Astronomy
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Astrophysics

Orbit correction without variational equations

The orbits of Caliban, 46P/Wirtanen and 67P/Churyumov-Gerasimenko[⋆]

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Abstract. Differential orbit correction is performed by computing an osculating orbit from observations from which the perturbations have been subtracted. No variational equations need to be integrated except to solve for non-gravitational forces. The perturbations in the observations are obtained by subtracting a two-body orbit from an integrated, perturbed orbit. The method is used to update the orbits of the Uranian satellite Caliban and of the comets 46P/Wirtanen and 67P/Churyumov-Gerasimenko with recent observations. The method is shown to agree very well with, but is considerably faster than, the conventional method based entirely on variational equations. Algorithms are given for both methods.

Key words. celestial mechanics – comets: general – planets and satellites: general

1. Introduction

This paper deals with orbit correction of solar system objects (comets, asteroids, satellites). Usually orbit correction is done by matching a perturbed orbit directly to the available observations, starting from an osculating orbit or state vector whose six elements are adjusted by least squares. This requires numerical integration of one orbit and six variational equations to derive the partial derivatives needed in the least squares solution. This method may be referred to as the *coordinate perturbation* method.

We present here a simpler and faster method which we shall call the *observation perturbation* method. The perturbations are first removed from the observations to obtain a set of unperturbed fictitious observations from which the osculating orbit is derived as a pure two-body orbit using analytic partial derivatives. The perturbations in the observations are obtained by subtraction between a numerically integrated, perturbed orbit and an analytic, unperturbed orbit in an iterative procedure. No variational equations need to be integrated except to solve for non-gravitational forces. The method was introduced by the first author who used it to compute the orbits of Jupiters's satellite Leda (Aksnes 1978) and Uranus' satellites Caliban and Sycorax (Marsden et al. 2000). The second author (Grav 2001) then applied the method to the comet 46P/Wirtanen and extended it by the inclusion of non-gravitational forces.

In the following we will give an exposition of the observation perturbation method and demonstrate its use through some examples. We will also compare the results with the corresponding ones obtained with the conventional coordinate perturbation method for which an algorithm is also given.

2. Orbit model

The equations of motion for a body of mass m attracted by a central mass m_0 and N perturbing masses m_1, m_2, ...m_N take the form

$$\ddot{\boldsymbol{r}} = -(\mu_0 + \mu)\frac{\boldsymbol{r}}{r^3} + \sum_{i=1}^{N} \mu_i \left(\frac{\boldsymbol{r}_i - \boldsymbol{r}}{\Delta_i^3} - \frac{\boldsymbol{r}_i}{r_i^3} \right) \quad (1)$$

where μ, μ_0 and μ_i are the gravitational constant times respectively the masses m, m_0 and m_i, $\boldsymbol{r}$ being the radius vector of the object whose distance from body i is Δ_i. Let $\boldsymbol{s}_0 = \{x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0\}$ be the initial state vector (column vector) which we seek to adjust by fitting an orbit to observations by least squares, the O-C residuals being represented as a variation $\Delta\boldsymbol{r} = \{\Delta x, \Delta y, \Delta z\}$,

$$\Delta\boldsymbol{r} = G\Delta\boldsymbol{s}_0 \quad (2)$$

where G is the 3 by 6 matrix

$$G = \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \cdots & \frac{\partial x}{\partial \dot{z}_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \cdots & \frac{\partial y}{\partial \dot{z}_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \cdots & \frac{\partial z}{\partial \dot{z}_0} \end{bmatrix}. \quad (3)$$

The elements of this matrix; i.e. the partial derivatives needed in the least squares adjustment, can be obtained by solving the differential equations (variational equations)

$$\ddot{G} = \frac{\partial \ddot{r}}{\partial r} G \tag{4}$$

where

$$\frac{\partial \ddot{r}}{\partial r} = (\mu_0 + \mu) D(r) + \sum_{i=1}^{N} \mu_i D(r_i - r), \tag{5}$$

D being the operator (Aksnes 1973)

$$D(r) = -\nabla(\frac{r}{r^3}) = \frac{3}{r^5} \begin{bmatrix} x^2 - r^2/3 & xy & xz \\ yx & y^2 - r^2/3 & yz \\ zx & zy & z^2 - r^2/3 \end{bmatrix}. \tag{6}$$

Initially all the elements of G and $\dot{G}$ are zero except for $G_{11} = G_{22} = G_{33} = \dot{G}_{14} = \dot{G}_{25} = \dot{G}_{36} = 1$. If the observables are right ascension α and declination δ, Eq. (2) is used to obtain

$$\begin{bmatrix} \Delta\alpha \cos\delta \\ \Delta\delta \end{bmatrix} = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\sin\delta\cos\alpha & -\sin\delta\sin\alpha & \cos\delta \end{bmatrix} \frac{\Delta r}{r}. \tag{7}$$

Let α, δ be the perturbed and α_0, δ_0 the unperturbed, computed right ascension and declination corresponding to the observed values $\alpha_{\text{obs}}, \delta_{\text{obs}}$. By subtracting the perturbations from the observations, we get fictitious observations

$$\alpha_{0,\text{obs}} = \alpha_{\text{obs}} - (\alpha - \alpha_0) = \alpha_0 + (\alpha_{\text{obs}} - \alpha) = \alpha_0 + \Delta\alpha$$
$$\delta_{0,\text{obs}} = \delta_{\text{obs}} - (\delta - \delta_0) = \delta_0 + (\delta_{\text{obs}} - \delta) = \delta_0 + \Delta\delta. \tag{8}$$

Through the method of least squares the osculating orbit at the initial epoch is adjusted to minimize the sum of squares of the residuals $\Delta\alpha \cos\delta_0$ and $\Delta\delta$. This leads to the following equations of condition for the observation perturbation method,

$$\sum_{i=1}^{6} \cos\delta_0 \frac{\partial\alpha_0}{\partial c_i} \Delta c_i = \Delta\alpha \cos\delta_0$$
$$\sum_{i=1}^{6} \frac{\partial\delta_0}{\partial c_i} \Delta c_i = \Delta\delta \tag{9}$$

where the Δc's represent corrections to the initial state vector s_0 or equivalently to orbital elements $a, e, M_0, I, \omega, \Omega$. The partial derivatives $\partial\alpha_0/\partial c_i$ and $\partial\delta_0/\partial c_i$ w.r.t. the orbital elements are derived by means of Eq. (7) and the two-body partials $\partial r/\partial c_i$ supplied in the Appendix. The integration and least squares solution are repeated with an improved initial orbit until convergence.

The above equations of condition also apply for the coordinate perturbation method if the unperturbed α_0, δ_0 are replaced by the perturbed α, δ. This then requires the partials in the G matrix obtained through numerical integration.

2.1. Non-gravitational perturbations

When dealing with comets, one frequently has to include forces due to outgassing from the nucleus. We here adopt a model due

to Marsden et al. (1973). To Eq. (1) is added an acceleration decomposed in a radial, transverse and normal direction:

$$\Delta\ddot{r} = g(r) \left(A_1 \frac{r}{r} + A_2 \frac{r\dot{r} - \dot{r}r}{h} + A_3 \frac{h}{h} \right) \tag{10}$$

where $h = r \times \dot{r}$ and

$$g(r) = \alpha \left(\frac{r}{r_0} \right)^{-m} \left(1 + \left(\frac{r}{r_0} \right)^n \right)^{-k}. \tag{11}$$

Beyond the distance r_0 the non-gravitational forces drop off very rapidly. Through curves describing the rate of vaporization of water ice from a rapidly rotating nucleus with a Bond albodo 0.1, the values $r_0 = 2.808$ AU, $m = 2.15$, $n = 5.093$ and $k = 4.6142$ were deduced. The constant $\alpha = 0.111262$ normalizes $g(r)$ to unity at 1 AU, the unit for the acceleration being AU/day^2.

The parameters A_1, A_2 and A_3 are assumed to be solved for along with the state vector or orbital elements of the comet at some epoch. We then need the partial derivatives of the observables w.r.t. these parameters. We define a column vector

$$p_i = \frac{\partial r}{\partial A_i} = \begin{bmatrix} \partial x/\partial A_i \\ \partial y/\partial A_i \\ \partial z/\partial A_i \end{bmatrix}, i = 1, 2, 3 \tag{12}$$

and derive the variational equations

$$\ddot{p}_i = \frac{\partial\ddot{r}}{\partial r} p_i + \frac{\partial\ddot{r}}{\partial A_i}, i = 1, 2, 3. \tag{13}$$

The partials $\partial\ddot{r}/\partial A_i$ can be immediately deduced from Eq. (10), while the partials $\partial\ddot{r}/\partial r$ may be found from Eq. (5). The partials p_i are all initially zero.

2.2. Implementation

The preceding formulas have been implemented in two FORTRAN programs which we shall refer to as Model 1, the *coordinate perturbation* method, and Model 2, the *observation perturbation* method. Both programs use the same input:

1. Epoch in Julian days.
2. State vector $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ or orbital elements $(T, \omega, \Omega, I, q, e)$ at the epoch.
3. ID's of central body and perturbing bodies (1 = Mercury, 2 = Venus, ... 9 = Pluto, 10 = Moon, 11 = Sun).
4. Initial values (usually zero) of the non-gravitational parameters A_1, A_2, A_3 for comets.
5. Initial and min step size in the integration and max number of iterations (normally stopped at a specified rms level).
6. Observation file in the format used at the Minor Planet Center.

The output is the improved state vector and orbital elements and,if applicable, non-gravitational parameters. The observation residuals and their overall rms value are also given.

The integration is done with the Bulirsch-Stoer (1980) first-order method and adaptive step size control. This is a very efficient scheme of numerical integration. No additional interpolations are required to match the output to the times of observation.

Table 1. Orbit of Caliban from 47 obs. 1984–1998.

Epoch	1999 Jan. 22.0 TT	
	Model 1	Model 2
T	1998 Apr. 18.729	Apr. 18.709
ω	338.792	338.780
Ω	175.070	175.071
I	139.697	139.697
q	0.0439947	0.0439950
e	0.0814373	0.0814311
rms	$0''63$	$0''63$

In both models the state vector is integrated in accordance with Eq. (1) and, for comets, Eq. (10), and the 9 components of the non-gravitational partial derivatives p_i are obtained by integrating Eqs. (13) with the aid of Eq. (5). Model 1 requires the additional integrations of the variational Eqs. (4) for the 18 elements of G. This sums up to the integration of 30 s-order differential equations in Model 1 but only 12 in Model 2. Without non-gravitational perturbations, these numbers are reduced to 21 and 3 respectively. Model 2 thus offers a very substantial savings of integration time, especially if non-gravitational forces are excluded, although to some extent this savings will be offset by the need to compute two-body partials. Model 2 also appears to converge somewhat faster and be more stable, but this may be due to solving for corrections to the orbital elements instead of to the state vector, as in Model 1.

3. Applications to satellites and comets

We will illustrate the use of both models for one satellite and two comets.

3.1. Caliban (Uranus XVI)

Our first example is orbit improvement for the Uranian satellite Caliban based on 47 observations between June 1984 and November 1998, covering a little over 9 revolutions of the satellite which has a period of 579 days. Perturbations were included due to the Sun, Jupiter and Saturn. We started from an orbit with an average rms residual of 104 arcsec. Note that this is a rather large *relative* error because the orbit spans only about 17 arcmin in the sky. Model 1 diverged, while Model 2 converged with an rms residual of 0.63 arcsec after 4 integrations. With a more accurate starting orbit (rms = 37 arcsec), Model 1 converged on very nearly the same orbit, but it took 6 integrations. Thus the observation perturbation method appears to be more robust and stable than the coordinate perturbation method.

Table 1 lists the orbital elements of the satellite according to both solutions. The differences between the two sets of elements are well within the standard errors of the fits.

3.2. 46P/Wirtanen

Our second example involves the comet 46P/Wirtanen which was the original target of ESA's Rosetta mission. The orbit of this comet has been studied by several authors. Grav (2001)

used the observation perturbation method to analyze observations between 1948 and 1999. He tried several models for the non-gravitational forces: (1) Marsden's model with and without a time shift τ making $g(r)$ in Eq. (11) peak after perihelion if $\tau > 0$, (2) Sitarski's rotating nucleus model (Sitarski 1990), and (3) Sekanina's forced precession model as implemented by Krolikowska & Szutowicz (1999).

This comet is affected rather strongly by non-gravitational forces which have undergone substantial changes during the 50-year interval considered, probably mainly due to close approaches to Jupiter in 1971 and 1984. Just prior to the 1971 approach the comet's period was 6.7 years which was reduced to about 5.5 years after the 1984 approach. The same set of non-gravitational parameters gives satisfactory residuals for at most three apparitions of the comet.

Table 2 gives some results obtained by several authors for the non- gravitational parameters A_1, A_2 and A_3 for different observation intervals.

The best result (Grav 2001) was obtained with the use of the forced precession model with a rotating nucleus, as introduced by Krolikowska & Szutowicz (1999) for 46P/Wirtanen; see Table 3. It was necessarry to use two different values for the main parameter A: A_{I} before 1971 and A_{II} after 1971. The inclusion of a time shift τ in the second run did not have a noticeable effect on the residuals. The non- gravitational parameters are in rough agreement with those obtained by Krolikowska and Szutowicz, except that they found a much larger negative value for τ (–23.476). However, they used observations only up to 1995 and a different epoch (Jan. 27 1998), so the two sets of results are not directly comparable.

In all of Grav's calculations, the available observations were screened using the the Bielicki criterion for rejecting and weighting observations, as outlined by Bielicki & Sitarski (1991). Note that in the examples that follow (Tables 4 and 5), the observations are all given equal weight and are rejected if they produce residuals larger than 4 arcsec. The rms values of the fits can therefore be expected to be somewhat larger than if the Bielicki selection criterion had been used.

The orbital elements in Table 3 are in good agreement with the orbital elements in Table 4 derived from 555 observations between July 8 1991 and April 10 2003, using both model 1 and model 2 with the three Marsden parameters. Perturbations were included from all the nine planets and the Moon. The very good internal agreement of the two models again shows that the observation perturbation method works equally well as the coordinate perturbation method.

3.3. 67P/Churyumov-Gerasimenko

Our last example is orbit computation for the comet 67P/Churyumov- Gerasimenko which has a period of 6.6 yr. This comet is of special interest since the Rosetta spacecraft is scheduled to rendezvous with the comet in 2014. Table 5 gives the orbital elements and the non-gravitational parameters obtained for this comet by means of model 1 and model 2 based on 1301 observations between Sep. 9, 1969 and June 20, 2004. Again all nine planets and the Moon were included as

Table 2. Non-gravitational parameters in units of 10^{-8} AU/day^2 for 46P/Wirtanen.

Source	Obs. interval	No. of obs.	A_1	A_2	A_3	rms
Marsden (1983)	1948–1975	36	+0.52	−0.0871	–	1″86
Krolikowska & Sitarski (1996)	1948–1975	45	+0.5735	−0.0872	+0.2019	1″90
Grav (2001)	1948–1975	45	+0.6969	−0.0877	+0.1712	1″87
Grav (2001)	1986–1999	338	+0.5029	−0.1690	−0.0097	1″27
Grav (2001)	1960–1999	360	+0.7218	−0.1661	−0.1866	3″15

Table 3. Forced precession model for 46P/Wirtanen 1948–1999.

Epoch	2003 Dec. 27.0 TT	
T	2002 Aug. 26.62046	Aug. 26.61928
ω	356.37262	356.37450
Ω	82.16978	82.16775
I	11.73824	11.73849
q	1.0586022	1.0586088
e	0.6577448	0.6577425
A_I	+0.5207±0.0067	+0.6010±0.0097
A_{II}	+0.7093±0.0021	+0.7629±0.0077
η	17.65±0.11	13.00±0.46
I_0	145.07±0.79	152.01±0.56
ϕ_0	284.46±2.19	317.34±3.36
f_p	+0.5293±0.0137	+0.7464±0.0189
s	+0.1861±0.0066	+0.2445±0.0120
τ	–	−7.75±0.84
rms	1″53	1″53

Notes: A_I and A_{II} are in 10^{-8} AU/day^2, the angular parameters η, I_0 and ϕ_0 are in degrees, f_p is in 10^6 day/AU, s and τ are in days. A_I was applied before and A_{II} after 1971.

Table 4. Results for 46P/Wirtanen from 555 obs. 1991–2003.

Epoch	2003 Dec. 27.0 TT	
	Model 1	Model 2
T	2002 Aug. 26.67881	Aug. 26.67877
ω	356.37739	356.37734
Ω	82.16756	82.16757
I	11.73852	11.73853
q	1.0586219	1.0586216
e	0.6577531	0.6577532
A_1	+0.3427±0.0106	+0.3487±0.0096
A_2	−0.1404±0.0001	−0.1404±0.0001
A_3	+0.1000±0.0072	+0.1027±0.0072
rms	1″37	1″37

Table 5. Results for 67P/Churyumov-Gerasimenko.

Epoch	2003 Dec. 27.0 TT		
	Model 1	Model 2	Krolikowska (2003)
T	2002 Aug. 18.28687	Aug. 18.28715	Aug. 18.28695
ω	11.40963	11.40974	11.40976
Ω	50.92877	50.92869	50.92865
I	7.12417	7.12415	7.12415
q	1.2906480	1.2906474	1.2906479
e	0.6317508	0.6317510	0.6317509
A_1	+0.05089±0.00271	+0.06120±0.00281	+0.05444±0.00267
A_2	+0.00979±0.00002	+0.00991±0.00001	+0.00981±0.00002
A_3	+0.03552±0.00214	+0.02961±0.00187	+0.03386±0.00215
rms	1″57	1″57	1″16

4. Conclusions

Our results show that the observation perturbation method is fully compatible with the conventional coordinate perturbation method. The former method is simpler and requires less computer time; it also seems to have better convergence when applied to satellites. One might suspect that this method will handle very strongly perturbed orbits less well than the conventional method, but this is not borne out in the case of 46P/Wirtanen which had two close approahes to Jupiter, although both methods had problems satisfying the observations near those times.

perturbers (inclusion of the Moon reduced the rms residual by 0.2 arcsec). For comparison, we also show the corresponding results obtained by Krolikowska (2003) based on 1169 observations between Sep. 9, 1969 and March 13, 2003. The fact that the 1.16 arcsec rms residual obtained by Krolikowska is smaller than the 1.57 arcsec in model 1 and model 2, is probably due to Krolikowska's use of the Bielicki observation selection criterion. Even so, the three sets of orbital elements and non-gravitational parameters are in excellent agreement.

The small values for A_1, A_2 and A_3 show that this comet is much less affected by non-gravitational forces than is 46P/Wirtanen. For a discussion of the orbital evolution of this comet, the reader is referred to the article by Krolikowska (2003).

References

Aksnes, K. 1973, Cel. Mech., 8, 99
Aksnes, K. 1978, AJ, 83, 1249
Bielicki, M., & Sitarski, G. 1991, Acta Astron., 41, 309
Grav, T. 2001, The Orbital Motion of Periodic Comet 46P/Wirtanen, Cand. Scient. (MS) thesis, University of Oslo
Krolikowska, M., & Sitarski, G. 1996, A&A, 310, 992
Krolikowska, M., & Szutowicz, S. L. 1999, A&A, 343, 997
Krolikowska, M. 2003, Acta Astron., 53, 195
Marsden, B. G., Sekanina, Z., & Yeomans, D. K. 1973, AJ, 78, 211
Marsden, B. G. 1983, Minor Planet Circular, 8274
Marsden, B. G., Williams, G. V., & Aksnes, K. 2000, in Minor Bodies in the Outer Solar System (Berlin: Springer Verlag), 187
Sitarski, G. 1990, Acta Astron., 44, 91
Stoer. J., & Bulirsch, R. 1980, in Introduction to Numerical Analysis (New-York: Springer Verlag), Chap. 7

Online Material

Appendix: Two-body partials

Let $a, e, M_0, I, \omega, \Omega$ be the orbital elements, referred to the ecliptic, of an object in an elliptic orbit. The object's Earth-equatorial coordinates x, y, z centered on the primary may then be expressed as

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}
\tag{14}
$$

where $\bar{x}$ and $\bar{y}$ are components of the radius vector r in the orbital system,

$$
\bar{x} = a(\cos E - e), \ \ \bar{y} = a \sqrt{1 - e^2} \sin E, \ \ r = a(1 - e \cos E) \tag{15}
$$

Here E is given from Kepler's equation,

$$
E - e \sin E = M = M_0 + n(t - t_0), \ \ n = \sqrt{\frac{\mu}{a^3}}, \tag{16}
$$

t_0 being the epoch of the orbital elements. The elements of the rotation matrix are

$$
\begin{aligned}
R_{11} &= \cos \Omega \cos \omega - \sin \Omega \cos I \sin \omega \\
R_{12} &= -\cos \Omega \sin \omega - \sin \Omega \cos I \cos \omega \\
R_{21} &= (\sin \Omega \cos \omega + \cos \Omega \cos I \sin \omega) \cos \epsilon - \sin \omega \sin I \sin \epsilon \\
R_{22} &= (-\sin \Omega \sin \omega + \cos \Omega \cos I \cos \omega) \cos \epsilon - \cos \omega \sin I \sin \epsilon \\
R_{31} &= \sin I \sin \omega \cos \epsilon + (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos I) \sin \epsilon \\
R_{32} &= \sin I \cos \omega \cos \epsilon + (-\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos I) \sin \epsilon
\end{aligned}
\tag{17}
$$

where $\epsilon = 23°43928$ (J2000) is the obliquity of the ecliptic. We need the partial derivatives of x, y, z w.r.t. $a, e, M_0, I, \omega, \Omega$:

$$
\begin{aligned}
\frac{\partial x}{\partial a} &= R_{11} \frac{\partial \bar{x}}{\partial a} + R_{12} \frac{\partial \bar{y}}{\partial a} \\
&\ \vdots \\
\frac{\partial z}{\partial \Omega} &= \frac{\partial R_{31}}{\partial \Omega} \bar{x} + \frac{\partial R_{32}}{\partial \Omega} \bar{y}
\end{aligned}
\tag{18}
$$

The partials of the R's w.r.t. I, ω, Ω are readily derived, while the required partials of $\bar{x}$ and $\bar{y}$ are found to be

$$
\begin{aligned}
\frac{\partial \bar{x}}{\partial a} &= \frac{\bar{x}}{a} + \frac{3a}{2r} n(t - t_0) \sin E \\
\frac{\partial \bar{x}}{\partial e} &= -a - \frac{\bar{y}^2}{r(1 - e^2)} \\
\frac{\partial \bar{x}}{\partial M_0} &= -\frac{a\bar{y}}{r \sqrt{1 - e^2}} \\
\frac{\partial \bar{y}}{\partial a} &= \frac{\bar{y}}{a} - \frac{3a}{2r} n(t - t_0) \sqrt{1 - e^2} \cos E \\
\frac{\partial \bar{y}}{\partial e} &= \frac{\bar{x}\bar{y}}{r(1 - e^2)} \\
\frac{\partial \bar{y}}{\partial M_0} &= \frac{a}{r} \sqrt{1 - e^2}(\bar{x} + ae)
\end{aligned}
\tag{19}
$$