (supposed to be a perfect sphere). In this paper, we lift this assumption and shift the location of the magnetic moment to a point $\boldsymbol{r}_{\rm s}(t) = (r_{\rm s} \neq 0, \varphi_{\rm s} = \Omega_* \, t, z_{\rm s})$ inside the star, such that $||\boldsymbol{r}_{\rm s}(t)|| = \sqrt{r_{\rm s}^2 + z_{\rm s}^2} \leq R_*$ where R_* is the stellar radius. We use cylindrical coordinates denoted by (r, φ, z). Nevertheless, the origin of the coordinate system coincides with the centre of the neutron star. Furthermore, in order to compute such a magnetic field structure analytically, we assume that the star is made of a homogeneous and isotropic matter everywhere with total mass M_* and that is spinning around its centre at an angular rate $\boldsymbol{\Omega}_* = \Omega_* \boldsymbol{e}_z$, aligned with the z-axis. The magnetic field induced by the dipolar source is, therefore,

$$\boldsymbol{B}(r, \varphi, z, t) = \frac{\mu_0}{4\pi} \left[\frac{3 (\boldsymbol{\mu} \cdot \boldsymbol{R}) \boldsymbol{R}}{R^5} - \frac{\boldsymbol{\mu}}{R^3} \right]. \quad (1)$$

The vector joining the source point $\boldsymbol{r}_{\rm s}(t) = (r_{\rm s}, \varphi_{\rm s}, z_{\rm s})$ to the observer point $\boldsymbol{r} = (r, \varphi, z)$ is

$$\boldsymbol{R}(t) = \boldsymbol{r} - \boldsymbol{r}_{\rm s}(t). \quad (2)$$

Using the cylindrical frame of reference, the distance between source point and observer is

$$R^2 = r^2 + r_{\rm s}^2 - 2\, r\, r_{\rm s} \cos\psi + (z - z_{\rm s})^2 \quad (3)$$

where the azimuth in the corotating frame is $\psi = \varphi - \Omega_*\, t$. The magnetic moment anchored in the neutron star rotates at the stellar speed such that

$$\boldsymbol{\mu}(t) = \mu\, [\sin\chi\, \{\cos(\Omega_*\, t)\, \boldsymbol{e}_x + \sin(\Omega_*\, t)\, \boldsymbol{e}_y\} + \cos\chi\, \boldsymbol{e}_z] \quad (4)$$

where the obliquity, i.e. the angle between $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}_*$, is denoted by χ. Moreover, each component of the magnetic field can be expressed as follows:

$$B_r = \frac{\mu_0}{4\pi R^3} \left[\frac{3 (\boldsymbol{\mu} \cdot \boldsymbol{R}) (r - r_{\rm s} \cos\psi)}{R^2} - \mu \sin\chi \cos\psi \right] \quad (5)$$

$$B_\varphi = \frac{\mu_0}{4\pi R^3} \left[\frac{3 (\boldsymbol{\mu} \cdot \boldsymbol{R})\, r_{\rm s} \sin\psi}{R^2} + \mu \sin\chi \sin\psi \right] \quad (6)$$

$$B_z = \frac{\mu_0}{4\pi R^3} \left[\frac{3 (\boldsymbol{\mu} \cdot \boldsymbol{R}) (z - z_{\rm s})}{R^2} - \mu \cos\chi \right] \quad (7)$$

$$\boldsymbol{\mu} \cdot \boldsymbol{R} = \mu \left[\sin\chi\, (r \cos\psi - r_{\rm s}) + \cos\chi\, (z - z_{\rm s}) \right]. \quad (8)$$

The total linear response of the disk is then the sum of each perturbation corresponding to one particular azimuthal mode m. Because the perturber is inside the star and the disk never reaches the stellar surface, the Fourier coefficients of each component B_i^m ($i = r, \varphi, z$) never diverge. It is convenient to introduce the Fourier decomposition of the magnetic field component, Eqs. (5)–(7), in order to describe the response of the test particle. Moreover, because evaluation of the Fourier coefficients requires integrating terms containing $\cos(m\psi)$ in the integrand, the value of these coefficients decreases rapidly with increasing azimuthal number m. As a result, only low azimuthal modes will significantly influence the evolution of the disk. Keeping only the few first terms in the expansion is sufficient for achieving reasonable accuracy. For discussing the results, we only keep the three first modes, namely, the dipolar, quadrupolar, and octupolar moments ($m = 1, 2, 3$, respectively).

2.2. Equation of motion for a charged test particle

All particles evolve in the gravitatomagnetic field imposed by the rotating neutron star. To keep things as simple as possible, their motion is described in the guiding centre approximation. The drift arising from the gyration around the local magnetic field is not an essential feature we want to discuss here. As a consequence, magnetic curvature and gradient, as well as gravity drift motions, are ignored in this study. Nevertheless, the main characteristic that consists of a periodic variation in the epicyclic frequencies is preserved. The equation of motion then reads

$$\ddot{\boldsymbol{G}} = \boldsymbol{g} + \frac{q_{\rm e}}{m_{\rm e}} \dot{\boldsymbol{G}} \wedge \boldsymbol{B} \quad (9)$$

where $\boldsymbol{G}$ is the location of the guiding centre, and the dot means time derivative d/dt. The gravitational field of the star M_* is denoted by $\boldsymbol{g} = \boldsymbol{\nabla}(G\, M_* / \sqrt{r^2 + z^2})$. The mass and the charge of the particle are denoted, respectively, by $m_{\rm e}$ and $q_{\rm e}$.

If, for instance, the magnetic gradient drift is taken into account, the term $-(\mu_{\rm e}/2\, m_{\rm e}\, B)\, \boldsymbol{\nabla} B^2$ should be added to the right hand side of Eq. (9) where $\mu_{\rm e}$ is an adiabatic invariant, namely the magnetic moment of the test particle gyrating along the local field line. This would introduce another modulation of the epicyclic frequencies, which is already included in the Lorentz force $q_{\rm e}\, \dot{\boldsymbol{G}} \wedge \boldsymbol{B}$. Thus, the physical behaviour is not changed by neglecting the drift motion of the guiding centre $\boldsymbol{G}$. Expressed in cylindrical coordinates, Eq. (9) develops into

$$\ddot{r} - r\, \dot{\varphi}^2 = g_r + \frac{q_{\rm e}}{m_{\rm e}} (r\, \dot{\varphi}\, B_z - \dot{z}\, B_\varphi) \quad (10)$$

$$2\, \dot{r}\, \dot{\varphi} + r\, \ddot{\varphi} = \frac{q_{\rm e}}{m_{\rm e}} (\dot{z}\, B_r - \dot{r}\, B_z) \quad (11)$$

$$\ddot{z} = g_z + \frac{q_{\rm e}}{m_{\rm e}} (\dot{r}\, B_\varphi - r\, \dot{\varphi}\, B_r). \quad (12)$$

The magnetic field $\boldsymbol{B}$ is assumed to be weak enough for the flow to remain essentially hydrodynamical (weakly magnetized thin disk approximation). We therefore treat $\boldsymbol{B}$ as a perturbation of order $\varepsilon \ll 1$. The perturbations induced in the flow are of the same order of magnitude as $\boldsymbol{B}$, i.e. of order ε. The perturbed orbit and velocity of the test particle in the radial and vertical direction are also of order ε, $\{\dot{r}, \dot{z}\} = {\rm O}(\varepsilon)$, whereas the azimuth varies as $\dot{\varphi} \approx \Omega_{\rm k} \approx \sqrt{G\, M_*/r_0^3} + {\rm O}(\varepsilon)$, the Keplerian orbital frequency at the radius of the orbit r_0. In the equations of motion (10), (11), and (12), terms such as $\{\dot{r}, \dot{z}\} \times B_i$ with $i = \{{\rm r}, \varphi, {\rm z}\}$ are second order ${\rm O}(\varepsilon^2)$, so we neglect them. According to this simplification, the right hand side of Eq. (11) vanishes. Equation (11) states the conservation of angular momentum of the particle and integrates into $L = m\, r^2\, \dot{\varphi} = $ const, where L is the angular momentum of the particle. This is an obvious integral of motion for this problem in the aforementioned approximation.

We are only interested, however in the vertical motion experienced by the test particles in response to the perturbed magnetic field. Indeed, the response of an accretion disc to an inclined rotating magnetic dipole has been studied by Terquem & Papaloizou (2000). They showed that the vertical displacement (i.e. the warping) is the dominant effect in the thin disc, while the horizontal perturbations can be neglected. Perturbing

Eq. (12) and developing to a first order in the perturbation around the equilibrium Keplerian orbit (contained in the equatorial plane) defined by $(r_0, \varphi_0 = \Omega_k t, z_0 = 0)$, the vertical motion reads

$$\ddot{z} = g_z - \frac{q_e}{m_e} r_0 \, \Omega_k \, B_r. \tag{13}$$

To avoid mathematically irrelevant complications, we assume the rotator to be "aligned" with the star in the sense that $\boldsymbol{\mu}$ and $\boldsymbol{\Omega}_*$ are parallel ($\chi = 0$). Therefore the radial component of the magnetic field reads

$$B_r = \frac{3 \, \mu_0 \, \mu}{4 \, \pi} \frac{(z - z_s) \, (r - r_s \, \cos \psi)}{R^5}. \tag{14}$$

By developing in a Fourier series, we obtain

$$B_r = (z - z_s) \sum_{m=0}^{+\infty} B_r^m(r, z) \, \cos(m \, \psi) \tag{15}$$

where the Fourier coefficients are given by

$$B_r^m(r, z) = \frac{3 \, \mu_0 \, \mu}{4 \, \pi} \frac{2 - \delta_m^0}{2 \, \pi} \int_0^{2\pi} \frac{r - r_s \, \cos \psi}{R^5} \times \cos(m \, \psi) \, \mathrm{d}\psi \tag{16}$$

where δ_m^0 is the Kronecker symbol. Note that B_r^m do not have the dimension of a magnetic field because of their definition Eq. (15). These coefficients, which are a function of the space position (r, z), decrease with increasing azimuthal number m (for (r, z) fixed). Keeping the first few coefficients is sufficient for reasonable accuracy (we retain the first three). Putting the expansion Eq. (15) into the vertical equation of the motion Eq. (13), we get the fundamental equation to describe vertical forced oscillations of a test particle as follows:

$$\left[\Omega_k^2 + \frac{q_e}{m_e} r_0 \, \Omega_k \sum_{m=0}^{+\infty} B_r^m(r_0, z_0) \, \cos\{m(\Omega_k - \Omega_*) \, t\} \right] z + \ddot{z} = \frac{q_e}{m_e} r_0 \, \Omega_k \, z_s \sum_{m=0}^{+\infty} B_r^m(r_0, z_0) \, \cos\{m \, (\Omega_k - \Omega_*) \, t\}. \tag{17}$$

Note that the Fourier coefficients $B_r^m(r_0, z_0)$ in this last equation are evaluated at the location of the unperturbed orbit and no longer depend on the perturbed position (r, z). To the lowest order of the expansion, this approximation is justified. We recognize a Hill equation (periodic variation of the eigenfrequency of the system, on the left hand side) with a periodic driving force (on the right hand side).

2.3. Resonance conditions

This equation is very similar to the one obtained in the case where solely gravity perturbation exists. The discussion is, therefore, exactly the same as in Paper I. Here we recall the main results, but adapted to the magnetic configuration. Equation (17) describes a harmonic oscillator with periodically varying eigenfrequency that is also excited by a driven force. It is well known that some resonances will therefore occur in this system, so we expect three kind of resonances corresponding to:

- a *corotation resonance* at the radius where the angular velocity of the test particle equals the rotation speed of the magnetic structure, which is equal to the stellar rotation rate. Corotation is only possible for a prograde motion. The resonance condition determining the corotating radius is simply $\Omega_k = \Omega_*$;
- a *driven resonance* at the radius where the vertical epicyclic frequency equals the frequency of each mode of the magnetic perturbation, as seen in the locally corotating frame. The resonance condition is $m \, |\Omega_* - \Omega_k| = \kappa_z$.
- a *parametric resonance* related to the time-varying vertical epicyclic frequency (Hill equation). The rotation of the magnetosphere induces a sinusoidal variation of the vertical epicyclic frequency leading to the well-known Mathieu equation for a given azimuthal mode m. The resonance condition is derived as

$$m \, |\Omega_* - \Omega_k| = 2 \, \frac{\kappa_z}{n} \tag{18}$$

where $n \geq 1$ is a natural integer.

Note that the driven resonance is a special case of the the parametric resonance for $n = 2$. However, their growth rate differs by the timescale of the amplitude magnification. Driving causes a linear growth in time, while parametric resonance causes an exponential growth. We also rewrite the vertical epicyclic frequency as κ_z instead of Ω_k, in order to apply the results to a more general case that could include stronger magnetic fields or general relativistic effects.

Consequently, the resonance conditions for the magnetized rotator are exactly the same as for the unmagnetized rotator of Paper I, as long as the oscillations remain in the linear regime i.e. in the thin disk approximation for which the vertical motion of the particle remains small with respect to the radius of the unperturbed orbit. For more details and a discussion of these results, we refer the reader to Paper I.

The dichotomy between the QPOs in fast and slow rotators has been explained by Lee et al. (2004) and is a consequence of the fact that the coupling between the neutron star spin and the modes of accretion disk oscillations excites possible resonances at different locations in the disk, either very close to the star or far away from it. While we agree with this result, we point out that (as found in Paper I) the spectrum of modes that *could be in a resonance* is more complex than the one considered by Lee et al. Their discussion is concentrated on the 3:2 resonance that occurs between the radial and vertical epicyclic modes *only in strong gravity*. We have identified possible forced resonances between different modes that may occur in both strong and weak gravity. Our analysis was done in a linear regime, so, at the moment, we are only able to say that such resonances theoretically exist[1].

[1] I was informed by M. Abramowicz, the referee, that the unpublished numerical results of Lee et al. fully confirms existence of these additional weak gravity resonances.

3. Conclusion

The toy-model discussed in this Research Note illustrates the idea (Kluźniak et al. 2004; Lee et al. 2004; Pétri 2005a) that the double peak QPOs in neutron stars' sources may be due to the coupling of the rotation of these sources to modes in accretion disk oscillations and, therefore, excites resonances.

The model is physically very specific. It explicitly shows how the rotating magnetic field of the neutron stars could couple with the disk dynamics and oscillations. It confirms the general discussion and numerical results of Kluźniak et al. (2004) and Lee et al. (2004), in particular the important one that the strongest resonant response occurs when the difference between frequencies of the two modes equals one-half of the spin frequency as observed in SAX J1808.4-3658 and other "fast rotators", and when it equals the spin frequency, as observed in "slow rotators" like XTE J1807-294.

When the MHD nature of the flow is taken into account, QPOs can be explained by a similar mechanism to those exposed here (Pétri 2005b). However, in an accreting system in which the neutron star is an oblique rotator, we expect a perturbation in the magnetic field to the same order of magnitude as the unperturbed one. Therefore, the linear analysis developed in this paper has to be extended to oscillations having non-negligible amplitude compared to the stationary state. Nonlinear oscillations therefore arise naturally in the magnetized accretion disk. Abramowicz et al. (2003) showed that the non-linear resonance for the geodesic motion of a test particle can lead to this 3:2 ratio for the two main resonances. Nevertheless, an extension to the 3D MHD flow in curved spacetime is required to make quantitatively accurate predictions of the peak frequencies variation correlated with the accretion rate.

Acknowledgements. I am grateful to the referee Marek A. Abramowicz for his valuable comments. This work was supported by a grant from the German-Israeli Foundation for Scientific Research and Development.

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