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RR Lyrae variables in Galactic globular clusters

IV. Synthetic HB and RR Lyrae predictions

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Abstract. We present theoretical predictions concerning horizontal branch stars in globular clusters, including RR Lyrae variables, as derived from synthetic procedures collating evolutionary and pulsational constraints. On this basis, we explore the predicted behavior of the pulsators as a function of the horizontal branch morphology and over the metallicity range $Z = 0.0001$ to 0.006, revealing an encouraging concordance with the observed distribution of fundamentalised periods with metallicity. Theoretical relations connecting periods to K magnitudes and BV or VI Wesenheit functions are presented, both appearing quite independent of the horizontal branch morphology only with $Z \geq 0.001$. Predictions concerning the parameter R are also discussed and compared under various assumptions about the horizontal branch reference luminosity level.

Key words. stars: variables: RR Lyrae – stars: evolution – stars: horizontal-branch

1. Introduction

The pulsational variability of RR Lyrae stars in Galactic globular clusters has represented for several decades intriguing evidence which has stimulated a large number of investigations. It was, indeed, early understood that the pulsation is governed by the physical structure of the stellar objects, providing independent access to the evolutionary features of low mass metal poor stellar structures.

In the first paper of this series (Castellani et al. 2003), we have revisited the present status of the art, presenting and discussing on purely empirical grounds the current observational scenario. Paper II (Marconi et al. 2003) tests the pulsational predictions for low mass structures with $Z = 0.001$, calibrating the theory on the rich sample of RR Lyrae stars in the Galactic globular M 3, while Paper III (Di Criscienzo et al. 2004) dealt with pulsational predictions covering the whole range of mass and metal content, as expected in the Galactic globular cluster family.

On this basis, we are now in the position to attempt a direct connection between pulsation and evolution theories by investigating the predicted behavior of suitable low mass stellar models evolving through the central He burning, horizontal branch (HB) evolutionary phase. We will perform this task making use of the well-known synthetic HB (SHB) procedure, i.e. by using evolutionary theory to predict the distribution of HB stars in globular clusters of selected metallicities. Adopting the instability strip boundaries given in Paper III, from each computed SHB one easily derives the number and the period of predicted RR Lyrae pulsators, to be compared with the observational scenario presented in Paper I.

As in Paper I, in this paper we will remove the problem of the pulsation mode by considering only fundamentalised periods ($P_{\rm f}$) to be compared with similar observational data. The problem of pulsational modes will be addressed in forthcoming papers, where we will approach also the discussion of RR Lyrae stars in selected clusters, such as the peculiar case of NGC 6441.

2. Synthetic horizontal branches

The main ingredient to produce SHBs is obviously given by suitable sets of HB stellar models. To this purpose, we computed a fine grid of HB models for the selected metallicities Z = 0.0001, 0.0003, 0.0006, 0.001, 0.003 and 0.006 assuming an original He content $Y = 0.23$, implemented with a grid for $Z = 0.006$ but $Y = 0.245$ to investigate the effects of the Galactic correlation between He and metals. All computations start from Zero Age HB (ZAHB) structures as obtained, for each given metallicity and helium content, by adopting the helium core mass and the envelope chemical abundance profile suitable for a Red Giant Branch (RGB) progenitor with mass equal to 0.8 $M_{\odot}$, with 5% by mass of ^{12}C added in the core to account for the nucleosynthesis during the He flash. The models have been followed through the whole phase of central He burning and He shell burning and, for those reaching

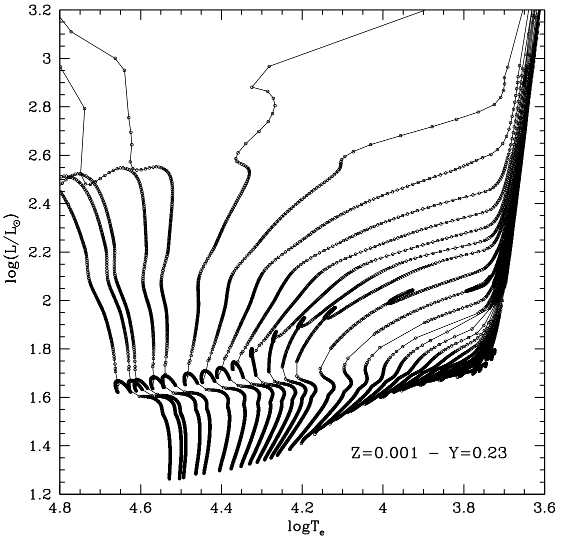

Fig. 1. The set of HB evolutionary tracks for metallicity $Z = 0.001$. Dots along the tracks mark the points used for the interpolation allowing to predict the luminosity and effective temperature of a HB model for each given value of mass and HB evolutionary time.

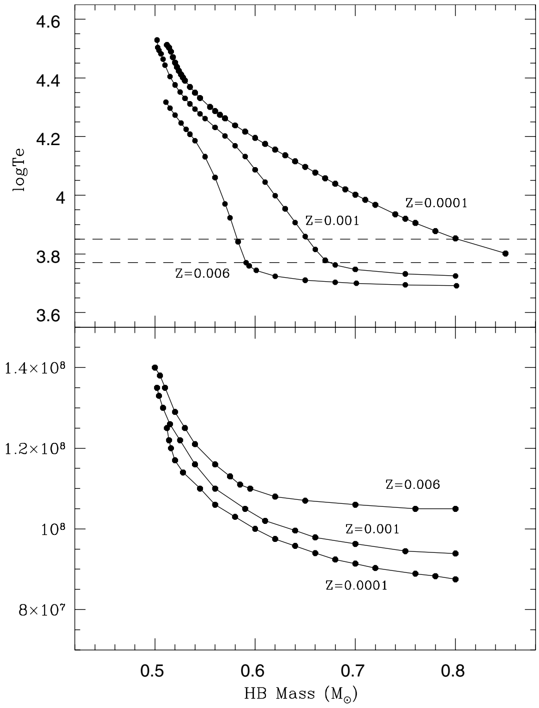

Fig. 2. *Upper panel*: the effective temperature of ZAHB models as a function of the stellar mass for three values of the metallicity Z. Dashed lines give an indication of the range of temperatures covered by the instability strip. *Lower panel*: as in the upper panel, but for the evolutionary times in the central He burning phase.

the Asymptotic Giant Branch (AGB) phase, until the onset of thermal pulses.

The models make use of OPAL radiative opacities (Iglesias & Rogers 1996) for temperatures higher than 10 000 K, while for lower temperatures, molecular opacity tables provided by Alexander & Ferguson (1994) have been adopted. Both high and low-temperature opacities have been computed by assuming a solar scaled heavy element distribution (Grevesse 1991). The equation of state (EOS) is from Straniero (1988), supplemented at lower temperatures by a Saha EOS. The outer boundary conditions have been fixed according to the $T(\tau)$ relation given by Krishna-Swamy (1966). Concerning the treatment of the superadiabatic layers, the mixing-length calibration provided by Salaris & Cassisi (1996) has been adopted. Other physical inputs are the same as in Cassisi & Salaris (1997).

As an example, Fig. 1 shows the data set of evolutionary tracks for $Z = 0.001$, where dots along the tracks mark the points used in the the interpolation which allows us to predict luminosity and effective temperature of a HB model for each given value of the mass and HB evolutionary time. Tables containing the whole set of HB evolutionary lines are available at `http://www.mporzio.astro.it/~marco/sintetici/`, together with detailed results of our synthetic HB simulations. Figure 2 shows the mass – effective temperature relation for ZAHB models at three selected metallicities. The figure reveals the well known occurrence in which for increasing the metallicity the ZAHB models tend to accumulate at the lower effective temperatures so that it decreases the range of masses predicted within the RR Lyrae instability strip (approximately, $3.77 \leq \log T_{\rm e} \leq 3.85$). The lower panel in the same figure gives the central He burning lifetimes again as a function of ZAHB masses and for the three selected metallicities.

Making use of the evolutionary models, synthetic HBs have been produced by assuming a Gaussian random distribution of masses centered on a central mass $\langle M_{\rm HB} \rangle$ together with a flat random distribution of post-ZAHB evolutionary time. According to the adopted procedure, a synthetic HB provides for each star its mass, luminosity and effective temperature. Adopting atmosphere models by Castelli et al. (1997), we finally evaluate U, B, V, R, I, J and K magnitudes for all the objects. Moreover, based on the edges of the pulsation region given in Paper III, we determine whether a star is crossing the RR Lyrae instability strip. As a result, for each given total number of stars in the He burning evolutionary phase, we eventually obtain the number of HB stars to the blue (B), within (V) or to the red (R) of the instability strip, together with the predicted number of stars in the early AGB phase. In this way, we evaluate the so-called "HB type" (Lee 1990), as given by the ratio $(B - R)/(B + V + R)$, and the number ratio $R^{\rm AGB}$ between AGB and HB stars. For the models falling within the pulsation region, i.e., for RR Lyrae candidates, we derive the predicted mean visual magnitude $\langle M_V^{\rm RR} \rangle$, together with their period frequency histogram and the value of the mean fundamental period $\langle \log P_{\rm f} \rangle$, as evaluated according to the relation given in Paper III.

Comparison with parallel evolutionary tracks for RGB structures, as computed for the various assumptions on Z, allows finally to predict the value of the parameter R (Iben 1968), as given by the number ratio of HB to RGB stars more luminous than the HB luminosity level. According to the different choices made in the literature, this last quantity is alternatively defined as the bolometric magnitude of the ZAHB at the mean temperature of the instability strip

$\log T_e = 3.83$ (R^I), as the absolute visual magnitude of the faintest RR Lyrae pulsators (R^{II}) or the mean RR Lyrae $\langle M_V^{RR} \rangle$ magnitude (R^{III}).

However, before proceeding one has to take into consideration the already known small discrepancy between evolutionary and pulsational predictions. As discussed in Caputo et al. (1999, but see also Paper II) a straightforward application of pulsational results to HB evolutionary models predicts periods slightly larger than observed in well studied clusters like M 5 and M 3. Using M 3 as a test, we now find that with $Z \sim 10^{-3}$ the predictions of our SHB computation would give the right interval $\Delta \log P_f = \log P_f^{max} - \log P_f^{min}$ (see Paper I), but with the period frequency histogram slightly shifted toward larger values. On this basis only, one would conclude that the adopted pulsational theory gives the right width in temperature of the instability strip, but either the predicted strip is too cool, or the models are too luminous, if not a combination of both.

In Paper II, we have presented what we regard as the reasonable indication that the dilemma should be solved in the sense that current evolutionary HB models are indeed too luminous, by an amount which depends on the input physics adopted in the various available evolutionary computations. Here we add that numerical experiments have shown that adopting the most recent OPAL EOS our HB models will become even more luminous. We feel that this indicates that the overluminosity is not a matter of EOS, but very likely of uncertainties in the sophisticated mechanisms governing the mass size of the He core in the Red Giant progenitors, such cooling by neutrinos and electron conduction. So, we decided to retain the already quoted theoretical framework, but calibrating the luminosity of the models to reproduce the period distribution as observed in M 3. On this basis, the luminosity of our He burning models has been decreased by $\Delta \log L = 0.04$. Such an empirical correction leaves substantially unchanged the evolutionary scenario, allowing a close agreement with observations, at the least in the case of M 3.

According to such a procedure, Synthetic HB have been produced, randomly distributing for all the adopted metallicites 800 stars in the He burning phases for selected values of the mean mass and adopting in all cases a Gaussian dispersion of masses with a standard deviation $\sigma = 0.02\ M_\odot$ to account for the evidence of differential mass loss in the ZAHB progenitor. As found in the seminal paper by Rood (1973), such a value of the standard deviation appears the most adequate to reproduce the observed distribution of stars along the HB of typical Galactic globulars (see also Caputo et al. 1984 and Fig. 3 in Brocato et al. 2000). However, numerical experiments show that the results of the present investigation are not critically dependent on this assumption, remaining substantially unchanged when the adopted dispersion is either increased or decreased by a factor of two.

For each metallicity and for each mass, we have performed 10 different simulations, thus deriving the mean values and the corresponding standard deviations for all the relevant quantities. The complete set of results is available at `http://www.mporzio.astro.it/~marco/sintetici/`, where one can find the data concerning the single HB simulations together with the plots of the corresponding $\log L$, $\log T_e$ or M_V, $B - V$ diagrams and the period frequency histograms. Table 1 gives selected results for RR Lyrae rich cases, intended as SHB simulations where the number of pulsators is ≥5% of the global HB star population.

Since this investigation is mainly devoted to the pulsational behavior of RR Lyrae variables, this issue will be discussed first in the next sections. However, the adopted SHB procedure offers the opportunity of clarifying some other relevant evolutionary parameters, such as the number ratio of RGB to HB stars. We will devote Sect. 6 to the discussion of such evolutionary features.

3. Pulsator predictions

For each assumed metal content, data in Table 1 show some details of the already predicted dependence of the mean RR Lyrae magnitude on the HB type (see, e.g., Lee et al. 1990; Caputo et al. 1993; Demarque et al. 2000). As expected, one finds that this magnitude becomes brighter when the HB morphology moves from red to blue, i.e., with HB type going from −1 to +1. This is easily understood according to the evidence that in very blue HBs the RR Lyrae instability strip is populated only with stars in their later phase of central He burning, crossing the pulsation region above the ZAHB luminosity level. A similar, but much more reduced, effect can appear in the case of very red HBs at $Z = 0.0001$: again the strip lacks ZAHB pulsators, being populated only by stars entering from the red and slightly more luminous than the local ZAHB level. This is shown in the upper panel in Fig. 3, where the dependence of the mean pulsator magnitude $\langle M_V^{RR} \rangle$ on the HB type for three selected metallicities is reported. From data in the figure, one can also infer the dependence of the same magnitude on the metallicity.

The lower panel in Fig. 3 shows the predicted dependence of the mean fundamentalised period on both the HB type and metal content. To discuss these data, one has to recall that, according to theory, the period decreases as luminosity decreases and/or mass or effective temperature increases. As for mean periods, one finds that at a fixed metallicity when going from blue HB types to the redder ones, the decrease in luminosity and the simultaneous increase in mass (see Table 1) both tend to decrease the pulsator periods. However, data in Fig. 3 show that such a decrease is successfully counteracted by a temperature effect: when the population in the cooler portion of the strip begins to dominate, the mean period starts to increase again, damping the tendency towards a continuous decrease.

Such an effect is more remarkable with $Z \leq 0.0006$ for the simple reason that at these metallicities the adopted mass dispersion ($\sigma = 0.02\ M_\odot$) produces the smallest dispersion of HB temperatures (see Fig. 2) and, thus, the largest efficiency of the temperature effect. Increasing the metallicity, the dispersion in HB temperatures increases, progressively smoothing away the evidence for the mechanism. Note that very metal poor globulars in the Galaxy all have blue HB type, thus falling along the descending branch of the mean period relation. Note also that, if Galactic globular cluster ages are comparable with the Hubble time (∼13 Gyr), presently evolving RG are predicted to be less massive than 0.8 $M_\odot$, allowing only HB types bluer than ∼0.4 in the most metal-poor clusters. However, the

Table 1. Selected results of synthetic HB simulations at $Y = 0.23$. Each row gives the metallicity, the mean HB mass, the HB type, the number of RR Lyrae stars together with their mean mass, mean visual magnitude and mean fundamentalised period, the AGB ratio $R^{\rm AGB} = N({\rm AGB})/N({\rm HB})$ and the predicted number ratio of HB to RGB stars more luminous than i) the ZAHB bolometric magnitude at the RR Lyrae gap ($R^{\rm I}$); ii) the absolute visual magnitude of the faintest RR Lyrae pulsator ($R^{\rm II}$); and iii) the mean $\langle M_V^{\rm RR}\rangle$ magnitude of the pulsators. Masses are in solar units. Absolute visual magnitudes of the ZAHB at $\log T_{\rm e} = 3.83$ are given in parentheses below the corresponding metallicity values.

Z	$\langle M_{\rm HB}\rangle$	HB type	N(RR)	$\langle M_{\rm RR}\rangle$	$\langle M_V^{\rm RR}\rangle$	$\langle \log P_{\rm f}\rangle$	$R^{\rm AGB}$	$R^{\rm I}$	$R^{\rm II}$	$R^{\rm III}$
0.0001	0.70	0.90	46.3	0.714	0.378	−0.233	0.11	1.16	1.45	1.56
(0.517 mag)	0.72	0.83	80.4	0.729	0.405	−0.237	0.11	1.14	1.41	1.50
	0.74	0.74	117.5	0.745	0.424	−0.245	0.11	1.14	1.38	1.47
	0.76	0.62	152.4	0.767	0.437	−0.259	0.11	1.12	1.32	1.41
	0.78	0.44	246.9	0.791	0.451	−0.297	0.10	1.12	1.32	1.39
	0.80	0.15	401.0	0.809	0.459	−0.309	0.11	1.10	1.30	1.36
	0.82	−0.09	532.3	0.822	0.460	−0.294	0.10	1.10	1.29	1.35
	0.84	−0.18	579.8	0.835	0.457	−0.266	0.10	1.10	1.29	1.35
0.0003	0.66	0.91	43.3	0.678	0.435	−0.248	0.11	1.16	1.40	1.58
(0.607 mag)	0.68	0.78	107.2	0.696	0.490	−0.275	0.11	1.14	1.35	1.46
	0.70	0.52	228.2	0.713	0.532	−0.297	0.11	1.14	1.33	1.42
	0.72	0.11	378.8	0.726	0.550	−0.293	0.12	1.13	1.32	1.38
	0.74	−0.29	394.6	0.738	0.556	−0.269	0.10	1.13	1.32	1.37
	0.76	−0.60	271.8	0.747	0.555	−0.239	0.11	1.12	1.30	1.37
	0.78	−0.82	131.4	0.756	0.552	−0.212	0.11	1.11	1.31	1.35
0.0006	0.64	0.89	50.8	0.664	0.478	−0.271	0.12	1.19	1.41	1.57
(0.630 mag)	0.66	0.70	146.9	0.677	0.535	−0.288	0.12	1.16	1.36	1.47
	0.68	0.28	272.9	0.688	0.568	−0.290	0.12	1.13	1.32	1.38
	0.70	−0.24	316.2	0.696	0.580	−0.278	0.12	1.13	1.32	1.37
	0.72	−0.69	188.7	0.703	0.585	−0.255	0.12	1.13	1.32	1.36
	0.74	−0.91	62.3	0.710	0.588	−0.231	0.13	1.11	1.32	1.34
0.001	0.62	0.89	55.3	0.645	0.513	−0.270	0.11	1.18	1.41	1.57
(0.668 mag)	0.64	0.64	155.1	0.657	0.583	−0.290	0.11	1.16	1.36	1.46
	0.66	0.11	261.8	0.665	0.609	−0.293	0.12	1.13	1.33	1.41
	0.68	−0.44	231.7	0.672	0.623	−0.278	0.10	1.13	1.33	1.39
	0.70	−0.80	113.4	0.678	0.631	−0.261	0.11	1.10	1.30	1.35
0.003	0.58	0.84	57.3	0.605	0.682	−0.328	0.12	1.14	1.72	1.92
(0.799 mag)	0.60	0.40	150.0	0.611	0.733	−0.342	0.12	1.12	1.69	1.80
	0.62	−0.25	156.2	0.615	0.755	−0.337	0.13	1.10	1.67	1.77
	0.64	−0.77	81.2	0.629	0.769	−0.334	0.11	1.10	1.66	1.74
0.006	0.56	0.79	57.0	0.584	0.787	−0.369	0.10	1.39	1.84	2.04
(0.906 mag)	0.58	0.29	119.0	0.588	0.840	−0.381	0.12	1.36	1.80	1.91
	0.60	−0.42	111.3	0.590	0.857	−0.385	0.12	1.34	1.76	1.85
	0.62	−0.86	40.6	0.592	0.877	−0.382	0.12	1.32	1.74	1.80

predictions for redder HB distributions may be of some interest in the case of some metal-poor dwarf spheroidal galaxies in the Local Group which are characterized by relatively red HBs and are thought to be a sort of "bridge" between Oosterhoff type I and type II Galactic globular clusters (see Mateo 1998; Dall'Ora et al. 2003, and references therein).

Importantly, data in Fig. 3 confirm the prediction (Caputo & Castellani 1975) that increasing the metallicity in the low metallicity ($Z \leq 0.001$) range, the decrease in luminosity (which will induce a decrease in period) is counteracted by the concurrent decrease in mass. As a result, over the range $Z = 10^{-4}$ to 10^{-3} the mean fundamentalised period is unaffected by metallicity, at fixed HB morphology. This is not the case for the higher metal contents, where the effect of luminosity dominates and mean periods are predicted to decrease when increasing the metallicity. This is exactly the behavior shown by Galactic globular clusters as discussed in Paper I. This can be better appreciated by looking at the upper panel in Fig. 4, where we report the predicted mean fundamentalised periods as a function of the hydrogen-to-iron content $[{\rm Fe/H}] = \log Z + 1.7$, in varying the HB type and assuming only HB type ≥ 0.5 in the case $Z = 0.0001$. Comparison with

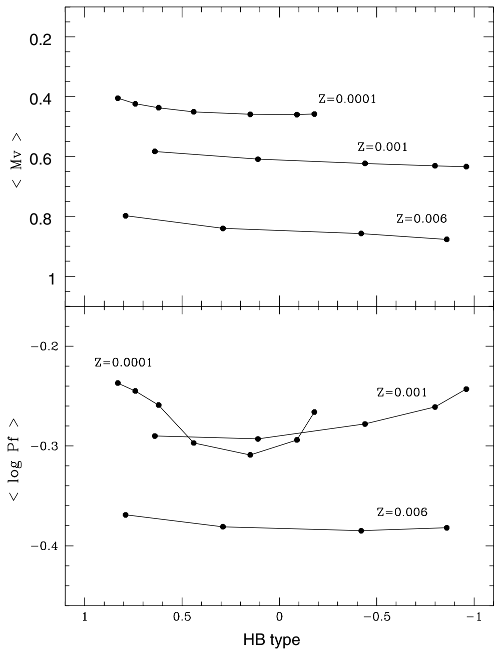

Fig. 3. *Upper panel*: predicted mean visual magnitude of RR pulsators as a function of the HB type and for the three labelled metallicities. *Lower panel*: as in the upper panel, but for the predicted mean fundamentalised period.

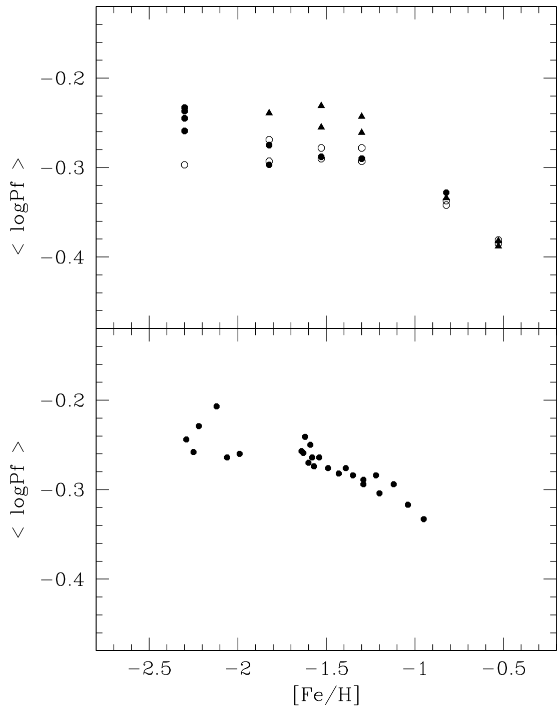

Fig. 4. *Upper panel*: predicted mean fundamentalised periods as a function of the hydrogen-to-iron content [Fe/H] = $\log Z + 1.7$. Filled circles: HB type > 0.5; open circles: 0.5 > HB type > −0.5; triangles: HB type < −0.5. *Lower panel*: observed mean fundamentalised periods for RR Lyrae rich Galactic globular clusters as a function of the cluster [Fe/H] metallicity (from Paper I).

the observed data, as presented in Paper I and now displayed in the lower panel of Fig. 4, shows an encouraging similarity, supporting the capability of the theoretical scenario, as obtained by collating stellar evolution and pulsation theories, to account for the actual behavior of RR pulsators in different simple stellar populations.

We note some relevant concordances between present predictions and observational data for Galactic globular clusters as displayed in Table 1 of Paper I. As an example, by relying on the Harris (1996) metallicity scale, the two RR rich globulars M 5 and M 62 have both metallicity $Z \sim 10^{-3}$, quite similar HB type (0.31 and 0.32, respectively) and, not surprising, quite similar mean fundamentalised periods, as given by $\langle \log P_{\rm f} \rangle = -0.294$ and -0.289, respectively. For this metallicity one can interpolate the data in Table 1 to derive for HB type ∼ 0.30 a predicted mean fundamentalised period $\langle \log P_{\rm f} \rangle \sim -0.290$. Going down to $Z \sim 10^{-4}$, we find M 15, with HB type 0.67 and $\langle \log P_{\rm f} \rangle = -0.258$, which is in impressive concordance with the predicted value $\langle \log P_{\rm f} \rangle$ −0.259 at HB type 0.62.

Data in Table 2 allow the comparison of present predictions with the observed mean $\log P_{\rm f}$ in the twelve clusters with more than 40 RR Lyrae listed in Table 1 of Paper I. Not surprisingly, predictions for some clusters appear less precise, possibly in connection with the uncertainties in the HB type, with the occurrence of non-negligible statistical fluctuations and with the still existing uncertainties in the evaluation of the actual cluster metallicity (see, e.g., Kraft & Ivans 2003; Asplund et al. 2004). From data in Table 1 one can indeed estimate that an uncertainty of 0.2 dex in [Fe/H] produces on average an uncertainty of ∼0.02 in the mean $\log P_{\rm f}$, and that such an uncertainty on the mean periods increases to ∼0.03 when adding an uncertainty of 0.1 in the HB type. Bearing this in mind, one finds that all the RR Lyrae-rich clusters have mean fundamentalised periods which agree with theoretical predictions within the quoted uncertainty, with the majority of clusters, but two (M 68 and NGC 6934), with discrepancies lower or of the order of 0.02.

Table 2. The comparison between observed and predicted mean periods for the sample of RR Lyrae rich Galactic globulars (see Paper I). Metal abundances and HB type (HB) are from Harris.

NGC /IC		[Fe/H]	HB	$\langle \log P_{\rm f} \rangle_{\rm ob}$	$\langle \log P_{\rm f} \rangle_{\rm pr}$
NGC 7078	M 15	−2.25	0.67	−0.258	−0.259
NGC 4590	M 68	−2.06	0.17	−0.264	−0.294
NGC 5024	M 53	−1.99	0.81	−0.260	−0.259
NGC 7006		−1.63	−0.28	−0.259	−0.270
IC 4499		−1.60	0.11	−0.270	−0.283
NGC 6715	M 54	−1.59	0.87	−0.250	−0.259
NGC 3201		−1.58	0.08	−0.264	−0.285
NGC 5272	M 3	−1.57	0.08	−0.274	−0.285
NGC 6934		−1.54	0.25	−0.264	−0.289
NGC 6402	M 14	−1.39	0.65	−0.276	−0.288
NGC 5904	M 5	−1.29	0.31	−0.294	−0.293
NGC 6266	M 62	−1.29	0.32	−0.289	−0.293

Catelan (2004) has recently discussed the evidence for an anomalous distribution of RR periods in M 3, an occurrence which could be at the origin of additional discrepancies, since our prediction assumes a smooth distribution of stars within the instability strip.

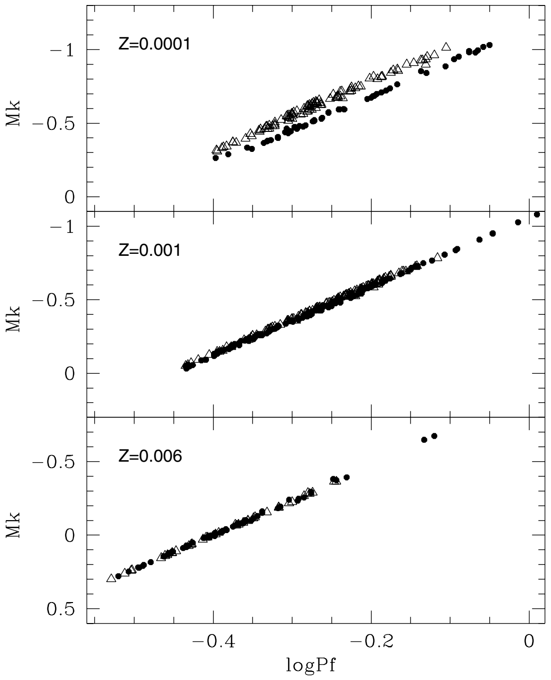

Fig. 5. Predicted PL_K relation for the three labelled values of metal content. Each panel shows predictions concerning the bluest (circles) and the reddest (triangles) RR Lyrae rich HB type.

4. The PL_K relations

The observational evidence has already shown a tight correlation between fundamental periods and near-infrared K magnitudes of RR Lyrae stars. Longmore et al. (1986) and Longmore et al. (1990) found that RR Lyrae stars in Galactic globular clusters follow a well-defined near-infrared period-luminosity (PL_K) relation, whose slope appears almost independent of the cluster metallicity. Bono et al (2001, 2003) have shown that such an occurrence is actually in agreement with the predictions of current pulsational theory: making use of suitable analytical approximations of evolutionary and pulsational results, these authors presented theoretical predictions on the PL_K in good agreement with observational constraints. We are now in the position of performing a much more detailed analysis because the adopted synthetic procedure does not require any assumption about the predicted distribution of star masses, luminosities and effective temperatures.

Figure 5 shows the synthetic $\log P_{\rm f} - M_K$ distribution for three selected assumptions about the star metallicity presenting, for each metallicity, predictions concerning the bluest and the reddest HB type allowing the occurrence of a substantial fraction of RR Lyrae pulsators (≥5% of the total number of HB stars). Data in this figure can be easily understood in terms of the mass distributions presented in the upper panel of Fig. 2. At the low metallicity limit $Z = 0.0001$, the ZAHB mass is a sensitive function of the effective temperature and the instability strip can be populated by a non-negligible range of ZAHB masses. When evolutionary effects are taken into account, one finds that in the case of the bluest or the reddest RR Lyrae-rich HB types the pulsator average mass is around $M \sim 0.71\, M_\odot$ and $\sim 0.84\, M_\odot$, respectively. In Bono et al. (2003) we have already presented the theoretical prediction

$$M_K \propto -2.102 \log P - 1.753 \log M$$

from which one derives, for each given $\log P$, a difference between the two cases of $\Delta M_K \sim 0.13$ mag, as actually found in the SHB simulations. Increasing the metallicity decreases the range of masses and the difference is only marginal for $Z = 0.001$ and vanishes for $Z = 0.006$.

The exploration of the synthetic results shows that the actual slope of the predicted $\log P_{\rm f} - M_K$ relation is a bit steeper than the analytical evaluation at constant mass. We also find that this slope is, within the uncertainties, not dependent on the HB type and with a minor dependence on the metal content. As a whole, putting the PL_K relation in the usual form

$$M_K = M_K^{-0.3} + b_K (\log\ P_{\rm f} + 0.3)$$

where $M_K^{-0.3}$ is the M_K value when $\log P_{\rm f} = -0.3$, we derive the predicted values of $M_K^{-0.3}$ and b_K for the various metallicities and HB types, as given in Table 3, where HB types in italics deal with scarce populations of RR Lyrae pulsators. Note that typical uncertainties for both $M_K^{-0.3}$ and b_K are ±0.02 mag.

5. The Wesenheit functions

The reddening-free algorithm originally suggested by van den Bergh (1975) has proved to be a useful tool for investigating variable stars (e.g., Classical Cepheids, see Madore 1982), including the RR Lyrae class (Kovács & Walker 2001, Paper III). As widely known, the use of Wesenheit functions, such as $W(BV) = V - 3.10(B - V)$, $W(VI) = V - 2.54(V - I)$, has the twofold advantage of getting rid of reddening but also of strongly reducing the luminosity dispersion due to the finite width of the instability strip. In this sense, the Period-Wesenheit relation (PW) works like the PL_K, although on the basis of quite different physical principles. This matter has been discussed in Paper III within a purely pulsational scenario and it seems interesting to improve that discussion on the basis of the present synthetic procedure.

The predicted $W(BV)$ and $W(VI)$ functions are plotted in Figs. 6 and 7, respectively, as a function of period and for the labelled choices on the pulsator metallicity. As in the PL_K case, one finds that the dispersion of pulsator masses can play a relevant role only at the lower metallicities, where the predicted distributions are significantly dependent on the HB type. On the contrary, for metal content larger or of the order of $Z = 0.001$ one expects a tight and univocal PW relation. Again one finds that the slope of the predicted relation is independent of the HB type, being only marginally dependent on the pulsator metallicity.

In all cases, putting the theoretical predictions in the form

$$W = W^{-0.3} + b^W (\log\ P_{\rm f} + 0.3)$$

we derive the values listed in Table 3 for both the $W(BV)$ and $W(VI)$ functions. Uncertainties on these values are of the same order of magnitude (±0.02 mag) as in the case of the PL_K relation.

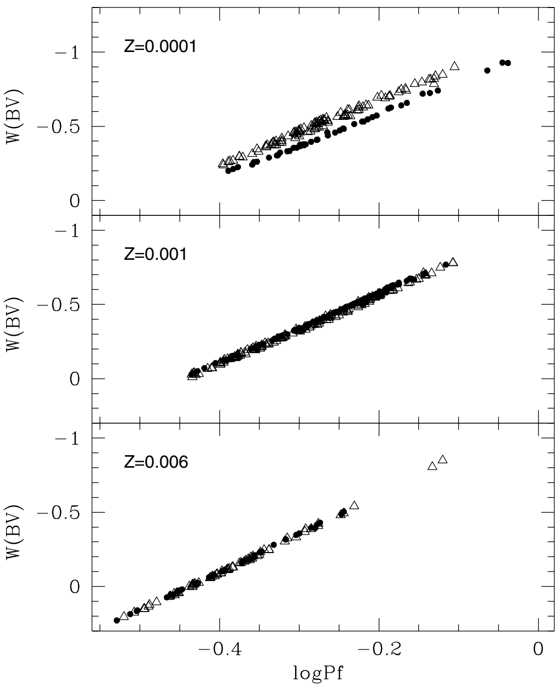

Fig. 6. Predicted $W(BV)$ functions versus periods for the labelled values of metal content. Each panel shows predictions concerning the bluest (triangles) and the reddest (circles) RR Lyrae rich HB type.

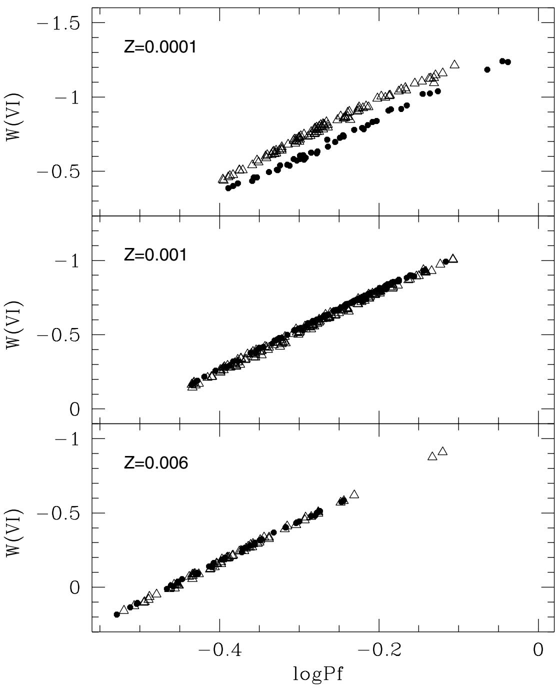

Fig. 7. As in Fig. 6, but with the $W(VI)$ function.

As a conclusion, we predict that, at least in moderately metal-rich clusters, both the PL_K and the PW relations are able to provide strong constraints to the cluster distance moduli independent of the HB type. Concerning the reliability of the presented theoretical result, we notice that the slope of both the PL_K and PW relations are dependent only on the relation between periods and stellar structural parameters, such as L and T_e, which represent a well-established and firm result of the current pulsational scenarios. Conversely, the zero points of the predicted relations depend on the assumption about the luminosity of HB models and on the adopted bolometric corrections, thus requiring firmer theoretical or empirical constraints. Here, we recall that all through this paper we made use of HB luminosities following the theoretical dependence on metallicity, as predicted by stellar evolution theory, and calibrated to fit the observed period distribution of RR Lyrae stars in M 3.

6. Evolutionary parameters

Since the pioneering paper by Iben (1968), the number ratio R between HB and RG stars more luminous than the HB luminosity level has been widely used as an evolutionary indicator of the star original He content. The theoretical calibration of this parameter can be performed by simply evaluating the ratio of theoretical lifetimes in the two quoted evolutionary phases. However, the lifetimes in the central He burning phase is a function of the mass of HB stellar structures, depending on the amount of mass lost by their RG progenitors. Since HB stars in actual globular clusters cover a non-negligible range of masses, the calibration of R requires an appropriate mean of the HB evolutionary times, especially when hot long-living HB stars are present (see lower panel in Fig. 2).

Since the first exhaustive analysis by Buzzoni et al. (1983), the calibration of R has been the subject of several investigations attempting to take into consideration the dependence of He burning lifetimes on the effective temperature of HB structures. However, as already stated in Zoccali et al. (2000), a correct measure of the absolute He content on the basis of the R-parameter would require the construction, for any given cluster, of synthetic CMD which properly reproduce the distribution of stars along the HB. The synthetic procedure adopted in this paper indeed allows a straightforward evaluation of this parameter by randomly populating both the RG and HB phases for any given assumption on the cluster metallicity and for any given assumption about the HB masses and, thus, about the cluster HB type.

According to the already-described procedure, we have evaluated the predicted values of R all along our sample of synthetic HB simulations. However, it is not intended to produce firm predictions on this parameter. As a fact, it has already brought to light (see, e.g., Brocato et al. 1998; Cassisi et al. 1998) the evidence that HB lifetimes, and thus the theoretical calibration of R, are sensitively affected by – at least – current uncertainties on the nuclear cross section for the He burning reaction $^{12}C(\alpha,\gamma)^{16}O$. Thus, the given evaluations of R are mainly intended to investigate the sensitivity of this parameter to the HB type at the various metallicities, putting on a more robust basis the results already presented on the matter by several authors (Caputo et al. 1987; but see also Zoccali et al. 2000; Cassisi et al. 2003).

However, one has also to account for the several slightly different definitions of R adopted in the literature as far the HB luminosity level is concerned. As an example, Zoccali et al. (2000) assumed as a reference the visual magnitude

Table 3. Slope and zero point (in magnitudes) of predicted PL_K and PW relations for various metallicity and HB type with significant numbers of RR Lyrae pulsators (≥5% the total number of HB stars), except the HB types in italics.

Z	HB type	b_K	$M_K^{-0.3}$	$b^W_{\rm BV}$	$W^{-0.3}_{\rm BV}$	$b^W_{\rm VI}$	$W^{-0.3}_{\rm VI}$
0.0001	*0.99*	−2.30	−0.439	−2.15	−0.359	−2.54	−0.532
	0.90	−2.30	−0.455	−2.15	− 0.370	−2.54	−0.594
	0.83	−2.30	−0.465	−2.15	−0.378	−2.54	−0.608
	0.62	−2.30	−0.498	−2.15	−0.409	−2.54	−0.649
	0.44	−2.30	−0.516	−2.15	−0.427	−2.54	−0.669
	0.15	−2.30	−0.532	−2.15	−0.442	−2.54	−0.689
	−0.09	−2.30	−0.543	−2.15	−0.451	−2.54	−0.704
	−0.18	−2.30	−0.553	−2.15	−0.459	−2.54	−0.716
0.0003	*0.97*	−2.34	−0.399	−2.25	−0.332	−2.61	−0.530
	0.91	−2.34	−0.404	−2.25	−0.337	−2.61	−0.545
	0.78	−2.34	−0.416	−2.25	−0.348	−2.61	−0.566
	0.52	−2.34	−0.427	−2.25	−0.358	−2.61	−0.583
	0.11	−2.34	−0.435	−2.25	−0.365	−2.61	−0.596
	−0.29	−2.34	−0.442	−2.25	−0.371	−2.61	−0.606
	−0.60	−2.34	−0.449	−2.25	−0.378	−2.61	−0.614
	−0.82	−2.34	−0.458	−2.25	−0.386	−2.61	−0.524
0.0006	*0.97*	−2.34	−0.382	−2.30	−0.333	−2.63	−0.513
	0.89	−2.34	−0.384	−2.30	−0.338	−2.63	−0.530
	0.70	−2.34	−0.389	−2.30	−0.343	−2.63	−0.543
	0.28	−2.34	−0.394	−2.30	−0.348	−2.63	−0.555
	−0.24	−2.34	−0.397	−2.30	−0.351	−2.63	−0.561
	−0.69	−2.34	−0.404	−2.30	−0.359	−2.63	−0.569
	−0.91	−2.34	−0.406	−2.30	−0.361	−2.63	−0.572
0.001	*0.97*	−2.34	−0.357	−2.33	−0.327	−2.65	−0.502
	0.89	−2.34	−0.359	−2.33	−0.331	−2.65	−0.508
	0.64	−2.34	−0.357	−2.33	−0.329	−2.65	−0.517
	0.11	−2.34	−0.362	−2.33	−0.334	−2.65	−0.525
	−0.44	−2.34	−0.362	−2.33	−0.334	−2.65	−0.525
	−0.80	−2.34	−0.367	−2.33	−0.340	−2.65	−0.536
	−0.96	−2.34	−0.370	−2.33	−0.346	−2.65	−0.539
0.003	0.9 to −0.9	−2.34	−0.280	−2.45	−0.327	−2.66	−0.459
0.006	0.9 to −0.9	−2.41	−0.242	−2.60	−0.356	−2.68	−0.438

of the lower envelope of the observed HB distribution at the temperature of the RR Lyrae gap, taken as representative of the ZAHB luminosity. In the same year, Sandquist (2000) took the bolometric luminosity corresponding to the average V magnitude of HB stars at the same temperatures. We note the already advanced warning about the use of the lower envelope of RR luminosity as indicative of the ZAHB luminosity (Ferraro et al. 1999, but see also Paper II). To investigate the consequences of different assumptions about the reference luminosity levels, Table 1 gives results for three different approaches, where $R^{\rm I}$ represent the number ratio of HB to RG stars more luminous than the predicted ZAHB luminosity at $\log T_{\rm e} = 3.85$, $R^{\rm II}$ the same ratio but for RG stars more luminous than the faintest RR luminosity level and $R^{\rm III}$ for RG stars more luminous than the RR Lyrae mean magnitude.

As expected, the three values are in increasing order, for the very simple reason that the reference luminosity level is progressively increasing from $R^{\rm I}$ to $R^{\rm III}$, driving a consequent decrease in the number of RG stars. Data in Table 1 should be read bearing in mind that, according to current calibrations, a variation in R by $\Delta R \sim 0.1$, if interpreted in terms of original He, would imply a ΔY a bit larger than 0.01 (Buzzoni et al. 1983, but see also Caputo et al. 1987; Bono et al. 1995). One finds that, in the extreme case, for each given metallicity the HB type can move the $R^{\rm II}$ value up to $\Delta R \sim 0.12$, i.e., with a variation larger than predicted for $R^{\rm I}$, where only the effects of HB evolutionary lifetimes are at work. In passing, note that the sudden increase of R values at the large metallicity follow the decreasing luminosity of the RGB bump (see the discussion in Zoccali et al. 2000).

To allow a more general discussion, Fig. 8 shows theoretical predictions with $Z = 0.0001$, as produced from the entire sample of synthetic HBs, i.e., not only for RR Lyrae rich cases. The adopted metallicity follows from the evidence that with such a choice one is maximizing the increase of HB time when increasing HB effective temperatures. One notices that in the

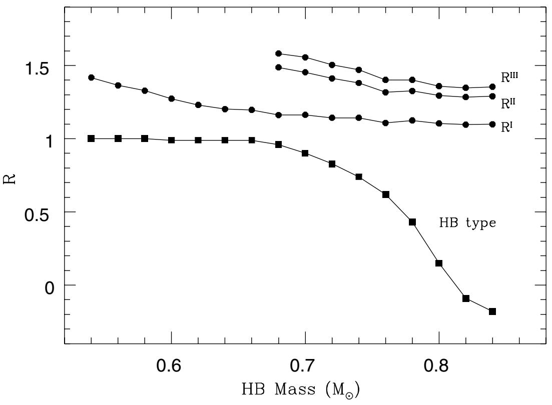

Fig. 8. Predicted values of the number ratios $R^{\rm I}$, $R^{\rm II}$ and $R^{\rm III}$ (see text) versus the mean HB mass, at $Z = 0.0001$. For comparison, the HB type is also reported (squares).

range of RR Lyrae rich HB types, the effects of lifetimes are of minor relevance. However, going toward hotter HB, the effects of HB lifetime grows, as expected, producing a variation of $R^{\rm I}$ up to ~0.3. Data in Table 1 shows that the difference between $R^{\rm II}$ and $R^{\rm III}$ is increasing with increasing metallicity. This is because synthetic HB computations predict that the width in luminosity is increasing with metallicity, thus increasing the difference between the lower luminosity level and $\langle M_V^{\rm RR}\rangle$. In this sense, the theory appears in agreement with the empirical correlation presented by Sandage (1987).

7. Discussion and final remarks

The main purpose of this paper is to show that synthetic HB procedures can be usefully adopted to explore not only globular cluster CM diagrams, but also to predict several relevant features of cluster RR Lyrae variables. We have presented the architecture and the results of the adopted procedure, showing in particular the encouraging concordance of several predictions with the observed distribution of fundamentalised periods at the various metallicities. In the following papers, we plan to make use of such a theoretical tool to discuss the variable populations in single well observed Galactic and extragalactic clusters.

However, we give a brief comparison of our predictions with available data for the well studied globular cluster M 3 for which, following the recent metallicity scale by Kraft & Evans (2003), we adopt $Z = 0.0006$. All the predicted relations are based on "static" magnitudes, whereas observed data deal with quantities averaged over the pulsation cycle. This issue has been discussed in Papers II and III and we wish only to recall that in the case of K magnitudes the discrepancy between static and mean values is at the most of the order of 0.01 mag, whereas it increases moving to visual and to blue photometric bands, as well as from symmetric (low amplitude) to asymmetric (high amplitude) light curves. On this basis, the observed mean K magnitudes can be directly compared with the predicted PL_K relation, whereas the observed $W(BV)$ and $W(VI)$ quantities need to be first corrected for the amplitude effect (see Papers II and III).

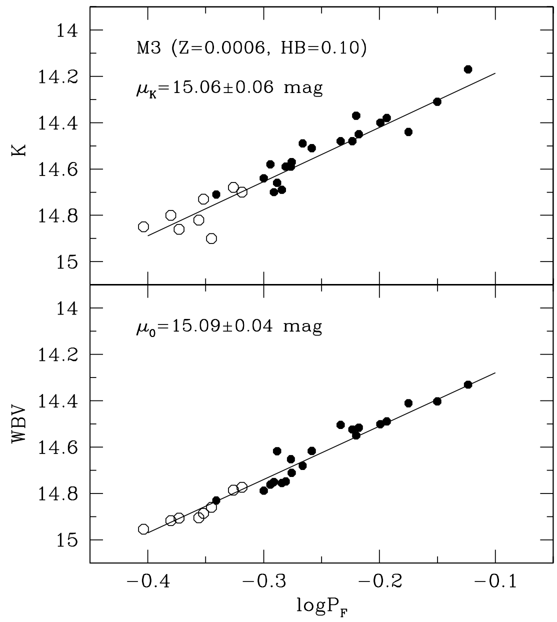

Fig. 9. Predicted PL_K and $PW(BV)$ relations in comparison with observed data in the globular cluster M 3.

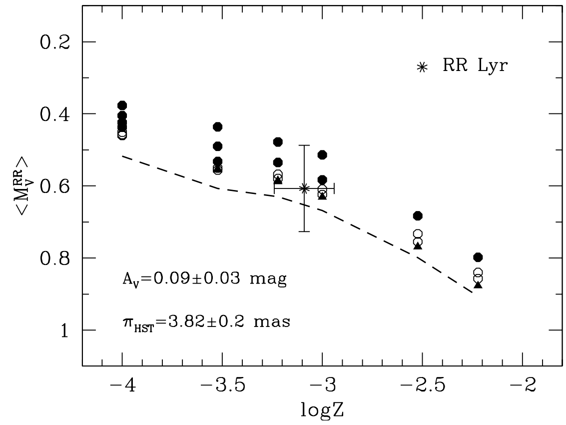

Fig. 10. Predicted mean magnitude of RR Lyrae stars versus metallicity for different HB types, with symbols as in Fig. 4. The dashed line depicts the ZAHB behavior. The location of the prototype RR Lyr is shown according to the labelled HST trigonometric parallax and visual extinction.

Figure 9 shows the comparison of M 3 observed data with present theoretical predictions (solid lines) both for the PL_K (upper panel) and the PW relation (lower panel). We regard the close agreement between empirical and predicted slopes as satisfactory evidence, whereas we notice that a discrepancy of only 0.03 mag in the derived cluster distance modulus is not only well within our estimated uncertainty but, possibly, also within the empirical uncertainty of the mean K magnitudes, as obtained in 1990 by Longmore et al. New and better K data would thus be needed to explore in more detail the internal consistency of the presented scenario.

We show in Fig. 10 the predicted mean magnitudes $\langle M_V^{\rm RR}\rangle$ listed in Table 1, as derived for the various RR Lyrae rich HB types, versus the metal content. For comparison, the level of ZAHB at the RR Lyrae gap is also drawn with a dashed line.

Looking at the results plotted in Fig. 10, one should agree that it is quite risky to predict a *unique* $\langle M_V^{RR} \rangle - \log Z$ relation independent of the HB type, a result which appears quite consistent with the significantly different empirical relations that have appeared in the relevant literature. Moreover, one should also consider that our predictions hold for a constant helium content $Y = 0.23$ and that accounting for $\Delta Y/\Delta Z \sim 2.5$ will produce not negligible effects on the results at $Z = 0.003$ and 0.006, leading to magnitudes brighter by ~0.04 and 0.08 mag, respectively.

However, our theoretical predictions agree with the absolute magnitude of the prototype RR Lyr, as estimated from the recent HST trigonometric parallax and visual extinction estimate (see Benedict et al. 2002). We are facing a relevant agreement between theory and firm observational constraints, disclosing the progresses performed since the first unsuccessful but pioneering attempts by Rood (1973).

The computed synthetic HB contains of course much more information than that summarized in this paper. Detailed results of the single synthetic models have been made available at the already quoted www site, together with the corresponding plots of the $\log L$, $\log T_e$ and $V, B - V$ predicted diagrams for the whole HB and the upper portion of the RG branch and with Tables summarizing the whole data set.

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