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Tidal and rotational effects in the perturbations of hierarchical triple stellar systems

I. Numerical model and a test application for Algol*

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Abstract. A new numerical integrator has been developed for studying the orbital and spin evolution of hierarchical triple stellar systems. The code includes equilibrium tide approximations with arbitrary direction of rotational axes. The variation of the orbital elements (e.g. the inclination of the close – eclipsing – binary) and its observational consequences according to the distorted models with different mass-distributions of the stars, as well as with and without dissipation are studied in the case of the well-known eclipsing triple system Algol. We found that, in the absence of the tidal dissipation, the presence of the third star may cause sudden fluctuations in the orbital elements and the stellar rotation of the binary members even in the previously synchronized case, too. The dissipation can eliminate these fluctuations, nevertheless some variations which would produce observable effects in the same order which have been measured in several eclipsing binaries are also present.

Key words. methods: numerical – celestial mechanics – stars: binaries: close – stars: individual: Algol

1. Introduction

Several close binary systems have third (or further) more distant companions. As is well-known, the dynamics of the members of such systems depart fairly much from pure Keplerian motion. Considering only the gravitational perturbations, these arise from two different aspects. On the one side the gravitational force of the more distant tertiary perturbs the orbital motion of the close binary, while on the other side the shape of the stars will not be spherical due to the strong gravitational interaction between the two stars. Furthermore, the members of very close binaries usually rotate so fast that their oblateness makes another non-spherical contribution to their gravitational field.

Both these effects may be strong enough to cause observable orbital changes on a time scale of years, or decades. Some well-known examples are the variation of the eclipse depths in some eclipsing binaries due to the orbital precession caused by a third star in an inclined orbit, as well as the apsidal motion forced by the distortion of the binary members, which has already been observed in more than a hundred systems.

In most of the previous studies these two different categories of orbital perturbations were studied separately. On the one hand, the perturbations of a third body were considered in the frame of the general three-body problem, i.e. all three stars were treated as mass points. Starting from lunar theories Harrington (1968, 1969) gave the first formulae for the perturbations for both the orbital elements of the binary orbit and the orbit of the tertiary. Söderhjelm (1975, 1982) reformulated these equations into a more explicit form, and used them to compare their predictions with the observational results for two well-known close triples, Algol (which is the subject of the present study, too) and λ Tau. Recently Ford et al. (2000) published a third order theory of perturbations in hierarchical triple systems.

On the other hand, the tidal interaction was mainly studied for close binaries without further companion(s). The main aim of such studies is to understand how the circularization and (both rotational and orbital) synchronization of the close binaries work, as well as to interpret those orbital perturbations which cannot be discussed in the frame of the mass-point

* Appendices are only available in electronic form at
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models. (We mainly refer to the apsidal motion in close, eccentric binaries.) While most recent studies (e.g., Claret 1999; Witte & Savonije 1999; Smeyers & Willems 2001; Claret & Willems 2002) in this field are based on the most recent stellar models, they do not consider the possible perturbative effects of a close third companion which can cause e.g. some precession.

While this second treatment seems to be correct at least in those cases where the system really does not contain a relatively close third companion, the mass-point approximation of hierarchical triple systems containing distorted components should be revised. As the observational interval of the closest triple systems becomes longer and longer, and the precision of the measurements in the near future will grow by several orders (we refer to the planned space missions, e.g. GAIA), it will be necessary to examine the different perturbations simultaneously.

Up to know only a very limited number of papers has been published in which the dynamics of hierarchical triple systems was discussed in the tidally distorted body model. After the first works of Kopal (1978, Chap. V) and Söderhjelm (1984) recently Kiseleva et al. (1998) and Eggleton & Kiseleva-Eggleton (2001) studied the effects of tidal friction in triple stars with numerical integration based on the model of Eggleton et al. (1998). Finally, Borkovits et al. (2002) integrated numerically the orbital motion of the eclipsing triple system IM Aur, and compared the perturbations of the orbital elements in the frame of the mass-point model and the disturbed star model. This integration was already based on the model and integrator code which is described in the present article.

In this work we use the model of equilibrium tides and tidal friction of Kopal (1978) to study simultaneously the orbital motion of hierarchical triple systems (containing two distorted stars consisting of linear viscous fluid rotating uniaxially around an arbitrarily directed axis, and a more distant third body treated as a mass point), and the rotation of the members of the close binary (which is the main novelty of this work). Through some macroscopic parameters the results of the new stellar models can be built into the code, at least with some limitation.

In the next section the equations of motion are presented. In Sect. 3 the integrator code is described, while in Sect. 4 we give an illustration of the results, applying the method to Algol itself.

2. Equations of motion

In formulating the equations of motion for the close pair we mainly followed the treatment of Kopal (1978). The stars were considered as liquid, viscous bodies. Furthermore, the mass distribution of the stars follows the instantaneous gravitational-centrifugal potential, only with a small departure in the case of the tidal friction. (This assumption gives the main limitation of the validity of the following calculations.) The initial orientation of the axis of rotation of each component can be arbitrary. The angular velocity of the rotation can differ from the Keplerian angular velocity. Only the first order terms were treated. For the more distant third companion only a mass-point

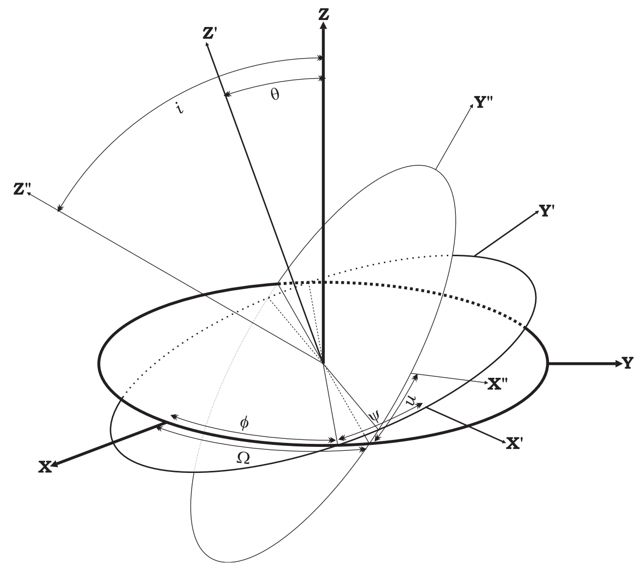

Fig. 1. The different coordinate systems, and the corresponding Eulerian angles. (See text for details.)

model was applied. This means that the effect of the third body was calculated only as a point mass-point mass interaction.

2.1. The coordinate systems

In order to describe the revolving and the rotating motions of the bodies we introduce different sets of rectangular Cartesian systems (see Fig. 1). These coordinate systems have the origin at the centre of the mass of the primary (or internally the secondary) star. The orientation of the axes in the unprimed x, y, z system is fixed. This is why we will refer to this system in the following as "fixed system", although this also orbits with the star, so it ceases to be an inertial system. The $x'y'$ plane of the co-rotating (primed) coordinate system coincides with the equatorial plane of the star, so this rotates (approximately uniaxially) around the z' axis. Finally, the x'', y'' plane of the co-revolving (doubly-primed) system lies in the orbital plane of the close binary, and the x'' axis is directed toward the centre of mass of the other member of the close binary. The a'_{ij}, a''_{ij} elements of the matrices which give the transformations between these coordinate systems are given e.g. in Kopal (1978), and for the sake of self-consistency of this paper we also list them in Appendix A.

In the doubly primed (co-revolving) coordinate system we introduce the usual set of orthogonal vectors $\boldsymbol{\rho}_1$, $\boldsymbol{\eta} = \boldsymbol{c} \times \boldsymbol{\rho}_1$, and $\boldsymbol{c} = \boldsymbol{\rho}_1 \times \dot{\boldsymbol{\rho}}_1$, (see Fig. 2), which are parallel to the x'', y'', and z'' axes, respectively, and so it is evident that in the fixed system

$$\boldsymbol{\rho}_1 = \rho_1(a''_{11}, a''_{21}, a''_{31}), \tag{1}$$

$$\boldsymbol{\eta} = c\rho_1(a''_{12}, a''_{22}, a''_{32}), \tag{2}$$

$$\boldsymbol{c} = c\left(a''_{13}, a''_{23}, a''_{33}\right). \tag{3}$$

The time derivatives of the vectors (1)–(3) can be written in the following forms:

$$\dot{\boldsymbol{\rho}}_1 = \boldsymbol{\rho}_1^{\circ} + \boldsymbol{w} \times \boldsymbol{\rho}_1, \tag{4}$$

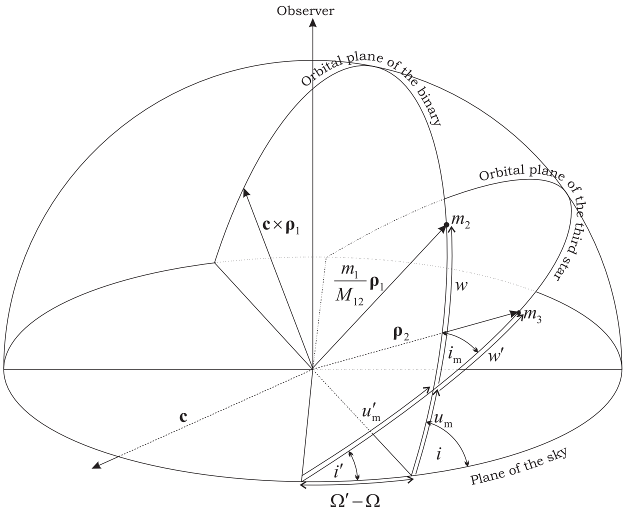

Fig. 2. The spatial orientation of the orbital planes. (See text for details.)

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\eta}^{\mathrm{o}} + \boldsymbol{w}\times\boldsymbol{\eta}, \tag{5}$$

$$\dot{\boldsymbol{c}} = \boldsymbol{c}^{\mathrm{o}} + \boldsymbol{w}\times\boldsymbol{c}, \tag{6}$$

where $^{\mathrm{o}}$ refers to the time derivatives in the co-orbiting (doubly-primed) coordinate system (expressed in the fixed system), while $\boldsymbol{w}$ denotes the angular velocity vector of the co-orbiting system with respect to the fixed one (see Appendix A)

$$\boldsymbol{w} = \frac{\boldsymbol{c}}{\rho_1^2} + \frac{\boldsymbol{c}\cdot\ddot{\boldsymbol{\rho}}_1}{c^2}\boldsymbol{\rho}_1. \tag{7}$$

2.2. Equations of the orbital motion

The equations of the orbital motion of the three bodies written in the usual Jacobian coordinates have the following form:

$$\ddot{\boldsymbol{\rho}}_1 = \frac{m_{12}}{m_1 m_2}\frac{\partial(U+R)}{\partial\boldsymbol{\rho}_1}, \tag{8}$$

$$\ddot{\boldsymbol{\rho}}_2 = \frac{m_{123}}{m_{12} m_3}\frac{\partial(U+R)}{\partial\boldsymbol{\rho}_2}, \tag{9}$$

where m_{12} is the total mass of the close binary, while m_{123} is the same for the triple system (see Fig. 2). Furthermore, U is the mass-point part of the three-body potential, while R is the perturbing potential, which up to first order contains the following four terms:

$$\begin{aligned} R_{\mathrm{T}} = {} & \frac{Gm_1m_2}{\rho_1}\sum_{j=2}^{4}\left\{\frac{m_2}{m_1}2k_j^{(1)}\left(\frac{R_1}{\rho_1}\right)^j\left(\frac{R_1}{r_{d_1}}\right)^{j+1}P_j(\lambda_1)\right. \\ & \left.+\frac{m_1}{m_2}2k_j^{(2)}\left(\frac{R_2}{\rho_1}\right)^j\left(\frac{R_2}{r_{d_2}}\right)^{j+1}P_j(\lambda_2)\right\}, \end{aligned} \tag{10}$$

$$\begin{aligned} R_{\mathrm{T3}} = {} & \sum_{i=1}^{2}\frac{Gm_im_3}{r_{i3}}\frac{m_{3-i}}{m_i}2k_2^{(i)}\left(\frac{R_i}{r_{i3}}\right)^2\left(\frac{R_i}{\rho_1}\right)^3P_2(\lambda_{i3}) \\ & +\frac{Gm_1m_2}{\rho_1}\sum_{i=1}^{2}\frac{m_3}{m_i}2k_2^{(i)}\left(\frac{R_i}{\rho_1}\right)^2\left(\frac{R_i}{r_{i3}}\right)^3P_2(\lambda_{i3}), \end{aligned} \tag{11}$$

$$R_{\mathrm{R}} = \frac{Gm_1m_2}{\rho_1}\sum_{i=1}^{2}\left\{\frac{k_2^{(i)}R_i^5}{Gm_i}\left[\frac{\omega_{z'_i}^2}{3\rho_1^2}-\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z'_i})^2}{\rho_1^4}\right]\right\}, \tag{12}$$

$$R_{\mathrm{R3}} = \sum_{i=1}^{2}\frac{Gm_im_3}{r_{i3}}\left\{\frac{k_2^{(i)}R_i^5}{Gm_i}\left[\frac{\omega_{z'_i}^2}{3r_{i3}^2}-\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\omega}_{z'_i})^2}{r_{i3}^4}\right]\right\}, \tag{13}$$

where the subscript $_{\mathrm{T}}$ refers to the terms arising from the tidal interaction between the two members of the close binary, as well as the binary members and the tertiary ($_{\mathrm{T3}}$), respectively, while $_{\mathrm{R}}$, $_{\mathrm{R3}}$ denote the rotational terms. In the above equations R_i stands for the radius of the ith star, $k_{2..4}^{(i)}$ denote the usual apsidal motion constants, $\boldsymbol{r}_{\mathrm{d}_i}$ represents the direction vector of the maximum amplitude of the tidal bulge in the given star, while λ_i is the direction cosine between this direction and the radius vector $\boldsymbol{\rho}_1$. Similarly, $\boldsymbol{r}_{i3}$ stands for the radius vector between the ith and the third star, while λ_{i3} is the direction cosine between this direction and the radius vector $\boldsymbol{\rho}_1$. Furthermore, we define the $\boldsymbol{\omega}_{z'_i}$ vector as the uniaxial rotational angular velocity of the ith star. It is clear that in the fixed system

$$\boldsymbol{\omega}_{z'_i} = \omega_{z'_i}\left(a'_{13}, a'_{23}, a'_{33}\right), \tag{14}$$

while the other letters have their usual meaning. The effect of the third companion manifests itself in two different ways. The first term of R_{T3} gives the effect of the tidal potential of the binary on the motion of the tertiary, while the second term represents the direct effect of the tidal interaction between the binary members and the third body for the motion of the binary. (In these terms the dissipation is cancelled, as well as the higher order tidal terms.) The explicit form of the orbital Eqs. (8) and (9) in vectorial form is the following:

$$\begin{aligned} \ddot{\boldsymbol{\rho}}_1 = {} & -\frac{Gm_{12}}{\rho_1^3}\boldsymbol{\rho}_1 + Gm_3\left(\frac{\boldsymbol{r}_{23}}{r_{23}^3}-\frac{\boldsymbol{r}_{13}}{r_{13}^3}\right) \\ & -\frac{Gm_{12}}{\rho_1^3}\left\{\sum_{i=1}^{2}\left\{\sum_{j=1}^{4}\frac{m_{3-i}}{m_i}2(j+1)k_j^{(i)}\left(\frac{R_i}{\rho_1}\right)^j\left(\frac{R_i}{r_{\mathrm{d}_i}}\right)^{j+1}\mathcal{P}_j(\lambda_i)\right.\right. \\ & \left.\left.\times\frac{k_2^{(i)}R_i^5}{Gm_i}\left\{\left[\frac{\omega_{z'_i}^2}{\rho_1^2}-5\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z'_i})^2}{\rho_1^4}\right]\boldsymbol{\rho}_1+\frac{2\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z'_i}}{\rho_1^2}\boldsymbol{\omega}_{z'_i}\right\}\right\}\right\} \\ & +m_3\sum_{i=1}^{2}(-1)^i\frac{k_2^{(i)}R_i^5}{m_ir_{i3}^5}\left\{\left[\omega_{z'_i}^2-5\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\omega}_{z'_i})^2}{r_{i3}^2}\right]\boldsymbol{r}_{i3}\right. \\ & \left.+2(\boldsymbol{r}_{i3}\cdot\boldsymbol{\omega}_{z'_i})\boldsymbol{\omega}_{z'_i}\right\} \\ & +Gm_3\sum_{i=1}^{2}(-1)^i\frac{3m_{3-i}k_2^{(i)}R_i^5}{m_ir_{i3}^5\rho_1^3}\left\{\left[5\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1)^2}{\rho_1^2r_{i3}^2}-1\right]\boldsymbol{r}_{i3}\right. \\ & \left.-2\frac{\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1}{\rho_1^2}\boldsymbol{\rho}_1\right\} \\ & -Gm_{12}\sum_{i=1}^{2}\frac{3m_3k_2^{(i)}R_i^5}{m_ir_{i3}^3\rho_1^5}\left\{\left[5\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1)^2}{\rho_1^2r_{i3}^2}-1\right]\boldsymbol{\rho}_1-2\frac{\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1}{r_{i3}^2}\boldsymbol{r}_{i3}\right\}, \end{aligned} \tag{15}$$

$$\ddot{\boldsymbol{\rho}}_2 = -\frac{Gm_{123}}{m_{12}}\left\{\left(\frac{m_1}{r_{13}^3}\boldsymbol{r}_{13} + \frac{m_2}{r_{23}^3}\boldsymbol{r}_{23}\right)\right.$$
$$+\sum_{i=1}^{2}\frac{k_2^{(i)}R_i^5}{r_{i3}^5}\left\{\left[\omega_{z_i'}^2 - 5\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\omega}_{z_i'})^2}{r_{i3}^2}\right]\boldsymbol{r}_{i3} + 2(\boldsymbol{r}_{i3}\cdot\boldsymbol{\omega}_{z_i'})\boldsymbol{\omega}_{z_i'}\right\}$$
$$+\sum_{i=1}^{2}\frac{3m_{3-i}k_2^{(i)}R_i^5}{r_{i3}^5\rho_1^5}\left\{\left[5\frac{(\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1)^2}{r_{i3}^2} - \rho_1^2\right]\boldsymbol{r}_{i3}\right.$$
$$\left.\left.-2(\boldsymbol{r}_{i3}\cdot\boldsymbol{\rho}_1)\boldsymbol{\rho}_1\right\}\right\}, \tag{16}$$

where

$$\boldsymbol{\mathcal{P}}_2(\lambda_i) = P_2(\lambda_i)\boldsymbol{\rho}_1 + \frac{\lambda_i}{\rho_1 r_{\mathrm{d}_i}}(\boldsymbol{r}_{\mathrm{d}_i}\times\boldsymbol{\rho}_1)\times\boldsymbol{\rho}_1$$
$$= \frac{1}{2}\left[5\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^2}{\rho_1^2 r_{\mathrm{d}_i}^2} - 1\right]\boldsymbol{\rho}_1 - \frac{\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i}}{r_{\mathrm{d}_i}^2}\boldsymbol{r}_{\mathrm{d}_i}, \tag{17}$$

$$\boldsymbol{\mathcal{P}}_3(\lambda_i) = P_3(\lambda_i)\boldsymbol{\rho}_1 + \frac{3}{2}\frac{5\lambda_i^2 - 1}{\rho_1 r_{\mathrm{d}_i}}(\boldsymbol{r}_{\mathrm{d}_i}\times\boldsymbol{\rho}_1)\times\boldsymbol{\rho}_1$$
$$= \left[10\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^3}{\rho_1^3 r_{\mathrm{d}_i}^3} - 3\frac{\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i}}{\rho_1 r_{\mathrm{d}_i}}\right]\boldsymbol{\rho}_1$$
$$-\frac{3}{2}\left[5\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^2}{\rho_1 r_{\mathrm{d}_i}^3} - \frac{\rho_1}{r_{\mathrm{d}_i}}\right]\boldsymbol{r}_{\mathrm{d}_i}, \tag{18}$$

$$\boldsymbol{\mathcal{P}}_4(\lambda_i) = P_4(\lambda_i)\boldsymbol{\rho}_1 + \frac{5}{8}\frac{7\lambda_i^3 - 3\lambda_i}{\rho_1 r_{\mathrm{d}_i}}(\boldsymbol{r}_{\mathrm{d}_i}\times\boldsymbol{\rho}_1)\times\boldsymbol{\rho}_1$$
$$= \frac{1}{8}\left[70\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^4}{\rho_1^4 r_{\mathrm{d}_i}^4} - 45\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^2}{\rho_1^2 r_{\mathrm{d}_i}^2} + 3\right]\boldsymbol{\rho}_1$$
$$-\frac{5}{8}\left[7\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i})^3}{\rho_1^2 r_{\mathrm{d}_i}^4} - 3\frac{\boldsymbol{\rho}_1\cdot\boldsymbol{r}_{\mathrm{d}_i}}{r_{\mathrm{d}_i}^2}\right]\boldsymbol{r}_{\mathrm{d}_i}. \tag{19}$$

In the case of a non-inclined tidal bulge (when $\boldsymbol{r}_{\mathrm{d}_i} = \pm\boldsymbol{\rho}_1$) the equations listed above become much simpler. This happens if the stars are inviscid or if their revolution and rotation are exactly synchronized. Nevertheless, while the first possibility is unphysical, the second can never happen in such a hierarchical triple system where the tertiary is close enough to significantly perturb the motion of the binary. Consequently, a model of the stellar dissipation is evidently necessary for the description of the dynamics of a hierarchical triple system. Since in the present work our aim is to model the behaviour of almost "relaxed" triple systems (with moderate eccentricities and weakly inclined rotational axes), here we use a very simplified model for the tidal friction. According to this model the star adjusts its form to the gravitational potential with a small time delay or lag time (Δt) which is proportional to the time scale of dissipation. The dissipation mechanism itself comes from the interaction between the convective motions and the tidal flow (Zahn 1966, 1989). This is the so-called "equilibrium-tide" approximation. (For a short description see Eggleton et al. 1998.) Although the physical background of this lag time is not exactly clear, Eggleton et al. (1998) showed that this mathematical description is equivalent with the assumption that the rate of the energy loss is some positive-definite function of the rate of change of the shape of the star (in the co-rotating coordinate system). In this case $\boldsymbol{r}_{\mathrm{d}}$ can be written as

$$\boldsymbol{r}_{\mathrm{d}} = \boldsymbol{\rho}_1 + \boldsymbol{\rho}_1'\Delta t + \frac{1}{2}\boldsymbol{\rho}_1''(\Delta t)^2 + ..., \tag{20}$$

where $'$ denotes the time derivate in the co-revolving (primed) system. It can be seen easily that for the angle ϵ of the tidal lag:

$$\cos\epsilon = 1 - \frac{1}{2}\frac{(\boldsymbol{p}\times\boldsymbol{\rho}_1)^2}{\rho_1^2}(\Delta t)^2, \tag{21}$$

$$\sin\epsilon = \frac{|\boldsymbol{p}\times\boldsymbol{\rho}_1|}{\rho_1}\Delta t, \tag{22}$$

where $\boldsymbol{p} = \boldsymbol{\omega} - \boldsymbol{w}$. Using this simple model Eq. (15) can be replaced by the following equation:

$$\ddot{\boldsymbol{\rho}}_1 = Gm_3\left(\frac{\boldsymbol{r}_{23}}{r_{23}^3} - \frac{\boldsymbol{r}_{13}}{r_{13}^3}\right) - \frac{Gm_{12}}{\rho_1^3}\left\{\boldsymbol{\rho}_1\right.$$
$$+\sum_{i=1}^{2}\left\{\frac{m_{3-i}}{m_i}k_2^{(i)}\left(\frac{R_i}{\rho_1}\right)^5 6\left[\boldsymbol{\rho}_1 - \left(3\frac{\boldsymbol{\rho}_1\cdot\dot{\boldsymbol{\rho}}_1}{\rho_1^2}\boldsymbol{\rho}_1 - \boldsymbol{p}_i\times\boldsymbol{\rho}_1\right)\Delta t_i\right]\right.$$
$$\left.\left.+\frac{k_2^{(i)}R_i^5}{Gm_i}\left\{\left[\frac{\omega_{z_i'}^2}{\rho_1^2} - 5\frac{(\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z_i'})^2}{\rho_1^4}\right]\boldsymbol{\rho}_1 + \frac{2\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z_i'}}{\rho_1^2}\boldsymbol{\omega}_{z_i'}\right\}\right\}\right\}. \tag{23}$$

(For the sake of simplicity we did not list the tidal terms multiplied by k_3, k_4.) Comparing this equation with Eq. (45) in Eggleton et al. (1998) it can be seen that the two expressions are equal if

$$\Delta t_i = -\frac{3}{2}\sigma_i\frac{k_2^{(i)}R_i^5}{G}, \tag{24}$$

where σ_i is the dissipation constant. For a better comparison we also express this small time lag by the dimensionless dissipation constant λ_i defined by Eq. (13) in Kiseleva et al. (1998):

$$\Delta t_i = -\frac{3}{8}\sqrt{\frac{R_i^3}{Gm_i}}\left(1 + 2k_2^{(i)}\right)^2\lambda_i. \tag{25}$$

Equations (15)–(23) are coupled with the (Eulerian) equations of the stellar rotation through the centrifugal and the dissipative terms. Consequently, we have to integrate these latter equations simultaneously with the orbital equations.

2.3. Equations of the stellar rotation

In order to get the equations of the rotation, let us start from the equation of the total angular momentum of the triple star in an inertial frame of reference, which has the following form (see e.g., Truesdell & Toupin 1960).

$$\boldsymbol{L} = \sum_{i=1}^{3}\left[\mathsf{I}_i'\boldsymbol{\omega}_i + \boldsymbol{L}_i' + \boldsymbol{x}_i\times(m_i\dot{\boldsymbol{x}}_i)\right], \tag{26}$$

where I_i' and $\boldsymbol{L}_i'$ denote the expressions of the inertia tensor and the angular momentum respectively with respect to the origin of the current star in the inertial coordinate system. As the total angular momentum is conserved, we have

$$\sum_{i=1}^{3}\left[\mathsf{I}'_i\dot{\boldsymbol{\omega}}_i+\dot{\mathsf{I}}'_i\boldsymbol{\omega}_i+\dot{\boldsymbol{L}}'_i+\boldsymbol{x}_i\times(m_i\ddot{\boldsymbol{x}}_i)\right]=0. \tag{27}$$

The last term of the above sum can be written in the more usual form by using Jacobian coordinates (hiding the centre of mass integrals) as follows:

$$\sum_{i=1}^{3}\boldsymbol{x}_i\times(m_i\ddot{\boldsymbol{x}}_i)=\frac{m_1m_2}{m_{12}}\boldsymbol{\rho}_1\times\ddot{\boldsymbol{\rho}}_1+\frac{m_{12}m_3}{m_{123}}\boldsymbol{\rho}_2\times\ddot{\boldsymbol{\rho}}_2. \tag{28}$$

Furthermore, as the third (more distant) companion is treated as a point mass, the summation in the first two terms must be taken only for the two members of the close binary. In the following we omit the primes and the i indices. We calculate the terms of Eq. (27) for the primary star only.

Equation (27) is nothing more than the extended form of the well-known Eulerian equations of a deformable body in the unprimed (fixed) system as follows (see, e.g., Tokis 1974).

$$\begin{aligned}\mathsf{I}\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times\mathsf{I}\boldsymbol{\omega}+\dot{\mathsf{I}}\boldsymbol{\omega}-\boldsymbol{\omega}\times\int\boldsymbol{r}\times\dot{\boldsymbol{r}}'_0\mathrm{d}m+\int\boldsymbol{r}\times\ddot{\boldsymbol{r}}'_0\mathrm{d}m=\\ \int\boldsymbol{r}\times\nabla\Psi\mathrm{d}m+\int\boldsymbol{r}\times\nabla\mathcal{I}\mathrm{d}V,\end{aligned} \tag{29}$$

where the different terms have the following meaning:
I – is the inertia tensor of the current star,
$\dot{\mathsf{I}}\boldsymbol{\omega}-\boldsymbol{\omega}\times\int\boldsymbol{r}\times\dot{\boldsymbol{r}}'_0\mathrm{d}m$ – is the gyroscopic term (which includes the Coriolis acceleration $2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}'_0$).
$\int\boldsymbol{r}\times\ddot{\boldsymbol{r}}'_0\mathrm{d}m$ – is arises from acceleration of deformation $\ddot{\boldsymbol{r}}'_0$,
$\int\boldsymbol{r}\times\nabla\Psi\mathrm{d}m$ – is arises from the centrifugal potential, and
$\int\boldsymbol{r}\times\nabla\mathcal{I}\mathrm{d}V$ – is contains the dissipative term.

The calculation of the explicit form of the following terms can be found in Kopal (1978, Chap. IV). Despite this, for the self-consistency of this paper we repeat here the basic steps of the calculations. Nevertheless, we mostly give only the results except for places where our approximations differ from the previous studies. These differences appear in the calculation of the rhs of Eq. (29).

2.3.1. The inertia tensor

As the configuration of the system changes in time, so does the shape of the bodies. Consequently, the moment of inertia of the stars is not constant. However, even if the shape of the bodies were unchanged, the elements of the inertia tensor would vary due to the variation of the stars' orientation with respect to the fixed coordinate system.

In our approximation every mass element of the star belongs to one and only one equipotential surface, so the r polar coordinate of an element can be written as

$$r=R\left(1+\sum_{j=2}^{4}f_j+g_2\right), \tag{30}$$

where $f_{2..4}$ are the amplitudes of the first order tidal deformations, which can be described with the following spherical harmonics:

$$f_j=(1+2k_j)\frac{m_2}{m_1}\left(\frac{R}{\rho_1}\right)^{j+1}P_j(\lambda'')=K_jR^{j+1}P_j(\lambda''), \tag{31}$$

for $j=2,3,4$, where P_j means the Legendre polynomial of jth order in λ'', which is the direction cosine between the direction of the tidal bulge (which is supposed to coincide with the x''-axis of the co-revolving system) and an arbitrary vector. (In this subsection we concentrate on the primary of the close binary, so we omit the subscript $_i$. Similar equations can be written for the secondary, too.) The first order centrifugal distortion can be described in the same way with the following term:

$$g_2=-\frac{\omega_{z'}^2R^3}{3Gm_1}P_2(\nu')=G_2R^3P_2(\nu'), \tag{32}$$

where ν' is the direction cosine between the z' axis of rotation and an arbitrary vector.

Here we show the method of calculation for one of the elements of the inertia tensor. Let this element be I_{11}, usually denoted by A. By definition

$$I_{11}=\int_m(y^2+z^2)\mathrm{d}m, \tag{33}$$

or replacing the Cartesian coordinates by spherical polar ones

$$I_{11}=\int_r\int_\theta\int_\phi r^2(1-\lambda^2)\rho r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\phi, \tag{34}$$

where

$$\lambda=\frac{x}{r}=\cos\phi\sin\theta. \tag{35}$$

According to Eq. (30) the mass element can be written as

$$\mathrm{d}m=R^2\left[1+\sum_{j=2}^{4}(j+4)f_j+6g_2\right]\mathrm{d}R\sin\theta\mathrm{d}\theta\mathrm{d}\phi. \tag{36}$$

Using this the integrand has the following form:

$$I_{11}=\int\rho R^4\mathrm{d}R\int\left[1+\sum_{j=2}^{4}(j+6)f_j+8g_2\right](1-\lambda^2)\mathrm{d}\sigma. \tag{37}$$

As the last step we have to express the direction cosines λ'' and ν' in the fixed system as follows:

$$\lambda''=a''_{11}\lambda+a''_{21}\mu+a''_{31}\nu, \tag{38}$$

and

$$\nu'=a'_{13}\lambda+a'_{23}\mu+a'_{33}\nu. \tag{39}$$

Substituting these into Eq. (37) and evaluating the integral we get the first member of the inertia tensor. Using the above method the elements of the inertia tensor will be the following:

$$\begin{aligned}I_{ii}=&\frac{8}{3}\pi\int_0^{R_1}\rho R^4\mathrm{d}R\\ &-\frac{64}{15}\pi[K_2P_2(a''_{i1})+G_2P_2(a'_{i3})]\int_0^{R_1}\rho R^7\mathrm{d}R,\end{aligned} \tag{40}$$

$$I_{ik}=-\frac{32}{5}\pi[K_2a''_{i1}a''_{k1}+G_2a'_{i3}a'_{k3}]\int_0^{R_1}\rho R^7\mathrm{d}R, \tag{41}$$

where R_1 denotes the surface value of R. The first expression on the rhs of Eq. (40) defines the so-called gyration radius as

$$m\mathcal{R}^2 = \frac{8}{3}\pi\int_0^{R_1}\rho R^4\mathrm{d}R. \tag{42}$$

This quantity in the present version of our code was calculated by using the crude approximation of Motz (1952), as

$$\log\mathcal{R} = 0.24\log k_2 - 0.13 + \log R_1. \tag{43}$$

Instead of the direct calculations of the other integrals we also used an approximate form which is valid in the case of strong central condensation, as

$$\int_0^{R_1}\rho R^{2j+3}\mathrm{d}R = m_1\frac{(2j+1)k_j}{4\pi(j+2)}R_1^{2j+1}. \tag{44}$$

Using the above equations, the first two terms in the lhs of Eq. (29) can be written into the following vectorial form:

$$\begin{aligned}\mathsf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathsf{I}\boldsymbol{\omega} = &\left[m\mathcal{R}^2 + \frac{8}{3}\alpha(K_2+G_2)\right]\dot{\boldsymbol{\omega}}\\ &-8\alpha\left\{K_2\left[\frac{\boldsymbol{\rho}_1\cdot\dot{\boldsymbol{\omega}}}{\rho_1^2}\boldsymbol{\rho}_1 + \frac{\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}}{\rho_1^2}\boldsymbol{\omega}\times\boldsymbol{\rho}_1\right]\right.\\ &\left.+G_2\left[\frac{\boldsymbol{\omega}_{z'}\cdot\dot{\boldsymbol{\omega}}}{\omega_{z'}^2}\boldsymbol{\omega}_{z'} + \frac{\boldsymbol{\omega}_{z'}\cdot\boldsymbol{\omega}}{\omega_{z'}^2}\boldsymbol{\omega}\times\boldsymbol{\omega}_{z'}\right]\right\},\end{aligned} \tag{45}$$

where

$$\alpha = \frac{1}{4}k_2 m R_1^5. \tag{46}$$

2.3.2. The gyroscopic and accelerating terms

The third and fourth terms of the lhs of Eq. (29) describe the Coriolis force raised by the tides on the rotating star. It can be easily shown (see, e.g., Tokis 1974) that the velocity of the deformation with respect to the co-rotating system (expressed in the fixed frame of reference) is the following:

$$\dot{\boldsymbol{r}}_0' = f'\boldsymbol{r} - f\boldsymbol{p}\times\boldsymbol{r}, \tag{47}$$

where

$$f' = \frac{\partial f}{\partial t} - \boldsymbol{p}\cdot(\boldsymbol{r}\times\nabla)f. \tag{48}$$

Furthermore, it is can also be easily seen that the acceleration of the deformation can be written as:

$$\ddot{\boldsymbol{r}}_0' = f''\boldsymbol{r} - 2f'\boldsymbol{p}\times\boldsymbol{r} - f\boldsymbol{p}'\times\boldsymbol{r} + f\boldsymbol{p}\times(\boldsymbol{p}\times\boldsymbol{r}). \tag{49}$$

After substitution and integration we get that the integral which contains the Coriolis force is the following:

$$\mathcal{I}_\mathrm{C} = 2\boldsymbol{\omega}\int f'r^2\mathrm{d}m - 2\int(\boldsymbol{r}\cdot\boldsymbol{\omega})(f'\boldsymbol{r} - f\boldsymbol{p}\times\boldsymbol{r})\mathrm{d}m, \tag{50}$$

while the accelerating terms become as follows:

$$\begin{aligned}\mathcal{I}_\mathrm{a} = &\ 2\int(\boldsymbol{r}\cdot\boldsymbol{p})(f'\boldsymbol{r} - f\boldsymbol{p}\times\boldsymbol{r})\mathrm{d}m - 2\boldsymbol{p}\int f'r^2\mathrm{d}m\\ &+\int f(\boldsymbol{r}\cdot\dot{\boldsymbol{p}})\boldsymbol{r}\mathrm{d}m - \dot{\boldsymbol{p}}\int fr^2\mathrm{d}m.\end{aligned} \tag{51}$$

2.3.3. The effect of the disturbing potential of the companion

Following the assumptions briefly listed in Sect. 2, it can be seen (see, e.g., Kopal 1978, pp. 168–170) that the contribution of the pressure and the self-gravitating potential vanish. The terms arising from the interaction of the distorted star and the outer potential can be calculated easily as the inertial momenta of the outer forces acting upon the star. E.g. substituting the lowest order term of the perturbing potential

$$V_i' = \frac{Gm_{3-i}}{2\rho_1^3}r^2\left[3\frac{(\boldsymbol{r}\cdot\boldsymbol{\rho}_1)^2}{r^2\rho_1^2} - 1\right] \tag{52}$$

into the first integral on the rhs of Eq. (29) we find that

$$\int\boldsymbol{r}\times\nabla V_i'\mathrm{d}m = \frac{8\alpha_i m_{3-i}}{m_i\rho_1^5}(\boldsymbol{\rho}_1\cdot\boldsymbol{\omega}_{z_i'})\boldsymbol{\rho}_1\times\boldsymbol{\omega}_{z_i'}. \tag{53}$$

When the tidal bulge does not point exactly in the direction of the companion this gives an additional momentum as

$$\int\boldsymbol{r}\times\nabla V_{\mathrm{d}_i}'\mathrm{d}m = \frac{24G\alpha_i K_2^{(i)}m_{3-i}}{\rho_1^5 r_{d_i}^2}(\boldsymbol{r}_{\mathrm{d}_i}\cdot\boldsymbol{\rho}_1)\boldsymbol{r}_{\mathrm{d}_i}\times\boldsymbol{\rho}_1 \tag{54}$$

$$= \frac{144\alpha_i^2\left(K_2^{(i)}\right)^2\sigma_i}{(1+2k_2^{(i)})\rho_1^2}\boldsymbol{\rho}_1\times(\boldsymbol{\rho}_1\times\boldsymbol{p}_i). \tag{55}$$

Furthermore, the distant third body also gives a non-zero contribution which is the following:

$$\begin{aligned}\int\boldsymbol{r}\times\nabla V_{i3}' = &\frac{8\alpha_i m_3}{r_{i3}^5}\left[\frac{1}{m_i}(\boldsymbol{\omega}_{z_i'}\cdot\boldsymbol{r}_{i3})\boldsymbol{r}_{i3}\times\boldsymbol{\omega}_{z_i'}\right.\\ &\left.+3\frac{K_2^{(i)}G}{r_{\mathrm{d}_i}^2}(\boldsymbol{r}_{\mathrm{d}_i}\cdot\boldsymbol{r}_{i3})\boldsymbol{r}_{\mathrm{d}_i}\times\boldsymbol{r}_{i3}\right].\end{aligned} \tag{56}$$

2.3.4. The effect of dissipation

Considering the stars as linear viscous fluids, the stress tensor which arises in the Eulerian equation Eq. (29) has the following form (Tokis 1974):

$$\begin{aligned}\nabla\mathcal{I} = &\ \mu\left[\nabla^2\dot{\boldsymbol{r}}_0' + \frac{1}{3}\nabla(\nabla\cdot\dot{\boldsymbol{r}}_0')\right] + 2(\nabla\mu\cdot\nabla)\boldsymbol{r}_0'\\ &+\nabla\mu\times(\nabla\times\dot{\boldsymbol{r}}_0') - \frac{2}{3}\nabla\mu(\nabla\cdot\dot{\boldsymbol{r}}_0'),\end{aligned} \tag{57}$$

where μ denotes the coefficient of viscosity. We note that in a fluid star the velocity of the deformation (Eq. (47)) is not the fluid velocity which enters the viscous stress tensor. They coincide only for the radial component (see the discussion by Scharlemann 1981). Consequently, substituting Eqs. (47)

and (48) into the above equation gives only an approximation for the correct value, but admittedly an acceptable one, considering the uncertainties in the determination of the turbulent viscosity. Supposing that

$$\mu = \mu(r) \tag{58}$$

we get that

$$\begin{aligned} \boldsymbol{r} \times \nabla \mathcal{I} = & \ \mu_{\mathrm{r}} \left\{ \left[(\boldsymbol{r} \cdot \boldsymbol{p}) \boldsymbol{r} - r^2 \boldsymbol{p} \right] f_{\mathrm{r}} + r (\boldsymbol{r} \times \nabla) f' \right\} \\ & + \mu \left\{ \left[(\boldsymbol{r} \cdot \boldsymbol{p}) \, \boldsymbol{r} - r^2 \boldsymbol{p} \right] \nabla^2 f \right. \\ & + 2 \left[(\boldsymbol{r} \cdot \boldsymbol{p}) \nabla f - r f_{\mathrm{r}} \boldsymbol{p} \right] \\ & \left. + \frac{1}{3} (\boldsymbol{r} \times \nabla)(10 f' - f_{\mathrm{t}} + r f_{\mathrm{r}}') \right\}, \end{aligned} \tag{59}$$

where the subscripts $_\mathrm{r}$ and $_\mathrm{t}$ refer to the partial derivatives with respect to radius and time. After the integration we get that

$$\int \boldsymbol{r} \times \nabla \mathcal{I} = \frac{4}{5} \pi \frac{K_2^{(i)}}{\rho_1^2} \mathcal{V} \left(3 \boldsymbol{\rho}_1 \times (\boldsymbol{\rho}_1 \times \boldsymbol{p}_i) + 2 \rho_1^2 \boldsymbol{p}_i \right), \tag{60}$$

where

$$\mathcal{V} = \int_0^{R_i} \left(\frac{\partial \mu}{\partial r} r^6 + 6 \mu r^5 \right) \mathrm{d}r. \tag{61}$$

This integral can be also calculated from the applied stellar models. Nevertheless, as in the convective envelope of the late type stars the turbulent viscosity is larger by several orders than the viscosity in the other regions of the star, we can integrate only for the convective zone. Furthermore, supposing that the viscosity coefficient is a continuous function of the radius, by partially integrating the above formula we obtain

$$\begin{aligned} \int \boldsymbol{r} \times \nabla \mathcal{I} = & \ \frac{4}{5} \pi \frac{K_2^{(i)}}{\rho_1^2} \left[\mu_{\mathrm{T}}(r) r^6 \right]_{R_{\mathrm{b}}}^{R_{\mathrm{u}}} \\ & \times \left[3 \boldsymbol{\rho}_1 \times (\boldsymbol{\rho}_1 \times \boldsymbol{p}) + 2 \rho_1^2 \boldsymbol{p} \right], \end{aligned} \tag{62}$$

where R_b and R_u denote the lower and upper boundary of the convective zone, respectively.

A comparison of Eq. (62) with the lag moment in Eq. (55) shows that the connection between $\mathcal{V}_i$ and σ_i has the following form:

$$\mathcal{V} = \frac{540}{\pi} \frac{\alpha_i^2 K_2^{(i)}}{1 + 2k_2^{(i)}} \sigma_i. \tag{63}$$

2.3.5. The final form of the fundamental equation of rotation

Substituting the results of the previous subsections into Eq. (29) we have the following form of the equation of stellar rotation.

$$\begin{aligned} & \left[m_i \mathcal{R}_i^2 + \frac{1}{3} \alpha \left(7 K_2^{(i)} + 8 G_2^{(i)} \right) \right] \dot{\boldsymbol{\omega}}_i \\ & \quad -8 \alpha_i \left(K_2^{(i)} \frac{\boldsymbol{\rho}_1 \cdot \dot{\boldsymbol{\omega}}_i}{\rho_1^2} \boldsymbol{\rho}_1 + G_2^{(i)} \frac{\boldsymbol{\omega}_{z_i'} \cdot \dot{\boldsymbol{\omega}}_i}{\omega_{z_i'}^2} \boldsymbol{\omega}_{z_i'} \right) = \alpha_i \boldsymbol{J}_i, \end{aligned} \tag{64}$$

where

$$\begin{aligned} \boldsymbol{J}_i = & \ K_2^{(i)} \left[\frac{(7 \boldsymbol{\omega}_i + \boldsymbol{w}) \cdot \boldsymbol{\rho}_1}{\rho_1} \boldsymbol{\omega}_i \times \boldsymbol{\rho}_1 + \frac{\boldsymbol{p}_i \cdot \boldsymbol{\rho}_1}{\rho_1^2} \boldsymbol{w} \times \boldsymbol{\rho}_1 \right. \\ & \left. + \frac{(\dot{\boldsymbol{w}} + 2 \boldsymbol{w} \times \boldsymbol{\omega}_i) \cdot \boldsymbol{\rho}_1}{\rho_1^2} \boldsymbol{\rho}_1 - \frac{1}{3} (\dot{\boldsymbol{w}} + 2 \boldsymbol{w} \times \boldsymbol{\omega}_i) \right] \\ & + 2 H_2^{(i)} \left(\frac{\boldsymbol{w} \cdot \boldsymbol{\rho}_1}{\rho_1^2} \boldsymbol{\rho}_1 - \frac{1}{3} \boldsymbol{w} \right) + 8 G_2^{(i)} \boldsymbol{\omega}_i \times \boldsymbol{\omega}_{z_i'} \\ & + 8 \frac{m_{3-i}}{m_i} \frac{\boldsymbol{\rho}_1 \cdot \boldsymbol{\omega}_{z_i'}}{\rho_1^5} \boldsymbol{\rho}_1 \times \boldsymbol{\omega}_{z_i'} + 8 \frac{m_3}{m_i} \frac{\boldsymbol{r}_{i3} \cdot \boldsymbol{\omega}_{z_i'}}{\boldsymbol{r}_{i3}^5} \boldsymbol{r}_{i3} \times \boldsymbol{\omega}_{z_i'} \\ & + \frac{288 \alpha_i \left(K_2^{(i)} \right)^2 \sigma_i}{(1 + 2 k_2^{(i)}) \rho_1^2} \left[\boldsymbol{\rho}_1 \times (\boldsymbol{\rho}_1 \times \boldsymbol{p}_i) + \frac{1}{3} \rho_1^2 \boldsymbol{p}_i \right]. \end{aligned} \tag{65}$$

In this latter equation H_2 denotes the partial time derivative of the tidal amplitude K_2, as

$$H_2^{(i)} = -3 K_2^{(i)} \frac{\boldsymbol{\rho}_1 \cdot \dot{\boldsymbol{\rho}}_1}{\rho_1^2}. \tag{66}$$

The above equations can be solved only numerically in their complexity. Nevertheless, after applying some simplifications analytical solutions can be given also. As we mainly concentrate on the perturbations in the orbital motion of the binary as well as of the third companion, the only importance from our point of view of these equations of rotation is that they give the instantaneous value of the direction of the rotational axes, as well as the angular velocity of the rotation.

Equation (64) is first order in the components of $\boldsymbol{\omega}$, nevertheless in the integration we calculate directly the Eulerian angles θ, ϕ, ψ. For these we get second order ordinary differential equations which are listed in Appendices B and C. These latter equations are integrated together with the orbital equations of motion Eqs. (15), or (23) and (16).

3. The method of integration

The method of integration was already described in Borkovits et al. (2002). Here we confine ourselves to the basic properties of the code. Our 24 time-dependent variables are as follows:

- the six components of the two Jacobian position vectors, and the six components of the corresponding velocities;
- the two sets of the usual Eulerian angles for both the primary and the secondary component, together with their time derivatives,

while the other input parameters are

- the three masses;
- the two radii;
- the six apsidal motion constants $\left(k_{2..4}^{(1,2)} \right)$;
- the two dimensionless dissipation constants $(\lambda_{1,2})$.

The values of the dependent variables were calculated by direct integration of the equations of motion by a seventh order Runge-Kutta-Nyström integrator (Fehlberg 1974).

The determination of the osculating orbital elements from the Cartesian coordinates and velocities poses no difficulty.

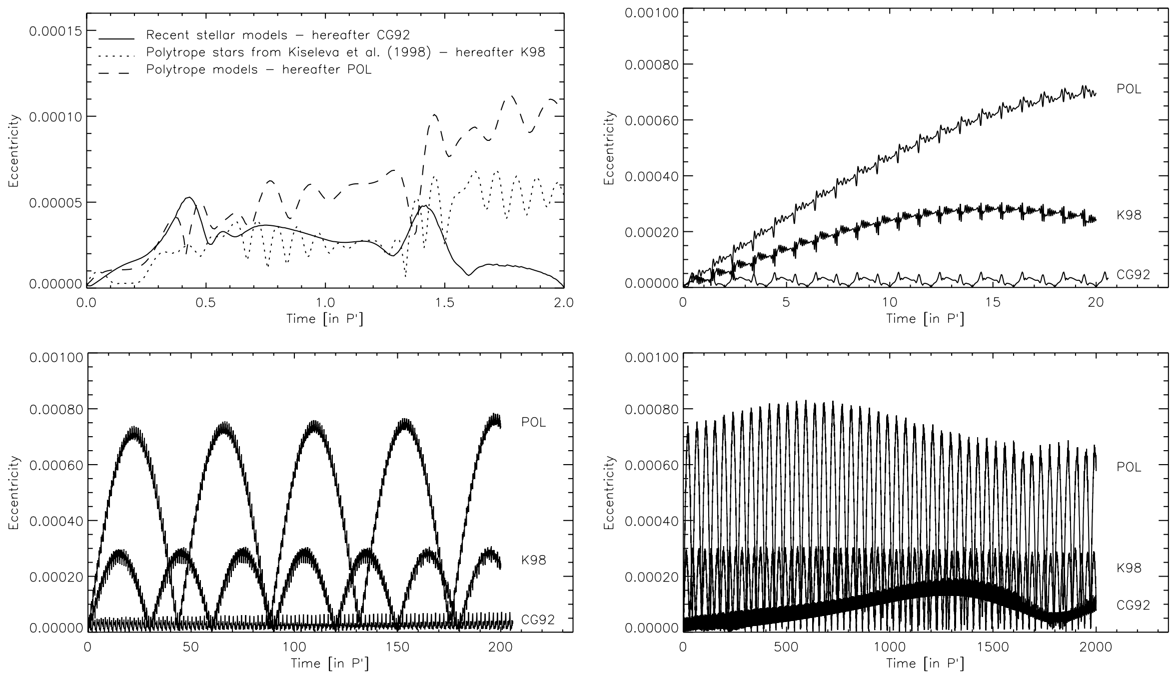

Fig. 3. The variation of the "super-osculating" eccentricity of the close binary of Algol during 2, 20, 200, and 2000 revolutions of the third companion, with three different sets of apsidal motion constants. (See text for details.) The eccentricity was sampled at the integration step that was the closest to the pericenter of the inner binary. In the first 10 000 days the sampling was done at every pericentrum passage, then at every tenth.

Two sets of orbital elements were calculated. Beside the commonly used osculating orbital elements we calculated a second set of "orbital elements", which give a better description of the real orbit. For the determination of this set of orbital parameters we formally replaced the mass parameter $\mu_1 = Gm_{12}$ by the $\mu_1^* = -\boldsymbol{\rho}_1 \cdot \ddot{\boldsymbol{\rho}}_1 \rho_1$ expression. The main advantage of this choice is that if the orbit is exactly circular, then these "elements" give the real orbit of the binary at any moment, which in turn is not true for the osculating elements (e.g. the eccentricity of the osculating orbit will never be zero), while further reasons are also described in Borkovits et al. (2002).

In this paper we mainly concentrate on the possible observational consequences of the perturbed motion of the eclipsing binary. For this reason, the angular orbital elements were calculated with respect to the plane of the sky throughout this work.

4. An application for Algol

First we applied the method to the well-known triple system Algol, one of the two stars considered in the similar work of Kiseleva et al. (1998). This gives an opportunity of checking our equations on the one side, and shows the differences between the methods on the other side. So for the first runs we used the parameters for the β Persei system which were given in Kiseleva et al. (1998). However, we have to note that setting the mutual inclination of the orbits to 100°, as was done in the above mentioned work, results in a too fast variation in the observable inclination of the close system, which is inconsistent with the observations (see, e.g., Söderhjelm 1980). For these first runs we used three different sets of the k_2, k_3 apsidal motion constants (see Table 1). In the first run

Table 1. The apsidal motion constants for the Algol AB stars in three different runs.

Code	$k_2^{(1)}$	$k_3^{(1)}$	$k_2^{(2)}$	$k_3^{(2)}$
Pol	0.0144	0.00368	0.1433	0.0529
K98	0.0288	0.00736	0.2866	0.1058
CG92	0.0038	0.0011	0.0240	0.0087

(denoted by "Pol"), a simple $n = 1.5$ polytrope model was applied for the primary, and one with $n = 3$ for the secondary (see, e.g., Finlay-Freundlich 1958). The k_2 values, which can be calculated from the Q parameters of Kiseleva et al. (1998) ("K98" run) differ from the polytropic ones with a factor of 2, which is a "result" of some mistakes in the equation of motion in Kiseleva et al. (1998), as was pointed out in Eggleton & Kiseleva-Eggleton (2001). Nevertheless, these values seem to be unrealistically large if one compares them with the tables of Claret & Gimènez (1992), which were used to run "CG92". Figure 3 is analogous to Figs. 3b–e of Kiseleva et al. (1998). Comparing the "K98" curve in our Figs. 3a to 3b of the aforementioned paper the agreement is almost perfect (apart from the different sampling). However, the next panels which cover an approx. 37 year-long time interval show completely different behaviour of the eccentricity variation. This is because the stellar equators are "frozen" into the orbital plane of the close pair in Kiseleva et al. (1998), i.e. no stellar precession is allowed. In our physically more realistic case the orbital precession of the plane of the close binary driven by the effect of the aligned tertiary forces a slight precession in the rotation of the members

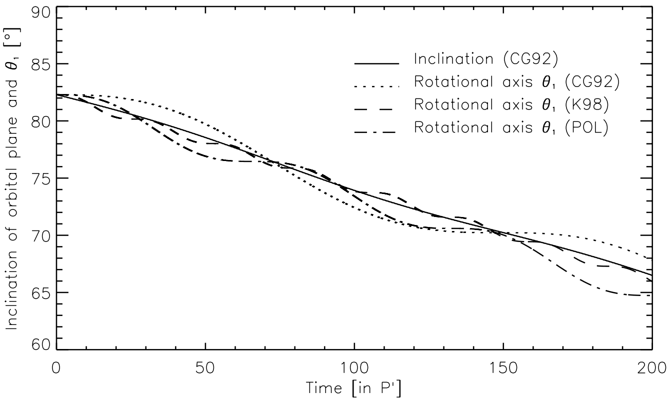

Fig. 4. The precession of component A of Algol. We plotted the variation of the inclination i of the binary together with the variation of θ_1, the inclination of the rotational axis of Algol A, for different mass distributions.

of the close binary which reacts to the orbital eccentricity (as well as to the other orbital elements). This effect in the present configuration can increase up the eccentricity variation even by one order of magnitude in the polytropic cases, although it remains almost inefficient in the recent centrally more condensed stellar models ("CG92" run). In Fig. 4 we plotted the variation of the inclination-equivalent Eulerian angle θ_1 of component A of Algol during 200 revolutions of the outer body in the different models. As a reference, the variation of the visible orbital inclination i of the close pair is also shown in the "CG92" run. (The variation of θ_2, as well as the inclination in the other runs are very close to the reference inclination line, so they are not drawn.) We carried out the integration on a longer time-scale, too. This is important, because the behaviour of such highly non-linear dynamical systems may be completely different on different time-scales. It is well-known e.g. that in the frame of mass-point dynamics the present configuration of this triple system would be no longer stable, due to the Kozai resonance (see, e.g., Kiseleva et al. 1998; Borkovits 2001, and further references therein). However, in close systems the oblateness of the components may eliminate this instability, as was found first by Söderhjelm (1984). This happens also in the present situation. Nevertheless, as can be seen in Fig. 5, in the case of the physically more realistic "CG92" run, departing from synchronized rotation and revolution, the rotational angular velocities of the components of Algol begin to show sudden fluctuations within 50 000 years, which react on some of the orbital elements, too. We note that the chaotic behaviour of the rotation of the planets in our Solar system has already been extensively studied since the early '90s, and it was found that the phase spaces of the terrestrial planets contain large chaotic regions, (see, e.g., Laskar & Robutel 1993).

In what follows we "switched on" the dissipation. We set $\lambda_1 = 10^{-4}$, $\lambda_2 = 10^{-3}$, which correspond to a tidal lag time of $\Delta t_1 \approx 2\overset{\mathrm{d}}{.}5 \times 10^{-6}$, and $\Delta t_2 \approx 5\overset{\mathrm{d}}{.}8 \times 10^{-5}$, respectively. The other stellar parameters were identical to the values of the "CG92" run, while the orbital and rotational parameters were chosen to be equal to the corresponding parameters of the "CG92" run at $t = 0$, and $t \approx 104\,000$ years, i.e. in the first case ("CG92d1") the close system departed from a synchronized state, while

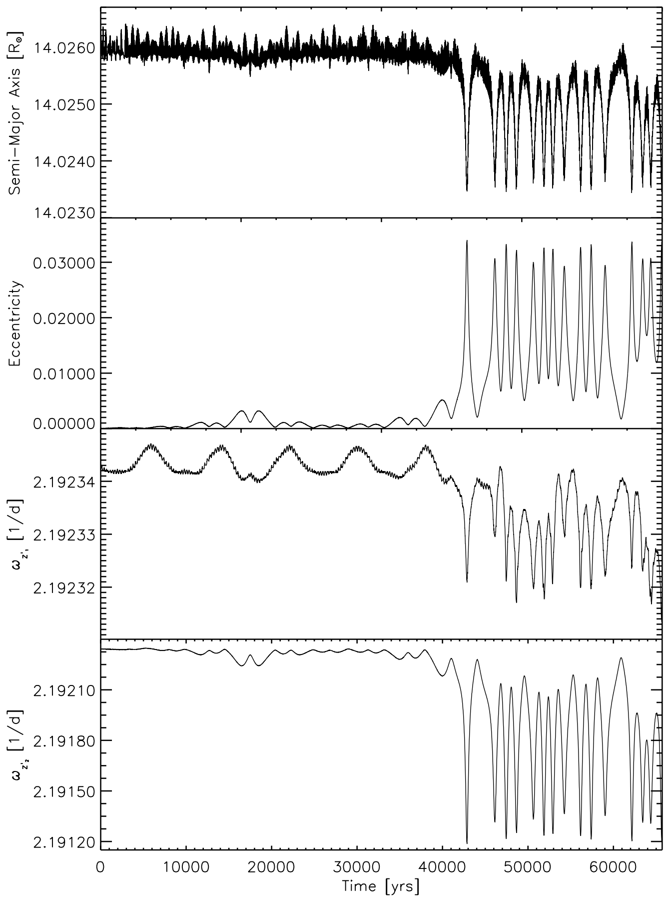

Fig. 5. The variation of the semi-major axis, eccentricity, and the rotational angular velocities of the components in the close pair of Algol in the "CG92" run.

in the second one ("CG92d2") the above mentioned fluctuations were already present. In the first case the large fluctuations of the non-dissipative run did not appear in the orbital elements, only small quasi-cyclic variations were found of the order of 10^{-5} in the eccentricity, and 10^{-5} $R_\odot$ in the semi-major axis. (Note that the perturbation of the semi-major axis of the binary in every pericenter and/or nodal-line passage of the tertiary may be larger by as much as two orders of magnitude.)

The second case can model the relaxation of such a triple system after some moderate perturbations (e.g. mass-transfer, or engulfing a giant planet, etc.). As is seen in Fig. 6, in this latter case the small lag nearly circularized the orbit, and resynchronized the rotation of the more dissipative secondary on a time-scale of 10 000 years, while for the less dissipative primary the synchronization time is naturally longer; in the present run we extrapolated it as some 10^6 years. Nevertheless, even after the whole time interval of our some 10^5 year-long integration a small periodic variation in the orbital as well as in the rotational angular momenta was present, with a period that was exactly the half of the precession period of the orbital plane. This change in the orbital angular momentum gives a period change rate $\dot{P}/P$ of the order of 10^{-10} d^{-1}. This magnitude of the secular period change is found in several Algol-type eclipsing binaries, and usually explained as a consequence of

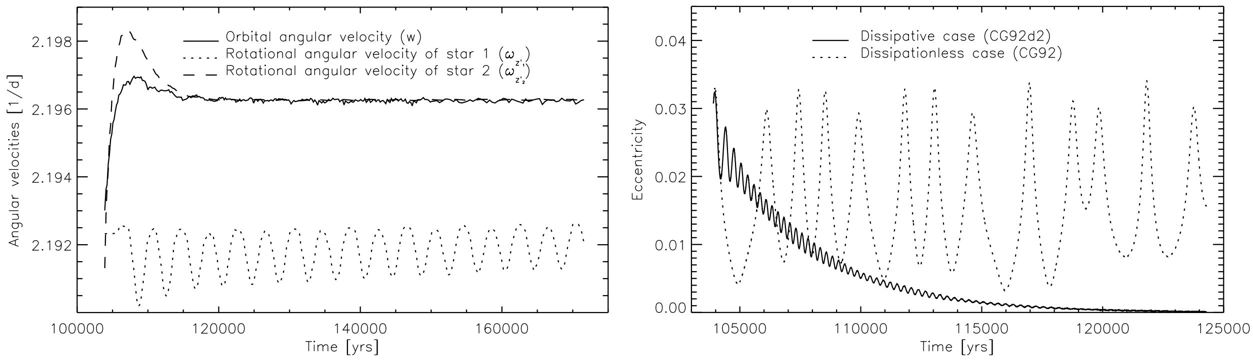

Fig. 6. The effect of the stellar viscosity in the "CG92d2" run. *Left panel*: the fast synchronization of the orbital and rotational angular velocities. *Right panel*: circularization of the inner orbit. Here we also plotted the variation of the eccentricity in the non-dissipative case.

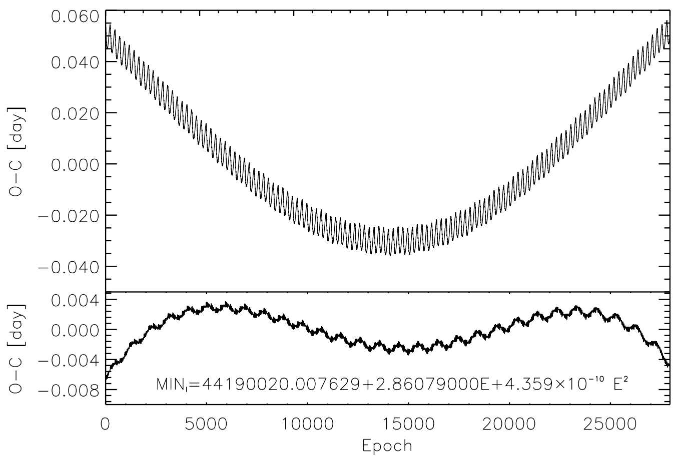

Fig. 7. The "O–C curve" of the Algol system derived from the "CG92d2" run (*upper part*). The thickness of the parabola mainly comes from the pure geometrical light-time effect of the tertiary. In the lower panel the residual curve is plotted, as obtained after the removal of the parabolic as well as the light-time terms.

some mass-transfer or mass-loss events, although in some systems this explanation does not seem to be plausible. (A recent case is e.g. IM Aur, see Borkovits et al. 2002.) We calculated eclipsing times of minima for an approx. 200 year-long interval between $t = 4\overset{\mathrm{d}}{.}419 \times 10^7$, and $t = 4\overset{\mathrm{d}}{.}427 \times 10^7$ from the "CG92d2" run. (We chose this region because during this time the observable inclination varied between $i \approx 82°$ and $i \approx 72°$, so the system was really an eclipsing one.) Then we determined the period variations by the use of the O–C diagram, as is usual. We plotted this curve as well as the residual curve after the simultaneous subtraction of the parabolic and light-time terms in Fig. 7.

Finally we also integrated the motion of this triple in the almost perpendicular case. This configuration is not so interesting dynamically, because as the mutual inclination tends to 90° the period of the orbital precession tends to infinity; furthermore, as the normal component of the perturbing force does not affect the semi-major axis and the eccentricity, the perturbations of the third body on these elements are minimal with respect to other configurations. Nevertheless, as was already mentioned, the observations suggest such a scenario. In our run the mutual inclination was set to $i_{\mathrm{m}} = 89°$, which produced an orbital precession with a period of $P_{\mathrm{node}} = 10^5$ y, which is about 12 times longer than in the previous runs. In this case the so-called long period perturbations of the tertiary were dominant. These perturbations were the subject of a previous work (Borkovits et al. 2003).

5. Conclusion

We studied the motion of close binaries in triple stellar systems combining the effects of the gravitational perturbations of the tertiary as well as the influence of the stellar distortion, arbitrary stellar rotation, and tidal friction. Beside the questions of the dynamical evolution of such systems we concentrated mainly on those short-term effects which might be observable in the present or at least in the near future. This is the main reason while an eclipsing system was chosen as an example. From that point of view the short fluctuations found in the Algol system in some of our integrations might be of importance. On one side, this shows that chaotic variations may also be present even in dynamically relaxed systems in the presence of a tertiary component, and so a new explanation of several non-understood eclipsing O–C diagrams, or other pecularities of specific systems may arise. We refer e.g. for the young eclipsing binary TY CrA which is also a member of a triple system, and is found to be circularized, but not synchronized. In this case Beust et al. (1997) have already shown that this situation may be a consequence of the combined effect of the tidal and third body interactions. On the other side, as was shown, the presence of dissipation eliminated this phenomenon. Nevertheless, in the present case we used a large, perhaps unrealistically large dissipation rate. We plan to perform further tests to study the effect of a smaller dissipation rate for this kind of perturbations. This would be especially important since the presence of this kind of perturbations could make it possible to measure the dissipation rate of stars in close triple systems.

In this paper we applied the method to a dynamically (almost) relaxed triple system. As was illustrated, even in this case the combined effects of the third-body and tidal perturbations may cause significant changes in the dynamical behaviour of the system which could produce observable phenomena. In the following paper we will extend our study to eccentric binaries.

Finally, we show a further aspect of why the short term, high-accuracy integration (which requires more and more complex dynamical models) of motion of concrete systems will be important in the near future. The small astronomical satellites of the next decades will produce enormous amounts of high precision data for millions of stars. From these satellites a precision of micro-magnitudes is expected. This (together with the relatively long observational interval) gives the possibility to observe physical effects that are so small that it has never before been possible to observe them. Of course this can be done only if all the presently known physical processes which can influence the observed light-curve (or radial velocity data, etc.) at the available accuracy are carefully taken into account. This is why the dynamical perturbations caused by a close third body, or some other effects (e.g. small stellar precession) already have to be considered. For example, in an earlier paper we showed that in the close hierarchical triple system IU Aur the perturbations of the third companion may affect the light-curve in the range of 10^{-4} mag within some months (Borkovits et al. 2004). Perhaps more important is the indirect problem, when the perturbations might be deduced from the light curve, and consequently we can get more exact third body parameters, or, what can be more important, further information on the tidal terms.

Keeping in mind this possibility we plan to improve the above described dynamical model with more accurate stellar models in the near future.

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Online Material

Appendix A: Direction cosines and angular velocities in the coordinate systems used

Let $\mathsf{A}'(t)$ and $\mathsf{A}''(t)$ be time-dependent orthogonal transformations which relate the coordinates of the unprimed and the single-primed and doubly-primed systems, respectively. Then it is well known that the elements of the A', A'' matrices are the following:

$$a'_{11} = \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta, \tag{A.1}$$
$$a'_{12} = -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta, \tag{A.2}$$
$$a'_{13} = \sin\phi\sin\theta, \tag{A.3}$$

$$a'_{21} = \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta, \tag{A.4}$$
$$a'_{22} = -\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta, \tag{A.5}$$
$$a'_{23} = -\cos\phi\sin\theta, \tag{A.6}$$

$$a'_{31} = \sin\psi\sin\theta, \tag{A.7}$$
$$a'_{32} = \cos\psi\sin\theta, \tag{A.8}$$
$$a'_{33} = \cos\theta, \tag{A.9}$$

where θ, ϕ, and ψ are the usual Eulerian angles (see Fig. 1), and

$$a''_{11} = \cos u\cos\Omega - \sin u\sin\Omega\cos i, \tag{A.10}$$
$$a''_{12} = -\sin u\cos\Omega - \cos u\sin\Omega\cos i, \tag{A.11}$$
$$a''_{13} = \sin\Omega\sin i, \tag{A.12}$$

$$a''_{21} = \cos u\sin\Omega + \sin u\cos\Omega\cos i, \tag{A.13}$$
$$a''_{22} = -\sin u\sin\Omega + \cos u\cos\Omega\cos i, \tag{A.14}$$
$$a''_{23} = -\cos\Omega\sin i, \tag{A.15}$$

$$a''_{31} = \sin u\sin i, \tag{A.16}$$
$$a''_{32} = \cos u\sin i, \tag{A.17}$$
$$a''_{33} = \cos i, \tag{A.18}$$

where i is the inclination of the close orbit, Ω is the longitude of the node, and u is the true longitude of the secondary component along its relative orbit. The corresponding "angular velocities" are the following:

$$\omega_x = \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi = \dot\theta\cos\phi + \dot\psi a'_{13} = a'_{2j}\dot a'_{3j} \tag{A.19}$$
$$\omega_y = \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi = \dot\theta\sin\phi + \dot\psi a'_{23} = a'_{3j}\dot a'_{1j} \tag{A.20}$$
$$\omega_z = \dot\psi\cos\theta + \dot\phi = \dot\psi a'_{33} + \dot\phi = a'_{1j}\dot a'_{2j} \tag{A.21}$$

$$w_x = i\cos\Omega + \dot u\sin i\sin\Omega = i\cos\Omega + \dot u a''_{13} = a''_{2j}\dot a''_{3j} \tag{A.22}$$
$$w_y = i\sin\Omega - \dot u\sin i\cos\Omega = i\sin\Omega + \dot u a''_{23} = a''_{3j}\dot a''_{1j} \tag{A.23}$$
$$w_z = \dot u\cos i + \dot\Omega = \dot u a''_{33} + \dot\Omega = a''_{1j}\dot a''_{2j} \tag{A.24}$$

Appendix B: Coefficients of the Eulerian angles

Equation (64) is a linear algebraic system of equations of the variables $\ddot\theta$, $\ddot\phi$, and $\ddot\psi$. Solving this we get three coupled non-linear second order differential equations for the Eulerian angles. It is convenient to introduce a further non-orthogonal coordinate system as follows:

$$\boldsymbol{n}' = \frac{\boldsymbol{z}\times\boldsymbol{z}'}{\sin\theta} = (\cos\phi, \sin\phi, 0), \tag{B.1}$$
$$\boldsymbol{z}' = (a'_{13}, a'_{23}, a'_{33}), \tag{B.2}$$
$$\boldsymbol{z} = (0, 0, 1). \tag{B.3}$$

The time-derivatives of these axes with respect to the fixed system are the following:

$$\dot{\boldsymbol{n}}' = \dot{\phi}\boldsymbol{z} \times \boldsymbol{n}', \tag{B.4}$$

$$\dot{\boldsymbol{z}}' = \boldsymbol{\omega} \times \boldsymbol{z}', \tag{B.5}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{0}, \tag{B.6}$$

where

$$\boldsymbol{z} \times \boldsymbol{n}' = (-\sin\phi, \cos\phi, 0). \tag{B.7}$$

Thus

$$\boldsymbol{\omega} = \dot{\theta}\boldsymbol{n}' + \dot{\psi}\boldsymbol{z}' + \dot{\phi}\boldsymbol{z} \tag{B.8}$$

$$\dot{\boldsymbol{\omega}} = (\ddot{\theta} + \dot{\psi}\dot{\phi}\sin\theta)\boldsymbol{n}' + \ddot{\psi}\boldsymbol{z}' + (\ddot{\phi} - \dot{\theta}\dot{\psi}\sin\theta)\boldsymbol{z} + \dot{\theta}(\dot{\phi} - \dot{\psi}\cos\theta)\boldsymbol{z} \times \boldsymbol{n}', \tag{B.9}$$

and finally

$$\begin{aligned}
&\ddot{\theta}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right]\boldsymbol{n}' - 7\alpha K_2(\boldsymbol{x}'' \cdot \boldsymbol{n}')\boldsymbol{x}''\right\} + \ddot{\phi}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right]\boldsymbol{z} - 7\alpha K_2(\boldsymbol{x}'' \cdot \boldsymbol{z})\boldsymbol{x}'' - 8\alpha G_2\cos\theta\boldsymbol{z}'\right\} \\
&\quad + \ddot{\psi}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)\right]\boldsymbol{z}' - 7\alpha K_2(\boldsymbol{x}'' \cdot \boldsymbol{z}')\boldsymbol{x}''\right\} + \dot{\psi}\dot{\phi}\sin\theta\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right]\boldsymbol{n}' - 7\alpha K_2(\boldsymbol{x}'' \cdot \boldsymbol{n}')\boldsymbol{x}''\right\} \\
&\quad + \dot{\theta}(\dot{\phi} - \dot{\psi}\cos\theta)\left\{-7\alpha K_2[\boldsymbol{x}'' \cdot (\boldsymbol{z} \times \boldsymbol{n}')]\boldsymbol{x}'' + 8\alpha G_2\sin\theta\boldsymbol{z}' + \left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right]\boldsymbol{z} \times \boldsymbol{n}'\right\} \\
&\quad - \dot{\theta}\dot{\psi}\sin\theta\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right]\boldsymbol{z} - 7\alpha K_2(\boldsymbol{x}'' \cdot \boldsymbol{z})\boldsymbol{x}'' - 8\alpha G_2\cos\theta\boldsymbol{z}'\right\} = \alpha\boldsymbol{J},
\end{aligned} \tag{B.10}$$

or writing this into components:

$$\begin{aligned}
&\ddot{\theta}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right] - 7\alpha K_2 x_{\boldsymbol{n}'}^{''2}\right\} + \ddot{\phi}\left\{-7\alpha K_2 a''_{31} x''_{\boldsymbol{n}'}\right\} + \ddot{\psi}\left\{-7\alpha K_2 x''_{\boldsymbol{z}'} x''_{\boldsymbol{n}'}\right\} + \dot{\psi}\dot{\phi}\sin\theta\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right] - 7\alpha K_2 x_{\boldsymbol{n}'}^{''2}\right\} \\
&\quad + \dot{\theta}(\dot{\phi} - \dot{\psi}\cos\theta)\left\{-7\alpha K_2 x''_{\boldsymbol{z}\times\boldsymbol{n}'} x''_{\boldsymbol{n}'}\right\} - \dot{\theta}\dot{\psi}\sin\theta\left\{-7\alpha K_2 a''_{31} x''_{\boldsymbol{n}'}\right\} = \alpha J_{\boldsymbol{n}'},
\end{aligned} \tag{B.11}$$

$$\begin{aligned}
&\ddot{\theta}\left\{-7\alpha K_2 x''_{\boldsymbol{n}'} x''_{\boldsymbol{z}'}\right\} + \ddot{\phi}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)a'_{33}\right] - 7\alpha K_2 a''_{31} x''_{\boldsymbol{z}'}\right\} + \ddot{\psi}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)\right] - 7\alpha K_2 x_{\boldsymbol{z}'}^{''2}\right\} \\
&\quad + \dot{\psi}\dot{\phi}\sin\theta\left\{-7\alpha K_2 x''_{\boldsymbol{n}'} x''_{\boldsymbol{z}'}\right\} + \dot{\theta}(\dot{\phi} - \dot{\psi}\cos\theta)\left\{-7\alpha K_2 x''_{\boldsymbol{z}\times\boldsymbol{n}'} \boldsymbol{x}''_{\boldsymbol{z}'} - \left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)\right]\sin\theta\right\} \\
&\quad - \dot{\theta}\dot{\psi}\sin\theta\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)a'_{33}\right] - 7\alpha K_2 a''_{31} x''_{\boldsymbol{z}'}\right\} = \alpha J_{\boldsymbol{z}'},
\end{aligned} \tag{B.12}$$

$$\begin{aligned}
&\ddot{\theta}\left\{-7\alpha K_2 x''_{\boldsymbol{n}'} a''_{31}\right\} + \ddot{\phi}\left\{m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2) - 7\alpha K_2 a_{31}^{''2} - 8\alpha G_2 a_{33}^{'2}\right\} \\
&\quad + \ddot{\psi}\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 - 16G_2)\right]a'_{33} - 7\alpha K_2 x''_{\boldsymbol{z}'} a''_{31}\right\} + \dot{\psi}\dot{\phi}\sin\theta\left\{-7\alpha K_2 x''_{\boldsymbol{n}'} a''_{31}\right\} + \dot{\theta}(\dot{\phi} - \dot{\psi}\cos\theta)\Big\{ - 7\alpha K_2 x''_{\boldsymbol{z}\times\boldsymbol{n}'} a''_{31} \\
&\quad + 8\alpha G_2 a'_{33}\sin\theta\Big\} - \dot{\psi}\dot{\theta}\sin\theta\left\{\left[m\mathcal{R}^2 + \frac{1}{3}\alpha(7K_2 + 8G_2)\right] - 7\alpha K_2 a_{31}^{''2} - 8\alpha G_2 a_{33}^{'2}\right\} = \alpha J_{\boldsymbol{z}}.
\end{aligned} \tag{B.13}$$

Appendix C: The $\sin\theta \sim 0$ case

When the equator of one of the rotating stars coincides with the plane of the fixed system, the (B.10) system of equations becomes overdetermined since it depends only upon $\ddot{\theta}$ and $\ddot{\psi} + \ddot{\phi}$. To solve this problem we introduce a new set of dependent variables as follows:

$$h = \tan\frac{\theta}{2}\cos\phi, \tag{C.1}$$

$$k = \tan\frac{\theta}{2}\sin\phi, \tag{C.2}$$

$$l = \psi + \phi, \tag{C.3}$$

furthermore,

$$\dot{h} = \frac{1+h^2+k^2}{2}\dot{\theta}\cos\phi - \dot{\phi}k, \tag{C.4}$$

$$\dot{k} = \frac{1+h^2+k^2}{2}\dot{\theta}\sin\phi + \dot{\phi}h, \tag{C.5}$$

$$\dot{l} = \dot{\psi} + \dot{\phi}. \tag{C.6}$$

As can be seen easily the inverse transformations are the following:

$$\phi = \tan\frac{k}{h}, \tag{C.7}$$

$$\theta = 2\tan^{-1}(h\cos\phi + k\sin\phi), \tag{C.8}$$

$$\psi = l - \phi, \tag{C.9}$$

$$\dot{\phi} = \frac{\dot{k}\cos\phi - \dot{h}\sin\phi}{h\cos\phi + k\sin\phi}, \tag{C.10}$$

$$\dot{\theta} = \frac{2}{1+h^2+k^2}(\dot{h}\cos\phi + \dot{k}\sin\phi), \tag{C.11}$$

$$\dot{\psi} = \dot{l} - \dot{\phi}. \tag{C.12}$$

Using these variables, after some algebra we get that

$$a'_{13} = \frac{2k}{1+h^2+k^2}, \tag{C.13}$$

$$a'_{23} = -\frac{2h}{1+h^2+k^2}, \tag{C.14}$$

$$a'_{33} = \frac{1-(h^2+k^2)}{1+h^2+k^2}, \tag{C.15}$$

and so

$$\omega_x = (1+a'_{33})\dot{h} + a'_{13}\dot{l}, \tag{C.16}$$

$$\omega_y = (1+a'_{33})\dot{k} + a'_{23}\dot{l}, \tag{C.17}$$

$$\omega_z = a'_{33}\dot{l} - (a'_{13}\dot{h} + a'_{23}\dot{k}), \tag{C.18}$$

while

$$\dot{\omega}_x = (1+a_{33})(\ddot{h} + \dot{k}\dot{l}) + a'_{13}\ddot{l} + (a'_{23}\dot{h} - a'_{13}\dot{k})\omega_x, \tag{C.19}$$

$$\dot{\omega}_y = (1+a'_{33})(\ddot{k} - \dot{h}\dot{l}) + a'_{23}\ddot{l} + (a'_{23}\dot{h} - a'_{13}\dot{k})\omega_y, \tag{C.20}$$

$$\dot{\omega}_z = a'_{33}\ddot{l} - (a'_{13}\ddot{h} + a'_{23}\ddot{k}) + (a'_{23}\dot{h} - a'_{13}\dot{k})(\dot{l} + \omega_z). \tag{C.21}$$

Furthermore,

$$\omega_{z'} = \dot{l} + a'_{13}\dot{h} + a'_{23}\dot{k}, \tag{C.22}$$

and finally

$$\dot{\boldsymbol{\omega}}\cdot\boldsymbol{z}' = \ddot{l} + a'_{13}\ddot{h} + a_{23}\ddot{k} + \left(a'_{23}\dot{h} - a'_{13}\dot{k}\right)\left(\omega_{z'} - \dot{l}\right). \tag{C.23}$$

Using Eqs. (C.16)–(C.23), (B.11)–(B.13) can be written as follows:

$$\begin{aligned}
&\ddot{h}\left\{\left(1+a'_{33}\right)\left[m\mathcal{R}^2 + \frac{1}{3}\alpha\left(7K_2+8G_2\right)\right] - 7\alpha K_2 a''_{11}\left[a''_{11}\left(1+a'_{33}\right) - a''_{31}a'_{13}\right] - 8\alpha G_2 a'^2_{13}\right\}\\
&\quad -\ddot{k}\left\{7\alpha K_2 a''_{11}\left[a''_{21}\left(1+a'_{33}\right) - a''_{31}a'_{23}\right] + 8\alpha G_2 a'_{13}a'_{23}\right\} + \ddot{l}\left\{a'_{13}\left[m\mathcal{R}^2 + \frac{1}{3}\alpha\left(7K_2 - 16G_2\right)\right] - 7\alpha K_2 x''_{z'}a''_{11}\right\}\\
&\quad +\dot{h}\dot{l}\left\{7\alpha K_2 a''_{11}\left[a''_{21}(1+a'_{33}) - a''_{31}a'_{23}\right] + 8\alpha G_2 a'_{13}a'_{23}\right\} + \dot{k}\dot{l}\left\{\left(1+a'_{33}\right)\left[m\mathcal{R}^2 + \frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad \left. -7\alpha K_2 a''_{11}\left[a''_{11}\left(1+a'_{33}\right) - a''_{31}a'_{13}\right] - 8\alpha G_2 a'^2_{13}\right\} + \left(a'_{23}\dot{h} - a'_{13}\dot{k}\right)\left\{\omega_x\left[m\mathcal{R}^2 + \frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad \left. -7\alpha K_2\omega_{x''}a''_{11} - 8\alpha G_2\omega_{z'}a'_{13}\right\} = \alpha J_x,
\end{aligned} \tag{C.24}$$

$$
\begin{aligned}
&\ddot{h}\left\{-7\alpha K_2 a''_{21}\left[a''_{11}\left(1+a'_{33}\right)-a''_{31}a'_{13}\right]-8\alpha G_2 a'_{13}a'_{23}\right\}+\ddot{k}\left\{\left(1+a'_{33}\right)\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad\left.-7\alpha K_2 a''_{21}\left[a''_{21}(1+a'_{33})-a''_{31}a'_{23}\right]-8\alpha G_2 a'^2_{23}\right\}+\ddot{l}\left\{a'_{23}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2-16G_2\right)\right]-7\alpha K_2 x''_{z'}a''_{21}\right\}\\
&\quad+\dot{h}\dot{l}\left\{-(1+a'_{33})\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]+7\alpha K_2 a''_{21}\left[a''_{21}\left(1+a'_{33}\right)-a''_{31}a'_{23}\right]+8\alpha G_2 a'^2_{23}\right\}\\
&\quad+\dot{k}\dot{l}\left\{-7\alpha K_2 a''_{21}\left[a''_{11}\left(1+a'_{33}\right)-a''_{31}a'_{13}\right]-8\alpha G_2 a'_{13}a'_{23}\right\}+\left(a'_{23}\dot{h}-a'_{13}\dot{k}\right)\left\{\omega_y\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad\left.-7\alpha K_2\omega_{x''}a''_{21}-8\alpha G_2\omega_{z'}a'_{23}\right\}=\alpha J_y,
\end{aligned}
\tag{C.25}
$$

and

$$
\begin{aligned}
&\ddot{h}\left\{-a'_{13}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]-7\alpha K_2 a''_{31}\left[a''_{11}\left(1+a'_{33}\right)-a''_{31}a'_{13}\right]-8\alpha G_2 a'_{13}a'_{33}\right\}\\
&\quad+\ddot{k}\left\{-a'_{23}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]-7\alpha+K_2 a''_{31}\left[a''_{21}\left(1+a'_{33}\right)-a''_{31}a'_{23}\right]-8\alpha G_2 a'_{23}a'_{33}\right\}\\
&\quad+\ddot{l}\left\{a'_{33}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2-16G_2\right)\right]-7\alpha K_2 x''_{z'}a''_{31}\right\}+\dot{h}\dot{l}\left\{a'_{23}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad\left.+7\alpha K_2 a''_{31}\left[a''_{21}\left(1+a'_{33}\right)-a''_{31}a'_{23}\right]+8\alpha G_2 a'_{23}a'_{33}\right\}+\dot{k}\dot{l}\left\{-a'_{13}\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad\left.-7\alpha K_2 a''_{31}\left[a''_{11}\left(1+a'_{33}\right)-a''_{31}a'_{13}\right]-8\alpha G_2 a'_{13}a'_{33}\right\}+\left(a'_{23}\dot{h}-a'_{13}\dot{k}\right)\left\{\omega_z\left[m\mathcal{R}^2+\frac{1}{3}\alpha\left(7K_2+8G_2\right)\right]\right.\\
&\quad\left.-7\alpha K_2\omega_{x''}a''_{31}-8\alpha G_2\omega_{z'}a'_{33}\right\}=\alpha J_z.
\end{aligned}
\tag{C.26}
$$