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Research Note

An orthonormal set of Stokes profiles

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Abstract. A family of well-known orthonormal functions, the set of Hermite functions, is proposed as a suitable basis for expanding the Stokes profiles of any spectral line. An expansion series thus provides different degrees of approximation to the Stokes spectrum, depending on the number of basis elements used (or on the number of coefficients). Hence, an usually large number of wavelength samples, may be substituted by a few such coefficients, thus reducing considerably the size of data files and the analysis of observable information. Moreover, since the set of Hermite functions is an universal basis, it promises to help in modern inversion techniques of the radiative transfer equation that infer the solar physical quantities from previously compiled look-up tables or artificial neural networks. These features appear to be particularly important in modern solar applications producing huge amounts of spectropolarimetric data and on near-future, on-line applications aboard spacecrafts.

Key words. line: profiles – magnetic fields – Sun: magnetic fields – stars: magnetic fields – radiative transfer

1. Introduction

Most of what we currently know about the Sun's magnetic fields is provided to us by the Stokes profiles of spectral lines (Lites et al. 1994). Spectropolarimetry appears to be the most useful among the current means devised to observe the solar and stellar atmospheres (del Toro Iniesta 2003a) and, certainly, inversion techniques (ITs) of the radiative transfer equation reveal themselves as the more suitable techniques devoted to infer the atmospheric physical quantities from the observables. Recent advances on instrumentation are challenging our abilities to deal with the huge amounts of data produced and some improvements or modifications of our currently available ITs will be welcome as well as any speed increase in numerically solving the radiative transfer equation, a procedure in which most ITs lay somehow. The main spectropolarimetric observable is the set of four Stokes parameters as functions of the wavelength: the Stokes profiles. Every inference technique pretends to reproduce the observable by means of some physical model of the stellar atmosphere and some known physics of transport of radiation. Certainly, if we are able to isolate repetitive features in the profiles and represent them as a linear superposition of these basic features, the analysis simplify and the computation time may reduce significantly as compared to other, more *classical*, inversion procedures (for reviews, see Socas-Navarro 2001; del Toro Iniesta 2003b; Bellot Rubio 2003). An expansion series in terms of basis profiles is the fundamental of the Principal Component Analysis (PCA, Rees et al. 2000). This method has proved to produce reliable results (Socas-Navarro et al. 2001) and its coefficients to have relations with the solar atmospheric parameters (Skumanich & López Ariste 2002) but, from a practice point of view, it shows the main drawback of having a non-unique set of basic profiles: this set must be calculated once for every observation or one must acknowledge the risk of using basic profiles that do not reproduce every particular observation. Here we propose a new means for Stokes profile data management by introducing a well-known universal basis for the vector space to which any Stokes profile belongs: $\mathcal{L}^2$, the space of square integrable functions. (More specifically, all three Stokes Q, U, and V, and Stokes I in line depression.) By means of this basis, several wavelength samples can be substituted by a few expansion coefficients, thus reducing the dimensionality of the inversion problem.

The structure of the paper is as follows: we introduce the set of Stokes basis functions in Sect. 2. In Sect. 3, a numerical experiment with a data base of more than 40 000 synthetic Stokes profiles is carried out in order to illustrate the various degrees of approximation depending on the number of basis profiles used. A fit to a real (i.e. observed) Stokes profile set is also presented in that section. Finally, the results are discussed in Sect. 4.

2. A new set of Stokes *basis functions*

Considered as functions of the wavelength, all three Stokes Q, U, and V profiles and Stokes I in line depression ($I_{\rm d} \equiv 1 - I/I_{\rm c}$,

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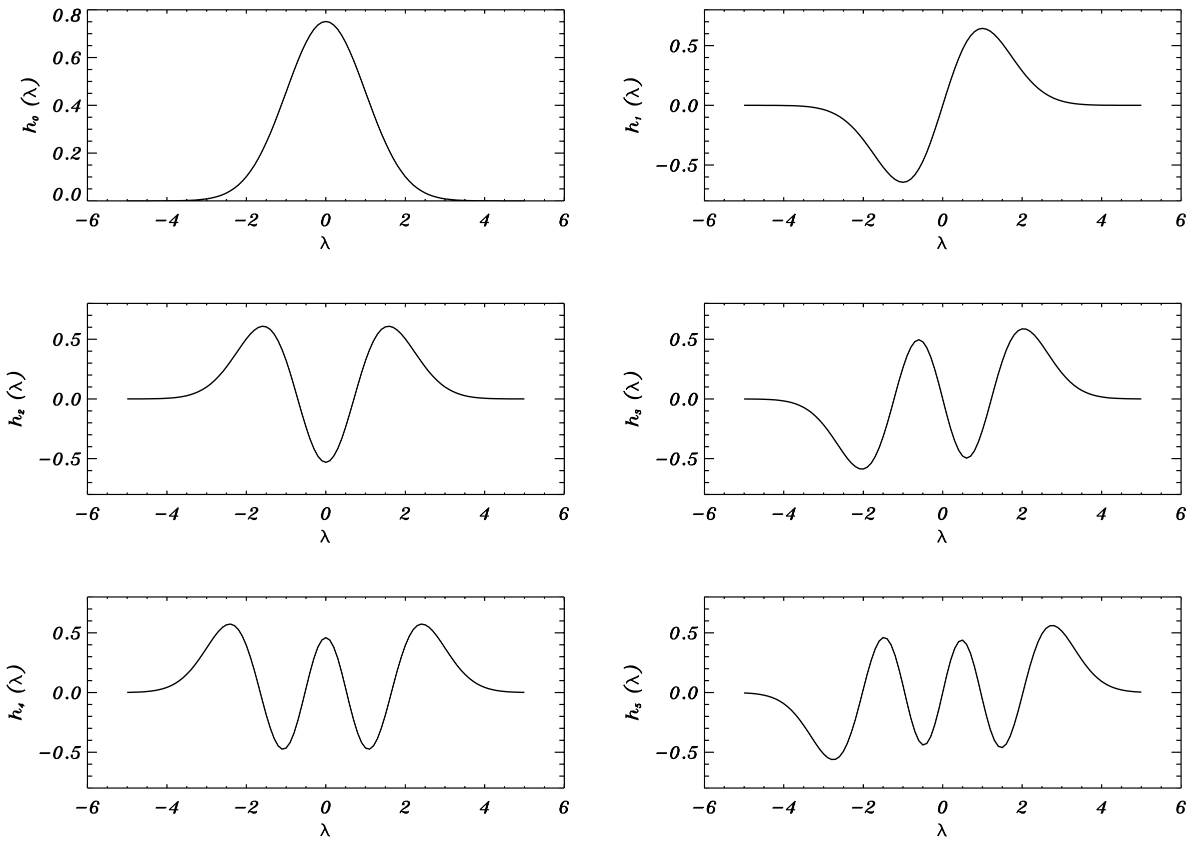

Fig. 1. First six Hermite functions.

where I_c is the continuum intensity) of any spectral line are square integrable functions of wavelength: I_d, Q, U, $V \in \mathcal{L}^2$. This property stems from the fact that all of them are continuous and drop asymptotically to zero. As is well known, $\mathcal{L}^2$ is a Hilbert space where a scalar product,

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f(\lambda)\, g(\lambda)\, \mathrm{d}\lambda, \tag{1}$$

is defined. Among the many families of functions that constitute an orthonormal basis for this space, let us pay special attention to the set of Hermite functions, the product of the Gaussian, $\exp(-\lambda^2/2)$, by the Hermite polynomials,

$$h_n(\lambda) = \frac{1}{\sqrt{2^n n!\, \sqrt{\pi}}}\, \mathrm{e}^{\frac{-\lambda^2}{2}}\, H_n(\lambda), \tag{2}$$

where the polynomials are defined by the recurrent relationships

$$\begin{aligned} H_0(\lambda) &= 1, \\ H_1(\lambda) &= 2\lambda, \\ H_{n+1}(\lambda) &= 2\lambda\, H_n(\lambda) - 2n\, H_{n-1}(\lambda), \end{aligned} \tag{3}$$

and the first fraction is a normalization factor. The Hermite functions of Eq. (2) are orthonormal ($|\langle h_i, h_j\rangle| = \delta_{i,j}$, with $\delta_{i,j}$ the Kronecker delta).

A glance to Fig. 1 readily illustrates our interest in this set of functions: $h_0(\lambda)$ has the shape of I_d, $h_1(\lambda)$ looks like a typical Stokes V profile due to the Zeeman effect, and $h_2(\lambda)$ is similar to Stokes Q or U. Certainly, these three first elements of the orthonormal basis have definite symmetry properties and would represent the Zeeman Stokes spectrum of a line formed in the absence of velocity gradients (see, e.g., Auer & Heasley 1978; Landi Degl'Innocenti & Landi Degl'Innocenti 1981). The asymmetries present in *real* Stokes profiles of such lines, formed in the presence of gradients, can be taken into account through the contribution of higher order functions. Since the set of Hermite functions is a basis of the $\mathcal{L}^2$ space, every Stokes profile can be represented exactly by an infinite linear combination of h_n functions. The strong similarities between the first Hermite functions and the actually observed profiles suggests that just a few such functions are needed for an expansion of the Stokes parameters as functions of the wavelength. If this is the case, we have the means for substituting the (usually) many wavelength samples of the observations by a few coefficients with the consequent reduction in dimensionality. Note that an expansion of the profiles as the one we propose is independent of any radiative transfer physics and, therefore, the results can later be processed with any other analysis technique that infers the atmospheric physical quantities.

3. Numerical experiments

The huge amounts of data currently produced by state-of-the-art spectropolarimeters and the even bigger data flow expected for the near-future polarimeters and magnetographs aboard balloons and spacecrafts, which will eventually require quasi-on-line reduction and analysis, makes

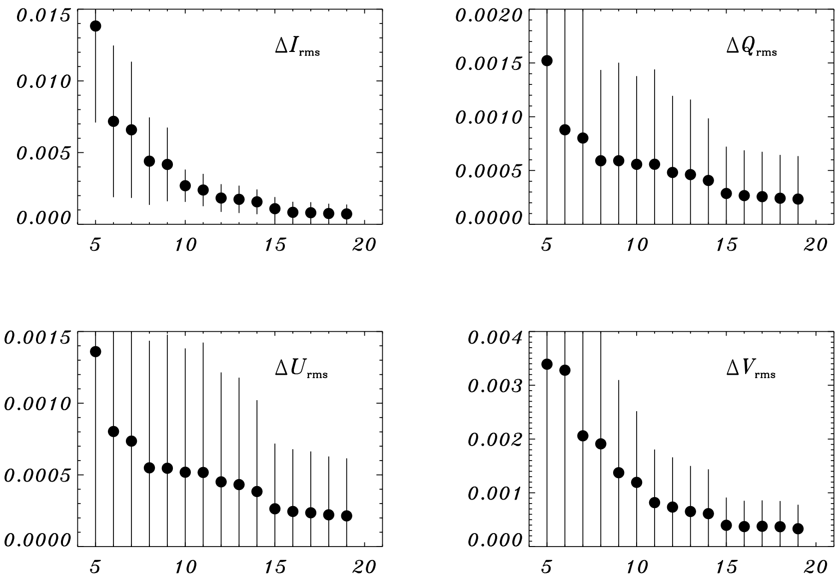

Fig. 2. Average root-mean-square deviations between Milne-Eddington profiles and their successive expansions in terms of Hermite functions. The number of terms is in the abscissae. The error bars represent the standard deviation among the 46 844 expansions.

it necessary to find very fast, still reliable techniques. The Milne-Eddington (ME) inversion technique of the HAO (Skumanich & Lites 1987; Lites & Skumanich 1990) has been for years the fastest way of inferring accurate values of the magnetic field vector (see Westendorp Plaza et al. 1998 for a thorough, comparative analysis). Most of its success stems from a significant dimensional simplification of the atmospheric modeling: only ten parameters are needed to identify the Stokes profiles. The ME inability to reproducing asymmetric profiles can be considered as a consequence of its low-order representation of the atmosphere (and hence, the profiles) as compared to other. It is this philosophy of successive approximation which is underlying any expansion approach. Therefore, it seems suitable to start with ME Stokes profiles to check the usefulness of an expansion in terms of Hermite functions. If all four Stokes profiles of a spectral line formed through the Zeeman effect in a ME atmosphere are fully characterized by just ten parameters, sampling each of them at more than fifty wavelengths looks disproportionate. This may also be the case of real profiles: although they are asymmetric and need more parameters to be fully reproduced, as many as 200 observables (say, 50 wavelengths times four Stokes parameters) may be too much. In this section we expand both ME and real Stokes profiles in terms of Hermite functions in order to get an idea of how many of them are necessary for a reliable fit.

The calculation of the expansion coefficients is very simple and based on the scalar product (1). In fact, if a generic Stokes profile is denoted by $S(\lambda)$ and is to be expanded to n Hermite functions,

$$S(\lambda) \simeq \sum_{i=0}^{n-1} c_i\, h_i(\lambda), \tag{4}$$

the c_i coefficients are given by

$$c_i = \langle S(\lambda), h_i(\lambda)\rangle = \int_{-\infty}^{\infty} S(\lambda)\, h_i(\lambda)\, \mathrm{d}\lambda. \tag{5}$$

We have used a set of 46 844 ME, Stokes profiles in a data base built upon the results of inverting a *real* ASP map of a sunspot and its surroundings. Thus, a pretty large span in the parameters is contained in this set. The Stokes profiles have been expanded in terms of Hermite functions. The number of such functions has been varied between 5 and 19. The root-mean-square difference between the profiles and the analytic expansion has then been calculated and averaged for the whole set. Figure 2 summarizes the results: the solid circles represent the average rms differences and the error bars the standard deviations. As is clearly seen in the figure, as few as ten Hermite functions provide (*analytic!*) fits to Stokes I differing less than 3×10^{-3}, to Stokes Q and Stokes U differing less than 6×10^{-4}, and to Stokes V differing less than 1.2×10^{-3} of the continuum intensity. These rms differences are quite compatible with the typical signal-to-noise ratios obtained with modern spectropolarimeters, but can be significantly decreased with a few more terms in the expansion.

Not only symmetric Stokes profiles like ME profiles can be represented with a few Hermite functions. As an example, we show in Fig. 3 two fits, with 11 ($n = 10$; dotted line) and 20 terms ($n = 19$; solid line), to real Stokes profiles observed in the penumbra of a sunspot where del Toro Iniesta et al. (2001) discovered supersonic downflows. The rms differences between observations and eleven-term fits are 5.3×10^{-3}, 1.4×10^{-3}, 1.6×10^{-3}, and 1.3×10^{-3} for I, Q, U, and V, respectively, clearly smaller than or equal to the noise levels. The differences with the twenty-term fits are even slightly smaller. This observation of the Fe I line at 1558.8257 nm has been selected as an extreme example of asymmetry.

4. Discussion and conclusions

We have shown the properties of the Hermite family of functions as an orthonormal set of Stokes profiles. Every Stokes profiles, symmetric or not, can be fitted with a few Hermite functions, hence providing an excellent means of reducing the dimensionality of data representation. As compared to other Stokes expansions like that performed within the PCA analysis, the advantages of the Hermite Stokes basis profiles is twofold.

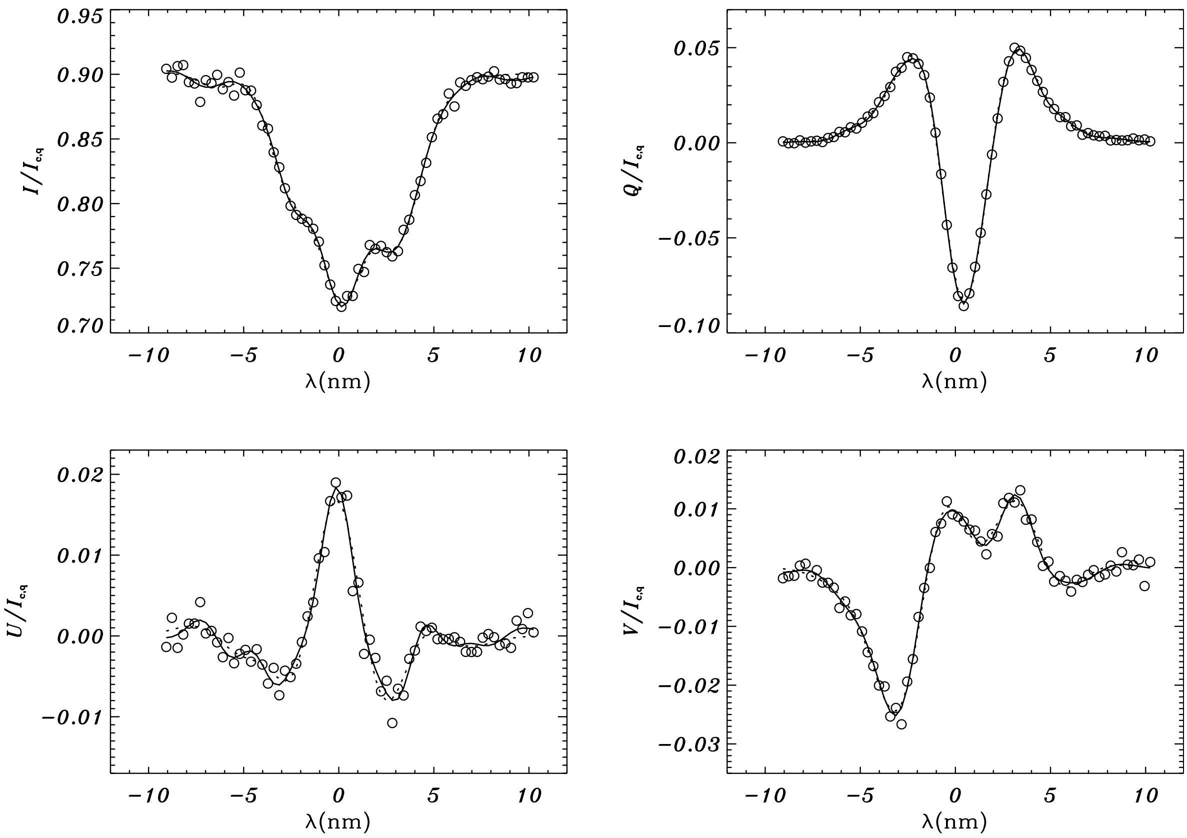

Fig. 3. Observed Stokes profiles (circles), and Hermite expansions up to $n = 10$ (dotted lines) and up to $n = 19$ (solid lines), of the Fe I line at 1558.8257 nm as formed in a sunspot penumbra.

On the one hand, the basis is universal and does not depend on the particular observation; we know it a priori and can even use this knowledge in on-line applications. On the other hand, the orthonormal basis is analytical and thus noise-free. These features allow us to consider the Hermite expansion of the profiles as an alternative representation of the observations and enable its use in whatever inversion technique of the radiative transfer equation, either current or future.

We can simply consider Hermite expansion as a tool for reducing the data dimensionality and use it, for example, to circumvent limited telemetry problems from spacecrafts. Or we can use the analytic approximations as to build the look-up tables needed in PCA-like inversion techniques, or even to train neural network machines (Carroll & Staude 2001; Socas-Navarro 2003). In both PCA and neural network cases, an acceleration of the procedures is guaranteed since we have to deal with much less observables: the set of Hermite coefficients. But the finite Hermite representation can even be used as an input to "regular" inversion techniques like ME or SIR. Therefore, we are facing an extremely useful means in practice.

Besides these practical features, the mathematical properties of Hermite functions may in principle be used with much more theoretical purposes. In fact, the possibilities must be explored that these Hermite functions help in solving the radiative transfer equation in a more efficient and accurate way than those currently used.

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