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Research note

Cosmic ray moderation of the thermal instability

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Abstract. We apply the Hermite-Bieler theorem in the analysis of the effect of cosmic rays on the thermal stability of an initially uniform, static background. The cosmic rays were treated in a fluid approximation and the diffusion coefficient was assumed to be constant in time and space. The inclusion of cosmic rays does not alter the criterion for the thermal stability of a medium subjected to isobaric perturbations. It does alter the criteria for the stability of a medium perturbed by small amplitude sound waves. In the limit of a high background cosmic ray pressure to thermal pressure ratio, the instability in response to high frequency sound waves is suppressed.

Key words. ISM: kinematics and dynamics – ISM: cosmic rays – hydrodynamics – instabilities

1. Introduction

Field (1965) deduced the criteria for the thermal stability of a uniform, non-magnetic, single-fluid medium. The interstellar medium's pressure has nonthermal contributions, including that due to cosmic rays, which modify the criteria for stability. Here, we investigate the effects of cosmic rays on the thermal stability of a uniform medium.

In Sect. 2, we give the basic one-dimensional equations governing the dynamics of a medium consisting of a thermal fluid and a cosmic ray fluid. Section 3 contains a description of the method based on the Hermite-Bieler theorem and the stability criteria derived for general conditions. Section 4 provides results for special limiting cases and Sect. 5 concludes the paper.

2. A one dimensional two-fluid description

To simplify the treatment of the cosmic rays, they are sometimes treated as a fluid (e.g. McKenzie & Völk 1982). The second velocity moment of the diffusion convection equation for cosmic rays gives

$$\frac{\partial P_{\rm c}}{\partial t} + u\frac{\partial P_{\rm c}}{\partial x} + \gamma_{\rm c} P_{\rm c}\frac{\partial u}{\partial x} - \chi\frac{\partial^2 P_{\rm c}}{\partial x^2} = 0 \quad (1)$$

(e.g. Drury & Falle 1986). x is the spatial coordinate, u is the mean speed of thermal particles, $\gamma_{\rm c}$ is an adiabatic constant equal to $4/3$ for ultrarelativistic cosmic rays, χ is a spatial diffusion coefficient, and $P_{\rm c}$ is the cosmic ray pressure. The thermal fluid obeys

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (2)$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial P_{\rm g}}{\partial x} + \frac{\partial P_{\rm c}}{\partial x} = 0 \quad (3)$$

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \frac{P_{\rm g}}{\gamma_{\rm g}-1}\right) + \frac{\partial}{\partial x}\left[\left(\frac{\rho u^2}{2} + \frac{P_{\rm g}}{\gamma_{\rm g}-1}\right)u\right] + \frac{\partial(uP_{\rm g})}{\partial x} + u\frac{\partial P_{\rm c}}{\partial x}$$
$$+\rho\mathcal{L}(\rho, T) = 0 \quad (4)$$

$$P_{\rm g} - \frac{\rho R T}{\mu} = 0. \quad (5)$$

ρ, $P_{\rm g}$, T, $\gamma_{\rm g}$, μ, R and $\rho\mathcal{L}$ are the mass density, pressure, temperature, ratio of specific heats, mean mass per particle, gas constant and net energy loss per unit volume per unit time, respectively of the thermal fluid. Equations (1) to (5) and an equation giving $\mathcal{L}$ as a function of ρ and T constitute a closed set.

3. The stability analysis

We assume that $\gamma_{\rm c}$, χ, $\gamma_{\rm g}$ and μ are constant and that the background values of $P_{\rm c}$, ρ, u, $P_{\rm g}$ and T are constant and are denoted by the subscript 0. $u_0 = 0$ and $\mathcal{L}(\rho_0, T_0) = 0$. $P_{\rm c1}$, ρ_1, u_1, $P_{\rm g1}$ and T_1 are small perturbations of the state variables and are assumed to vary as $\exp\big(\mathrm{i}(\omega t + kx)\big)$.

Substitution of the expressions for the state variables, appropriate for the assumptions stated in the preceding paragraph, into Eqs. (1) to (5) yields the dispersion relation

$$G(z) \equiv z^4 - \mathrm{i}z^3\left(\frac{k}{k_\mathrm{c}} + \frac{k_\mathrm{T}}{k}\right) - z^2\left(\frac{k_\mathrm{T}}{k_\mathrm{c}} + \phi + 1\right) + \mathrm{i}z\left(\frac{k_\mathrm{T}}{k}\phi + \frac{k}{k_\mathrm{c}} + \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g} k}\right) + \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g} k_\mathrm{c}} = 0. \tag{6}$$

Here

$$z \equiv \frac{\omega}{ak} \tag{7}$$

$$a \equiv \left(\frac{\gamma_\mathrm{g} P_\mathrm{g0}}{\rho_0}\right)^{\frac{1}{2}} \tag{8}$$

$$k_\mathrm{T} \equiv \frac{\mu(\gamma_\mathrm{g} - 1)}{Ra} \left.\frac{\partial \mathcal{L}}{\partial T}\right|_{T=T_0} \tag{9}$$

$$k_\rho \equiv \frac{\mu(\gamma_\mathrm{g} - 1)\rho_0}{R a T_0} \left.\frac{\partial \mathcal{L}}{\partial \rho}\right|_{\rho=\rho_0} \tag{10}$$

$$k_\mathrm{c} \equiv \frac{a}{\chi} \tag{11}$$

$$\phi \equiv \frac{\gamma_\mathrm{c} P_\mathrm{c0}}{\gamma_\mathrm{g} P_\mathrm{g0}}. \tag{12}$$

Like Tytarenko et al. (2002) we make use of the Hermite-Bieler theorem (e.g. Levin 1964, Chapter VII) in the analysis of the relevant dispersion relation. We write $G(z) = G_\mathrm{r}(z) + \mathrm{i}G_\mathrm{i}(z)$ where G_r and G_i are both real functions. The theorem implies that the physical system is stable if G_r and G_i have only simple real roots, between two successive roots of one of these two polynomials there lies exactly one root of the other, and at some point $z = z'$ on the real axis

$$\left.\frac{\mathrm{d}G_\mathrm{i}}{\mathrm{d}z}\right|_{z=z'} G_\mathrm{r}(z') - G_\mathrm{i}(z') \left.\frac{\mathrm{d}G_\mathrm{r}}{\mathrm{d}z}\right|_{z=z'} < 0. \tag{13}$$

The Hermite-Bieler theorem applied to the polynomial

$$E(z) = z^4 - \mathrm{i}az^3 - bz^2 + \mathrm{i}cz + d = 0 \tag{14}$$

gives the following stability conditions

$$d > 0 \tag{15}$$

$$\frac{c}{a} > \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - d} \tag{16}$$

$$\frac{c}{a} < \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - d}. \tag{17}$$

Direct comparison of Eq. (14) and the dispersion relation Eq. (6) leads us to conclude that three conditions must be satisfied for the system to be stable with respect to the types of perturbations we have specified. Those conditions are

$$\frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g} k_\mathrm{c}} > 0 \tag{18}$$

$$\frac{k_\mathrm{T}\phi + \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g}} + \frac{k^2}{k_\mathrm{c}}}{k_\mathrm{T} + \frac{k^2}{k_\mathrm{c}}} > \frac{1}{2}\left(\phi + \frac{k_\mathrm{T}}{k_\mathrm{c}} + 1\right) - \sqrt{\frac{1}{4}\left(\phi + \frac{k_\mathrm{T}}{k_\mathrm{c}} + 1\right)^2 - \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g} k_\mathrm{c}}} \tag{19}$$

$$\frac{k_\mathrm{T}\phi + \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g}} + \frac{k^2}{k_\mathrm{c}}}{k_\mathrm{T} + \frac{k^2}{k_\mathrm{c}}} < \frac{1}{2}\left(\phi + \frac{k_\mathrm{T}}{k_\mathrm{c}} + 1\right) + \sqrt{\frac{1}{4}\left(\phi + \frac{k_\mathrm{T}}{k_\mathrm{c}} + 1\right)^2 - \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g} k_\mathrm{c}}}. \tag{20}$$

Condition (18) is closely related to the condition for the stability of a medium containing no cosmic rays to be thermally stable with respect to isobaric perturbations; that condition for the one fluid medium is

$$k_\mathrm{T} - k_\rho > 0 \tag{21}$$

(Field 1965).

4. Limiting cases

Equations (19) and (20) are complicated for the general case but simplify in various limits. We first define

$$\Delta \equiv \frac{k_\mathrm{T} - k_\rho}{\gamma_\mathrm{g}}. \tag{22}$$

For the case in which $\phi \gg 1$, $\left|\frac{k_\mathrm{T}}{k_\mathrm{c}}\right|$, $\left|\frac{k_\rho}{k_\mathrm{c}}\right|$, $\left|\frac{k}{k_\mathrm{c}}\right|$ conditions (19) and (20) give

$$0 \lesssim \frac{k_\mathrm{T}}{k_\mathrm{T} + \frac{k^2}{k_\mathrm{c}}} \lesssim 1 \tag{23}$$

for stability. As k_c is always positive, this implies that when the cosmic ray pressure is very high and the perturbation is on a small scale relative to $\sqrt{|k_\mathrm{c} k_\mathrm{T}|}$, sound waves will not grow irrespective of the nature of $\mathcal{L}$.

In the case that $1 \gg \phi, \left|\frac{k_\mathrm{T}}{k_\mathrm{c}}\right|, \left|\frac{k_\rho}{k_\mathrm{c}}\right|, \left|\frac{k}{k_\mathrm{c}}\right|$ conditions (19) and (20) imply that sound waves are stable if

$$-\frac{\Delta k_\mathrm{T} k_\mathrm{c}}{k_\mathrm{T} - \Delta} \lesssim k^2 \lesssim k_\mathrm{T} k_\mathrm{c} \qquad \text{when } k_\mathrm{T} - \Delta > 0 \tag{24}$$

$$k_\mathrm{T} k_\mathrm{c} \lesssim k^2 \lesssim -\frac{\Delta k_\mathrm{T} k_\mathrm{c}}{k_\mathrm{T} - \Delta} \qquad \text{when } k_\mathrm{T} - \Delta < 0. \tag{25}$$

We note that $k_\mathrm{T} - \Delta > 0$ is the condition for the stability of sound waves and $\Delta > 0$ is the condition for the stability of isobaric perturbations in a non-magnetic thermal fluid with no cosmic rays (Field 1965). When $k_\mathrm{T} > 0$ and $\Delta > 0$ condition (24) is always satisfied and condition (25) is never satisfied. Hence, $k_\mathrm{T} - \Delta < 0$ is a sufficient condition for instability when no cosmic rays are present and when they are but have a low pressure relative to the thermal pressure and the lengthscale associated with cosmic ray diffusion is small compared to the lengthscale associated with cooling and the wavelength of the sound wave.

In the case that $\left|\frac{k_\mathrm{T}}{k_\mathrm{c}}\right|, \left|\frac{k_\rho}{k_\mathrm{c}}\right|, \left|\frac{k}{k_\mathrm{c}}\right| \gg 1$, ϕ condition (19) is satisfied if

$$k^2 \gtrsim k_\mathrm{T} k_\mathrm{c} \frac{k_\mathrm{T}(1-\phi) - \Delta}{k_\mathrm{T} - \Delta} \qquad \text{when } k_\mathrm{T} - \Delta > 0 \tag{26}$$

$$k^2 \lesssim k_\mathrm{T} k_\mathrm{c} \frac{k_\mathrm{T}(1-\phi) - \Delta}{k_\mathrm{T} - \Delta} \qquad \text{when } k_\mathrm{T} - \Delta < 0 \tag{27}$$

and condition (20) is satisfied if

$$k^2 \gtrsim k_\mathrm{c}^2 \left(\phi - 1 + \frac{\Delta}{k_\mathrm{T}}\right) \qquad \text{when } k_\mathrm{T} > 0 \tag{28}$$

$$k^2 \lesssim k_\mathrm{c}^2 \left(\phi - 1 + \frac{\Delta}{k_\mathrm{T}}\right) \qquad \text{when } k_\mathrm{T} < 0. \tag{29}$$

Thus, when the lengthscale associated with the cosmic ray diffusion is long compared to the lengthscale associated with the cooling and the wavelength of the perturbation, the criteria for the stability of sound waves are somewhat more complicated than in the simplest limiting cases.

5. Conclusions

Thermal instability may play a role in the formation of condensations in many diffuse astrophysical media, a fact that motivated Field's (1965) classic study. These media include the interstellar medium, where the cosmic ray pressure in many regions is comparable to the thermal pressure (e.g. Boulares & Cox 1990; Webber 1987; Ferrière 1998) and might be expected to play a role in structure formation due to thermal instability.

We have adopted a simplified treatment of the cosmic rays by using a fluid description and assuming a diffusion coefficient that is constant in time and space. Given the uncertainties and difficulties in the calculation of the diffusion coefficient from first principles, a simple treatment of the problem of cosmic ray moderation of the thermal stability is justified as a first step. We have found that the criterion for the growth of isobaric perturbations is unaffected by the inclusion of cosmic rays, irrespective of the diffusion coefficient and the ratio of cosmic ray to thermal pressure. The criteria for a medium to be stable to perturbations by small ampitude sound waves are altered by the inclusion of cosmic rays. In general these criteria are rather complicated. In the simple limit of a very high cosmic ray to thermal pressure, a medium is stable to all small amplitude sound wave perturbations on scales small compared to the geometric mean of the scale on which cosmic ray diffusion occurs and the scale associated with cooling. Thus, not surprisingly, in cases in which the cosmic ray pressure is high compared to the thermal pressure the instability of high frequency sound waves is suppressed.

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