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The three-point correlation function of cosmic shear

II. Relation to the bispectrum of the projected mass density and generalized third-order aperture measures

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Abstract. Cosmic shear, the distortion of images of high-redshift sources by the intervening inhomogeneous matter distribution in the Universe, has become one of the essential tools for observational cosmology since it was first measured in 2000. Since then, several surveys have been conducted and analyzed in terms of second-order shear statistics. Current surveys are on the verge of providing useful measurements of third-order shear statistics, and ongoing and future surveys will provide accurate measurements of the shear three-point correlation function which contains essential information about the non-Gaussian properties of the cosmic matter distribution.

We study the relation of the three-point cosmic shear statistics to the third-order statistical properties of the underlying convergence, expressed in terms of its bispectrum. Explicit relations for the natural components of the shear three-point correlation function (which we defined in an earlier paper) in terms of the bispectrum are derived. The behavior of the correlation function under parity transformation is obtained and found to agree with previous results. We find that in contrast to the two-point shear correlation function, the three-point function at a given angular scale θ is not affected by power in the bispectrum on much larger scales. These relations are then inverted to obtain the bispectrum in terms of the three-point shear correlator; two different expressions, corresponding to different natural components of the shear correlator, are obtained and can be used to separate E and B-mode shear contributions. These relations allow us to explicitly show that correlations containing an odd power of B-mode shear vanish for parity-symmetric fields. Generalizing a recent result by Jarvis et al., we derive expressions for the third-order aperture measures, employing multiple angular scales, in terms of the (natural components of the) three-point shear correlator and show that they contain essentially all the information about the underlying bispectrum. We discuss the many useful features these (generalized) aperture measures have that make them convenient for future analyses of the skewness of the cosmic shear field (and any other polar field, such as the polarization of the Cosmic Microwave Background).

Key words. cosmology: large-scale structure of the Universe

1. Introduction

Recent surveys have measured second-order cosmic shear statistics with high accuracy, owing to the large sky area covered, and thus the large number of faint galaxy images (e.g., van Waerbeke et al. 2001, 2002; Jarvis et al. 2003a; Hoekstra et al. 2002). With surveys of this size, it now becomes feasible to obtain higher-order cosmic shear statistics which probe the non-Gaussian features of the cosmic shear field. These higher-order statistics are particularly useful in breaking near-degeneracies in cosmological parameters which are present at the level of second-order statistics. Bernardeau et al. (1997), van Waerbeke et al. (1999) and others pointed out that the skewness of the convergence underlying the cosmic shear field can break the degeneracy between the density parameter $\Omega_{\rm m}$ and the normalisation of the matter power spectrum, expressed in terms of the rms density fluctuations σ_8 on a scale of 8 h^{-1} Mpc. However, the convergence cannot be observed directly, but needs to be inferred from the observed galaxy image ellipticities which yield an estimate of the local shear. The dispersion of the shear in a (circular) aperture, frequently used to quantify second-order shear statistics, cannot be generalized to a third-order statistics. Schneider et al. (1998, hereafter SvWJK) have defined an alternative cosmic shear measure, the aperture mass, which is a scalar quantity that can be directly obtained from the shear, and therefore is particularly suited to define higher-order statistics.

Recently, interest in higher-order cosmic shear statistics has been revived. The three-point correlation function (3PCF henceforth) of the shear contains all the information on the third-order statistical properties of the shear field, and therefore is of prime

interest. In addition, it can be obtained directly from the observed image ellipticities and, in contrast to the aperture mass statistics, is insensitive to holes and gaps in the data field. However, the shear 3PCF is a function with $2 \times 2 \times 2 = 8$ components (since each shear has two independent components) and 3 variables (e.g., the sides of the triangle formed by the three points) and therefore difficult to handle. Bernardeau et al. (2003) defined a specific integral over the 3PCF and applied that to the VIRMOS-DESCART survey in Bernardeau et al. (2002) to obtain the first detection of a non-zero third-order cosmic shear signal. Using the same observational data, Pen et al. (2003) calculated the skewness of the aperture mass, where the latter has been obtained from integrating the shear 3PCF. Jarvis et al. (2003b, JBJ hereafter) obtained an alternative expression for the aperture mass skewness in terms of the shear 3PCF and applied this to the CTIO cosmic shear survey, finding a signal at about the 2.3-σ level.

Following a different approach, the 3PCF was considered directly in a number of recent papers. Schneider & Lombardi (2003, hereafter Paper I) defined special combinations of the shear 3PCF which we termed the "natural components", because they obey simple transformation laws under coordinate rotations. In particular, we derived the behavior of the 3PCF under parity transformations, and showed that all eight components are expected to be non-zero for a general triangle configuration. Zaldarriaga & Scoccimarro (2003) and Takada & Jain (2003a) obtained analytic approximations and numerical results, using ray-tracing simulations, for the 3PCF. Takada & Jain (2003b) provide an extensive study of the expectation for the shear 3PCF in terms of the halo model of the large-scale structure, and they verified the accuracy of their analytical results with numerical simulations. Schneider (2003) investigated the transformation properties of a general 3PCF of a polar under parity transformations and showed that the expectation value of any quantity containing an odd power of B-modes vanishes for parity-invariant shear fields.

In this paper, we first consider the relation between the shear 3PCF and the bispectrum of the underlying convergence (or projected density) field. If the shear field is derivable from a scalar potential (that is, a pure E-mode field), as expected for cosmic shear in the absence of intrinsic galaxy alignments and systematics in the observing process, the bispectrum of the convergence fully encodes the third-order information of the random field[1]. In this first part of the paper, we therefore generalize the relations between the power spectrum of the convergence and the various second-order shear statistics (derived in Crittenden et al. 2002; Schneider et al. 2002) to third-order shear statistics. After some preliminaries in Sect. 2, we derive the shear 3PCF in terms of the bispectrum of the convergence. From these explicit relations, general transformation laws of the 3PCF can be directly seen; for example, the behavior of the 3PCF under parity inversion as studied before in Paper I and in Schneider (2003) can be explicitly verified, as will be shown in Sect. 4. In Sect. 5 we invert these relations, i.e., we express the bispectrum in terms of the 3PCF of the shear. We obtain two formally different expressions for the bispectrum which must, however, be identical in the case of a pure E-mode shear field.

In the second part of this paper (Sect. 6), we consider the third-order aperture mass statistics as a particularly convenient integral over the shear 3PCF; in fact, this part of the paper will quite likely be most relevant for future studies of higher-order cosmic shear statistics. We first express $\left\langle M_{\rm ap}^3(\theta)\right\rangle$ in terms of the bispectrum and then replace the bispectrum in terms of the 3PCF. This procedure yields the same result for $\left\langle M_{\rm ap}^3(\theta)\right\rangle$ in terms of the shear 3PCF as derived by JBJ. We then argue that the third-order aperture measures contain only part of the information about the bispectrum of the underlying convergence, and generalize the aperture measures to the case of three different scale lengths. We show that this generalization allows us to obtain essentially the full information about the bispectrum. These generalized aperture measures are then expressed in terms of the shear 3PCF. Section 7 summarizes and discusses our results.

It must be stressed that all our results are valid for other random fields which share the properties of that of cosmic shear: A homogeneous, isotropic, parity-symmetric random field of a polar. The most obvious example of such a field in the cosmological context, apart from cosmic shear, is the polarization field of the Cosmic Microwave Background (e.g., Zaldarriaga et al. 1997)

2. Preliminaries

In the first part of this paper (through Sect. 5) we shall consider a shear field γ which is caused by an underlying projected density (or convergence) field κ, as is expected for a shear field produced by light propagation in an inhomogeneous Universe (e.g., Blandford et al. 1991; Miralda-Escudé 1991; Kaiser 1992; see also the reviews by Mellier 1999 and Bartelmann & Schneider 2001). The relation between the shear (expressed throughout this paper as a complex number) and the convergence is most simply given in Fourier space,

$$\hat\gamma(\boldsymbol\ell) = {\rm e}^{2{\rm i}\beta}\,\hat\kappa(\boldsymbol\ell), \tag{1}$$

where β is the polar angle of $\boldsymbol\ell$, and $\boldsymbol\ell$ is the Fourier transform variable of the angular position vector on the sky.

In Paper I we considered the 3PCF of the shear. Since the shear is a two-component quantity, the 3PCF has 8 independent components. Since one cannot form a scalar from the product of three shears alone, one needs to project the shear with respect to some reference directions. The three points at which the shear is considered form a triangle, and one can project the shear along

[1] It should be noted that even for a pure cosmic shear field, B-modes do occur if source galaxies are clustered (Bernardeau 1998; Schneider et al. 2002); furthermore, they can occur from slight violations of the lowest-order approximations employed when considering light propagation through a slightly inhomogeneous Universe – see, e.g., Bernardeau et al. (1997), SvWJK, Jain et al. (2000).

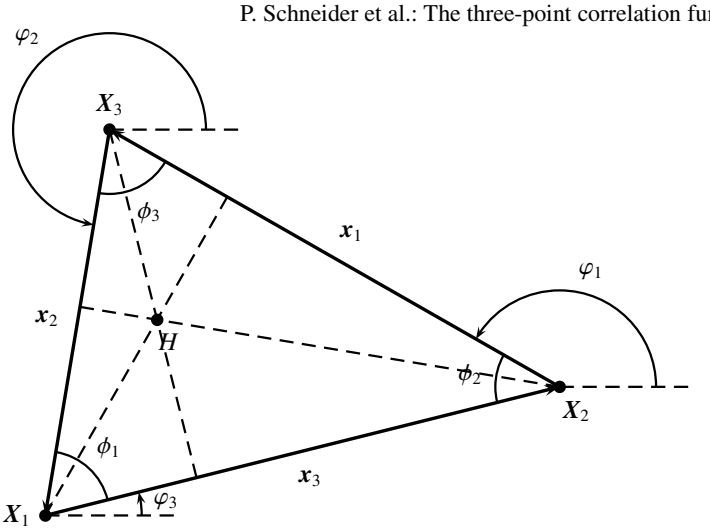

Fig. 1. Definitions of the geometry of a triangle. The $\boldsymbol{X}_l$ are the vertices of the triangle, the $\boldsymbol{x}_l$ the corresponding sides, φ_l are the orientations of the sides relative to the positive x_1-direction, the ϕ_l are the interior angles of the triangle. The figure also shows the orthocenter H, i.e. the intersection of the heights of the triangle.

directions attached to such a triangle, i.e., which rotate with the triangle in coordinate rotations. We have considered a number of such obvious projections, namely with respect to the directions of the vertices towards one of the centers of a triangle. Let ζ_i be the polar angle of the vector connecting the point $\boldsymbol{X}_i$ with the chosen center, then the Cartesian components $\gamma_{1,2}$ of the shear $\gamma = \gamma_1 + \mathrm{i}\gamma_2$ are used to define the tangential and cross components of the shear relative to the chosen direction ζ_i,

$$\gamma(\boldsymbol{X}_i;\zeta_i) \equiv \gamma_{\rm t}(\boldsymbol{X}_i;\zeta_i) + \mathrm{i}\gamma_\times(\boldsymbol{X}_i;\zeta_i) = -\left[\gamma_1(\boldsymbol{X}_i) + \mathrm{i}\gamma_2(\boldsymbol{X}_i)\right]\,\mathrm{e}^{-2\mathrm{i}\zeta_i} \equiv -\gamma(\boldsymbol{X}_i)\,\mathrm{e}^{-2\mathrm{i}\zeta_i}. \tag{2}$$

If the reference directions are changed, the tangential and cross components of the shear will change correspondingly. In particular, defining the 3PCF of the shear components, they will change if a different center of the triangle is chosen. In Paper I we defined four complex "natural components" of the shear 3PCF which show a simple behavior under such transformations; they have been termed $\Gamma^{(\mu)}(x_1,x_2,x_3)$, $\mu = 0,1,2,3$. The $\Gamma^{(i)}$, $i = 1,2,3$ can be obtained from one another by permutations of the arguments x_i, which represent the sides of the triangle.

We specify our geometry in the same way as in Paper I (see Fig. 1): let $\boldsymbol{X}_i$ be three points at which the shear is considered. The connecting vectors are $\boldsymbol{x}_1 = \boldsymbol{X}_3 - \boldsymbol{X}_2$, $\boldsymbol{x}_2 = \boldsymbol{X}_1 - \boldsymbol{X}_3$, and $\boldsymbol{x}_3 = \boldsymbol{X}_2 - \boldsymbol{X}_1$. We denote the polar angle of the vector $\boldsymbol{x}_i$ by φ_i, and the interior angle of the triangle at the point $\boldsymbol{X}_i$ by ϕ_i. Furthermore, we assume that the triangle is oriented such that $\boldsymbol{x}_1\times\boldsymbol{x}_2 = \boldsymbol{x}_2\times\boldsymbol{x}_3 = \boldsymbol{x}_3\times\boldsymbol{x}_1 > 0$, where for two two-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we defined $\boldsymbol{a}\times\boldsymbol{b} = a_1b_2 - a_2b_1$. Hence, the points $\boldsymbol{X}_i$ are ordered counter-clockwise around the triangle.

In the first part of this paper, we shall consider the projection of the shear relative to the orthocenter of the triangle. As the vector connecting the point $\boldsymbol{X}_i$ and the orthocenter is perpendicular to the vector $\boldsymbol{x}_i$ (see Fig. 1), one has $\zeta_i = \varphi_i + \pi/2$ in this case, so that

$$\gamma^{(\rm o)}(\boldsymbol{X}_i) \equiv \gamma_{\rm t}^{(\rm o)}(\boldsymbol{X}_i) + \mathrm{i}\gamma_\times^{(\rm o)}(\boldsymbol{X}_i) = \left[\gamma_1(\boldsymbol{X}_i) + \mathrm{i}\gamma_2(\boldsymbol{X}_i)\right]\,\mathrm{e}^{-2\mathrm{i}\varphi_i} \equiv \gamma(\boldsymbol{X}_i)\,\mathrm{e}^{-2\mathrm{i}\varphi_i}. \tag{3}$$

The shear 3PCF depends linearly on the 3PCF of the convergence, or equivalently on its Fourier transform, the bispectrum. The bispectrum of the surface mass density is defined as (see, e.g., van Waerbeke et al. 1999)

$$\langle\hat\kappa(\boldsymbol{\ell}_1)\hat\kappa(\boldsymbol{\ell}_2)\hat\kappa(\boldsymbol{\ell}_3)\rangle = (2\pi)^2\left[B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2) + B(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3) + B(\boldsymbol{\ell}_3,\boldsymbol{\ell}_1)\right]\,\delta(\boldsymbol{\ell}_1+\boldsymbol{\ell}_2+\boldsymbol{\ell}_3), \tag{4}$$

hence, it is non-zero only for closed triangles in ℓ-space. This follows from the assumed statistical homogeneity of the random field κ. Furthermore, if κ is an isotropic random field, the function $B(\boldsymbol{\ell},\boldsymbol{\ell}')$ depends only on $|\boldsymbol{\ell}|$, $|\boldsymbol{\ell}'|$, and the angle φ enclosed by $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$. We shall therefore write $B(\boldsymbol{\ell},\boldsymbol{\ell}') = b(|\boldsymbol{\ell}|,|\boldsymbol{\ell}'|,\varphi)$. If, in addition, the statistical properties of the field κ are invariant under parity transformation (as we shall assume throughout), then b is an even function of φ, or, equivalently, b is invariant against exchanging ℓ and ℓ',

$$b(\ell,\ell',-\varphi) = b(\ell,\ell',\varphi) = b(\ell',\ell,\varphi). \tag{5}$$

3. The shear three-point correlation function in terms of the bispectrum

In this section, we will derive the shear 3PCF in terms of the bispectrum B of the convergence. As it turns out, the calculations are fairly cumbersome, owing to the rich mathematical structure of the 3PCF with its three arguments, compared to the 2PCF which has only one argument (and for which the direction along which the shear components are measured is uniquely given by the connecting vector between any pair of points).

3.1. The case of $\Gamma^{(0)}$

The natural component $\Gamma^{(0)}$ of the shear 3PCF measured relative to the orthocenter reads

$$\Gamma^{(0)}(x_1,x_2,x_3) := \left\langle\gamma^{(\mathrm{o})}(\boldsymbol{X}_1)\gamma^{(\mathrm{o})}(\boldsymbol{X}_2)\gamma^{(\mathrm{o})}(\boldsymbol{X}_3)\right\rangle = \int\frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_2}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_3}{(2\pi)^2}\exp\left[-\mathrm{i}\left(\boldsymbol{\ell}_1\cdot\boldsymbol{X}_1+\boldsymbol{\ell}_2\cdot\boldsymbol{X}_2+\boldsymbol{\ell}_3\cdot\boldsymbol{X}_3\right)\right]$$
$$\times\exp\left[2\mathrm{i}\left(\sum\beta_i-\sum\varphi_i\right)\right]\langle\hat\kappa(\boldsymbol{\ell}_1)\hat\kappa(\boldsymbol{\ell}_2)\hat\kappa(\boldsymbol{\ell}_3)\rangle\,, \tag{6}$$

where we made use of the relation (1) between the Fourier transforms of the shear and the convergence, and the definition (3) of the shear components relative to the orthocenter. Inserting the bispectrum (4) into Eq. (6), performing for each of the resulting three terms the integration over the $\boldsymbol{\ell}$-vector which is not in the argument of the B function, by making use of the delta-"function" in (4), and using the relations between the corner points $\boldsymbol{X}_i$ and the side vectors $\boldsymbol{x}_i$, one obtains after renaming the dummy integration variables

$$\Gamma^{(0)}(x_1,x_2,x_3) = \int_0^\infty\frac{\mathrm{d}\ell_1\,\ell_1}{(2\pi)^2}\int_0^\infty\frac{\mathrm{d}\ell_2\,\ell_2}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\beta_1\int_0^{2\pi}\mathrm{d}\beta_2\;b(\ell_1,\ell_2,\beta_1-\beta_2)\;\exp\left[2\mathrm{i}\left(\sum\beta_i-\sum\varphi_i\right)\right]$$
$$\times\left[\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_3)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_3-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\right]. \tag{7}$$

The angle β_3 occurring in (7) is the polar angle of the vector $\boldsymbol{\ell}_3=-\boldsymbol{\ell}_1-\boldsymbol{\ell}_2$, so that

$$\cos 2\beta_3=\frac{\ell_1^2\cos 2\beta_1+\ell_2^2\cos 2\beta_2+2\ell_1\ell_2\cos(\beta_1+\beta_2)}{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|^2};\quad \sin 2\beta_3=\frac{\ell_1^2\sin 2\beta_1+\ell_2^2\sin 2\beta_2+2\ell_1\ell_2\sin(\beta_1+\beta_2)}{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|^2}.$$

We next rename the angles in the following way:

$$\beta_1=\theta+\varphi/2;\quad \beta_2=\theta-\varphi/2;\quad \beta_3=\theta+\bar\beta, \tag{8}$$

so that φ is the angle between $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$, as previously defined, and $\bar\beta$ is the angle between the direction of $\boldsymbol{\ell}_3$ and the mean of the directions of $\boldsymbol{\ell}_1$ and ℓ_2. Since B is independent of θ, a further integration can be carried out in Eq. (7), using $\int\mathrm{d}\beta_1\int\mathrm{d}\beta_2=\int\mathrm{d}\theta\int\mathrm{d}\varphi$. Owing to the symmetric form of the three terms occurring, only one of the three terms has to be calculated explicitly; we shall consider the first term in the following. From the foregoing equations one finds that

$$\cos 2\bar\beta=\frac{(\ell_1^2+\ell_2^2)\cos\varphi+2\ell_1\ell_2}{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|^2};\quad \sin 2\bar\beta=\frac{(\ell_1^2-\ell_2^2)\sin\varphi}{|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|^2};\quad \text{with}\quad |\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|^2=\ell_1^2+\ell_2^2+2\ell_1\ell_2\cos\varphi. \tag{9}$$

Next, we consider the argument of the exponential,

$$\begin{aligned}\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2 &= \ell_2x_1\cos\left(\theta-\varphi/2-\varphi_1\right)-\ell_1x_2\cos\left(\theta+\varphi/2-\varphi_2\right)\\ &= -\ell_2x_1\sin\left[\theta'-(\varphi+\phi_3)/2\right]-\ell_1x_2\sin\left[\theta'+(\varphi+\phi_3)/2\right],\end{aligned} \tag{10}$$

where we have defined $\theta=\theta'+(\varphi_1+\varphi_2)/2$ and used the fact that $\varphi_2-\varphi_1=\pi-\phi_3$. Therefore, we can write

$$\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2=-A_3\sin(\theta'+\alpha_3), \tag{11}$$

from which one finds, after expanding the trigonometric functions in Eqs. (10) and (11),

$$A_3\cos\alpha_3=(\ell_1x_2+\ell_2x_1)\cos\left(\frac{\varphi+\phi_3}{2}\right);\quad A_3\sin\alpha_3=(\ell_1x_2-\ell_2x_1)\sin\left(\frac{\varphi+\phi_3}{2}\right);$$
$$A_3=\sqrt{(\ell_1x_2)^2+(\ell_2x_1)^2+2\ell_1\ell_2x_1x_2\cos(\varphi+\phi_3)}. \tag{12}$$

Finally, we consider the sums over the angles that occur in (7),

$$\sum\beta_i-\sum\varphi_i=3\theta+\bar\beta-\sum\varphi_i=3\theta'+\bar\beta+(\varphi_1+\varphi_2)/2-\varphi_3=3\theta'+\bar\beta+(\phi_1-\phi_2)/2. \tag{13}$$

We can now perform the θ-integration of the first term in Eq. (7) as follows:

$$\int_0^{2\pi}\mathrm{d}\theta\;\exp\left[2\mathrm{i}\left(\sum\beta_i-\sum\varphi_i\right)\right]\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2)}=\mathrm{e}^{2\mathrm{i}\bar\beta}\,\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}\int_0^{2\pi}\mathrm{d}\theta'\;\mathrm{e}^{6\mathrm{i}\theta'}\;\mathrm{e}^{-\mathrm{i}A_3\sin(\theta'+\alpha_3)}$$
$$=\mathrm{e}^{2\mathrm{i}\bar\beta}\,\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}\int_0^{2\pi}\mathrm{d}\vartheta\;\mathrm{e}^{6\mathrm{i}(\vartheta-\alpha_3)}\,\mathrm{e}^{-\mathrm{i}A_3\sin\vartheta}=2\pi\,\mathrm{e}^{2\mathrm{i}\bar\beta}\,\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2)}\,\mathrm{e}^{-6\mathrm{i}\alpha_3}\;\mathrm{J}_6(A_3), \tag{14}$$

where J_n is the Bessel function of the first kind. Therefore, Eq. (7) becomes

$$\Gamma^{(0)}(x_1,x_2,x_3)=(2\pi)\int_0^\infty\frac{\mathrm{d}\ell_1\,\ell_1}{(2\pi)^2}\int_0^\infty\frac{\mathrm{d}\ell_2\,\ell_2}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\varphi\;b(\ell_1,\ell_2,\varphi)\;\mathrm{e}^{2\mathrm{i}\bar\beta}$$
$$\times\left[\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2-6\alpha_3)}\mathrm{J}_6(A_3)+\mathrm{e}^{\mathrm{i}(\phi_2-\phi_3-6\alpha_1)}\mathrm{J}_6(A_1)+\mathrm{e}^{\mathrm{i}(\phi_3-\phi_1-6\alpha_2)}\mathrm{J}_6(A_2)\right], \tag{15}$$

where the A_i and α_i are obtained from Eq. (12) by cyclic permutations of the x_1,x_2,x_3.

3.2. The case of $\Gamma^{(1)}$

Next, we calculate the natural component

$$\Gamma^{(1)}(x_1,x_2,x_3) := \left\langle \gamma^{(\mathrm{o})*}(\boldsymbol{X}_1)\gamma^{(\mathrm{o})}(\boldsymbol{X}_2)\gamma^{(\mathrm{o})}(\boldsymbol{X}_3)\right\rangle = \int\frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_2}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_3}{(2\pi)^2}\exp\left[\mathrm{i}\left(\boldsymbol{\ell}_1\cdot\boldsymbol{X}_1-\boldsymbol{\ell}_2\cdot\boldsymbol{X}_2-\boldsymbol{\ell}_3\cdot\boldsymbol{X}_3\right)\right]$$
$$\times\mathrm{e}^{2\mathrm{i}(\beta_2+\beta_3-\beta_1+\varphi_1-\varphi_2-\varphi_3)}\left\langle\hat\kappa(-\boldsymbol{\ell}_1)\hat\kappa(\boldsymbol{\ell}_2)\hat\kappa(\boldsymbol{\ell}_3)\right\rangle, \tag{16}$$

where we made use of the fact that $\hat\kappa^*(\boldsymbol{\ell}) = \hat\kappa(-\boldsymbol{\ell})$, since $\kappa(\boldsymbol{x})$ is a real field. Next, we change the integration variable $\boldsymbol{\ell}_1 \to -\boldsymbol{\ell}_1$; as a consequence, $\beta_1 \to \beta_1 + \pi$, but this does not change the exponential in Eq. (16). Inserting the bispectrum in the form Eq. (4), and appropriately renaming the dummy integration variables, one finds

$$\Gamma^{(1)}(x_1,x_2,x_3) = \mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2-\varphi_3)}\int_0^\infty\frac{\mathrm{d}\ell_1\,\ell_1}{(2\pi)^2}\int_0^\infty\frac{\mathrm{d}\ell_2\,\ell_2}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\beta_1\int_0^{2\pi}\mathrm{d}\beta_2\;b(\ell_1,\ell_2,\beta_1-\beta_2)$$
$$\times\left[\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2)}\mathrm{e}^{2\mathrm{i}(\beta_2+\beta_3-\beta_1)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_3)}\mathrm{e}^{2\mathrm{i}(\beta_1+\beta_2-\beta_3)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_3-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\mathrm{e}^{2\mathrm{i}(\beta_3+\beta_1-\beta_2)}\right]. \tag{17}$$

The further calculation proceeds in the same way as in the case of $\Gamma^{(0)}$. Specifically, we employ the change of angular integration variables given in Eq. (8), evaluate the three exponentials containing products of the form $\boldsymbol{x}_i\cdot\boldsymbol{\ell}_j$ using Eqs. (10), (11), and their analogous expressions obtained by cyclic permutations of the x_i, and calculating the angular sums in the exponentials. The final result reads

$$\Gamma^{(1)}(x_1,x_2,x_3) = 2\pi\int_0^\infty\frac{\mathrm{d}\ell_1\,\ell_1}{(2\pi)^2}\int_0^\infty\frac{\mathrm{d}\ell_2\,\ell_2}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\varphi\;b(\ell_1,\ell_2,\varphi)\left[\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2+2\phi_3)}\mathrm{e}^{2\mathrm{i}(\bar\beta-\varphi-\alpha_3)}\,\mathrm{J}_2(A_3)\right.$$
$$\left.+\mathrm{e}^{\mathrm{i}(\phi_3-\phi_2)}\mathrm{e}^{-2\mathrm{i}(\bar\beta+\alpha_1)}\,\mathrm{J}_2(A_1)+\mathrm{e}^{\mathrm{i}(\phi_3-\phi_1-2\phi_2)}\mathrm{e}^{2\mathrm{i}(\bar\beta+\varphi-\alpha_2)}\,\mathrm{J}_2(A_2)\right]. \tag{18}$$

The expressions for the other two natural components $\Gamma^{(2)}$ and $\Gamma^{(3)}$ of the shear 3PCF, which are defined in analogy to Eq. (16) by placing the complex conjugation of the shear at point $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$, respectively, are obtained from Eq. (18) by applying the transformation laws given in Paper I, i.e., even permutations of the arguments.

The resulting expressions (15) and (18) are not only relatively complicated, but their numerical evaluation also is quite cumbersome. Recalling that the relation between the 2PCF of the shear and the power spectrum $P_\kappa(\ell)$ of the convergence involves a convolution integral over a Bessel function, one should perhaps not be too surprised that in the case of third-order statistics there are three such oscillating factors in the transformation between the shear 3PCF and the bispectrum. In a future work, we will investigate numerical procedures with which the integration can be carried out accurately; first attempts, using Gaussian quadrature for the two ℓ-integrations and an equidistant grid for the φ-integration already yielded satisfactory results. Hence, despite the apparent complexity, the foregoing equations can be applied in practice.

4. Transformation laws

In Paper I the behavior of the natural components under a change of the order of the arguments was discussed, using simple geometrical arguments. We shall now consider this behavior explicitly, using the expressions (15) and (18). First, consider $\Gamma^{(0)}$.

Taking an even permutation of the arguments of $\Gamma^{(0)}$ just changes the order of the terms in the integral of Eq. (15) and therefore leaves $\Gamma^{(0)}$ unchanged. Taking an odd permutation of the arguments means that two of the arguments are interchanged, e.g., x_1 and x_2. Interchanging x_1 and x_2 corresponds to an interchange of ϕ_1 and ϕ_2. Using the property Eq. (5), one can also interchange ℓ_1 and ℓ_2. From Eq. (9) one sees that these changes imply that $\bar\beta \to -\bar\beta$. Furthermore, from Eq. (12), one sees that these changes lead to $A_3 \to A_3$, and $\alpha_3 \to -\alpha_3$. Together this implies that these transformations lead to a complex conjugation of the first term in Eq. (15). From the expressions for A_l and α_l obtained from Eq. (12) by cyclic permutations of the x_k, one finds that the above interchanges of x_1 and x_2 leads to $A_2 \to A_1$, $A_1 \to A_2$, $\alpha_2 \to -\alpha_1$, $\alpha_1 \to -\alpha_2$. This then implies that the second term in Eq. (15) becomes the complex conjugate of the third term, and vice versa. Taken together, we see that an odd permutation of the arguments changes $\Gamma^{(0)} \to \Gamma^{(0)*}$, as already argued from parity considerations in Paper I.

Note that an odd permutation of the arguments in geometric terms means that the orientation of the triangle is reversed. We shall now show that this is equivalent to replacing all ϕ_l by $-\phi_l$. The motivation for this observation comes from the fact that for a triangle with odd orientation, the relations (see Eq. (1) of Paper I) between the orientations φ_i of the $\boldsymbol{x}_i$ and the interior angles ϕ_i formally yield $\phi_i \in [-\pi, 0]$ (modulo 2π), whereas for a triangle with even orientation, $\phi_i \in [0, \pi]$ (modulo 2π). If we apply this transformation, $\phi_i \to -\phi_i$, we can change the integration variable $\varphi \to -\varphi$ in Eq. (15), noting from Eq. (5) that $b(\ell_1, \ell_2, \varphi)$ is unaffected by this change. These two changes together then imply that $\bar\beta \to -\bar\beta$, $A_i \to A_i$, and $\alpha_i \to -\alpha_i$ (see Eqs. (9) and (12), respectively). Hence, all three terms of Eq. (15) are transformed to their complex conjugates, as was claimed above. Note that this transformation behavior directly implies that $\Gamma^{(0)}$ is real if two of its arguments are equal.

Next one can consider the transformation behavior of $\Gamma^{(1)}$. Cyclic permutations of the arguments transform $\Gamma^{(1)}$ into $\Gamma^{(2)}$ and $\Gamma^{(3)}$, yielding the transformation behavior derived in Paper I. Interchanging x_2 and x_3 (and thus ϕ_2 and ϕ_3) should yield the complex conjugate of $\Gamma^{(1)}$. Again, we interchange ℓ_1 and ℓ_2, which then yields $\bar\beta \to -\bar\beta$, $A_1 \to A_1$, $\alpha_1 \to -\alpha_1$, and so the second term in Eq. (18) is complex conjugated. Furthermore, these transformations yield $A_2 \to A_3$, $\alpha_2 \to -\alpha_3$, $A_3 \to A_2$, $\alpha_3 \to -\alpha_2$, which shows that the first term in Eq. (18) becomes the complex conjugate of the third, and vice versa, so that $\Gamma^{(1)} \to \Gamma^{(1)*}$, as was to be shown. This transformation implies that $\Gamma^{(1)}$ is real if the last two arguments are equal.

Equivalently, we can also let $\phi_l \to -\phi_l$, and change the integration variable $\varphi \to -\varphi$. This implies $A_l \to A_l$, $\alpha_l \to -\alpha_l$, $\bar\beta \to -\bar\beta$, and each term in Eq. (18) is transformed into its complex conjugate.

5. Bispectrum in terms of the 3PCF

Recall the situation for the two-point correlation of the shear: there, the relation between the correlation functions and the power spectrum of the projected density fluctuations can be inverted (e.g., Schneider et al. 2002; hereafter SvWM), and thus the power spectrum can be expressed in terms of the two-point correlation function. It will be shown here that in analogy, the bispectrum $B(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2)$ can be expressed in terms of the 3PCF.

5.1. Bispectrum in terms of $\Gamma^{(0)}$

From (1) one finds that

$$\begin{aligned}\langle\hat\kappa(\boldsymbol{\ell}_1)\hat\kappa(\boldsymbol{\ell}_2)\hat\kappa(\boldsymbol{\ell}_3)\rangle &= \mathrm{e}^{-2\mathrm{i}\sum\beta_i}\,\langle\hat\gamma(\boldsymbol{\ell}_1)\hat\gamma(\boldsymbol{\ell}_2)\hat\gamma(\boldsymbol{\ell}_3)\rangle\\ &= \mathrm{e}^{-2\mathrm{i}\sum\beta_i}\int\mathrm{d}^2X_1\int\mathrm{d}^2X_2\int\mathrm{d}^2X_3\;\mathrm{e}^{2\mathrm{i}\sum\varphi_i}\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{X}_1+\boldsymbol{\ell}_2\cdot\boldsymbol{X}_2+\boldsymbol{\ell}_3\cdot\boldsymbol{X}_3)}\left\langle\gamma^{(\mathrm{o})}(\boldsymbol{X}_1)\gamma^{(\mathrm{o})}(\boldsymbol{X}_2)\gamma^{(\mathrm{o})}(\boldsymbol{X}_3)\right\rangle,\end{aligned}\tag{19}$$

where in the second step we used Eq. (3). We now split the right-hand side of the foregoing equation into three identical terms, each of which is thus one third of the above expression, and substitute $\boldsymbol{X}_1 = \boldsymbol{X}_3 + \boldsymbol{x}_2$, $\boldsymbol{X}_2 = \boldsymbol{X}_3 - \boldsymbol{x}_1$ in the first of these, and similar substitutions, obtained by cyclic permutations of the $\boldsymbol{X}_l$ and $\boldsymbol{x}_l$ in the other two terms. This then yields

$$\langle\hat\kappa(\boldsymbol{\ell}_1)\hat\kappa(\boldsymbol{\ell}_2)\hat\kappa(\boldsymbol{\ell}_3)\rangle = \frac{1}{3}\mathrm{e}^{-2\mathrm{i}\sum\beta_i}\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\int\mathrm{d}^2X_3\;\mathrm{e}^{2\mathrm{i}\sum\varphi_i}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)}\mathrm{e}^{\mathrm{i}\boldsymbol{X}_3\cdot(\boldsymbol{\ell}_1+\boldsymbol{\ell}_2+\boldsymbol{\ell}_3)}\Gamma^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2) + 2\text{ terms},\tag{20}$$

where we have used the notation $\Gamma^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2)$ for the 3PCF $\Gamma^{(0)}(x_1, x_2, x_3)$, with $x_3 = |\boldsymbol{x}_1 + \boldsymbol{x}_2|$, if $\boldsymbol{x}_1 \times \boldsymbol{x}_2 \geq 0$, and $\Gamma^{(0)*}(x_1, x_2, x_3)$ if $\boldsymbol{x}_1 \times \boldsymbol{x}_2 < 0$. Hence, if expressed in terms of the arguments $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, the information about the orientation of the three points $\boldsymbol{X}_l$ is included. Another notation to be used later on is $\Gamma^{(0)}(x_1, x_2, \phi_3)$, which also includes the orientation of the three points. The $\boldsymbol{X}_3$-integration in the previous equation yields a delta-function; comparison with Eq. (4) then results in

$$B(\boldsymbol{\ell}_1, \boldsymbol{\ell}_2) = \frac{1}{3}\mathrm{e}^{-2\mathrm{i}\sum\beta_i}\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\;\mathrm{e}^{2\mathrm{i}\sum\varphi_i}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)}\,\Gamma^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,.\tag{21}$$

It is easy to show that Eq. (21) is compatible with Eq. (7), since

$$\begin{aligned}\Gamma^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2) &= \int\frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_2}{(2\pi)^2}\,B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)\,\exp\Big[2\mathrm{i}\Big(\sum\beta_i-\sum\varphi_i\Big)\Big]\Big[\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_3)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_3-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\Big]\\ &= \frac{1}{3}\exp\Big(-2\mathrm{i}\sum\varphi_i\Big)\int\frac{\mathrm{d}^2\ell_1}{(2\pi)^2}\int\frac{\mathrm{d}^2\ell_2}{(2\pi)^2}\int\mathrm{d}^2y_1\int\mathrm{d}^2y_2\exp\Big(2\mathrm{i}\sum\varphi_i'\Big)\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{y}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{y}_1)}\\ &\quad\times\Big[\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_3)}+\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_3-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\Big]\Gamma^{(0)}(\boldsymbol{y}_1,\boldsymbol{y}_2),\end{aligned}\tag{22}$$

where the φ_i' are the polar angles of the $\boldsymbol{y}_i$, and $\boldsymbol{y}_3 = -\boldsymbol{y}_1 - \boldsymbol{y}_2$. The $\boldsymbol{\ell}_i$-integrations can be carried out, yielding delta-"functions": for the first term, e.g., they yield $(2\pi)^2\delta(\boldsymbol{y}_2 - \boldsymbol{x}_2)(2\pi)^2\delta(\boldsymbol{x}_1 - \boldsymbol{y}_1)$, so that also $\sum\varphi_i = \sum\varphi_i'$. Together, this yields

$$\Gamma^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{1}{3}\Big[\Gamma^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2)+\Gamma^{(0)}(\boldsymbol{x}_2,\boldsymbol{x}_3)+\Gamma^{(0)}(\boldsymbol{x}_3,\boldsymbol{x}_1)\Big],\tag{23}$$

but since $\Gamma^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \Gamma^{(0)}(\boldsymbol{x}_2, \boldsymbol{x}_3) = \Gamma^{(0)}(\boldsymbol{x}_3, \boldsymbol{x}_1)$, the compatibility of Eqs. (7) and (21) has been shown.

Next, we want to calculate $b(\ell_1, \ell_2, \varphi)$ from Eq. (21), i.e. taking a further integration. For that purpose, we write $\varphi_1 = \mu + \psi/2$, $\varphi_2 = \mu - \psi/2$, where $\varphi_1 - \varphi_2 = \psi = \phi_3 - \pi$. Then,

$$\begin{aligned}\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1 &= \ell_1x_2\cos(\theta+\varphi/2-\mu+\psi/2)-\ell_2x_1\cos(\theta-\varphi/2-\mu-\psi/2)\\ &= -\ell_1x_2\sin\left[\mu'-(\varphi+\phi_3)/2\right]-\ell_2x_1\sin\left[\mu'+(\varphi+\phi_3)/2\right] = -A_3\sin(\mu'-\alpha_3),\end{aligned}\tag{24}$$

where we defined $\mu' = \mu - \theta$ in the second step, and A_3 and α_3 are given in Eq. (12). Writing the polar angle of $\boldsymbol{x}_3$ as $\varphi_3 = \mu + \bar\varphi$, one finds that $\sum\varphi_i - \sum\beta_i = 3\mu' + \bar\varphi - \bar\beta$, where

$$\cos 2\bar\varphi = \frac{-(x_1^2+x_2^2)\cos\phi_3 + 2x_1x_2}{|\boldsymbol{x}_1+\boldsymbol{x}_2|^2}; \quad \sin 2\bar\varphi = -\frac{(x_1^2-x_2^2)\sin\phi_3}{|\boldsymbol{x}_1+\boldsymbol{x}_2|^2}, \tag{25}$$

and $\bar\beta$ is given in Eq. (9). Taken together, Eq. (21) becomes

$$\begin{aligned} b(\ell_1,\ell_2,\varphi) &= \frac{1}{3}\mathrm{e}^{-2\mathrm{i}\bar\beta}\int_0^\infty \mathrm{d}x_1\, x_1 \int_0^\infty \mathrm{d}x_2\, x_2 \int_0^{2\pi}\mathrm{d}\phi_3\, \Gamma^{(0)}(x_1,x_2,\phi_3)\,\mathrm{e}^{2\mathrm{i}\bar\varphi}\int_0^{2\pi}\mathrm{d}\mu'\mathrm{e}^{6\mathrm{i}\mu'}\,\mathrm{e}^{-\mathrm{i}A_3\sin(\mu'-\alpha_3)} \\ &= \frac{2\pi}{3}\mathrm{e}^{-2\mathrm{i}\bar\beta}\int_0^\infty \mathrm{d}x_1\, x_1 \int_0^\infty \mathrm{d}x_2\, x_2 \int_0^{2\pi}\mathrm{d}\phi_3\, \Gamma^{(0)}(x_1,x_2,\phi_3)\,\mathrm{e}^{2\mathrm{i}\bar\varphi}\,\mathrm{e}^{6\mathrm{i}\alpha_3}\,\mathrm{J}_6(A_3). \end{aligned} \tag{26}$$

It is easy to see that $b(\ell_1,\ell_2,\varphi)$ as given by Eq. (26) is real: since $b^*(\ell_1,\ell_2,\varphi) = b^*(\ell_1,\ell_2,-\varphi)$ – see Eq. (5) – and $\Gamma^{(0)*}(x_1,x_2,\phi_3) = \Gamma^{(0)}(x_1,x_2,-\phi_3)$, taking the complex conjugate of Eq. (26) and simultaneously replacing $\varphi\to-\varphi$, and the integration variable $\phi_3\to-\phi_3$ yields the same expression as Eq. (26), $b^*(\ell_1,\ell_2,\varphi) = b(\ell_1,\ell_2,\varphi)$.

5.2. Bispectrum in terms of the other $\Gamma^{(i)}$

Next we calculate the bispectrum as a function of the other three natural components of the shear 3PCF, starting from

$$\begin{aligned} \langle\hat\kappa_1(\boldsymbol{\ell}_1)\,\hat\kappa_1(\boldsymbol{\ell}_2)\,\hat\kappa_1(\boldsymbol{\ell}_3)\rangle &= \langle\hat\kappa_1^*(-\boldsymbol{\ell}_1)\,\hat\kappa_1(\boldsymbol{\ell}_2)\,\hat\kappa_1(\boldsymbol{\ell}_3)\rangle \\ &= \mathrm{e}^{-2\mathrm{i}(-\beta_1+\beta_2+\beta_3)}\int\mathrm{d}^2X_1\int\mathrm{d}^2X_2\int\mathrm{d}^2X_3\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{X}_1+\boldsymbol{\ell}_2\cdot\boldsymbol{X}_2+\boldsymbol{\ell}_3\cdot\boldsymbol{X}_3)}\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}\left\langle\gamma^{(\mathrm{o})*}(\boldsymbol{X}_1)\gamma^{(\mathrm{o})}(\boldsymbol{X}_2)\gamma^{(\mathrm{o})}(\boldsymbol{X}_3)\right\rangle \end{aligned} \tag{27}$$

where we used

$$\hat\gamma^*(-\boldsymbol{\ell}) = \int\mathrm{d}^2X\,\mathrm{e}^{\mathrm{i}\boldsymbol{\ell}\cdot\boldsymbol{X}}\gamma^{(\mathrm{o})*}(\boldsymbol{X})\,\mathrm{e}^{-2\mathrm{i}\varphi}. \tag{28}$$

In complete analogy to Eq. (21), we split the right-hand side into three terms and make appropriate substitutions. From a comparison with Eq. (4) we then obtain

$$\begin{aligned} B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)+B(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)+B(\boldsymbol{\ell}_3,\boldsymbol{\ell}_1) &= \frac{1}{3}\mathrm{e}^{-2\mathrm{i}(-\beta_1+\beta_2+\beta_3)}\Bigg[\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)} \\ &\quad+\int\mathrm{d}^2x_2\int\mathrm{d}^2x_3\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_3-\boldsymbol{\ell}_3\cdot\boldsymbol{x}_1)}+\int\mathrm{d}^2x_1\int\mathrm{d}^2x_3\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_3\cdot\boldsymbol{x}_1-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_3)}\Bigg]\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}. \end{aligned} \tag{29}$$

Here, we wrote the 3PCF as $\left\langle\gamma^{(\mathrm{o})*}(\boldsymbol{X}_1)\gamma^{(\mathrm{o})}(\boldsymbol{X}_2)\gamma^{(\mathrm{o})}(\boldsymbol{X}_3)\right\rangle \equiv \Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2)$. When renaming the integration variables, we have to apply the transformation rules to the 3PCFs (see Paper I). For the second term, we perform the substitutions $\boldsymbol{x}_2\to\boldsymbol{x}_1, \boldsymbol{x}_3\to\boldsymbol{x}_2$, so that $\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\to\Gamma^{(1)}(\boldsymbol{x}_3,\boldsymbol{x}_1) = \Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2)$. The third term is transformed similarly, and we get

$$\begin{aligned} B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)+B(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)+B(\boldsymbol{\ell}_3,\boldsymbol{\ell}_1) &= \frac{1}{3}\mathrm{e}^{-2\mathrm{i}(-\beta_1+\beta_2+\beta_3)}\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\Big[\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)}\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\ &\quad+\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2-\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_3\cdot\boldsymbol{x}_1)}\Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2)+\mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_3\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\Gamma^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\Big]. \end{aligned} \tag{30}$$

Unfortunately, the three terms on the right-hand side are not equal, as was the case for Eq. (20). Therefore, we repeat the above procedure with $\left\langle\hat\kappa_1(\boldsymbol{\ell}_1)\hat\kappa_1^*(-\boldsymbol{\ell}_2)\hat\kappa_1(\boldsymbol{\ell}_3)\right\rangle$ and $\left\langle\hat\kappa_1(\boldsymbol{\ell}_1)\hat\kappa_1(\boldsymbol{\ell}_2)\hat\kappa_1^*(-\boldsymbol{\ell}_3)\right\rangle$. The two resulting equations are

$$\begin{aligned} B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)+B(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)+B(\boldsymbol{\ell}_3,\boldsymbol{\ell}_1) &= \frac{1}{3}\mathrm{e}^{-2\mathrm{i}(\beta_1-\beta_2+\beta_3)}\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\,\Big[\mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)}\Gamma^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\ &\quad+\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_3\cdot\boldsymbol{x}_1)}\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2)+\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2-\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_3\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\Big], \end{aligned} \tag{31}$$

and

$$\begin{aligned} B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)+B(\boldsymbol{\ell}_2,\boldsymbol{\ell}_3)+B(\boldsymbol{\ell}_3,\boldsymbol{\ell}_1) &= \frac{1}{3}\mathrm{e}^{-2\mathrm{i}(\beta_1+\beta_2-\beta_3)}\int\mathrm{d}^2x_1\int\mathrm{d}^2x_2\,\Big[\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2-\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)}\Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2) \\ &\quad+\mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_2\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_3\cdot\boldsymbol{x}_1)}\Gamma^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2)+\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}\mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_3\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_1\cdot\boldsymbol{x}_1)}\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\Big]. \end{aligned} \tag{32}$$

Now, we can sum Eqs. (30)–(32), after moving the β-phase factors to the left-hand side. Then, we indeed get three equal terms, therefore

$$B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2) = \frac{1}{3g(\beta_1,\beta_2,\beta_3)}\int \mathrm{d}^2x_1 \int \mathrm{d}^2x_2\; \mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)} \times\left[\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)}\Gamma^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_2) + \mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2+\varphi_3)}\Gamma^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) + \mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2-\varphi_3)}\Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2)\right] \tag{33}$$

with

$$g(\beta_1,\beta_2,\beta_3) = \mathrm{e}^{2\mathrm{i}(-\beta_1+\beta_2+\beta_3)} + \mathrm{e}^{2\mathrm{i}(\beta_1-\beta_2+\beta_3)} + \mathrm{e}^{2\mathrm{i}(\beta_1+\beta_2-\beta_3)}. \tag{34}$$

Equation (33) can be written as a function of $\Gamma^{(1)}$ only, again using the transformation properties of the 3PCFs $\Gamma^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Gamma^{(1)}(\boldsymbol{x}_2,-\boldsymbol{x}_1-\boldsymbol{x}_2)$ and $\Gamma^{(3)}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Gamma^{(1)}(-\boldsymbol{x}_1-\boldsymbol{x}_2,\boldsymbol{x}_1)$.

We use Eq. (17) and its counterparts for $\Gamma^{(2)}$ and $\Gamma^{(3)}$ and insert them into Eq. (33). The φ-phase factors cancel, so that one obtains

$$\begin{aligned} B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2) = {} & \frac{1}{3g(\beta_1,\beta_2,\beta_3)}\int \mathrm{d}^2x_1 \int \mathrm{d}^2x_2\; \mathrm{e}^{\mathrm{i}(\boldsymbol{\ell}_1\cdot\boldsymbol{x}_2-\boldsymbol{\ell}_2\cdot\boldsymbol{x}_1)} \int \frac{\mathrm{d}^2k_1}{(2\pi)^2}\int\frac{\mathrm{d}^2k_2}{(2\pi)^2}\, B(\boldsymbol{k}_1,\boldsymbol{k}_2) \\ & \times\Big[\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_1-\boldsymbol{k}_1\cdot\boldsymbol{x}_2)}\mathrm{e}^{2\mathrm{i}\lambda_{-++}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_2-\boldsymbol{k}_1\cdot\boldsymbol{x}_3)}\mathrm{e}^{2\mathrm{i}\lambda_{++-}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_3-\boldsymbol{k}_1\cdot\boldsymbol{x}_1)}\mathrm{e}^{2\mathrm{i}\lambda_{+-+}} \\ & + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_2-\boldsymbol{k}_1\cdot\boldsymbol{x}_3)}\mathrm{e}^{2\mathrm{i}\lambda_{-++}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_3-\boldsymbol{k}_1\cdot\boldsymbol{x}_1)}\mathrm{e}^{2\mathrm{i}\lambda_{++-}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_1-\boldsymbol{k}_1\cdot\boldsymbol{x}_2)}\mathrm{e}^{2\mathrm{i}\lambda_{+-+}} \\ & + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_3-\boldsymbol{k}_1\cdot\boldsymbol{x}_1)}\mathrm{e}^{2\mathrm{i}\lambda_{-++}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_1-\boldsymbol{k}_1\cdot\boldsymbol{x}_2)}\mathrm{e}^{2\mathrm{i}\lambda_{++-}} + \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2\cdot\boldsymbol{x}_2-\boldsymbol{k}_1\cdot\boldsymbol{x}_3)}\mathrm{e}^{2\mathrm{i}\lambda_{+-+}}\Big], \end{aligned} \tag{35}$$

with $\lambda_{\pm,\pm,\pm} \equiv \pm\lambda_1 \pm \lambda_2 \pm \lambda_3$, where λ_i is the polar angle of the vector $\boldsymbol{k}_i$. The x_1- and x_2-integrals can be performed and yield δ-functions. This makes the k_1- and k_2-integrals trivial, yielding $\boldsymbol{k}_i = \boldsymbol{\ell}_i$ and $\lambda_i = \beta_i$. All the phase exponentials add up to give $3g$, canceling the pre-factor in Eq. (35), leaving only $B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)$ on the right hand side and thus verifying Eq. (32).

As was the case for Eq. (26), one additional angular integral can be carried out. With

$$g(\beta_1,\beta_2,\beta_3) = \mathrm{e}^{2\mathrm{i}\theta}\left[\mathrm{e}^{2\mathrm{i}(\bar\beta-\varphi)} + \mathrm{e}^{2\mathrm{i}(\bar\beta+\varphi)} + \mathrm{e}^{2\mathrm{i}(-\bar\beta)}\right] \equiv \mathrm{e}^{2\mathrm{i}\theta}\bar g, \tag{36}$$

we find for the three terms:

$$g^{-1}\mathrm{e}^{2\mathrm{i}(-\varphi_1+\varphi_2+\varphi_3)} = \bar g^{-1}\mathrm{e}^{2\mathrm{i}(\mu'+\bar\varphi-\phi_3)};\quad g^{-1}\mathrm{e}^{2\mathrm{i}(\varphi_1-\varphi_2+\varphi_3)} = \bar g^{-1}\mathrm{e}^{2\mathrm{i}(\mu'+\bar\varphi+\phi_3)};\quad g^{-1}\mathrm{e}^{2\mathrm{i}(\varphi_1+\varphi_2-\varphi_3)} = \bar g^{-1}\mathrm{e}^{2\mathrm{i}(\mu'-\bar\varphi)}. \tag{37}$$

In all three cases, we get the integral $\int_0^{2\pi}\mathrm{d}\mu'\exp(2\mathrm{i}\mu')\exp\left[-\mathrm{i}A_3\sin(\mu'-\alpha_3)\right] = 2\pi\exp(2\mathrm{i}\alpha_3)\mathrm{J}_2(A_3)$. Finally,

$$b(\ell_1,\ell_2,\varphi) = \frac{2\pi}{3\bar g}\int_0^\infty \mathrm{d}x_1\, x_1\int_0^\infty \mathrm{d}x_2\, x_2\int_0^{2\pi}\mathrm{d}\phi_3\; \mathrm{e}^{2\mathrm{i}\alpha_3}\,\mathrm{J}_2(A_3) \times\left[\mathrm{e}^{2\mathrm{i}(\bar\varphi-\phi_3)}\Gamma^{(1)}(x_1,x_2,\phi_3) + \mathrm{e}^{2\mathrm{i}(\bar\varphi+\phi_3)}\Gamma^{(2)}(x_1,x_2,\phi_3) + \mathrm{e}^{-2\mathrm{i}\bar\varphi}\Gamma^{(3)}(x_1,x_2,\phi_3)\right]. \tag{38}$$

5.3. Comments

After having seen the relations of the 3PCF in terms of the bispectrum, one is not surprised to find that their inversion derived in this section also is of considerable complexity. This can again be compared to the case of second-order statistics, where the power spectrum can be written in terms of the correlation function through an integration. The integration extends over all angular scales (as is also the case here), and so the direct inversion will always be of limited accuracy since the correlation functions can only be measured on a finite range of angular scales. In order to see the range of application of the previous relations, numerical simulations are probably required.

The foregoing equations also allow us in principle to express the 3PCF $\Gamma^{(1)}$ in terms of $\Gamma^{(0)}$, by using the expression (17) for $\Gamma^{(1)}$ and substituting the bispectrum in this equation by $\Gamma^{(0)}$, using Eq. (21). Whereas the corresponding equations can be reduced to a three-dimensional integral, they are fairly complicated; therefore, we shall not reproduce them here.

Given the measured correlation functions from a cosmic shear survey, there is no guarantee that the bispectrum estimates from Eqs. (26) and (38) will agree, even if we ignore noise and measurement errors. The two results will agree only if the shear field is derivable from an underlying convergence field, i.e., if the shear is a pure E-mode field. Significant differences between the two estimates would then signify that there is a B-mode contribution to the shear. In the case of second-order statistics, the separation between E- and B-modes is most conveniently done in terms of the aperture statistics (see Crittenden et al. 2002, hereafter CNPT); we shall therefore turn to the aperture measures of the third-order shear statistics in the next section.

At first sight, it may appear surprising that the expression (26) for the bispectrum is always real, even though we have not constrained $\Gamma^{(0)}$ to correspond to a pure E-mode field (in fact, we would not really be able to put this constraint on the 3PCF –

compare the 2PCF: only by combining the two correlation functions $\xi_\pm$ can one separate E- from B-modes). The only assumption we made was that the shear field is parity invariant. This can be understood as follows: We can describe a general shear field by the Fourier transform relation (1) if we formally replace the convergence by $\kappa(\boldsymbol{X}) = \kappa^{\rm E}(\boldsymbol{X}) + {\rm i}\kappa^{\rm B}(\boldsymbol{X})$, where $\kappa^{\rm E}$ gives rise to a pure E-mode shear field, and $\kappa^{\rm B}$ corresponds to a pure B-mode shear (SvWM). Considering the triple correlator of this complex κ, one finds that its real part consists of terms $\left\langle(\kappa^{\rm E})^3\right\rangle$ and $\left\langle\kappa^{\rm E}(\kappa^{\rm B})^2\right\rangle$, whereas the imaginary part has contributions $\left\langle(\kappa^{\rm E})^2\kappa^{\rm B}\right\rangle$ and $\left\langle(\kappa^{\rm B})^3\right\rangle$. As shown by Schneider (2003), the latter two terms are strictly zero for a parity-invariant shear field, so that the fact that Eq. (26) is real is fully consistent with the vanishing of the imaginary part of the triple correlator of the complex κ – both are due to the assumed parity invariance. This argument then also implies that the resulting expression (26) for b contains both E- and B-modes. One can separate E- and B-modes of the bispectrum by suitably combining the expressions (26) and (38). As we shall discuss in Sect. 7, the E-mode bispectrum is obtained by

$$b^{\rm E} = \left[b(\text{Eq. (26)}) + 3b(\text{Eq. (38)})\right]/4. \tag{39}$$

6. Aperture statistics

We have seen that it is possible to calculate the 3PCF in terms of the bispectrum, and in principle also to invert this relation. However, the resulting integrals are very cumbersome to evaluate numerically, owing to the various oscillating factors. It therefore would be useful to find some statistics that can be easily calculated in terms of the directly measurable 3PCF, but which can also be easily related to the bispectrum. In their very interesting paper, JBJ considered the aperture measures, which have been demonstrated to be very useful in the case of second-order statistics. The aperture mass centered on the origin of the coordinate system is defined as

$$M_{\rm ap}(\theta) = \int {\rm d}^2\vartheta\; U_\theta(|\boldsymbol{\vartheta}|)\,\kappa(\boldsymbol{\vartheta}) = \int {\rm d}^2\vartheta\; Q_\theta(|\boldsymbol{\vartheta}|)\,\gamma_{\rm t}(\boldsymbol{\vartheta}), \tag{40}$$

where $U_\theta(\vartheta)$ is a filter function of characteristic radius θ, the filter function

$$Q_\theta(\vartheta) = \frac{2}{\vartheta^2}\int_0^\vartheta {\rm d}\vartheta'\,\vartheta'\,U_\theta(\vartheta') - U_\theta(\vartheta) \tag{41}$$

is related to $U_\theta(\vartheta)$, and the second equality in Eq. (40) is true as long as U_θ is a compensated filter, i.e. $\int\vartheta\,\vartheta\,U_\theta(\vartheta) = 0$, as has been shown by Kaiser et al. (1994) and Schneider (1996). $\gamma_{\rm t}$ is the shear component tangent to the center of the aperture, i.e., the origin. Hence, $\gamma_{\rm t}(\boldsymbol{\vartheta}) + {\rm i}\gamma_\times(\boldsymbol{\vartheta}) = -\gamma\,\breve{\vartheta}^{*2}/|\boldsymbol{\vartheta}|^2$, where here and in the following we use the notation that a vector $\boldsymbol{x} = (x_1, x_2)$ can also be represented by a complex number $\breve{x} = x_1 + {\rm i}x_2$. Hence, $\breve{\vartheta}^{*2}/|\boldsymbol{\vartheta}|^2$ is nothing but the phase factor ${\rm e}^{-2{\rm i}\phi}$, where ϕ is the polar angle of $\boldsymbol{\vartheta}$.

The aperture mass as a statistics for cosmic shear was introduced by SvWJK who showed that the dispersion $\left\langle M_{\rm ap}^2(\theta)\right\rangle$ of the aperture mass is given as the integral over the power spectrum of the projected mass density κ, convolved with a filter function which is the square of the Fourier transform of U_θ. SvWJK derived this filter function for a family of functions U_θ which have a finite support. These filter functions turn out to be quite narrow, so that $\left\langle M_{\rm ap}^2(\theta)\right\rangle$ provides very localized information about the power spectrum (see also Bartelmann & Schneider 1999). Furthermore, SvWJK calculated the skewness of $M_{\rm ap}(\theta)$ in the frame of second-order perturbation theory for the growth of structure. As it turned out, the resulting equations are quite cumbersome, which is in part related to the fact that the Fourier transform of the functions U_θ chosen contains a Bessel function.

CNPT suggested an alternative form of the function U_θ. When we write $U_\theta(\vartheta) = \theta^{-2}\,u(\vartheta/\theta)$, then the filter used by CNPT is

$$u(x) = \frac{1}{2\pi}\left(1 - \frac{x^2}{2}\right){\rm e}^{-x^2/2};\quad \hat{u}(\eta) = \int {\rm d}^2x\; u(|\boldsymbol{x}|)\,{\rm e}^{-{\rm i}\boldsymbol{\eta}\cdot\boldsymbol{x}} = \frac{\eta^2}{2}\,{\rm e}^{-\eta^2/2};\quad Q_\theta(\vartheta) = \frac{\vartheta^2}{4\pi\theta^4}\exp\left(-\frac{\vartheta^2}{2\theta^2}\right)\cdot \tag{42}$$

Hence, this filter function does not have finite support; this is, however, only a small disadvantage for employing it since it cuts off very quickly for distances larger than a few θ. This disadvantage is more than compensated by the convenient analytic properties of this filter.

A further advantage of using aperture measures is that $M_{\rm ap}$, as calculated from the rightmost expression in (40), is sensitive only to an E-mode shear field (see CNPT and SvWM for a discussion of the E/B-mode decomposition of shear fields). Hence, defining the complex number

$$M(\theta) := M_{\rm ap}(\theta) + {\rm i}M_\perp(\theta) = \int {\rm d}^2\vartheta\; Q_\theta(|\boldsymbol{\vartheta}|)\left[\gamma_{\rm t}(\boldsymbol{\vartheta}) + {\rm i}\gamma_\times(\boldsymbol{\vartheta})\right] = -\int {\rm d}^2\vartheta\; Q_\theta(|\boldsymbol{\vartheta}|)\gamma(\boldsymbol{\vartheta})\,\breve{\vartheta}^{*2}/|\boldsymbol{\vartheta}|^2, \tag{43}$$

$M_{\rm ap}(\theta)$ vanishes identically for B-modes, whereas $M_\perp(\theta)$ yields zero for a pure E-mode field. Thus, the aperture measures are ideally suited to separating E- and B-modes of the shear.

CNPT and SvWM have shown that the dispersions $\left\langle M_{\rm ap}^2(\theta)\right\rangle$ and $\left\langle M_\perp^2(\theta)\right\rangle$ can be expressed as an integral over the two-point correlation functions of the shear. Since the correlation functions are the best measured statistics on real data (as they are

insensitive to the gaps and holes in the data field), this property allows an easy calculation of the aperture dispersions from the data. JBJ showed that the third-order moments of the aperture measures can likewise be expressed by the shear 3PCF, and they derived the corresponding relations explicitly – they are remarkably simple. The fact that such explicit results can be obtained is tightly related to the choice of the filter function (42); for a filter function with strictly finite support, the resulting expressions are very messy (indeed, we have derived such an expression for the filter function used in SvWJK, but it is so complicated that it will most likely be useless for any practical work).

6.1. An alternative derivation of the third-order aperture mass

We shall here rederive one of the results from JBJ making use of the results obtained in Sect. 5; the agreement of the resulting expression with that of JBJ provides a convenient check for the correctness of the results in Sect. 5. In a first step, we express $\left\langle M_{\rm ap}^3(\theta)\right\rangle$ in terms of the bispectrum. Using the first definition in Eq. (40), we find that

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = \int {\rm d}^2\vartheta_1\, U_\theta(|\boldsymbol{\vartheta}_1|) \int {\rm d}^2\vartheta_2\, U_\theta(|\boldsymbol{\vartheta}_2|) \int {\rm d}^2\vartheta_3\, U_\theta(|\boldsymbol{\vartheta}_3|) \times \int \frac{{\rm d}^2\ell_1}{(2\pi)^2} \int \frac{{\rm d}^2\ell_2}{(2\pi)^2} \int \frac{{\rm d}^2\ell_3}{(2\pi)^2} {\rm e}^{-{\rm i}(\boldsymbol{\ell}_1\cdot\boldsymbol{\vartheta}_1+\boldsymbol{\ell}_2\cdot\boldsymbol{\vartheta}_2+\boldsymbol{\ell}_3\cdot\boldsymbol{\vartheta}_3)} \langle \hat\kappa(\boldsymbol{\ell}_1)\,\hat\kappa(\boldsymbol{\ell}_2)\,\hat\kappa(\boldsymbol{\ell}_3)\rangle\,. \tag{44}$$

When carrying out the ϑ_i-integrations, the Fourier transforms $\hat U_\theta$ are obtained. Inserting the bispectrum in the form Eq. (4) and integrating out the corresponding delta function, one obtains three identical terms. With $\hat U_\theta(\ell) = \hat u(\theta\ell)$ one finds

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = 3\int \frac{{\rm d}^2\ell_1}{(2\pi)^2} \int \frac{{\rm d}^2\ell_2}{(2\pi)^2}\, B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)\; \hat u(\theta|\boldsymbol{\ell}_1|)\,\hat u(\theta|\boldsymbol{\ell}_2|)\,\hat u(\theta|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|) \tag{45}$$

$$= \frac{3}{(2\pi)^3}\int {\rm d}\ell_1\,\ell_1 \int {\rm d}\ell_2\,\ell_2 \int {\rm d}\varphi\; b(\ell_1,\ell_2,\varphi)\;\hat u(\theta\ell_1)\;\hat u(\theta\ell_2)\;\hat u\left(\theta\sqrt{\ell_1^2+\ell_2^2+2\ell_1\ell_2\cos\varphi}\right). \tag{46}$$

We shall discuss this result in the next subsection; here, we want to use Eq. (45) and obtain an explicit equation for $\left\langle M_{\rm ap}^3\right\rangle$ in terms of the 3PCF of the shear. For better comparison with JBJ, we shall slightly change our notation for the 3PCF. Up to now we have labeled the sides of the triangle formed by the three points $\boldsymbol{X}_i$ by the vectors $\boldsymbol{x}_i$ as defined in Sect. 2. The corresponding natural components of the 3PCF were then denoted by $\Gamma^{(n)}(x_1,x_2,x_3) = \Gamma^{(n)}(\boldsymbol{x}_1,\boldsymbol{x}_2)$. We shall now define the three points $\boldsymbol{X}_i$ in the form $\boldsymbol{X}_1 = \boldsymbol{X}_3+\boldsymbol{y}_1$, $\boldsymbol{X}_2 = \boldsymbol{X}_3+\boldsymbol{y}_2$, and then define

$$\left\langle \gamma^{(\rm x)}(\boldsymbol{X}_1)\,\gamma^{(\rm x)}(\boldsymbol{X}_2)\,\gamma^{(\rm x)}(\boldsymbol{X}_3)\right\rangle = \tilde\Gamma_{\rm x}^{(0)}(\boldsymbol{y}_1,\boldsymbol{y}_2);\quad \left\langle \gamma^{(\rm x)}(\boldsymbol{X}_1)\,\gamma^{(\rm x)}(\boldsymbol{X}_2)\,\gamma^{(\rm x)*}(\boldsymbol{X}_3)\right\rangle = \tilde\Gamma_{\rm x}^{(3)}(\boldsymbol{y}_1,\boldsymbol{y}_2), \tag{47}$$

where the "x" denotes an arbitrary projection of the shear components, i.e. relative to an arbitrary choice of reference directions. The relation between the Γ and the $\tilde\Gamma$ follows simply from the definitions of the separation vectors between the points $\boldsymbol{X}_i$, which is $\boldsymbol{y}_1 = \boldsymbol{x}_2$, $\boldsymbol{y}_2 = -\boldsymbol{x}_1$, so that

$$\tilde\Gamma_{\rm x}^{(0)}(\boldsymbol{y}_1,\boldsymbol{y}_2) = \tilde\Gamma_{\rm x}^{(0)}(\boldsymbol{x}_2,-\boldsymbol{x}_1) = \Gamma_{\rm x}^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_2), \tag{48}$$

and analogously for the other components of the 3PCF. Now, from combining Eqs. (21) with (45) and using our new notation for the 3PCF, one finds

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = \int {\rm d}^2y_1 \int {\rm d}^2y_2\; \tilde\Gamma_{\rm cart}^{(0)}(\boldsymbol{y}_1,\boldsymbol{y}_2) \int \frac{{\rm d}^2\ell_1}{(2\pi)^2} \int \frac{{\rm d}^2\ell_2}{(2\pi)^2}\; \hat u(\theta|\boldsymbol{\ell}_1|)\,\hat u(\theta|\boldsymbol{\ell}_2|)\,\hat u(\theta|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|)\,{\rm e}^{-2{\rm i}\sum\beta_i}\,{\rm e}^{{\rm i}(\boldsymbol{\ell}_1\cdot\boldsymbol{y}_1+\boldsymbol{\ell}_2\cdot\boldsymbol{y}_2)}, \tag{49}$$

where the subscript "cart" denotes the Cartesian components of the 3PCF. Inserting the Fourier transforms of u from (42) and using $\hat u(\theta|\boldsymbol{\ell}_i|)\,{\rm e}^{-2{\rm i}\beta_i} = \hat u(\theta|\boldsymbol{\ell}_i|)\breve\ell_i^{*2}/|\breve\ell_i|^2 = (\theta^2\breve\ell_i^{*2}/2)\,{\rm e}^{-|\breve\ell_i|^2\theta^2/2}$ (where we used, as before, the notation $\breve\ell = \ell_1+{\rm i}\ell_2$), (49) becomes

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = \int {\rm d}^2y_1 \int {\rm d}^2y_2\; \tilde\Gamma_{\rm cart}^{(0)}(\boldsymbol{y}_1,\boldsymbol{y}_2) \int \frac{{\rm d}^2\ell_1}{(2\pi)^2} \int \frac{{\rm d}^2\ell_2}{(2\pi)^2}\; \breve\ell_1^{*2}\,\breve\ell_2^{*2}\,(\breve\ell_1^*+\breve\ell_2^*)^2 \times \exp\left[-\frac{\left(|\breve\ell_1|^2+|\breve\ell_2|^2+|\breve\ell_1+\breve\ell_2|^2\right)\theta^2}{2} + {\rm i}\,(\boldsymbol{y}_1\cdot\boldsymbol{\ell}_1+\boldsymbol{y}_2\cdot\boldsymbol{\ell}_2)\right]. \tag{50}$$

The ℓ_i-integrations can now be carried out, by noting that the exponential is just a quadratic function of the integration variables, and it is multiplied by a polynomial. The straightforward, but tedious calculation has been carried out with Mathematica (Wolfram 1999); the result of the integration then depends on the $\boldsymbol{y}_i$. Substituting the $\boldsymbol{y}_i$ in favor of the vectors

$$\boldsymbol{q}_1 = \frac{2\boldsymbol{y}_1-\boldsymbol{y}_2}{3};\quad \boldsymbol{q}_2 = \frac{2\boldsymbol{y}_2-\boldsymbol{y}_1}{3};\quad \boldsymbol{q}_3 = -\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{3}, \tag{51}$$

which are the vectors connecting the center of mass of the triangle with its three corners, one can put the result in the form

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = -\frac{1}{24(2\pi)^2}\int\frac{{\rm d}^2y_1}{\theta^2}\int\frac{{\rm d}^2y_2}{\theta^2}\,\tilde\Gamma^{(0)}_{\rm cart}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,\frac{\breve q_1^{*2}\,\breve q_2^{*2}\,\breve q_3^{*2}}{\theta^6}\,\exp\!\left(-\frac{|\breve q_1|^2+|\breve q_2|^2+|\breve q_3|^2}{2\theta^2}\right). \tag{52}$$

Employing now the transformation laws of the natural components of the 3PCF as derived in Paper I, one sees that the squares of the complex conjugates of the q_i can be used to obtain the 3PCF with the shear projected along the direction towards the center of mass of the triangle, i.e.,

$$\tilde\Gamma^{(0)}_{\rm cart}(\boldsymbol{y}_1,\boldsymbol{y}_2)\breve q_1^{*2}\,\breve q_2^{*2}\,\breve q_3^{*2} = -\tilde\Gamma^{(0)}_{\rm cen}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,|\breve q_1|^2\,|\breve q_2|^2\,|\breve q_3|^2. \tag{53}$$

After this projection of the 3PCF, the integrand depends only on the absolute values of the $\boldsymbol{y}_1$ and the angle ψ between them. By carrying out one more integration, one finally obtains

$$\left\langle M_{\rm ap}^3(\theta)\right\rangle = \frac{1}{24}\int\frac{{\rm d}y_1\,y_1}{\theta^2}\int\frac{{\rm d}y_2\,y_2}{\theta^2}\int\frac{{\rm d}\psi}{(2\pi)}\,\tilde\Gamma^{(0)}_{\rm cen}(y_1,y_2,\psi)\,\frac{|\breve q_1|^2\,|\breve q_2|^2\,|\breve q_3|^2}{\theta^6}\,\exp\!\left(-\frac{|\breve q_1|^2+|\breve q_2|^2+|\breve q_3|^2}{2\theta^2}\right), \tag{54}$$

with

$$|\breve q_1|^2 = \frac{4y_1^2-4y_1y_2\cos\psi+y_2^2}{9};\quad |\breve q_2|^2 = \frac{y_1^2-4y_1y_2\cos\psi+4y_2^2}{9};\quad |\breve q_3|^2 = \frac{y_1^2+2y_1y_2\cos\psi+y_2^2}{9}, \tag{55}$$

and thus

$$|\breve q_1|^2+|\breve q_2|^2+|\breve q_3|^2 = \frac{2}{3}\,\left(y_1^2+y_2^2-y_1y_2\cos\psi\right). \tag{56}$$

The result (54) agrees with Eq. (44) of JBJ, after the projection of the 3PCF has been accounted for. It should be noted that in the case considered here, where the 3PCF was explicitly obtained in terms of the bispectrum of the convergence which, owing to parity invariance, is real (cf. the discussion in Sect. 5.3), the expression (46) is real, and hence Eq. (54) also is real in this case. This can be seen explicitly, since the value of $\tilde\Gamma^{(0)}_{\rm cen}$ at ψ is just the complex conjugate one of that at $-\psi$.

We like to point out that this derivation has shown two interesting aspects: first, the natural component $\Gamma^{(0)}$ of the shear 3PCF arises naturally in this context, confirming the hypothesis of Paper I that this combination of components of the correlation functions is indeed useful. Second, the derivation shows that the projection of the shear onto the centroid is the most convenient projection in this particular application.

6.2. Generalization: Third-order aperture statistics with different filter radii

How important is the third-order aperture statistics for investigating the third-order statistical properties of the cosmic shear? In order to discuss this question, we shall first consider the analogous situation for the second-order statistics. There, as mentioned before, the aperture mass dispersion is a filtered version of the power spectrum $P_\kappa(\ell)$ of the underlying convergence; for the function U_θ considered here, one has

$$\left\langle M_{\rm ap}^2(\theta)\right\rangle = \int\frac{{\rm d}\ell\,\ell}{(2\pi)}\,P_\kappa(\ell)\,\frac{\theta^4\,\ell^4}{4}\,{\rm e}^{-\theta^2\ell^2}; \tag{57}$$

hence, the filter function relating $P_\kappa(\ell)$ and $\left\langle M_{\rm ap}^2(\theta)\right\rangle$ is very narrow, and unless the power spectrum exhibits sharp features, the function $\left\langle M_{\rm ap}^2(\theta)\right\rangle$ contains basically all the information available for second-order shear statistics (not quite – see below). The analogous equation to Eq. (57) for third-order statistics is given in Eq. (46). The function $\hat u$ is very narrowly peaked at around $\ell\theta\sim 1$, and there is one factor of $\hat u$ for each of the three sides of a triangle in ℓ-space. This implies that in the integration of Eq. (46) the bispectrum is probed only in regions of ℓ-space where $\ell_1\sim\ell_2\sim|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|\sim 1/\theta$. Thus, $\left\langle M_{\rm ap}^3(\theta)\right\rangle$ probes the bispectrum essentially only for equilateral triangles in Fourier space. For this reason, the function $\left\langle M_{\rm ap}^3(\theta)\right\rangle$ cannot carry the full information of the bispectrum; it merely yields part of this information.

On the other hand, (46) immediately suggests how to improve this situation: if we define the aperture mass statistics with three different filter radii θ_i, we can probe the bispectrum at wavevectors whose lengths are $\ell_i\sim 1/\theta_i$, and by covering a wide range of θ_i, one can essentially probe the bispectrum over the full ℓ-space. Indeed,

$$\begin{aligned}\left\langle M_{\rm ap}(\theta_1)M_{\rm ap}(\theta_2)M_{\rm ap}(\theta_3)\right\rangle &= \int\frac{{\rm d}^2\ell_1}{(2\pi)^2}\int\frac{{\rm d}^2\ell_2}{(2\pi)^2}\,B(\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)\,\Big[\,\hat u(\theta_1|\boldsymbol{\ell}_1|)\,\hat u(\theta_2|\boldsymbol{\ell}_2|)\,\hat u(\theta_3|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|)\\ &\quad+\hat u(\theta_2|\boldsymbol{\ell}_1|)\,\hat u(\theta_3|\boldsymbol{\ell}_2|)\,\hat u(\theta_1|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|)+\hat u(\theta_3|\boldsymbol{\ell}_1|)\,\hat u(\theta_1|\boldsymbol{\ell}_2|)\,\hat u(\theta_2|\boldsymbol{\ell}_1+\boldsymbol{\ell}_2|)\,\Big]\\ &= \frac{1}{(2\pi)^3}\int{\rm d}\ell_1\ell_1\int{\rm d}\ell_2\ell_2\int{\rm d}\varphi\;b(\ell_1,\ell_2,\varphi)\left[\hat u(\theta_1\ell_1)\,\hat u(\theta_2\ell_2)\,\hat u\left(\theta_3\sqrt{\ell_1^2+\ell_2^2+2\ell_1\ell_2\cos\varphi}\right)+2\text{ terms}\right],\end{aligned} \tag{58}$$

which illustrates what was said above. Thus, this third-order statistics is expected to be as important for the third-order shear statistics as is the aperture mass dispersion for second-order shear statistics. The fact that we do not gain additional information by considering different filter scales for the second-order $M_{\rm ap}$ statistics follows from the fact that

$$\left\langle M_{\rm ap}(\theta_1)M_{\rm ap}(\theta_2)\right\rangle = \int \frac{{\rm d}\ell\,\ell}{(2\pi)}\,P_\kappa(\ell)\,\frac{\theta_1^2\theta_2^2\,\ell^4}{4}\,{\rm e}^{-(\theta_1^2+\theta_2^2)\ell^2/2} = \frac{4\theta_1^2\theta_2^2}{\left(\theta_1^2+\theta_2^2\right)^2}\left\langle M_{\rm ap}^2\left(\sqrt{\frac{\theta_1^2+\theta_2^2}{2}}\right)\right\rangle. \tag{59}$$

Whereas the fact that the mixed correlator can be expressed exactly in terms of the dispersion at an average angle depends on the special filter function considered here, it nevertheless shows that one does not gain additional information when considering the covariance of $M_{\rm ap}$.

We shall now calculate the triple correlator of $M_{\rm ap}$ for three different filter radii in terms of the shear 3PCF, essentially using the same method as JBJ. For that, we first calculate the third-order statistics of the complex aperture measure M, as defined in Eq. (43):

$$\langle M(\theta_1)M(\theta_2)M(\theta_3)\rangle \equiv \left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3) = -\int {\rm d}^2X_1\int {\rm d}^2X_2\int {\rm d}^2X_3\;Q_{\theta_1}(|\boldsymbol{X}_1|)Q_{\theta_2}(|\boldsymbol{X}_2|)Q_{\theta_3}(|\boldsymbol{X}_3|)$$
$$\times\langle\gamma(\boldsymbol{X}_1)\gamma(\boldsymbol{X}_2)\gamma(\boldsymbol{X}_3)\rangle\;{\rm e}^{-2{\rm i}(\phi_1+\phi_2+\phi_3)}, \tag{60}$$

where the ϕ_i are the polar angles of the vectors $\boldsymbol{X}_i$. Writing, as before, $\boldsymbol{X}_1 = \boldsymbol{X}_3+\boldsymbol{y}_1$, $\boldsymbol{X}_2 = \boldsymbol{X}_3+\boldsymbol{y}_2$, replacing the phase factors by ${\rm e}^{-2{\rm i}\phi_i} = \check{X}_i^{*2}/|\check{X}_i|^2$, and inserting the definitions of the Q_θ, one obtains

$$\left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3) = \frac{-1}{(4\pi)^3\theta_1^4\theta_2^4\theta_3^4}\int {\rm d}^2y_1\int {\rm d}^2y_2\;\tilde\Gamma^{(0)}_{\rm cart}(\boldsymbol{y}_1,\boldsymbol{y}_2)$$
$$\times\int {\rm d}^2Y\,\check{Y}^{*2}(\check{Y}^*+\check{y}_1^*)^2(\check{Y}^*+\check{y}_2^*)^2\exp\left[-\left(\frac{|\boldsymbol{Y}+\boldsymbol{y}_1|^2}{2\theta_1^2}+\frac{|\boldsymbol{Y}+\boldsymbol{y}_2|^2}{2\theta_2^2}+\frac{|\boldsymbol{Y}|^2}{2\theta_3^2}\right)\right], \tag{61}$$

where we set for ease of notation the dummy variable $\boldsymbol{X}_3\equiv\boldsymbol{Y}$. The Y-integration is again over the exponential of a second-order polynomial in the integration variable, times a polynomial, and thus straightforward to integrate, but tedious. Employing Mathematica does most of the job, though its output needed to be further simplified. The result is

$$\left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3) = \frac{S}{24}\int\frac{{\rm d}y_1\,y_1}{\Theta^2}\int\frac{{\rm d}y_2\,y_2}{\Theta^2}\int_0^{2\pi}\frac{{\rm d}\psi}{(2\pi)}\,\tilde\Gamma^{(0)}_{\rm cen}(y_1,y_2,\psi)\frac{|\check q_1|^2\,|\check q_2|^2\,|\check q_3|^2}{\Theta^6}\,f_1^{*2}\,f_2^{*2}\,f_3^{*2}\,{\rm e}^{-Z}, \tag{62}$$

where

$$\Theta^2 = \sqrt{\frac{\theta_1^2\theta_2^2+\theta_1^2\theta_3^2+\theta_2^2\theta_3^2}{3}};\quad S = \frac{\theta_1^2\theta_2^2\theta_3^2}{\Theta^6}, \tag{63}$$

$$Z = \frac{(-\theta_1^2+2\theta_2^2+2\theta_3^2)|\check q_1|^2+(2\theta_1^2-\theta_2^2+2\theta_3^2)|\check q_2|^2+(2\theta_1^2+2\theta_2^2-\theta_3^2)|\check q_3|^2}{6\Theta^4}, \tag{64}$$

$$f_1 = \frac{\theta_2^2+\theta_3^2}{2\Theta^2}+\frac{(\check q_2-\check q_3)\check q_1^*}{|\check q_1|^2}\,\frac{\theta_2^2-\theta_3^2}{6\,\Theta^2},\quad f_2 = \frac{\theta_1^2+\theta_3^2}{2\Theta^2}+\frac{(\check q_3-\check q_1)\check q_2^*}{|\check q_2|^2}\,\frac{\theta_3^2-\theta_1^2}{6\,\Theta^2},\quad f_3 = \frac{\theta_1^2+\theta_2^2}{2\Theta^2}+\frac{(\check q_1-\check q_2)\check q_3^*}{|\check q_3|^2}\,\frac{\theta_1^2-\theta_2^2}{6\,\Theta^2}. \tag{65}$$

The choice of the various quantities defined above was made such that S, Z, and the f_i are dimensionless, and that they become very simple if all θ_i are equal. Consider this special case next, i.e., let $\theta_1=\theta_2=\theta_3=\theta$. Then, $\Theta=\theta$, $S=1$, $f_1=f_2=f_3=1$, and $Z=\left(|\check q_1|^2+|\check q_2|^2+|\check q_3|^2\right)/(2\theta^2)$. Thus, we recover the result Eq. (54) in this case which was shown above to agree with the result from JBJ. The difference between Eqs. (54) and (62) is that the former has been derived in this paper from the bispectrum of the convergence, and therefore is strictly real, whereas Eq. (62) has been calculated directly in terms of the shear 3PCF and thus applies to arbitrary shear fields, containing both E- and B-modes. For reference, we explicitly give the combinations of the $\check q_i$ appearing in the f_i above,

$$\frac{(\check q_2-\check q_3)\check q_1^*}{|\check q_1|^2} = \frac{3y_2(2y_1\,{\rm e}^{{\rm i}\psi}-y_2)}{4y_1^2-4y_1y_2\cos\psi+y_2^2};\quad \frac{(\check q_3-\check q_1)\check q_2^*}{|\check q_2|^2} = \frac{3y_1(y_1-2y_2\,{\rm e}^{-{\rm i}\psi})}{y_1^2-4y_1y_2\cos\psi+4y_2^2};\quad \frac{(\check q_1-\check q_2)\check q_3^*}{|\check q_3|^2} = 3\,\frac{y_2^2-y_1^2+2{\rm i}y_1y_2\sin\psi}{y_1^2+2y_1y_2\cos\psi+y_2^2}.$$

One expects that $\left\langle M^3\right\rangle$ is symmetric with respect to any permutation of its arguments. Indeed, one can show explicitly that Eq. (62) is symmetric with respect to interchanging θ_1 and θ_2. Performing this interchange, changing the variables of integration as $y_1\to y_2$, $y_2\to y_1$, $\psi\to-\psi$, and making use of the fact that $\tilde\Gamma^{(0)}_{\rm cen}(y_1,y_2,\psi) = \tilde\Gamma^{(0)}_{\rm cen}(y_2,y_1,2\pi-\psi)$, one finds that these transformations lead to $f_1\to f_2$, $f_2\to f_1$, $f_3\to f_3$, and Z is unchanged. To show the symmetry with respect to even permutations

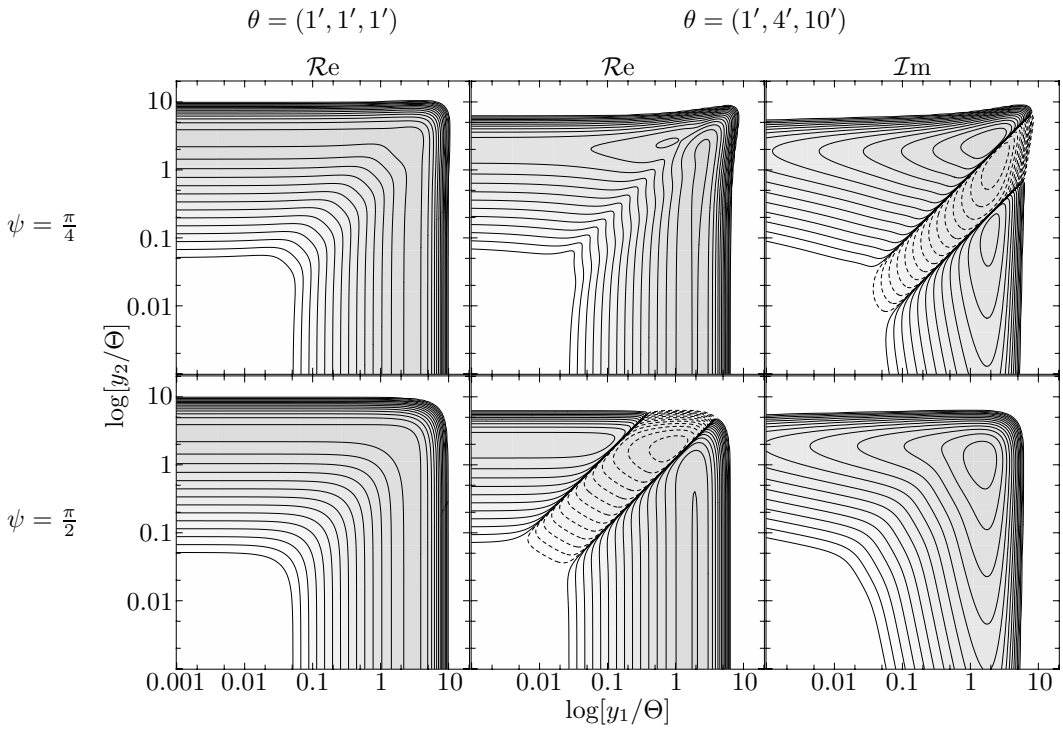

Fig. 2. Contours of the integration function $\frac{|\breve{q}_1|^2\,|\breve{q}_2|^2\,|\breve{q}_3|^2}{\Theta^6}\,f_1^{*2}\,f_2^{*2}\,f_3^{*2}\,\mathrm{e}^{-Z}$ – see (62) – as a function of y_1 and y_2 for fixed ψ (*upper row*: $\psi = \pi/4$, *lower row*: $\psi = \pi/2$). The left-most of the three columns represents the case where all three aperture radii are equal. The function scales with the aperture radius. Note that the imaginary part vanishes here because of symmetry. The two right columns show the real and imaginary part of the integrand for three different filter radii. The contour lines are logarithmically spaced with a factor of 5 between successive lines, starting with 10^{-10}. Dashed lines correspond to negative values.

of the arguments, one needs to employ the symmetry of $\Gamma^{(0)}(x_1, x_2, x_3) = \Gamma^{(0)}(x_2, x_3, x_1)$, and then use either X_1 or X_2 as the reference point in the derivation. This then leads to a cyclic permutation of the q_i and the f_i, and thus leaves Eq. (62) invariant.

In Fig. 2 we show the latter part of the integrand in Eq. (62) for the case of three equal apertures ($\theta_1 = \theta_2 = \theta_3$) and for different aperture sizes. Its zeros, if any, are lines of constant y_2/y_1, because the function only depends on the ratio of y_2 and y_1.

We next consider the combination of aperture measures

$$\langle M(\theta_1)M(\theta_2)M^*(\theta_3)\rangle \equiv \left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) = -\int \mathrm{d}^2X_1 \int \mathrm{d}^2X_2 \int \mathrm{d}^2X_3\; Q_{\theta_1}(|\boldsymbol{X}_1|)Q_{\theta_2}(|\boldsymbol{X}_2|)Q_{\theta_3}(|\boldsymbol{X}_3|) \times \langle\gamma(\boldsymbol{X}_1)\gamma(\boldsymbol{X}_2)\gamma^*(\boldsymbol{X}_3)\rangle\;\mathrm{e}^{-2\mathrm{i}(\phi_1+\phi_2-\phi_3)}, \tag{66}$$

where the semicolon in the arguments of $\left\langle M^2M^*\right\rangle$ indicates that this expression is symmetric with respect to interchanging the first two arguments, but not the third one, of course. Using the same conventions for labeling the vertices $\boldsymbol{X}_i$ of the triangle as before, we obtain

$$\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) = \frac{-1}{(4\pi)^3\theta_1^4\theta_2^4\theta_3^4}\int \mathrm{d}^2y_1 \int \mathrm{d}^2y_2\;\tilde{\Gamma}^{(3)}_{\mathrm{cart}}(\boldsymbol{y}_1,\boldsymbol{y}_2) \times \int \mathrm{d}^2Y\,\breve{Y}^2(\breve{Y}^*+\breve{y}_1^*)^2(\breve{Y}^*+\breve{y}_2^*)^2\exp\left[-\left(\frac{|\boldsymbol{Y}+\boldsymbol{y}_1|^2}{2\theta_1^2}+\frac{|\boldsymbol{Y}+\boldsymbol{y}_2|^2}{2\theta_2^2}+\frac{|\boldsymbol{Y}|^2}{2\theta_3^2}\right)\right]. \tag{67}$$

After performing the Y-integration and a few manipulations to express the $\boldsymbol{y}_i$ in terms of the $\boldsymbol{q}_i$, we obtain

$$\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) = -\frac{S}{(2\pi)^2}\int\frac{\mathrm{d}^2y_1}{\Theta^2}\int\frac{\mathrm{d}^2y_2}{\Theta^2}\;\tilde{\Gamma}^{(3)}_{\mathrm{cart}}(\boldsymbol{y}_1,\boldsymbol{y}_2)\,\mathrm{e}^{-Z} \times\left[\frac{1}{24}\,\frac{\breve{q}_1^{*2}\breve{q}_2^{*2}\breve{q}_3^2}{\Theta^6}\,f_1^{*2}f_2^{*2}f_3^2 - \frac{1}{9}\,\frac{\breve{q}_1^*\breve{q}_2^*|\breve{q}_3|^2}{\Theta^4}\,f_1^*f_2^*f_3\,g_3^* + \frac{1}{27}\left(\frac{\breve{q}_3^{*2}}{\Theta^2}\,g_3^{*2} + \frac{2\theta_1^2\theta_2^2}{\Theta^4}\,\frac{\breve{q}_1^*\breve{q}_2^*}{\Theta^2}\,f_1^*f_2^*\right)\right], \tag{68}$$

where we have defined

$$g_3 = \frac{\theta_1^2\theta_2^2}{\Theta^4} + \frac{(\breve{q}_1-\breve{q}_2)\breve{q}_3^*}{|\breve{q}_3|^2}\,\frac{\theta_3^2(\theta_2^2-\theta_1^2)}{3\,\Theta^4}, \tag{69}$$

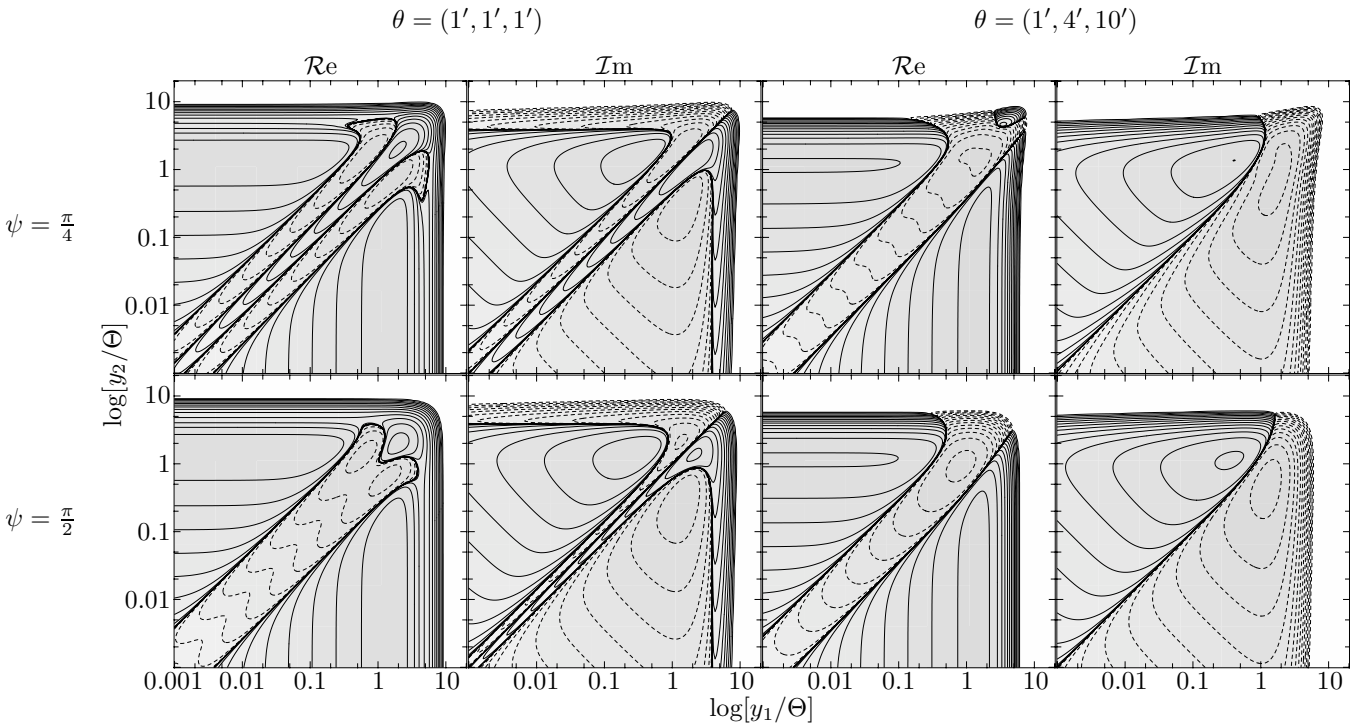

Fig. 3. Contours of the latter part of the integrand in Eq. (71). The plotted function is e^{-Z} times the term in square brackets. The upper and lower row correspond to fixed values of $\psi = \pi/4$ and $\pi/2$ respectively. In the left two columns, the real and imaginary part of the function is shown for the three aperture radii being equal. The right two panels correspond to three different radii. The contours are the same as in Fig. 2.

and the f_i are as before. This form of the equation is easily compared with the result obtained by JBJ, by setting $\theta_1 = \theta_2 = \theta_3 = \theta$, so that $\Theta = \theta$, $S = 1$, $f_1 = f_2 = f_3 = g_3 = 1$, and $Z = \left(|\breve{q}_1|^2 + |\breve{q}_2|^2 + |\breve{q}_3|^2\right)^2/2$. This then reproduces their Eq. (49), except for a different labeling of the q_i (we considered the complex conjugate shear at the point $\boldsymbol{X}_3$, whereas JBJ did this at $\boldsymbol{X}_1$).

We now employ again the relation between the natural components of the shear 3PCF in the Cartesian reference frame and those measured relative to the center of mass of the triangle (see Paper I),

$$\tilde{\Gamma}^{(3)}_{\rm cart}(\boldsymbol{y}_1, \boldsymbol{y}_2)\breve{q}_1^{*2}\,\breve{q}_2^{*2}\,\breve{q}_3^2 = -\tilde{\Gamma}^{(3)}_{\rm cen}(\boldsymbol{y}_1, \boldsymbol{y}_2)\,|\breve{q}_1|^2\,|\breve{q}_2|^2\,|\breve{q}_3|^2, \tag{70}$$

and make the corresponding replacements in Eq. (68), after which one more angular integration can be carried out, to obtain our final result

$$\begin{aligned}\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) = S\int\frac{\mathrm{d}y_1\,y_1}{\Theta^2}\int\frac{\mathrm{d}y_2\,y_2}{\Theta^2}\int_0^{2\pi}\frac{\mathrm{d}\psi}{(2\pi)}\,\tilde{\Gamma}^{(3)}_{\rm cen}(y_1,y_2,\psi)\,\mathrm{e}^{-Z}\Bigg[\frac{1}{24}\,\frac{|\breve{q}_1|^2|\breve{q}_2|^2|\breve{q}_3|^2}{\Theta^6}\,f_1^{*2}f_2^{*2}f_3^2\\ -\frac{1}{9}\,\frac{\breve{q}_1\breve{q}_2\breve{q}_3^{*2}}{\Theta^4}\,f_1^*f_2^*f_3\,g_3^* + \frac{1}{27}\left(\frac{\breve{q}_1^2\breve{q}_2^2\breve{q}_3^{*4}}{|\breve{q}_1|^2|\breve{q}_2|^2|\breve{q}_3|^2\Theta^2}\,g_3^{*2} + \frac{2\theta_1^2\theta_2^2}{\Theta^4}\,\frac{\breve{q}_1\breve{q}_2\breve{q}_3^{*2}}{|\breve{q}_3|^2\Theta^2}\,f_1^*f_2^*\right)\Bigg],\end{aligned} \tag{71}$$

which generalizes the result of JBJ for unequal aperture radii. The proof that this last expression is symmetric with respect to interchanging θ_1 and θ_2 is the same as the one given above. See Fig. 3 for an exemplary plot of the latter part of the integrand.

The product of the four q_i's can be written as follows,

$$\breve{q}_1\breve{q}_2\breve{q}_3^{*2} = \frac{1}{27}\left\{2\left[y_1^4 + y_2^4 + y_1^2y_2^2(2\cos 2\psi - 5)\right] - y_1y_2\left[(y_1^2 + y_2^2)\cos\psi + 9\,\mathrm{i}(\mathrm{y}_1^2 - \mathrm{y}_2^2)\sin\psi\right]\right\}. \tag{72}$$

From the two complex triple correlators $\left\langle M^3\right\rangle$ and $\left\langle M^2M^*\right\rangle$, we can now calculate the four real third-order aperture statistics, in analogy to what was done in JBJ,

$$\begin{aligned}\left\langle M_{\rm ap}^3\right\rangle(\theta_1,\theta_2,\theta_3) &= \mathcal{R}\mathrm{e}\left[\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) + \left\langle M^2M^*\right\rangle(\theta_1,\theta_3;\theta_2) + \left\langle M^2M^*\right\rangle(\theta_2,\theta_3;\theta_1) + \left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3)\right]/4,\\ \left\langle M_{\rm ap}^2M_\perp\right\rangle(\theta_1,\theta_2;\theta_3) &= \mathcal{I}\mathrm{m}\left[\left\langle M^2M^*\right\rangle(\theta_1,\theta_3;\theta_2) + \left\langle M^2M^*\right\rangle(\theta_2,\theta_3;\theta_1) - \left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) + \left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3)\right]/4,\\ \left\langle M_{\rm ap}M_\perp^2\right\rangle(\theta_1;\theta_2,\theta_3) &= \mathcal{R}\mathrm{e}\left[\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) + \left\langle M^2M^*\right\rangle(\theta_1,\theta_3;\theta_2) - \left\langle M^2M^*\right\rangle(\theta_2,\theta_3;\theta_1) - \left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3)\right]/4,\\ \left\langle M_\perp^3\right\rangle(\theta_1,\theta_2,\theta_3) &= \mathcal{I}\mathrm{m}\left[\left\langle M^2M^*\right\rangle(\theta_1,\theta_2;\theta_3) + \left\langle M^2M^*\right\rangle(\theta_1,\theta_3;\theta_2) + \left\langle M^2M^*\right\rangle(\theta_2,\theta_3;\theta_1) - \left\langle M^3\right\rangle(\theta_1,\theta_2,\theta_3)\right]/4,\end{aligned} \tag{73}$$

with the same notational convention as used before, e.g., $\left\langle M_{\rm ap}M_\perp^2\right\rangle(\theta_1;\theta_2,\theta_3) \equiv \left\langle M_{\rm ap}(\theta_1)M_\perp(\theta_2)M_\perp(\theta_3)\right\rangle$, which is symmetric in the last two arguments, as indicated by the semicolon. These four expressions have very different physical interpretations. A significant non-zero value of $\left\langle M_{\rm ap}^3\right\rangle$ indicates that the E-mode of the shear field corresponds to a convergence field κ which has significant skewness. This is the signal one wants to measure in future cosmic shear surveys, and this term contains the information about the underlying cosmic density field, and thus about cosmology. A significant non-zero value of $\left\langle M_{\rm ap}M_\perp^2\right\rangle$ indicates the presence of a B-mode in the shear field which is correlated with the E-mode. Although lensing can generate such a term with small amplitude, by higher-order lensing effects (caused by source clustering, violation of the Born approximation in studying light propagation in the Universe, or multiple light deflections – see SvWJK for a discussion of these latter effects), these are probably too small to be detectable. Therefore, a detection of a $\left\langle M_{\rm ap}M_\perp^2\right\rangle$ most likely will indicate the presence of a "shear" not coming from lensing, but from, e.g., intrinsic alignment of the galaxies (see, e.g., Catelan et al. 2000; Heavens et al. 2000; Crittenden et al. 2001; Croft & Metzler 2001; Jing 2002). A significant non-zero value of $\left\langle M_\perp^3\right\rangle$ indicates that the shear field violates parity invariance, as a B-mode shear cannot have odd moments if it is parity-symmetric (Schneider 2003). Finally, a significant non-zero value of $\left\langle M_{\rm ap}^2M_\perp\right\rangle$ indicates a parity invariance violation which is correlated with the E-mode shear field. Neither of these two latter terms can be explained by cosmic effects which are expected to by parity-invariant, but either indicates an underestimate of the statistical errors (coming from the intrinsic ellipticity distribution of the sources and from cosmic variance), or the presence of instrumental systematics or artifacts from data reduction (cf. the analogous situation for second-order statistics, where a non-zero value of $\left\langle M_{\rm ap}M_\perp\right\rangle$ would indicate significant systematics).

As stated above, measuring the third-order aperture statistics (through measuring the shear 3PCF and then using the foregoing relations) yields essentially all the information about the bispectrum, provided the latter has no sharp features in ℓ-space. The analogous statement for the second-order statistics is not really true: if one considers a cosmic shear survey consisting of several unrelated fields of size Φ each, one can calculate the aperture dispersion from the shear 2PCF for scales, say, $\theta \lesssim \Phi/4$. However, the cosmic shear field contains information about the power spectrum of the convergence from all scales; in particular, due to the fact that the shear 2PCF is obtained from the power spectrum through a filter function which tends to constant for $\theta\ell \to 0$, it contains information over the integrated power on large scales. Hence, the second-order aperture statistics do not recover the full information about the power spectrum contained in the shear 2PCF for a survey of a given size. In order to make better use of the shear data, one should take into account a shear measure which contains the large-scale power, such as the top-hat shear dispersion on an angular scale comparable to the size of the observed fields, say at $\theta = \Phi/4$, which can also be obtained in terms of the shear 2PCF (CNPT, SvWM). An analogous situation does not exist for the third-order statistics. This can be understood intuitively in the following way: consider again the survey geometry mentioned above, and assume that to each of the independent fields a constant shear is added, corresponding to very large-scale power and/or power in the bispectrum. The aperture measures will be unable to measure this constant shear, whereas the shear 2PCF will be sensitive to it, as will be the top-hat shear dispersion. However, since one cannot form a third-order shear statistics which contains the shear only, i.e., without reference directions (such as the direction to the centers of triangles), such a constant shear is expected to leave no trace on the shear 3PCF. This can be seen geometrically as follows: consider a triangle of points in a constant shear field. Rotation of this triangle by 90 degrees changes the sign of all shear components, and thus the triple product changes sign, for which reason a constant shear yields no shear 3PCF. This can also be seen algebraically from Eqs. (15) and (18): the occurrence of the Bessel functions, which behave like ℓ^6 and ℓ^2, respectively, for small ℓ (at fixed x_i) removes all large-scale contributions of the bispectrum in the 3PCF. This fact suggests that indeed the third-order aperture measures recover essentially all information about the bispectrum which is present in the shear field.

One might argue that the skewness of the convergence field, top-hat weighted in a circular aperture, is sensitive to long wavelength modes, and so third-order statistics on small scales knows about large scales. This is true, and may sound like a contradiction to what has been said above. Looking at the second-order statistics first, $\left\langle\kappa^2\right\rangle(\theta)$ is sensitive to the power spectrum on all scales $\ell \lesssim 2\pi/\theta$, and it can be expressed by the 2PCF ξ_+ on angular scales $\le 2\theta$ (CNPT, SvWM). This is due to the fact that the 2PCF of κ is the same as that of ξ_+. Indeed, if one expresses $\left\langle\kappa^2\right\rangle$ in terms of ξ_-, the resulting convolution kernel has infinite support; hence, $\left\langle\kappa^2\right\rangle$ cannot be expressed through ξ_- over a finite range, because ξ_- is *not* sensitive to the power spectrum on large scales. Something analogous happens for the three-point statistics. Whereas one can express $\left\langle\kappa^3\right\rangle$ in terms of the shear 3PCF (this in fact is easily done, e.g. by first expressing $\left\langle\kappa^3\right\rangle$ in terms of the bispectrum, and then replacing the bispectrum in terms of the shear 3PCF, using the relations in Sect. 5), the integration range is infinite. One cannot calculate $\left\langle\kappa^3\right\rangle$ from the shear over a finite region – in fact, the mass-sheet degeneracy does prevent this. Only on very large fields, where the mean of κ can be set to zero, can one in principle measure $\left\langle\kappa^3\right\rangle$, and that means, one needs information from much larger scales than the size of the aperture.

7. Summary and discussion

In this paper we have considered the relation between the 3PCF of the cosmic shear and the bispectrum of the underlying convergence field. Explicit expressions for the (natural components of the) shear 3PCF in terms of the bispectrum have been derived.

These expressions are fairly complicated, and their explicit numerical evaluation non-trivial. The transformation properties of the 3PCF under parity reversal can be directly studied using these explicit relations and confirm those derived in Paper I by geometrical reasoning. We have then inverted these relations, i.e., derived the bispectrum in terms of the shear 3PCF. Two different expressions were obtained, corresponding to the two types of natural 3PCF components: one the one hand $\Gamma^{(0)}$, and $\Gamma^{(i)}$, $i = 1, 2, 3$ on the other hand. If the shear is due to an underlying convergence field, these two expressions should yield the same result for the bispectrum; in general, however, if a B-mode contribution is present, these two results will differ. Drawing the analogy to the E/B-mode decomposition for the aperture measures in Sect. 6.2, we have conjectured a linear combination of the two expressions for the bispectrum which yields the E-mode only. The orthogonal linear combination then yields the cross-bispectrum of the E-mode with the square of the B-mode shear. The fact that the bispectrum is real, provided the 3PCF obeys parity invariance, reaffirms the result of Schneider (2003) that for a parity-symmetric field, all statistics with an odd power of B-modes have to vanish.

We have then turned to the aperture statistics, using the filter function that was suggested by CNPT and also used by JBJ. As a first step we have used the previously derived expressions for the bispectrum in terms of the 3PCF to rederive one of the results in JBJ. Then, by considering the third-order aperture statistics in terms of the underlying bispectrum we have argued that the third-order aperture statistics with a single filter radius probes the bispectrum only along a one-dimensional cut through its three-dimensional range of definition, namely that of equilateral triangles in ℓ-space. Generalizing the aperture statistics to three different filter radii, the full range of the bispectrum can be probed, and, in analogy to JBJ, we have derived explicit equations for the generalized third-order aperture statistics in terms of the directly measurable shear 3PCF. We showed that using different filter radii did not yield additional information in the case of second-order statistics.

The filter function used in the definition of the aperture measures was that suggested by CNPT. Whereas it does not strictly have finite support, this disadvantage compared to the filter function defined in SvWJK is outweighed by the convenient algebraic properties it has; these enabled the explicit derivation of fairly simple expressions.

Let us summarize the features of the aperture statistics which render them so useful as a quantity for characterizing cosmic shear (and other polar fields):

- The aperture measures can be directly calculated in terms of the shear correlation functions. It is the latter that can be measured best from a real cosmic shear survey, as they are not affected by the geometry of the survey and holes and gaps in the data field. The expressions of the aperture measures in terms of the shear correlation function are easy to evaluate by simple sums over the bins for which the correlation functions have been measured.
- The aperture measures provide very localized information about the underlying power spectrum (in the case of second-order statistics) and the bispectrum (for third-order statistics) and therefore contain essentially the full information about the properties of the underlying convergence field, unless its power in Fourier space has sharp features (which is not expected for a cosmological mass distribution, since there is no sharply defined characteristic length scale).
- One can easily calculate the aperture measures in terms of the power spectrum and the bispectrum, and hence their expected dependence on the cosmological parameters can be derived and compared to the measurements. Whereas the aperture measures are just one particular way to form integral measures of the shear correlation functions – a different integral measure was defined by Bernardeau et al. (2003) and applied to a cosmic shear survey in Bernardeau et al. (2002) – it is a particularly convenient one owing to its simple relation to the bispectrum.
- The aperture measures are the easiest way to separate E- and B-modes of the shear field. Essentially all E/B-mode decompositions for the second-order shear statistics have been performed using the aperture measures, and we expect that they will play the same role for the third-order statistics. Furthermore, since two of the four independent combinations (73) of the aperture measures are expected to vanish because of parity invariance, they provide a very convenient way to detect remaining systematics in the observing, data reduction and analysis process.
- The aperture statistics are also easily obtained from numerical ray-tracing simulations, since they are defined in terms of the underlying convergence in the first place. Hence, in these simulations one can work directly in terms of the convergence instead of the more complicated (due to the various components) shear field.

From the derivation of the 3PCF as a function of the bispectrum, it becomes clear that the definition of the natural components have eased the algebra considerably, compared to the case in which one would have tried to calculate its individual components (like $\gamma_{\rm tt\times}$). Furthermore, the derivation of the third-order aperture statistics directly requires the combination of the shear 3PCF provided by the natural components. As discussed in Paper I there are various ways to define the natural components of the shear 3PCF, corresponding to the different centers of a triangle. The derivation of the aperture measures in terms of the 3PCF has yielded the result that the projection with respect to the center-of-mass of a triangle is the most convenient definition (at least in this connection).

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