A&A 428, 365–371 (2004)
DOI: 10.1051/0004-6361:20041882

Astronomy & Astrophysics

Magnetic field generation in relativistic shocks

An early end of the exponential Weibel instability in electron-proton plasmas

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Received 21 April 2004 / Accepted 26 August 2004

Abstract. We discuss magnetic field generation by the proton Weibel instability in relativistic shocks, a situation that applies to the external shocks in the fireball model for Gamma-ray Bursts, and possibly also to internal shocks. Our analytical estimates show that the linear phase of the instability ends well before it has converted a significant fraction of the energy in the proton beam into magnetic energy: the conversion efficiency is much smaller (of order m_e/m_p) in electron-proton plasmas than in pair plasmas. We find this estimate by modelling the plasma in the shock transition zone with a waterbag momentum distribution for the protons and with a background of hot electrons.

For ultra-relativistic shocks we find that the wavelength of the most efficient mode for magnetic field generation equals the electron skin depth, that the relevant nonlinear stabilization mechanism is magnetic trapping, and that the presence of the hot electrons limits the typical magnetic field strength generated by this mode so that it does not depend on the energy content of the protons. We conclude that other processes than the linear Weibel instability must convert the free energy of the protons into magnetic fields.

Key words. plasmas – magnetic fields – instabilities – shock waves – gamma rays: bursts

1. Introduction

Magnetic field generation in relativistic shocks in a hydrogen (electron-proton) plasma is important for the fireball model for Gamma-ray Bursts (Rees & Mészáros 1992). This model proposes that the non-thermal radiation we observe in the prompt and afterglow emission from the Gamma-ray Burst is synchrotron radiation from collisionless relativistic shocks. To explain the observed intensity of the afterglows as synchrotron emission, the models need a magnetic field strength B of at least ten per cent of the equipartition field strength (Gruzinov & Waxman 1999; Panaitescu & Kumar 2002). This means that the magnetic energy density must contribute about one to ten per cent to the total energy density of the plasma behind the shock: $B^2/8\pi \sim (0.01-1) \times e$, with e the post-shock thermal energy density. In electron-proton plasmas this implies a much stronger magnetic field than in pair plasmas because the energy density in an ultra-relativistic shock propagating into a cold medium (where $nmc^2 \gg P$, with n the number density, m the rest mass and P the thermal pressure) is roughly proportional to the rest mass m of the particles. In this paper, we look at what happens in a plasma consisting of particles of widely different mass.

In collisionless shocks, plasma instabilities can generate magnetic fields. Within the shock transition layer the relative motion of the mixing pre- and post-shock plasma produces very anisotropic velocity distributions for all particle species concerned. Fluctuating electromagnetic fields deflect the incoming charged particles and act as the effective collisional process needed to complete the shock transition (e.g., Hoshino et al. 1992). These fluctuating fields occur naturally because anisotropic velocity distributions are unstable against several plasma instabilities, such as the electrostatic two-stream instability and the electromagnetic Weibel instability. The first is an instability of (quasi-)longitudinal charge density perturbations and leads to fluctuating electric fields satisfying $|\boldsymbol{E}| \gg |\boldsymbol{B}|$ where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the fluctuating electric and magnetic fields. The second is an instability of the advection currents (proportional to the beam velocity) that result from charge bunching in the beams, and leads to spontaneously growing transverse waves (Weibel 1959) with $|\boldsymbol{B}| \geq |\boldsymbol{E}|$. In a relativistic shock, where the relative velocity of the pre- and post-shock plasma approaches the velocity of light, the Weibel instability dominates because it has the largest growth rate (Califano et al. 2002).

As both analytical estimates and numerical simulations show, the Weibel instability in pair plasmas can produce a magnetic field of near-equipartition strength (Fonseca et al. 2003; Haruki & Sakai 2003; Kazimura et al. 1998; Medvedev & Loeb 1999; Yang et al. 1994). In numerical simulations, the magnetic field generation always undergoes an exponentially growing phase that agrees with the estimates from linear analytical theory, and enters a nonlinear phase after that. Yang et al. (1994) have shown that in pair plasmas, the magnetic field strength reaches its maximum value at the end of the linear phase of the instability. The question arises whether the same holds true for the Weibel instability operating in an electron-proton plasma (Medvedev & Loeb 1999). Numerical simulations (Frederiksen et al. 2004) show that the nonlinear phase may be more important in electron-ion plasmas.

In this paper we present an analytical estimate that shows that the linear phase of the instability ends much earlier for proton beams in a hydrogen plasma than for electron(-positron) beams in a pair plasma. We do not present a full self-consistent shock model: rather we consider the plasma processes that could generate a magnetic field in a plasma with properties such as one expects near the front of a collisionless relativistic shock. The paper is organised as follows. In Sect. 2 we define a model for the shock situation in terms of the momentum distributions of the particles. We calculate the conditions for the instability in Sect. 3.1 and the magnetic field strength at the end of the linear phase of the instability in Sect. 3.2. In Sect. 3.3, we compare the energy density associated with this magnetic field strength with the total available energy density associated with the beams. Section 4 contains the discussion and Sect. 5 the conclusions.

2. A simple model for a relativistic shock transition

In an electron-proton plasma, the protons dominate the shock energetics because they have a much larger rest mass than the electrons. Therefore, we will study the proton-driven Weibel instability. In this section we present a simple model for the plasma in the transition layer at the front of the shock.

The plasma in an astrophysical relativistic shock does not necessarily behave as a *single* fluid. Coulomb collisions between electrons and protons are not sufficiently fast to create thermal equilibrium between the protons and electrons. This problem of the non-equilibration of the electron and ion energies already exists in the much slower ($\sim$1000 km s^{-1}) shocks associated with Supernova Remnants (Draine & McKee 1993; Vink 2004).

We assume that scattering by plasma waves is far more efficient for the light electrons than for the heavy ions so that when the trajectories of the incoming protons start to become significantly perturbed, the electrons have already undergone the fast-growing electron Weibel instability (Frederiksen et al. 2004; Medvedev & Loeb 1999), which has converted the kinetic energy of their bulk motion into the thermal energy of a relativistically hot electron plasma with an (almost) isotropic thermal velocity distribution. The incoming protons form, seen from the rest frame of the hot electrons, a relativistic beam. We also assume that part of the protons are reflected further downstream although that assumption is not critical for our final conclusions (see Appendix A).

The electron-driven Weibel instability produces a weak fluctuating magnetic field with $B^2/8\pi \sim e_{\rm e}$, with $e_{\rm e}$ the energy density of the shocked electrons. We ignore this magnetic field in the calculations for proton beams, but it could serve as a seed perturbation for the proton-driven Weibel instability.

2.1. The proton velocity distribution

A simple model for the anisotropic proton velocity distribution within the shock transition layer is a *waterbag distribution* (Silva et al. 2002; Yoon & Davidson 1987). We consider a similar situation as in Fig. 6 of Frederiksen et al. (2004): we take two counter-streaming proton beams moving along the x-direction, with a small velocity spread in the z-direction to model thermal motions:

$$\begin{aligned} F(\boldsymbol{p}) \;=\; & \frac{n_{\rm p}}{4p_{z0}}\left[\delta(p_x - p_{x0}) + \delta(p_x + p_{x0})\right] \\ & \times\delta(p_y)\left[\Theta(p_z + p_{z0}) - \Theta(p_z - p_{z0})\right]. \end{aligned} \tag{1}$$

Here $n_{\rm p}$ is the total proton density, p_{x0} is the bulk momentum of the proton beams, p_{z0} is the maximum momentum in thermal motions and $\Theta(x) = (1+x/|x|)/2$ is the Heavyside step function. The assumption of two beams of equal strength is mathematically convenient, but not essential (see Appendix A).

This is a simple model that mimics the properties of non-relativistic collisionless shocks (see the *Microstructure* Section in Tsurutani & Stone 1985) in which (partial) reflection of the ions occurs as a result of deflection by an electrostatic potential jump in the shock transition, or by "overshoots" in the strong magnetic field in the wake of the shock. In addition, the waterbag model accounts for partial ion heating by including a velocity dispersion in the direction perpendicular to both the beam direction and the wave magnetic field. This direction lies along the wave vector of the unstable modes (the z-direction in our configuration).

2.2. The shock conditions for the electrons

We assume that the electrons have (almost) completed the shock transition so that their properties obey the relativistic shock conditions (Blandford & McKee 1976), which follow from the generally valid conservation laws for particle number, energy and momentum.

Here and below, we will label properties of the post-shock electron plasma with subscript 2, and those of the pre-shock plasma with subscript 1. We will assume that the pre-shock plasma is cold in the sense that $e_{\rm e,1} \ll n_{\rm e,1}m_{\rm e}c^2$. Then the shock conditions for the proper density $n_{\rm e}$ and the proper energy density $e_{\rm e}$ for the electrons are:

$$\begin{aligned} n_{\rm e,2} &= (4\gamma_{\rm rel} + 3)n_{\rm e,1}, \\ e_{\rm e,2} &= (4\gamma_{\rm rel} + 3)\gamma_{\rm rel}n_{\rm e,1}m_{\rm e}c^2, \end{aligned} \tag{2}$$

where $\gamma_{\rm rel}^2 = 1 + u_{x0}^2$ is the Lorentz factor of the relative velocity between the pre- and post-shock plasma (with $u_{x0} = p_{x0}/m_{\rm p}c$).

We neglect the dynamical influence of a pre-shock magnetic field on the electron-fluid jump conditions. This influence will be small if $V_{\rm Ae,1} \ll c$ where $V_{\rm Ae,1} \equiv B_1/\sqrt{4\pi n_{\rm e,1}m_{\rm e}}$ is the Alfvén speed based on the electron mass (instead of the proton mass). We also neglect any large-scale electrostatic field in the shock that might accelerate the electrons to higher energies while decelerating the incoming protons.

3. The Weibel instability

3.1. The dispersion relation

We consider the instability of a purely transverse electromagnetic wave with wave vector $\boldsymbol{k} = k\boldsymbol{e}_z$ and frequency ω. The instability will grow with a rate equal to the imaginary part of the wave frequency: $\sigma = \Im(\omega)$.

The Weibel instability for the model of the previous section obeys a dispersion relation of the form (e.g., Silva et al. 2002)

$$k^2c^2 - \omega^2\left[\, 1 + \chi_{xx}(\omega, k)\,\right] = 0, \tag{3}$$

where $\chi_{xx}(\omega, k) \equiv \chi_{xx,\mathrm{e}}(\omega, k) + \chi_{xx,\mathrm{p}}(\omega, k)$ is the xx-component of the plasma susceptibility tensor, which contains contributions of both the electrons and the protons.

Since the electrons have a relativistically hot thermal velocity distribution their contribution is

$$\chi_{xx,\mathrm{e}}(\omega, k) = -\frac{\tilde{\omega}^2_{\mathrm{pe}}}{\omega^2}, \tag{4}$$

where $\tilde{\omega}_{\mathrm{pe}}$ is the electron plasma frequency (in Gaussian units):

$$\tilde{\omega}^2_{\mathrm{pe}} = \frac{4\pi q^2 n_{\mathrm{e},2}}{m_{\mathrm{e}} h}, \tag{5}$$

with n_{e} the electron proper density, q the electron charge, m_{e} the electron mass and $h = (e_{\mathrm{e}} + P_{\mathrm{e}})/(n_{\mathrm{e}} m_{\mathrm{e}} c^2) \simeq 4e_{\mathrm{e}}/(3n_{\mathrm{e}} m_{\mathrm{e}} c^2)$ the electron enthalpy per unit rest energy for the relativistically hot electrons with $e_{\mathrm{e}} \simeq 3P_{\mathrm{e}} \gg n_{\mathrm{e}} m_{\mathrm{e}} c^2$. If we assume that the electrons are fully shocked, the shock conditions (2) enable us to express $\tilde{\omega}_{\mathrm{pe}}$ in terms of the pre-shock electron number density:

$$\tilde{\omega}^2_{\mathrm{pe}} = \frac{12\pi q^2 n_{\mathrm{e},1}}{m_{\mathrm{e}}}\left(\frac{4\gamma_{\mathrm{rel}} + 3}{4\gamma_{\mathrm{rel}}}\right). \tag{6}$$

The factor between brackets approaches unity in ultra-relativistic shocks with $\gamma_{\mathrm{rel}} \gg 1$.

The proton contribution to dispersion relation (3) is (see Silva et al. 2002)

$$\chi_{xx,\mathrm{p}}(\omega, k) = -\frac{\omega^2_{\mathrm{pp}}}{\gamma_{\mathrm{b0}}\,\omega^2}\left(\mathcal{F} + \frac{k^2 v^2_{x0}}{\omega^2 - k^2 v^2_{z0}}\right), \tag{7}$$

with

$$\mathcal{F} = \frac{c}{2v_{z0}}\,\ln\!\left(\frac{c + v_{z0}}{c - v_{z0}}\right) - \frac{u^2_{x0}}{1 + u^2_{x0}}. \tag{8}$$

Here $\omega_{\mathrm{pp}} = \sqrt{4\pi q^2 n_{\mathrm{p}}/m_{\mathrm{p}}}$ is the (non-relativistic) proton plasma frequency based on the density in the lab frame, m_{p} is the proton rest mass, $u_i = p_i/(m_{\mathrm{p}}c)$, $\gamma_{\mathrm{b0}} = (1 + u^2_{x0} + u^2_{z0})^{1/2}$ and $v_i = u_i c/\gamma_{\mathrm{b0}}$. To ensure quasi-neutrality of the plasma we must have

$$n_{\mathrm{p}} \approx n_{\mathrm{e},2}, \tag{9}$$

and the associated plasma frequency is

$$\omega^2_{\mathrm{pp}} = (4\gamma_{\mathrm{rel}} + 3)\,\frac{4\pi q^2 n_{\mathrm{e},1}}{m_{\mathrm{p}}}. \tag{10}$$

For what follows it is convenient to introduce the frequency $\hat{\omega}_{\mathrm{pp}}$ defined by

$$\hat{\omega}^2_{\mathrm{pp}} = \frac{\omega^2_{\mathrm{pp}}}{\gamma_{\mathrm{b0}}}. \tag{11}$$

If the velocity dispersion in the beam is small, which is always true for a Weibel-unstable proton distribution (see the end of this section), we have $\gamma_{\mathrm{b0}} \simeq \gamma_{\mathrm{rel}}$ so that

$$\frac{\hat{\omega}^2_{\mathrm{pp}}}{\tilde{\omega}^2_{\mathrm{pe}}} \simeq \frac{4m_{\mathrm{e}}}{3m_{\mathrm{p}}}. \tag{12}$$

Note that one can get the equations for an electron-positron beam in an electron-positron plasma (see also Silva et al. 2002; Yang et al. 1994) by replacing n_{p} and m_{p} with the beam density and electron mass respectively. We will use this in what follows to compare results for electron-proton plasmas with those for electron-positron plasmas. In those cases we assume that the density of the electron-positron beams is comparable to the density of the background electron-positron plasma.

Substituting the contributions (4) and (7) in (3) we can write the dispersion relation as a biquadratic equation for ω:

$$\omega^4 - \mathcal{B}\omega^2 + C = 0, \tag{13}$$

with

$$\mathcal{B} \;=\; k^2\left(c^2 + v^2_{z0}\right) + \Omega^2, \tag{14}$$

$$C \;=\; k^2\left\{v^2_{z0}\left(k^2c^2 + \Omega^2\right) - \hat{\omega}^2_{\mathrm{pp}} v^2_{x0}\right\}, \tag{15}$$

where

$$\Omega^2 \equiv \tilde{\omega}^2_{\mathrm{pe}} + \hat{\omega}^2_{\mathrm{pp}}\,\mathcal{F}. \tag{16}$$

Since $\mathcal{B} > 0$ the wave is unstable for $C < 0$ with $\omega^2 \equiv -\sigma^2 < 0$ where the growth rate σ follows from

$$\sigma^2 = \frac{\sqrt{\mathcal{B}^2 - 4C} - \mathcal{B}}{2}. \tag{17}$$

For a given set of shock parameters (γ_{b0}, v_{z0}, n_0), the growth rate is a function of the wave number (Fig. 1).

Anticipating our results for a proton beam in a background of (relativistically) hot electrons, we will assume that the growth rate of the unstable modes satisfies

$$\sigma \ll kc \quad \text{and} \quad \sigma \ll \tilde{\omega}_{\mathrm{pe}}, \tag{18}$$

and that the characteristic plasma frequencies satisfy

$$\hat{\omega}^2_{\mathrm{pp}} \ll \Omega^2, \tag{19}$$

see Eq. (27) below.

Under these assumptions we can approximate the solution of the dispersion relation with $\sigma^2 \simeq -C/\mathcal{B}$, which leaves the instability criterion ($C < 0$) unchanged. We will also make the approximation of a small beam velocity dispersion: $v^2_{z0} \ll c^2$. Then the dispersion relation reduces to

$$\sigma^2 = \frac{k^2\left(\hat{\omega}^2_{\mathrm{pp}}\,v^2_{x0} - \Omega^2\,v^2_{z0} - k^2c^2\,v^2_{z0}\right)}{k^2c^2 + \Omega^2}. \tag{20}$$

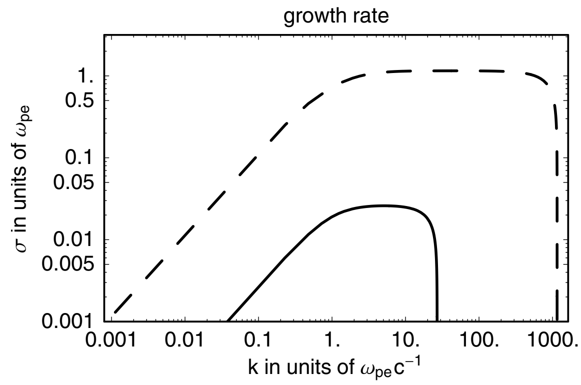

Fig. 1. Growth rate as a function of wave number for a shock with $\gamma_{\rm b0} = 1000$ and $v_{z0} = 0.001c$. The solid line is for proton beams in a background of hot electrons; the dashed line is for electron-positron beams in an electron-positron plasma.

For further analysis we introduce the following dimensionless quantities:

$$\kappa = \frac{kc}{\Omega}, \quad \nu = \frac{\sigma}{\Omega}, \quad \alpha = \frac{\hat{\omega}_{\rm pp}}{\Omega}\left(\frac{v_{x0}}{v_{z0}}\right). \tag{21}$$

Expressed in these quantities the dispersion relation (20) reads

$$\begin{aligned}\nu^2 &= \left(\frac{v_{z0}}{c}\right)^2 \frac{\kappa^2\left(\kappa_{\max}^2 - \kappa^2\right)}{1+\kappa^2} \\ &= \left(\frac{\hat{\omega}_{\rm pp}}{\Omega}\right)^2 \left(\frac{v_{x0}}{c}\right)^2 \frac{\kappa^2(\kappa_{\max}^2 - \kappa^2)}{(1+\kappa^2)\left(1+\kappa_{\max}^2\right)},\end{aligned} \tag{22}$$

where we have eliminated v_{z0} using the definition of α and we define

$$\kappa_{\max} \equiv \sqrt{\alpha^2 - 1}, \tag{23}$$

which is the limiting wavenumber of the Weibel instability: the instability condition is satisfied for $\kappa < \kappa_{\max}$. The parameter α is a measure for the range of unstable wave numbers. The Weibel instability occurs only if $\alpha > 1$. This condition poses a restriction on the velocity spread v_{z0}:

$$\frac{v_{z0}}{v_{x0}} < \frac{\hat{\omega}_{\rm pp}}{\Omega}. \tag{24}$$

The quantity Ω depends on v_{z0} through Eqs. (16) and (8), but a quick inspection shows that $\mathcal{F}$ and Ω are increasing functions of v_{z0} so if condition (24) holds, then we also have

$$\frac{v_{z0}}{v_{x0}} < \frac{\hat{\omega}_{\rm pp}}{\Omega|_{v_{z0}=0}} = \frac{\hat{\omega}_{\rm pp}}{(\tilde{\omega}_{\rm pe}^2 + \hat{\omega}_{\rm pp}^2/\gamma_{\rm rel}^2)^{1/2}}. \tag{25}$$

It follows that for relativistic shocks with $v_{x0} \to c$ the proton-driven instability requires $v_{z0} \ll c$ so that

$$\mathcal{F} \simeq \frac{1}{\gamma_{\rm b0}^2} + \frac{v_{z0}^2}{3c^2} \ll 1, \tag{26}$$

and

$$\Omega \simeq \tilde{\omega}_{\rm pe} \gg \hat{\omega}_{\rm pp} \tag{27}$$

(see Eq. (12)).

The largest growth rate occurs for a mode with wave number k_*, which follows from $({\rm d}\sigma/{\rm d}k)_{k=k_*} = 0$. Using the approximated dispersion relation (22) we find that this wavenumber equals

$$\frac{k_* c}{\Omega} \equiv \kappa_*(\alpha) = \sqrt{\alpha - 1}, \tag{28}$$

with the corresponding growth rate

$$\begin{aligned}\frac{\sigma(k_*)}{\Omega} &\equiv \nu_* = (\alpha - 1)\,\frac{v_{z0}}{c} \\ &= \frac{\alpha-1}{\alpha}\left(\frac{\hat{\omega}_{\rm pp}}{\Omega}\right)\frac{v_{x0}}{c}.\end{aligned} \tag{29}$$

In the last equality we have used the definition of α. For a strong, relativistic shock we find $\sigma(k_*) \simeq \hat{\omega}_{\rm pp}$.

In view of this fact and Eq. (27), conditions (18) and (19) automatically hold for the proton-driven Weibel instability.

3.2. Stabilization of the Weibel instability

The linear phase of the Weibel instability (during which perturbations grow exponentially with time) ends when the generated electromagnetic fields significantly perturb the trajectories of the particles taking part in the instability. Because the magnetic fields generated by the instability are inhomogeneous, the particles will quiver under the influence of the Lorentz force. When the amplitude of these quiver motions exceeds the wavelength of the instability, the linear theory breaks down. Yang et al. (1994) give a full treatment of these quiver motions and show that this criterion agrees with the magnetic trapping argument, which says that the instability will stop when the wave magnetic field becomes so strong that it traps the beam particles.

The linearized equation describing the quiver motions in the z-direction for a beam particle in a magnetic field $B(z,t)\,\boldsymbol{e}_y$ reads:

$$\frac{{\rm d}^2\xi_z}{{\rm d}t^2} = \frac{q v_{x0}\,B(z,t)}{\gamma_{\rm b0} m_{\rm p} c}, \tag{30}$$

where ξ_z is the displacement. In the linear stage of the instability the wave magnetic field varies as $B(z,t) = |\boldsymbol{B}|\exp(\sigma t)\sin(kz)$ for a wave with wave number k and growth rate σ, so the amplitude of the quiver motion is

$$|\xi_z| \sim \frac{q|\boldsymbol{B}|v_{x0}}{\gamma_{\rm b0} m_{\rm p} c \sigma^2}. \tag{31}$$

The trapping criterion $k|\xi_z| < 1$ corresponds to $|\boldsymbol{B}| < B_{\rm trap}$ with

$$B_{\rm trap} = \frac{\gamma_{\rm b0} m_{\rm p} c \sigma^2(k)}{k v_{x0} q}. \tag{32}$$

This corresponds to Eq. (18) of Yang et al. (1994). Assuming that trapping saturates the Weibel instability at *all* wavelengths we get the typical field amplitude as a function of wavenumber (Fig. 2).

The maximum field amplitude is reached at those wave numbers where σ^2/k has the maximum value. For dispersion

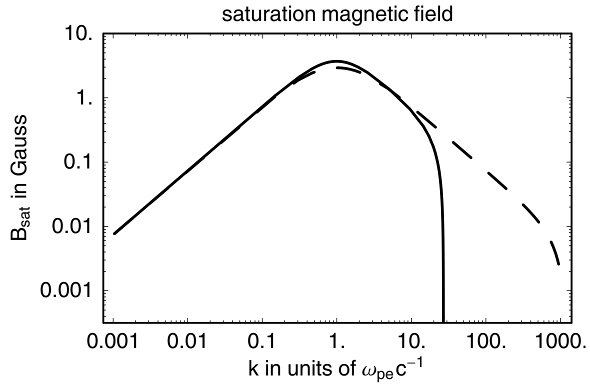

Fig. 2. Saturation magnetic field as a function of wave number for a proton-driven instability and for an instability in an electron-positron plasma, using the same parameters as in Fig. 1 with $n_{\mathrm{e,1}} = 1\,\mathrm{cm}^{-3}$.

relation (22) this maximum is reached at a wavenumber $k^\dagger$ that follows from

$$\frac{k^\dagger c}{\Omega} \equiv \kappa^\dagger(\alpha) = \left\{\frac{\alpha}{2}\sqrt{8+\alpha^2} - 1 - \frac{\alpha^2}{2}\right\}^{1/2}. \tag{33}$$

If the instability is strong, so that $\alpha \gg 1$, we find that this wavenumber and the associated growth rate are

$$\frac{k^\dagger c}{\Omega} \simeq 1, \quad \frac{\sigma(k^\dagger)}{\Omega} \simeq \frac{\alpha}{\sqrt{2}}\frac{v_{z0}}{c} = \left(\frac{\hat{\omega}_{\mathrm{pp}}}{\sqrt{2}\,\Omega}\right)\frac{v_{x0}}{c}. \tag{34}$$

Note that the trapping criterion predicts the largest field amplitude at a wavenumber $k^\dagger$, which is not equal to the wavenumber k_* of the fastest growing mode (Eq. (28)): $k^\dagger \simeq \Omega/c \ll k_* \simeq \sqrt{\alpha}\Omega/c$ for $\alpha \gg 1$. Eventually, this slower growing long-wavelength mode reaches a higher magnetic field than the faster growing mode (see also the discussion in Yang et al. 1994). If $\alpha \gg 1$ the saturation field strength due to trapping at $k^\dagger \simeq \Omega/c$ is

$$\begin{aligned} B_{\mathrm{trap}} &\simeq \frac{\gamma_{\mathrm{b0}} m_{\mathrm{p}} v_{x0}}{q}\frac{\hat{\omega}_{\mathrm{pp}}^2}{2\Omega} \\ &\simeq \frac{2\gamma_{\mathrm{b0}} m_{\mathrm{e}} \tilde{\omega}_{\mathrm{pe}} c}{3q}, \end{aligned} \tag{35}$$

where we have used Eqs. (12) and (27) to eliminate $\hat{\omega}_{\mathrm{pp}}$ and Ω.

For parameters typical for the external shock associated with Gamma-ray Bursts we have

$$B_{\mathrm{trap}} \simeq 3.7\sqrt{n_1}\,\gamma_{1000}\ \mathrm{Gauss}, \tag{36}$$

with $n_1 = n_0/(1\,\mathrm{cm}^{-3})$ and $\gamma_{1000} = \gamma_{\mathrm{b0}}/1000$.

We should note that Medvedev & Loeb (1999) use a different method for estimating B: they propose that the instability saturates when the beam ions become magnetized. This happens when the Larmor radius $r_{\mathrm{L}} = \gamma v m c^2/qB$ of the beam particles in the generated magnetic field becomes smaller than the wavelength of the fastest growing mode of the instability. This criterion $k_* r_{\mathrm{L}} < 1$ corresponds to $|\boldsymbol{B}| < B_{\mathrm{magn}}$ with

$$B_{\mathrm{magn}} = \frac{\gamma_{\mathrm{b0}} v_{x0} m_{\mathrm{p}} c k_*}{q}, \tag{37}$$

with k_* given by Eq. (28). However, the trapping argument predicts the smallest saturation amplitude $|\boldsymbol{B}|$. In particular, we have

$$\frac{B_{\mathrm{trap}}}{B_{\mathrm{magn}}} = \frac{\sigma^2(k^\dagger)}{k^\dagger k_* v_{x0}^2} = \left(\frac{\hat{\omega}_{\mathrm{pp}}}{\Omega}\right)^2 \Phi(\alpha), \tag{38}$$

where

$$\Phi(\alpha) \equiv \frac{\kappa^\dagger(\alpha)}{\alpha^2\sqrt{\alpha-1}}\left(\frac{3\alpha - \sqrt{8+\alpha^2}}{\sqrt{8+\alpha^2}-\alpha}\right). \tag{39}$$

To derive this relation we have used definition (21) to write $v_{z0}/v_{x0} = \hat{\omega}_{\mathrm{pp}}/(\alpha\Omega)$. For a proton beam in a hot electron background we have $\hat{\omega}_{\mathrm{pp}}^2/\Omega^2 \ll 1$ and $\Phi(\alpha) < 0.3$ for all $\alpha \geq 1$. For a strong instability with $\alpha \gg 1$ we have $\Phi(\alpha) \simeq (4\alpha)^{-1/2} \ll 1$. Therefore, trapping occurs well before the field can totally magnetize a proton beam with a density comparable to the density of the hot background electrons. In view of this we will use B_{trap} as an estimate of the saturation magnetic field strength.

The criteria (32) and (37) predict the typical amplitude of the magnetic field as one particular wave mode k saturates. In a realistic situation the instability will involve a superposition of wave modes and one should interpret B as the amplitude that follows from the power spectrum $\mathcal{I}_B(k)$ of the field fluctuations: $B^2/8\pi \sim k\mathcal{I}_B(k)$ with $k \approx k_*$ for magnetization and $k \approx k^\dagger$ for trapping. The total magnetic energy in the unstable modes is

$$U_B = \frac{B^2}{8\pi} = \int_0^{k_{\max}} \mathrm{d}k\,\mathcal{I}_B(k). \tag{40}$$

3.3. The equipartition parameter

A measure of the strength of the magnetic field is the equipartition parameter, which compares the energy density in the magnetic field with the total energy density.

The protons dominate the energy budget, and the total available energy density is

$$e_{\mathrm{p}} = \int \mathrm{d}\boldsymbol{p}\,F(\boldsymbol{p})\gamma(\boldsymbol{p}) m_{\mathrm{p}} c^2 \simeq \gamma_{\mathrm{b0}} n_{\mathrm{p}} m_{\mathrm{p}} c^2, \tag{41}$$

where the approximation is valid for $v_{z0} \ll v_{x0}$.

We define the proton equipartition parameter as

$$\epsilon_B = \frac{B_{\mathrm{trap}}^2}{8\pi e_{\mathrm{p}}}. \tag{42}$$

Using Eq. (35) for the magnetic field, with the definition (11) for $\hat{\omega}_{\mathrm{pp}}$ and the approximation $v_{x0} \simeq c$ for relativistic shocks we get

$$\epsilon_B = \frac{\hat{\omega}_{\mathrm{pp}}^2}{8\Omega^2}. \tag{43}$$

Then from Eqs. (12) and (27) we have

$$\epsilon_B \simeq \frac{m_{\mathrm{e}}}{6m_{\mathrm{p}}} \sim 10^{-4}. \tag{44}$$

4. Discussion

The small value of the equipartition parameter (44) implies that the proton-driven Weibel instability in a background of relativistically hot electrons saturates long before the magnetic field reaches equipartition with the available proton free energy. Equation (25) demonstrates that the proton-driven Weibel instability in a hydrogen plasma is not a suitable candidate for the mechanism responsible for "thermalization" of the incoming protons in the shock layer. In electron(-positron) beams in a hot pair background the instability condition, $\alpha > 1$, allows a large beam velocity dispersion: $v_{z0}/v_{x0} \leq 1$ if the number densities in the beam and in the hot background are of similar magnitude (see also Yoon & Davidson 1987). Therefore, the Weibel instability is in principle capable of randomizing a significant fraction of the beam momentum of an electron(-positron) beam, as asserted in Sect. 2, whereas this is not true for a proton beam in a hydrogen plasma.

In the limit of a cold, relativistic proton beam the properties of the electrons and not those of the protons determine many of the results. The electron plasma frequency determines the wave number (33) of the mode with the maximum magnetic field so that the electron skin depth $c/\tilde{\omega}_{\rm pe}$ sets the length scale of the dominant mode. This happens because the low inertia of the electrons makes them very responsive to the perturbations of the protons. The dispersion relation for the Weibel modes (Fig. 1), also supports this view: the plateau around the maximum growth rate starts roughly at a wave number $k \sim \tilde{\omega}_{\rm pe}/c$. Studies that do not include the response of the background electrons (by treating the protons as an isolated system) miss this point. The peak magnetic field (35) does not contain any parameters connected with the protons. Therefore, proton beams in a hydrogen plasma generate nearly the same magnetic field strength as electron(-positron) beams in an electron-positron plasma (Fig. 2) *despite* the larger kinetic energy of the protons.

Gruzinov (2001) anticipated this when he excluded the case where a small parameter in the theory might be important in his analysis of the Weibel instability: our analysis shows that the relevant small parameter is $\hat{\omega}_{\rm pp}^2/\Omega^2 \sim 4m_{\rm e}/3m_{\rm p}$. The result is a small equipartition parameter (44). In this respect our result is similar to the one found by Sagdeev (1966, p. 88), who argued for the Weibel instability in a non-relativistic plasma that the electrons have a quenching effect on the ion instability.

In our analysis we have excluded electrostatic waves, which could also play an important role in the shock transition zone (Schlickeiser et al. 2002). In that case electrostatic Bremsstrahlung could be an alternative explanation for the Gamma-ray Burst afterglow emission (Schlickeiser 2003), relaxing the need in synchrotron models for a high magnetic field strength.

5. Conclusions

We have presented an analytical estimate of the magnetic field produced at the end of the linear phase of the Weibel instability at the front of an ultra-relativistic shock propagating into a cold hydrogen plasma, a situation that applies to the external shocks that produce gamma-ray burst afterglows in the fireball model (Rees & Mészáros 1992). The magnetic field strength that we find is too weak to explain the observed synchrotron radiation (Gruzinov & Waxman 1999): the equipartition parameter (Eq. (44)) is at least two orders of magnitude too small. This is radically different from the results for the Weibel instability in pair plasmas. The reason is that the contribution of the electrons to the electromagnetic response of the plasma inhibits the instability of the protons.

The saturation magnetic field (35) which this low equipartition parameter corresponds to is the magnetic field at the point where the linear approximation breaks down and where nonlinear trapping effects start to limit the growth of the unstable Weibel mode. After this happens, it is likely that the instability enters a nonlinear phase or that another type of instability takes over: numerical simulations (Frederiksen et al. 2004) of similar plasmas show near-equipartition magnetic fields behind the Weibel-unstable region in the shock transition. The nonlinear phase would then be the dominant phase in electron-ion plasmas and deserves further study to determine the physical mechanism and the properties of the resulting magnetic field.

Acknowledgements. This research is supported by the Netherlands Research School for Astronomy (NOVA).

Appendix A: Asymmetric beams

We consider asymmetric proton beams and show that we can neglect the effect of the asymmetry for typical parameters. We replace the proton distribution (1) by

$$F(\boldsymbol{p}) = \frac{n_{\rm p}}{2p_{z0}}\left[\frac{1+\Delta}{2}\,\delta(p_x - p_{x0}) + \frac{1-\Delta}{2}\,\delta(p_x + p_{x0})\right] \times\delta(p_y)\left[\Theta(p_z + p_{z0}) - \Theta(p_z - p_{z0})\right], \quad \text{(A.1)}$$

with $0 \leq \Delta \leq 1$ the parameter measuring the asymmetry between the two beams. This asymmetry changes the dispersion relation of the waves to (e.g., Akhiezer et al. 1975):

$$k^2c^2 - \omega^2\left[1 + \chi_{xx}(\omega,\,k) - \frac{\chi_{xz}^2(\omega,\,k)}{1 + \chi_{zz}(\omega,\,k)}\right] = 0. \quad \text{(A.2)}$$

The extra term $\propto \chi_{xz}^2$ appears in relation (A.2) because the waves are no longer transverse: the charge bunching of the asymmetric beams produces a net charge density, leading to a component of the wave electric field along the wave vector.

We will give the extra components of the susceptibility tensor and publish a derivation elsewhere (Achterberg & Wiersma, in preparation). The proton contribution to $\chi_{xx}(\omega, k)$ is the same as in the symmetric case. For a thermal electron background with an isotropic momentum distribution, only the two proton beams contribute to the off-diagonal components of the susceptibility tensor so that

$$\chi_{xz}(\omega, k) = \chi_{xz,{\rm p}}(\omega, k) = \frac{\Delta\,\hat{\omega}_{\rm pp}^2}{\omega^2 - k^2v_{z0}^2}\,\frac{kv_{x0}}{\omega}\cdot \quad \text{(A.3)}$$

This off-diagonal component vanishes in the symmetric case ($\Delta = 0$). The longitudinal response of the beam-plasma system is contained in

$$1 + \chi_{zz}(\omega,\,k) = 1 - \frac{\tilde{\omega}_{\rm pe}^2}{\omega^2 - k^2C_{\rm e}^2} - \frac{\hat{\omega}_{\rm pp}^2}{\omega^2 - k^2v_{z0}^2}\cdot \quad \text{(A.4)}$$

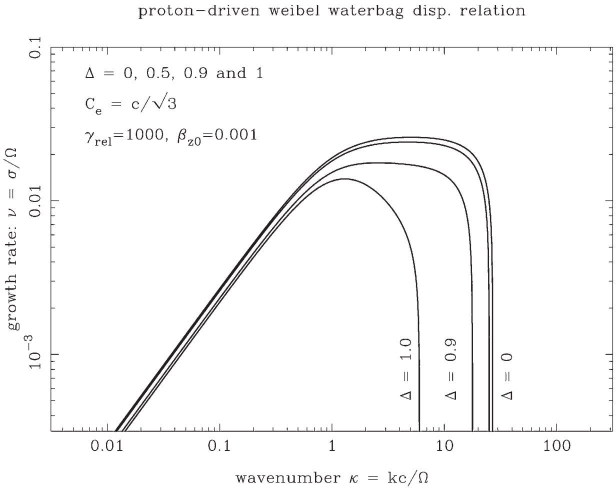

Fig. A.1. The growth rate as a function of wavenumber for asymmetric proton beams in a relativistically hot electron background with sound speed $C_e = c/\sqrt{3}$. Different lines correspond to different values of the parameter Δ, which measures the asymmetry between the beams.

Here $C_e^2 = 4P_e/3n_em_eh$ is the effective sound speed in the hot electron gas. In the ultra-relativistic case one has $C_e \simeq c/\sqrt{3}$. For proton beams in a hot electron background one has $\tilde{\omega}_{pe}^2 \gg \hat{\omega}_{pp}^2$ and $|\omega| \leq \hat{\omega}_{pp} \ll kC_e$ for the Weibel instability (see Sect. 3.1). This implies that

$$1 + \chi_{zz} \simeq 1 + \frac{\tilde{\omega}_{pe}^2}{k^2C_e^2} - \frac{\hat{\omega}_{pp}^2}{\omega^2 - k^2v_{z0}^2}. \tag{A.5}$$

The resulting dispersion relation can be written in terms of the dimensionless variable

$$\mathcal{Z}(\omega,\, k) \equiv \frac{\omega^2 - k^2v_{z0}^2}{\hat{\omega}_{pp}^2}. \tag{A.6}$$

One finds

$$(\mathcal{Z} - \mathcal{Z}_1)(\mathcal{Z} + \mathcal{Z}_2) - \Delta^2\, \mathcal{Z}_1\mathcal{Z}_2 = 0. \tag{A.7}$$

Here $\mathcal{Z}_{1,2}$ are defined by

$$\mathcal{Z}_1 = \frac{k^2C_e^2}{k^2C_e^2 + \tilde{\omega}_{pe}^2},\quad \mathcal{Z}_2 = \frac{k^2v_{x0}^2}{k^2c^2 + \Omega^2} \tag{A.8}$$

with $\mathcal{Z}_2 > \mathcal{Z}_1$ for the parameters considered here. The solution of this quadratic equation,

$$\mathcal{Z}_\pm = \frac{\mathcal{Z}_1 - \mathcal{Z}_2}{2} \pm \sqrt{\left(\frac{\mathcal{Z}_1 + \mathcal{Z}_2}{2}\right)^2 - \Delta^2\, \mathcal{Z}_1\mathcal{Z}_2}, \tag{A.9}$$

determines the frequency through

$$\omega_\pm^2 = \hat{\omega}_{pp}^2\, \mathcal{Z}_\pm + k^2v_{z0}^2. \tag{A.10}$$

The Weibel-unstable branch corresponds to the solution branch ω_- as $\mathcal{Z}_- < 0$. In the symmetric case ($\Delta = 0$) one has $\mathcal{Z}_- = -\mathcal{Z}_2$ which follows from Eq. (A.7), and one recovers dispersion relation (20). The stable branch ω_+ is a modified (largely electrostatic) ion-acoustic wave.

Although the asymmetry decreases the range of unstable wave numbers (Fig. A.1) and lowers the growth rate with respect to the symmetric case $\Delta = 0$, the change is small unless $\Delta \approx 1$: the case where there is almost no reflection. For a single beam, $\Delta = 1$, one has $\mathcal{Z}_- = \mathcal{Z}_1 - \mathcal{Z}_2$, which gives the following dispersion relation for $\nu^2 = -\omega_-^2/\Omega^2$ in the ultra-relativistic limit with $C_e = c/\sqrt{3}$ and $\Omega^2 \approx \tilde{\omega}_{pe}^2$:

$$\nu^2 = \frac{\hat{\omega}_{pp}^2}{\Omega^2}\, \frac{\kappa^2\left\{\left(\frac{3v_{x0}^2}{c^2} - 1\right) - \kappa^2\left(1 - \frac{v_{x0}^2}{c^2}\right)\right\}}{(3+\kappa^2)(1+\kappa^2)} - \kappa^2\frac{v_{z0}^2}{c^2}. \tag{A.11}$$

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