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Research Note

Tube waves: Exact and approximate

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Abstract. This note deals with magnetohydrodynamic body waves in a magnetic cylinder. It is shown that the solution obtained by the thin-tube expansion is, term by term, identical to the Taylor expansion of the exact solution. Each level of approximation adds a pair of modes, a slow and a fast one, and corrects the frequencies and eigenfunctions of the previous approximation. All eigenfrequencies, approximate and exact, can be read off from a single graph. All slow modes have phase velocities between the tube speed c_{T} and $\sqrt{2}\,c_{\mathrm{T}}$, all fast modes have phase velocities above $\sqrt{2}\,c_{\mathrm{T}}$.

Key words. magnetohydrodynamics (MHD) – waves – methods: analytical – Sun: magnetic fields

1. Introduction

Concentrations of magnetic flux abound on the solar surface, as first documented in the seventies by the work of Howard & Stenflo (1972) and Frazier & Stenflo (1972). Stenflo (1973) and Frazier & Stenflo (1978) found that the field strength in those concentrations often reaches 150 mT, which indicates an equilibrium between the magnetic pressure and the gas pressure.

The magnetic flux concentrations, or *flux tubes*, support a variety of magnetohydrodynamic waves. Theoretically, such waves have been treated in the *thin-tube limit* (Defouw 1976; Spruit 1981; Hasan 1984), which is equivalent to the leading order of an expansion in terms of the distance from the tube axis (Roberts & Webb 1978). Other applications of the thin-tube limit include the convective collapse of magnetic flux concentrations in a super-adiabatic environment (Spruit & Zweibel 1979), the Evershed flow in a sunspot penumbra (Schlichenmaier et al. 1998), and the heating of coronal loops that arrises from the non-linearity of the tube waves (Zhugzhda & Nakariakov 1997).

Second-order truncations of the expansion have been used by Browning & Priest (1983) and by Pneuman et al. (1986) to calculate equilibrium configurations of flux tubes. Including time-dependence, Ferriz Mas & Schüssler (1989) have considered the general case, in particular in the context of the exact solution of the magnetohydrodynamic equations which is known in a special case: waves of small amplitude in a straight circular magnetic cylinder; these waves are represented by Bessel functions (Roberts & Webb 1978). The latter authors drew confidence to the expansion procedure from the vicinity of the exact phase speed to the tube speed c_{T} that characterizes the leading order. Ferriz Mas et al. (1989), using a second-order approximation, concluded that body waves are less well represented than surface waves. Zhugzhda (2002) realized that the dispersion relation for body waves is the same at all levels of approximation, and suggests a recurrence procedure for calculating the eigenfunctions of the diverse orders. Applying such a procedure to the case of body waves in a magnetic cylinder, I show in this note that the expansion is identical to the Taylor expansion of the exact solution and that, therefore, the body waves can be represented with any accuracy. In addition, I shall give a scheme that allows to evaluate the wave frequencies at any level of truncation from a single graph.

In this note I use the terms *magnetic cylinder* and *magnetic flux tube* as synonyms. I call the *thin-tube limit*, or *leading order* what is commonly known as the thin-tube approximation, while *any* truncated expansion is called a *thin-tube approximation*. *Any* wave that is supported by the magnetic tube is called a *tube wave*. As I do not consider waves that ow their existence to the existence of boundaries, the tube waves of this note are all body waves, not surface waves. The latter have been discussed in a number of contributions, including Ferriz Mas et al. (1989) and Zhugzhda & Goossens (2001).

2. Equilibrium and perturbation

The thin-tube approximations make use of an expansion of all dependent variables in terms of powers of s, the distance from the axis of the tube. In order to clarify the convergence problem of that expansion, I shall consider the simplest possible case, a circular magnetic cylinder. Gravity is absent, the equilibrium as well as the perturbations are assumed to be axisymmetric and

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poloidal (no azimuthal vector components), and the medium external to the cylinder does not participate in the perturbation.

The equilibrium shall consist of a homogeneous magnetic field $\overline{B}$ inside the cylinder, of radius $\overline{R}$, and constant values $\overline{P}$ and $\overline{\rho}$ of pressure and density. This is a special solution of the equations describing the static case. Small perturbations of this equilibrium will be marked with a tilde. Thus

$$P = \overline{P} + \sum_{0,2\ldots} \tilde{P}_n(z,t)\, s^n, \tag{1}$$

$$\rho = \overline{\rho} + \sum_{0,2\ldots} \tilde{\rho}_n(z,t)\, s^n, \tag{2}$$

$$v_z = \sum_{0,2\ldots} \tilde{v}_{zn}(z,t)\, s^n, \qquad v_s = \sum_{1,3\ldots} \tilde{v}_{sn}(z,t)\, s^n, \tag{3}$$

$$B_z = \overline{B} + \sum_{0,2\ldots} \tilde{B}_{zn}(z,t)\, s^n, \tag{4}$$

$$B_s = \sum_{1,3\ldots} \tilde{B}_{sn}(z,t)\, s^n. \tag{5}$$

These expansions have either even or odd terms, as Ferriz Mas & Schüssler (1989) have shown by means of geometrical considerations.

3. Two- and four-mode approximations

The expansions (1)–(5) are substituted into the equations of ideal magnetohydrodynamics, under the assumption of adiabatic perturbations. This procedure has been described often (e.g., Ferriz Mas & Schüssler 1989; Stix 1989), hence I can rely on such earlier work. Comparing equal powers of s one obtains the Eqs. (3.7)–(3.10), (3.13), and (3.15) of Ferriz Mas & Schüssler (1989), which in the present case yield the following linearized system:

$$\dot{\tilde{\rho}}_n + (n+2)\overline{\rho}\,\tilde{v}_{sn+1} + \overline{\rho}\,\tilde{v}'_{zn} = 0, \tag{6}$$

$$\overline{\rho}\,\dot{\tilde{v}}_{zn} + \tilde{P}'_n = 0, \tag{7}$$

$$\overline{\rho}\,\dot{\tilde{v}}_{sn} + (n+1)\tilde{P}_{n+1} - \frac{\overline{B}}{\mu}\left[\tilde{B}'_{sn} - (n+1)\tilde{B}_{zn+1}\right] = 0, \tag{8}$$

$$\dot{\tilde{B}}_{zn} + (n+2)\overline{B}\,\tilde{v}_{sn+1} = 0, \tag{9}$$

$$(n+2)\tilde{B}_{sn+1} + \tilde{B}'_{zn} = 0, \tag{10}$$

$$\overline{\rho}\,\dot{\tilde{P}}_n - \gamma\overline{P}\,\dot{\tilde{\rho}}_n = 0, \tag{11}$$

where the evolution equation for $\tilde{B}_{sn}$ has been replaced by the solenoidality condition (10). A dot means time derivative, a prime means derivative with respect to z; γ is the ratio of the specific heats. The system (6)–(11) will be truncated at an even index number. For an "N-mode approximation", Eq. (8) must be included for $n = 1, 3, \ldots N-1$, while all other equations must be taken for $n = 0, 2, \ldots N-2$.

In addition to these equations, there is the condition of pressure continuity at the cylinder boundary, $s = \overline{R}$, which in the linearized form is

$$\sum_{0,2\ldots}^{N} \overline{R}^n\, \tilde{\Pi}_n = 0, \tag{12}$$

where

$$\tilde{\Pi}_n = \tilde{P}_n + \frac{\overline{B}}{\mu}\tilde{B}_{zn}. \tag{13}$$

The "two-mode approximation" of Zhugzhda (1996, 2002) is the case $N = 2$. As noticed by Zhugzhda, the seven Eqs. (6)–(12) suffice to derive the dispersion relation for two body waves, a slow and a fast one, because the two functions $\tilde{P}_2$ and $\tilde{B}_{z2}$ appear only in the particular combination $\tilde{\Pi}_2$, given by (13). The two-mode approximation is identical to the full second-order solution studied earlier by Ferriz Mas et al. (1989). However, the latter allows, in addition, to determine $\tilde{P}_2$ and $\tilde{B}_{z2}$ separately. The full set of second-order equations also yields a wave that propagates exactly with the tube speed $c_{\rm T}$, but this is a spurious solution, attributed to the truncation, as Ferriz Mas et al. have shown).

The dispersion relation for the two body waves is Eq. (53) of Ferriz Mas et al. (1989), or Eq. (31) of Zhugzhda (2002). It appears to me that it has not been realized that, once the phase velocity $V = \omega/k = \Omega c_{\rm T}$ is related to the tube speed $c_{\rm T}$, the dispersion relation for Ω can be written in terms of a single parameter ε, namely

$$\Omega^4 - 2\varepsilon\Omega^2 + 2\varepsilon = 0, \tag{14}$$

where

$$\varepsilon = \varepsilon_2 = \frac{c_{\rm S}^2 + c_{\rm A}^2}{2c_{\rm T}^2}\left(1 + \frac{4}{\alpha^2}\right). \tag{15}$$

The sound, Alfvén, and tube speeds are given by

$$c_{\rm S}^2 = \frac{\gamma\overline{P}}{\overline{\rho}}, \qquad c_{\rm A}^2 = \frac{\overline{B}^2}{\mu\overline{\rho}}, \qquad c_{\rm T}^2 = \frac{c_{\rm S}^2 c_{\rm A}^2}{c_{\rm S}^2 + c_{\rm A}^2}, \tag{16}$$

and α is a dimensionless wave number,

$$\alpha = k\overline{R}. \tag{17}$$

There is always $\varepsilon_2 > 2$. The solution of (14),

$$\Omega^2 = \varepsilon \pm \sqrt{\varepsilon^2 - 2\varepsilon}, \tag{18}$$

is shown in Fig. 1. As is well-known, the phase velocity of the slow mode approaches $c_{\rm T}$ for $\alpha \to 0$ (i.e., $\varepsilon \to \infty$), whereas the phase velocity of the fast mode tends to infinity. If ε becomes very large because $c_{\rm S} \ll c_{\rm A}$ (at finite α), then the phase velocity approaches $c_{\rm T} \approx c_{\rm S}$.

For $N = 4$ the system of perturbation equations defines the "four-mode approximation". As will be clear from the fourth-order dispersion formula (27) below, the dimensionless phase velocities of the four modes again obey (14), with only a single parameter. Hence these phase velocities can again be read off Fig. 1. However, the definition of the parameter is now

$$\varepsilon = \varepsilon_4 = \frac{c_{\rm S}^2 + c_{\rm A}^2}{2c_{\rm T}^2}\left(1 + \frac{8}{\alpha^2}\right). \tag{19}$$

The four-mode approximation is degenerate in that each of the two solutions (18) is counted twice, see below. Comparison of (15) and (19) shows that a particular mode of the four-mode approximation has the same phase velocity as the corresponding mode in the two-mode approximation whose wavelength is larger by a factor $\sqrt{2}$.

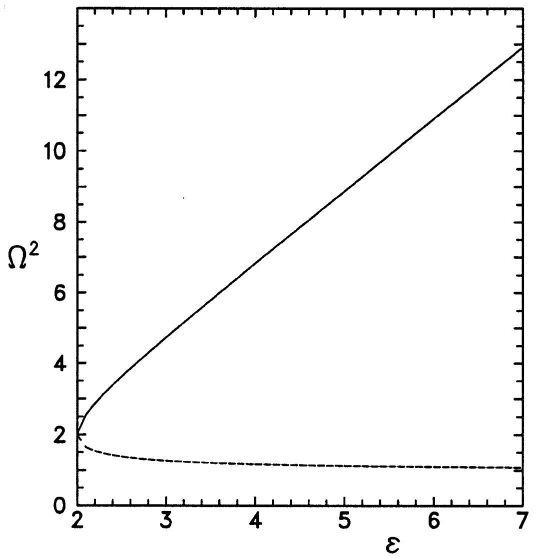

Fig. 1. Dimensionless squared phase velocity $\Omega = V/c_{\rm T}$ as function of the parameter ε. The lower and upper branches mark the slow and fast body waves, respectively.

4. General case

Zhugzhda (2002) has pointed out that the truncation of the perturbation equations at level N can be used to derive recurrence relations for the solutions, and that the eigenfunctions converge to Bessel functions of orders 0 and 1 (although he suggests the constant $2/\overline{R}$ in place of the frequency-dependent wave number). Here I eliminate the functions $\tilde{P}_n, \tilde{\rho}_n, \tilde{v}_{zn}, \tilde{v}_{sn}, \tilde{B}_{zn}$, and $\tilde{B}_{sn}$, in order to derive a recurrence relation for $\tilde{\Pi}_n$. This straightforward algebra yields

$$\tilde{\Pi}_n = -\frac{k_0^2}{n^2}\,\tilde{\Pi}_{n-2}, \tag{20}$$

where

$$k_0^2 = \frac{\left(\omega^2 - k^2 c_{\rm S}^2\right)\left(\omega^2 - k^2 c_{\rm A}^2\right)}{\left(c_{\rm S}^2 + c_{\rm A}^2\right)\left(\omega^2 - k^2 c_{\rm T}^2\right)} \tag{21}$$

is a function of frequency; it is the same squared wave number that also occurs in the exact treatment of waves in a magnetic cylinder (Roberts & Webb 1978; Edwin & Roberts 1983; Ferriz Mas et al. 1989; there denoted n_0^2). Since the perturbation equations are linear and homogeneous, we may set $\tilde{\Pi}_0 = 1$ (apart from the phase $\exp[{\rm i}(kz - \omega t)]$), and use the recurrence relation (20) to obtain, for all even n,

$$\tilde{\Pi}_n = \frac{\left(-k_0^2/4\right)^{n/2}}{[(n/2)!]^2}. \tag{22}$$

Therefore, the complete solution is

$$\tilde{\Pi} = \sum_{0,2\ldots}^{\infty} \tilde{\Pi}_n\, s^n = \sum_{0,2\ldots}^{\infty} \frac{(-s^2 k_0^2/4)^{n/2}}{[(n/2)!]^2} = \sum_{0,1\ldots}^{\infty} \frac{(-s^2 k_0^2/4)^{n}}{[(n)!]^2}. \tag{23}$$

The last series is the convergent expansion of the Bessel function J_0, e.g., Abramowitz & Stegun (1964), No. 9.1.12. Hence

$$\tilde{\Pi} = J_0(k_0 s)\cdot \tag{24}$$

Reversing the elimination procedure one can verify that $\tilde{P}, \tilde{\rho}, \tilde{v}_z$, and $\tilde{B}_z$ all converge to constants times $J_0(k_0 s)$, while $\tilde{v}_s$ and $\tilde{B}_s$ converge to constants times $J_1(k_0 s)$. The convergence is uniform over the entire interval $(0, \overline{R})$. The truncated series are, term by term, identical to the truncated Taylor expansions of those Bessel functions. Thus the expansion converges to the exact solution of waves in a magnetic cylinder, which are Bessel functions as already observed by Roberts & Webb (1978).

The appearance of the functions $\tilde{P}_n$ and $\tilde{B}_{zn}$ in the combination (13) does not mean that these two unknowns cannot be determined separately. For the N-mode approximation the process of elimination yields, for $n \le N-2$,

$$\tilde{P}_n = \frac{c_{\rm T}^2 \Omega^2}{c_{\rm A}^2(\Omega^2 - 1)}\,\tilde{\Pi}_n, \tag{25}$$

$$\tilde{B}_{zn} = \frac{c_{\rm T}^2\Omega^2 - c_{\rm S}^2}{c_{\rm A}^2(\Omega^2 - 1)}\,\frac{\overline{B}}{\overline{P}\gamma}\,\tilde{\Pi}_n. \tag{26}$$

If Eqs. (6), (7), (9), and (11) for $n = N$ are added to the equations of the N-mode approximation, it is seen that (25) and (26) can even be applied for $n = N$; this had been done by Ferriz Mas et al. (1989) in the case $N = 2$.

Using (22) we may evaluate the boundary condition (12) at any level of truncation. For $N = 2$ we recover the dispersion relation (14), for $N = 4$ we obtain

$$1 - \frac{(k_0\overline{R})^2}{4} + \frac{(k_0\overline{R})^4}{64} = 0, \tag{27}$$

which is a complete square and thus reduces to double solutions satisfying

$$1 - \frac{(k_0\overline{R})^2}{8} = 0. \tag{28}$$

This is equivalent to (14) above, with ε defined by (19).

At each level of truncation a pair of modes is being added, and the frequencies of the modes found in the previous approximation are corrected. Since the approximate dispersion relations converge to the exact dispersion relation, their zeros (and the eigenfrequencies) must also converge to the exact zeros (eigenfrequencies). It is easy to see that all approximate frequencies as well as all frequencies of the converged series can be obtained by solving (14). The recipe is: take the νth root $z^2 = z_\nu^2$ of the $N/2$ roots of the N-mode dispersion relation (written as a $N/2$-grade polynomial in $z^2 = (k_0\overline{R})^2$), or take the νth zero $z = j_{0\nu}$ of J_0; evaluate

$$\varepsilon = \frac{c_{\rm S}^2 + c_{\rm A}^2}{2c_{\rm T}^2}\left(1 + \frac{z^2}{\alpha^2}\right), \tag{29}$$

and then solve (14), or read the pair of mode frequencies off Fig. 1. It should be noticed that, in spite of the general character of Fig. 1, the frequencies obtained in the diverse approximations of course differ from each other, and from the exact eigenfrequencies, since the corresponding values of z^2 are different. It is clear from (14) and Fig. 1 that all slow modes, approximate and exact, have phase velocities between the tube speed $c_{\rm T}$ and $\sqrt{2}c_{\rm T}$, and all fast modes have phase velocities above $\sqrt{2}c_{\rm T}$.

5. Discussion

It seems to me that the value of the thin-tube expansion has not always been fully appreciated. In this note it has been demonstrated for a special case – a straight, circular, and homogeneous magnetic cylinder – that the expansion indeed converges to the exact solution. Moreover, at each level of truncation the solutions are precisely the truncated Taylor series of the Bessel function that represent the exact solution. Therefore each level of truncation can be considered as an improvement. This is true for the eigenfrequencies as well as for the eigenfunctions. Hence, in order to improve an approximation, it seems not advisable to restrict the result to a very thin tube, or to introduce a fictitious wave number or effective tube radius, depending on the level of approximation (Ferriz Mas et al. 1989; Zhugzhda 1996); it is better simply to go to an approximation of higher order. I concede, however, that the method of obtaining eigenfrequencies via the parameter ε is related to introducing those effective parameters.

The strength of the expansion method lies in its applicability to situations that are more involved than the present example. Such may include stratification, bending, twist, nonlinearity, and the interaction with the external medium. I suppose that the convergence prove of the present simple case adds confidence to the approximations used in those more complicated cases, and that similar proves might be possible there.

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