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Systematic bias in interstellar magnetic field estimates

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Abstract. Faraday rotation of the polarization plane in magnetized thermal plasma provides one of the most efficient methods to deduce regular magnetic fields from radio astronomical observations. Since the Faraday rotation measure RM is proportional to an integral, along the line of sight, of magnetic field weighted with thermal electron density, RM is believed to yield the regular magnetic field averaged over large volume. Here we show that this is not the case in a turbulent medium where fluctuations in magnetic field and electron density are not statistically independent, and so contribute to RM. For example, in the case of pressure equilibrium, magnetic field can be anticorrelated with plasma density to produce a negative contribution. As a result, the strength of the regular magnetic field obtained from RM can be *underestimated* if the fluctuations in electron density and magnetic field are neglected. The anticorrelation also reduces the standard deviation of RM. We further discuss the effect of the positive correlations where the standard treatment of RM leads to an *overestimated* magnetic field. Because of the anisotropy of the turbulent magnetic field, the regular magnetic fields strength, obtained from synchrotron emission using standard formulae, can be *overestimated*. A positive correlation between cosmic-ray number density and magnetic field leads to an overestimate of the strengths of the regular and total fields. These effects can explain the difference between the strengths of the regular Galactic magnetic field as indicated by RM and synchrotron emissivity data and reconcile the magnetic field strength in the Solar vicinity with typical strength of regular magnetic fields in external galaxies.

Key words. magnetic fields – polarization – turbulence – ISM : magnetic fields – galaxies: ISM

1. Introduction

Estimates of magnetic field strength in the diffuse interstellar medium (ISM) of the Milky Way and other galaxies are most efficiently obtained from the intensity and Faraday rotation of synchrotron emission. Other methods are only sensitive to relatively strong magnetic fields that occur in dense clouds (Zeeman splitting) or are difficult to quantify (optical polarization of star light by dust grains). The total I and polarized P synchrotron intensities and the Faraday rotation measure RM are integrals over the path length L, so they provide a measure of the average magnetic field in the emitting or magneto-active volume:

$$I = K \int_L n_{\rm cr} B_\perp^2 \, {\rm d}s,$$
$$P = K \int_L n_{\rm cr} \overline{B}_\perp^2 \, {\rm d}s, \qquad (1)$$
$$RM = K_1 \int_L n_{\rm e} B_\parallel \, {\rm d}s,$$

where $n_{\rm cr}$ and $n_{\rm e}$ are number densities of relativistic and thermal electrons, $\boldsymbol{B}$ is the total magnetic field comprising a regular $\overline{\boldsymbol{B}}$ and random $\boldsymbol{b}$ parts, $\boldsymbol{B} = \overline{\boldsymbol{B}}+\boldsymbol{b}$ with $\langle \boldsymbol{B}\rangle = \overline{\boldsymbol{B}}$, $\langle \boldsymbol{b}\rangle = 0$ and $\langle B^2\rangle = \overline{B}^2 + \langle b^2\rangle$, angular brackets denote averaging, subscripts $\perp$ and $\parallel$ refer to magnetic field components perpendicular and parallel to the line of sight, and K and $K_1 = 0.81\,{\rm rad\,m^{-2}\,cm^3\,\mu G^{-1}\,pc^{-1}}$ are certain dimensional constants. The degree of polarization p is related to the ratio $\langle b^2\rangle/\overline{B}^2$,

$$p = \frac{P}{I} \approx p_0 \frac{\overline{B}_\perp^2}{B_\perp^2} = p_0 \frac{\overline{B}_\perp^2}{\overline{B}_\perp^2 + \frac{2}{3}\langle b^2\rangle}, \qquad (2)$$

where the random field $\boldsymbol{b}$ has been assumed to be isotropic, $n_{\rm cr}$ is assumed to be a constant, and $p_0 \approx 0.75$ weakly depends on the spectral index of the emission (Burn 1966; Sokoloff et al. 1998). This is an approximate relation. In particular, it does not allow for any anisotropy of the random magnetic field (see Sect. 4.1), for equipartition between magnetic fields and cosmic rays (see Sect. 4.2) and for depolarization effects; some generalizations are discussed by Sokoloff et al. (1998).

Since $n_{\rm cr}$ is difficult to measure, it is most often assumed that magnetic fields and cosmic rays are in energy equipartition; this allows one to express $n_{\rm cr}$ in terms of B. The physical basis of this assumption is the fact that cosmic rays are confined by magnetic fields. An additional assumption involved is that the energy density of relativistic electrons responsible

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for synchrotron emission (and so $n_{\rm cr}$) is proportional to the total energy density of cosmic rays; it is usually assumed that the energy density of relativistic electrons in the relevant energy range (i.e. several GeV) is one percent of the proton energy density in the same energy interval (Chapter 19 in Longair 1994). The validity of this assumption requires that the diffusion/escape losses of the cosmic-ray electrons dominate over radiative losses, so that the energy spectrum of the electrons is not steeper than that of the protons (Beck 1997).

Estimates of B in our Galaxy via $n_{\rm cr}$ determined from γ-ray emission (Strong et al. 2000) are generally consistent with the equipartition values (E. M. Berkhuijsen, in Beck 2001). However, Eq. (2) is not consistent with the equipartition or pressure balance between cosmic rays and magnetic fields insofar as it assumes that $n_{\rm cr}$ = const. Therefore, the regular magnetic field strength obtained using Eq. (2) can be inaccurate (see Sect. 4).

The thermal electron density $n_{\rm e}$ in the ISM can be obtained from emission measure $EM \propto \int_L n_{\rm e}^2\, {\rm d}s$, although this involves additional assumptions regarding the filling factor of interstellar clouds. In the Milky Way, the dispersion measures of pulsars, $DM = \int_L n_{\rm e}\, {\rm d}s$ provide information about the mean thermal electron density, but the accuracy is limited by our uncertain knowledge of distances to pulsars. Estimates of the strength of the regular magnetic field in the Milky Way are often obtained from the Faraday rotation measures of pulsars simply as

$$\overline{B}_{\|} \simeq \frac{RM}{K_1\, DM}. \qquad (3)$$

This estimate is meaningful if magnetic field and thermal electron density are *uncorrelated*. If the fluctuations in magnetic field and thermal electron density are correlated with each other, they will contribute positively to RM and Eq. (3) will yield overestimated $\overline{B}_{\|}$. In the case of anticorrelated fluctuations, their contribution is negative and Eq. (3) is an underestimate. In order to quantify this effect, one needs a suitable model for the relation between magnetic fields and thermal electron density. As we show in Sect. 3, physically reasonable assumptions about the statistical relation between magnetic field strength and electron density can lead to Eq. (3) being in error by a factor of 2–3 even in a statistically homogeneous magneto-ionic medium. Lerche (1970) discussed the effects of correlated fluctuations in magnetic field and electron density on Faraday rotation measures. Some results of that paper are presented in a questionable form, although the general conclusion agrees with that proposed here.

Equation (3) can also lead to significantly underestimated strength of the regular magnetic field if it is enhanced in the interarm regions whereas electron density is maximum within the arms, as in galaxies with magnetic arms (Beck 2001).

2. Magnetic field estimates

The observable quantities (1) provide extensive data on magnetic field strengths in both the Milky Way and external galaxies (Ruzmaikin et al. 1988; Beck et al. 1996; Beck 2000, 2001). The average total field strengths in nearby spiral galaxies obtained from total synchrotron intensity I range from $B \simeq 4\,\mu$G in the galaxy M 31 to $\simeq 15\,\mu$G in M 51, with the mean for the sample of 74 galaxies of $B \simeq 9 \pm 3\,\mu$G (Beck 2000). The typical degree of polarization of synchrotron emission from galaxies at short radio wavelengths is p = 10%–20%, so Eq. (2) gives $\overline{B}/B$ = 0.4–0.5; these are always lower limits due to limited resolution of the observations. Most existing polarization surveys of synchrotron emission from the Milky Way, having much better spatial resolution, suffer from Faraday depolarization effects and missing large-scale emission and cannot provide reliable values for p (see Sect. 5). Phillipps et al. (1981) obtained $\overline{B}/B$ = 0.6–0.7 from analysis of the total synchrotron emission from the Milky Way along and perpendicular to the spiral arms. Heiles (1996) derived similar values from starlight polarization data. The total equipartition magnetic field in the Solar neighbourhood is estimated as $B = 6 \pm 2\,\mu$G from the synchrotron intensity of the large-scale, diffuse Galactic radio background (E. M. Berkhuijsen, in Beck 2001). Combined with $\overline{B}/B = 0.65$, this yields a strength of the local regular field of $\overline{B} = 4 \pm 1\,\mu$G. Hence, the typical strength of the local Galactic random magnetic fields, $\langle b^2\rangle^{1/2} = (B^2 - \overline{B}^2)^{1/2} = 5 \pm 2\,\mu$G, exceeds that of the regular field by a factor $\langle b^2\rangle^{1/2}/\overline{B} = 1.3 \pm 0.6$. RM data yield similar values for this ratio (Sect. 4 in Ruzmaikin et al. 1988; Ohno & Shibata 1993).

Meanwhile, the values of $\overline{B}$ in the Milky Way obtained from Faraday rotation measures seem to be systematically lower than the above values. RM of pulsars and extragalactic radio sources yield $\overline{B} = 1.4 \pm 0.3\,\mu$G in the local (Orion) (Rand & Lyne 1994; Frick et al. 2001), $\overline{B} = 1.7 \pm 0.3\,\mu$G in the Sagittarius–Carina spiral arm (Frick et al. 2001); for the Perseus arm, Frick et al. (2001) obtained $\overline{B} = 1.4 \pm 1.2\,\mu$G, and Mitra et al. (2003), $\overline{B} = 1.7 \pm 1.0\,\mu$G. The median value of $\overline{B}$ is about twice smaller than that inferred from other methods.

There can be several reasons for the discrepancy between the estimates of the regular magnetic field strength from Faraday rotation and synchrotron intensity. Both methods suffer from systematic errors due to our uncertain knowledge of thermal and relativistic electron densities, so one cannot be sure if the difference is significant. Nevertheless, the discrepancy seems to be worrying enough to consider carefully its possible reasons. (We should emphasize that the main results of this paper are independent of whether or not the discrepancy is real.)

The discrepancy can be explained, at least in part, if the methods described above sample different volumes. The observation depth of total synchrotron emission, starlight polarization and of Faraday rotation measures are all of the order of a few kpc. Polarized emission, however, may emerge from more nearby regions (see Sect. 5). However, there are more fundamental reasons for the discrepancy that we discuss in what follows.

3. The effects of magneto-ionic fluctuations on *RM*

In this section we show that a statistical correlation between electron density fluctuations and turbulent magnetic fields can affect significantly regular magnetic field estimates obtained from Faraday rotation measures. The effect of fluctuations in the magneto-ionic medium on RM can be the main reason for the discrepancy between magnetic field estimates discussed in Sect. 2. Note that the effect discussed here arises from correlations at scales $\lesssim$100 pc (i.e., less than the basic turbulent scale), rather than from any relations between the averaged quantities (e.g., arising from the vertical stratification of the ISM or galactic density waves).

In order to quantify the effect, one needs a specific physical model for the connection between thermal gas density and magnetic field. An appealing idea is based on the assumption that the ISM is in *pressure balance* involving not only thermal pressure (Field et al. 1969; McKee & Ostriker 1977) but also turbulent, magnetic and cosmic ray pressures (e.g., Parker 1979; Boulares & Cox 1990; Fletcher & Shukurov 2001). Pressure balance can be maintained at the scales of interest since turbulence is subsonic, $v \lesssim c_s$ where v is the turbulent velocity, and so the sound crossing time l/c_s (across the correlation length of the turbulent magnetic fields l) is comparable to or shorter than the presumed correlation time of interstellar turbulence l/v. Numerical simulations of the multiphase ISM driven by supernova explosions indicate statistical pressure equilibrium over a wide range of physical conditions in the ISM, both between different phases and within a region occupied by a single phase (Rosen et al. 1995; Korpi et al. 1999; Gazol et al. 2001). Anyway, any system that deviates from pressure equilibrium on average must either expand or collapse; thus, any statistically steady state of the ISM must involve statistical pressure equilibrium.

Treatment of pressure equilibrium involving magnetic field requires some caution since magnetic pressure only contributes to force balance across the field lines. However, splitting magnetic field into an (isotropic) random part and a large scale field, as we do here, admits the reasonable assumption that magnetic pressure due to the random part is isotropic as long as we consider quantities averaged over an intermediate scale which is larger than the turbulent correlation scale (l = 50–100 pc) but smaller than the scale of the regular field ($\gtrsim$1 kpc).

The peculiar nature of the Lorentz force still remains important, e.g. magnetic buoyancy would produce gas motions even under perfect pressure balance. However, magnetic buoyancy is important at relatively large scales of the order of 1 kpc (Parker 1979), and so can be neglected here. Indeed, its characteristic time, equal to the Alfvén crossing time over the density scale height, cannot be shorter than the sound crossing time over the turbulent correlation length because both the turbulent scale is smaller than the density scale height and the Alfvén speed is smaller than the speed of sound (again assuming that the turbulence is subsonic). It is therefore not unreasonable that studies of magnetic buoyancy often start with a pressure-equilibrium state as an initial condition. Hence, we can neglect any effects related to anisotropy of the pressure produced by the large-scale magnetic field.

Pressure equilibrium in the ISM naturally leads to an anticorrelation between gas density and magnetic fields because regions with enhanced magnetic field must have lower density (if only the cooling time is longer than the sound crossing time, so that a transient pressure excess cannot be removed by cooling). If pressure equilibrium is only maintained in the statistical sense, the anticorrelation will also be only statistical.

On the other hand, there are systematic deviations from pressure balance in the ISM. For example, the total pressure is larger than average in self-gravitating clouds, expanding supernova remnants, stellar wind regions, and in travelling galactic density waves. Such overpressured regions can result in a positive correlation between magnetic field and gas density, leading to enhanced Faraday rotation.

We derive an expression for the Faraday rotation measure under the assumption of pressure equilibrium in Sect. 3.1 and generalize it in Sect. 3.2. Here we employ the assumption of isotropic random magnetic field. As discussed in Sect. 4.1, this assumption is only approximate, but deviations from statistical isotropy can hardly affect significantly the effects discussed in this section.

3.1. Magneto-ionic medium in pressure balance

In order to evaluate a correction to RM resulting from the anticorrelation of b and n_e, we first obtain n_e as a function of b from pressure equilibrium equation. We introduce Cartesian coordinates (x, y, z) with the x-axis parallel to $\overline{\boldsymbol{B}}$, so that $\overline{\boldsymbol{B}} = (\overline{B}, 0, 0)$, and the line-of-sight vector $\boldsymbol{s} = (s_x, s_y, s_z)$ directed towards the observer, with L the distance from the source of polarized emission to the observer. Assuming that cosmic ray pressure is equal to the magnetic pressure, we start with pressure equilibrium equation,

$$\frac{B^2}{4\pi} + \mathcal{P}_{\rm gas} = \mathcal{P}, \tag{4}$$

where $\mathcal{P}_{\rm gas}$ is the gas pressure comprising thermal and turbulent components, the magnetic contribution has been doubled to allow for cosmic ray pressure, and $\mathcal{P}$ is the total pressure. Since $\mathcal{P}_{\rm gas}$ is proportional to the total gas density, we write

$$\mathcal{P}_{\rm gas} = n_e F,$$

where $F = \mathcal{P}_{\rm gas}/(X n_t)$ with n_t the total gas density and X the degree of ionization. This representation is convenient if the medium is isothermal, which is a good approximation since RM is mainly produced in a single phase of the ISM, the warm ionized medium (Heiles 1976). A more consistent theory should include fluctuations in the total pressure and variable degree of ionization, together with a more realistic equation of state (e.g., polytropic). We leave these generalizations for future work, and only briefly discuss the polytropic equation of state in Sect. 3.2.

Consider magnetic field B consisting of a regular $\overline{B}$ and turbulent parts, and similarly for electron density:

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}, \quad n_e = N + n, \quad \langle \boldsymbol{b} \rangle = \langle n \rangle = 0;$$

we shall assume that the random field $\boldsymbol{b}$ is isotropic. Averaging Eq. (4) yields

$$\overline{B}^2 + \langle b^2 \rangle = 4\pi(\mathcal{P} - NF), \tag{5}$$

where we have assumed that $\mathcal{P}$ and F do not fluctuate. Subtract (5) from (4) to obtain the following balance equation for the fluctuations:

$$b^2 - \langle b^2\rangle + 2\overline{\boldsymbol{B}}\cdot\boldsymbol{b} = -4\pi nF, \tag{6}$$

or

$$n = -\frac{1}{4\pi F}(b^2 - \langle b^2\rangle) - \frac{1}{2\pi F}\overline{\boldsymbol{B}}\cdot\boldsymbol{b}. \tag{7}$$

This equation shows that electron density is smaller ($n < 0$) where magnetic field is stronger, because either b^2 is larger than average or $\overline{\boldsymbol{B}}$ and $\boldsymbol{b}$ are similarly directed, $\overline{\boldsymbol{B}}\cdot\boldsymbol{b} > 0$.

Now we calculate RM identifying the integral in Eq. (1) with ensemble average:

$$RM/K_1L = \langle\overline{\boldsymbol{B}}\cdot\boldsymbol{s}n_\mathrm{e}\rangle = (\overline{\boldsymbol{B}}\cdot\boldsymbol{s})N + \langle(\boldsymbol{b}\cdot\boldsymbol{s})n\rangle; \tag{8}$$

where $\boldsymbol{s}$ is the line-of-sight vector. In order to calculate the last term on the right-hand side, we use Eq. (7) to obtain

$$\langle n\boldsymbol{b}\cdot\boldsymbol{s}\rangle = -\frac{1}{2\pi F}\langle(\boldsymbol{b}\cdot\boldsymbol{s})(\overline{\boldsymbol{B}}\cdot\boldsymbol{b})\rangle, \tag{9}$$

since $\left\langle(\boldsymbol{b}\cdot\boldsymbol{s})\left(b^2 - \langle b^2\rangle\right)\right\rangle = 0$ because $\langle\boldsymbol{b}\cdot\boldsymbol{s}\rangle = 0$ and $\langle(\boldsymbol{b}\cdot\boldsymbol{s})b^2\rangle = \langle b^3\cos\alpha\rangle = 0$ (with α the random angle between $\boldsymbol{b}$ and $\boldsymbol{s}$) as $\boldsymbol{b}$ is an isotropic random vector.

The average in Eq. (9) can be calculated expanding the dot products into Cartesian components:

$$\begin{aligned}\langle(\boldsymbol{b}\cdot\boldsymbol{s})(\overline{\boldsymbol{B}}\cdot\boldsymbol{b})\rangle &= \langle(b_xs_x + b_ys_y + b_zs_z)(\overline{B}b_x)\rangle\\ &= \overline{B}s_x\langle b_x^2\rangle + \overline{B}s_y\langle b_xb_y\rangle + \overline{B}s_z\langle b_xb_z\rangle.\end{aligned}$$

We assume that b_x, b_y and b_z are uncorrelated when evaluated at the same position, so $\langle b_xb_y\rangle = \langle b_xb_z\rangle = 0$. This condition is not restrictive as it is satisfied by any random magnetic field $\boldsymbol{b}$ with symmetric probability distributions of b_x, b_y and b_z (and for an isotropic $\boldsymbol{b}$ in particular). Then

$$\langle(\boldsymbol{b}\cdot\boldsymbol{s})(\overline{\boldsymbol{B}}\cdot\boldsymbol{b})\rangle = \frac{1}{3}\overline{B}_\parallel\langle b^2\rangle, \tag{10}$$

where $\overline{B}_\parallel = \overline{\boldsymbol{B}}\cdot\boldsymbol{s} \equiv \overline{B}s_x$ is the line-of-sight regular magnetic field and $\frac{1}{3}\langle b^2\rangle = \langle b_x^2\rangle$ due to the isotropy of $\boldsymbol{b}$.

We note that the assumption that the components of $\boldsymbol{b}$ are statistically point-wise independent is compatible with the solenoidality of magnetic field because the latter only requires that magnetic field components are correlated when taken at distinct positions. In particular, the components of a random, isotropic and homogeneous magnetic field must be statistically independent when taken at the same position.

Using Eqs. (9) and (8), we obtain

$$\begin{aligned}RM &= K_1\left[\overline{B}_\parallel NL - \frac{1}{6\pi F}\overline{B}_\parallel L\langle b^2\rangle\right]\\ &= RM^{(0)}\left(1 - \frac{2}{3}\,\frac{\langle b^2\rangle}{4\pi\langle\mathcal{P}_\mathrm{gas}\rangle}\right),\end{aligned} \tag{11}$$

where $RM^{(0)} = K_1\overline{B}_\parallel NL$ is the Faraday rotation measure that would arise if magnetic field and electron density fluctuations were uncorrelated. So, RM has a negative contribution from the turbulent magnetic field. We can rewrite this formula in terms of magnetic field alone assuming that the average magnetic pressure is equal to the gas pressure (equipartition between magnetic and gas energies),

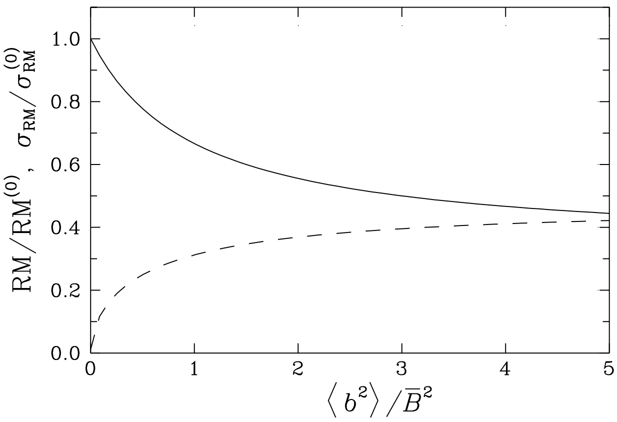

Fig. 1. The ratio of RM (solid) and σ_{RM} (dashed) to their standard estimates $RM^{(0)}$ and $\sigma_{RM^{(0)}}$ as given by Eqs. (13) and (15), respectively. In Eq. (15), we have taken $f = 0.1$ and $\overline{B}_\perp^2 = \overline{B}_\parallel^2 = \frac{1}{2}\overline{B}^2$.

$$\overline{B}^2 + \langle b^2\rangle = 4\pi\langle\mathcal{P}_\mathrm{gas}\rangle, \tag{12}$$

which yields

$$RM = RM^{(0)}\left(1 - \frac{2}{3}\,\frac{\langle b^2\rangle}{\overline{B}^2 + \langle b^2\rangle}\right)\cdot \tag{13}$$

The equipartition expressed by Eq. (12) is assumed to hold at scales $\gtrsim$1 kpc, so that it does not contradict the assumption of pressure equilibrium (Eq. (4)) that holds at small scales, $\lesssim$1 kpc.

The dependence of $RM/RM^{(0)}$ on $\langle b^2\rangle/\overline{B}^2$ is shown in Fig. 1. In the limiting case $\langle b^2\rangle/\overline{B}^2 \gg 1$, we have

$$RM = \frac{1}{3}RM^{(0)},$$

i.e., $\overline{B}$ would be underestimated by a factor of 3 if calculated as $\overline{B} = RM/K_1NL$ as usually done. As discussed in Sect. 2, the relative strength of the random magnetic field in the Milky Way is $\langle b^2\rangle^{1/2}/\overline{B} = 1.3 \pm 0.6$; we adopt $\langle b^2\rangle/\overline{B}^2 = 2$ as a representative value. Then Eq. (13) shows that the standard estimate of $\overline{B}$ is about two times too small. In other words, the regular magnetic field strength near the Sun consistent with the Faraday rotation measures of pulsars and extragalactic radio sources is $\overline{B} \simeq 3$–$4\,\mu$G (twice the value usually inferred from RM data), in agreement with the estimates from equipartition between cosmic rays and magnetic fields.

Apart from the reduction in the observed RM, quantified by Eq. (13), if there were an anticorrelation between magnetic field and electron density fluctuations, then the observed fluctuations in Faraday rotation measure would be reduced as well. When fluctuations in magnetic field and electron density are uncorrelated, the standard deviation of RM is given by

$$\sigma_{RM}^{(0)} = K_1\left(2\langle n^2\rangle\langle b^2\rangle Ld\right)^{1/2}, \tag{14}$$

where d is the common correlation length of magnetic and electron fluctuations (see, e.g., Appendix A in Sokoloff et al. 1998). As shown in Appendix A, under conditions where Eq. (13) applies, the standard deviation of RM is given by

$$\frac{\sigma_{RM}}{\sigma_{RM}^{(0)}} = \frac{40}{9\pi}\left(\frac{f}{1-f}\right)^{1/2} \frac{\langle b^2\rangle}{\overline{B}^2 + \langle b^2\rangle} \times\left(1 + \frac{9}{20}\frac{\overline{B}_\perp^2}{\langle b^2\rangle} - \frac{3}{40}\frac{\overline{B}_\parallel^2}{\langle b^2\rangle}\right)^{1/2}, \tag{15}$$

where f is the filling factor of thermal electrons, defined as $f = N^2/\langle n_e^2\rangle$. (Of course numerical coefficients in this formula are model dependent.) Since $f \ll 1$ (Berkhuijsen 1999 and references therein), the anticorrelation suppresses fluctuations in Faraday rotation, $\sigma_{RM}/\sigma_{RM}^{(0)} \lesssim 1$. This can be clearly seen in Fig. 1. For example, $\sigma_{RM}/\sigma_{RM}^{(0)} \simeq 0.4$ for $\langle b^2\rangle/\overline{B}^2 = 2$ and $f = 0.1$ (Berkhuijsen 1999). In the limiting case of strong magnetic fluctuations, $\langle b^2\rangle/\overline{B}^2 \gg 1$, the right-hand side of Eq. (15) is approximately equal to $f^{1/2}$. An important result of the reduction of the standard deviation of the Faraday rotation measure is that this reduces the depolarization effect of internal Faraday dispersion in comparison with standard estimates.

Another implication is that the value of σ_{RM} depends on the direction, it is minimum along the regular magnetic field and maximum in the directions across it. We stress that this anisotropy in σ_{RM} arises even in an isotropic random magnetic field. Brown & Taylor (2001) find that the scatter in RM is larger along the Orion arm than across it, contrary to our result. As we discuss in Sect. 4.1, it is natural to expect that the turbulent magnetic field in the ISM is anisotropic in a manner consistent with the observations of Brown & Taylor, and that the anisotropy in $\boldsymbol{b}$ masks the more subtle effect discussed in this section.

3.2. Overpressured regions and the polytropic equation of state

Systematic and random deviations from pressure equilibrium are widespread in the ISM, and this affects the relation between local magnetic field and electron density. For example, compression leads to enhancement in both magnetic field and gas density, and these variables can be correlated. (We note that the anticorrelation discussed in Sect. 3.1 is difficult to detect because observational estimates are biased towards dense regions.) In this section we discuss the effect of the positive correlation between b and n_e at scales $\lesssim$100 pc, i.e., where the correlation (if any) can have a random character; such regions are not likely to be widespread in the diffuse ISM. As might be expected, this leads to enhanced Faraday rotation. Next we consider the polytropic equation of state, where the assumption of isothermal medium of Sect. 3.1 is relaxed.

Suppose that the total magnetic field scales with the total electron density as

$$\frac{B^2}{4\pi} = a n_e^\kappa = a(N+n)^\kappa \simeq a(N^\kappa + \kappa n N^{\kappa-1}), \tag{16}$$

where a and κ are constants. The last equality is only valid for weak fluctuations, $n/N \ll 1$. Averaging Eq. (16) yields

$$\overline{B}^2 + \langle b^2\rangle = 4\pi a N^\kappa, \tag{17}$$

$$b^2 - \langle b^2\rangle + 2\overline{\boldsymbol{B}}\cdot\boldsymbol{b} = 4\pi a\kappa n N^{\kappa-1}, \tag{18}$$

which are useful to compare with Eqs. (5) and (6), respectively. Calculations similar to those in Sect. 3.1 (with F replaced by $a\kappa N^{\kappa-1}$) then yield

$$RM = RM^{(0)}\left(1 + \frac{2}{3}\kappa\frac{\langle b^2\rangle}{\overline{B}^2 + \langle b^2\rangle}\right), \tag{19}$$

which can be compared with Eq. (13). Thus, a positive (negative) correlation between B and n_e, i.e., $\kappa > 0$ ($\kappa < 0$) results in a positive (negative) bias in the observed RM. For example, we get $RM = \frac{5}{3}RM^{(0)}$ for $\kappa = 1$ and $\langle b^2\rangle/\overline{B}^2 \gg 1$, i.e., the standard estimate of $\overline{B}$ is by a factor 5/3 too large.

The corresponding effect on the standard deviation of RM is an additional factor κ on the right-hand side in Eq. (15).

Now consider a polytropic equation of state, $\mathcal{P}_{gas} = n_e^\gamma F_1$, where the factor F_1 is similar to F in Sect. 3.1. In order to simplify the calculations, we assume that density fluctuations are weak, $n \ll N$, so that $n_e^\gamma \simeq N^\gamma + \gamma n N^{\gamma-1}$. Averaging Eq. (4) now yields

$$\overline{B}^2 + \langle b^2\rangle \simeq 4\pi(P - \gamma n N^{\gamma-1}F_1),$$

and instead of Eq. (13) we obtain Eq. (19) with κ replaced by $-\gamma$.

4. Magnetic field estimates from synchrotron intensity

In this section we discuss how anisotropy of the interstellar random magnetic field and the dependence of the cosmic ray number density on magnetic field affect the estimates of the regular magnetic field from synchrotron intensity.

4.1. Anisotropic magnetic fields

Equation (2) applies only to isotropic random magnetic fields, but their anisotropy results in stronger polarization for a given $b/\overline{B}$ (Laing 1981, 2002; Sokoloff et al. 1998). Galactic differential rotation can make the interstellar turbulent magnetic field anisotropic by extending turbulent cells along azimuth so that their azimuthal and radial sizes are related via $l_\phi/l_r \simeq 1 + \Delta\Omega R\Delta t \simeq 1.3$ at $R = R_\odot$, where $\Delta\Omega$ is the angular velocity increment across the turbulent cell, R is the galactocentric radius with $R_\odot$ its Solar value, and $\Delta t \simeq 10^7$ yr is the lifetime of a turbulent eddy. This produces anisotropy in the turbulent magnetic field, with $b_\phi/b_r \simeq l_\phi/l_r$.

A more fundamental reason of anisotropy in $\boldsymbol{b}$ is the anisotropic nature of magnetohydrodynamic turbulence (see Goldreich & Sridhar 1997 and references therein), where turbulent "cells" are elongated along the mean magnetic field (Goldreich & Sridhar 1995), i.e. roughly in the azimuthal direction in the case of Galactic magnetic fields.

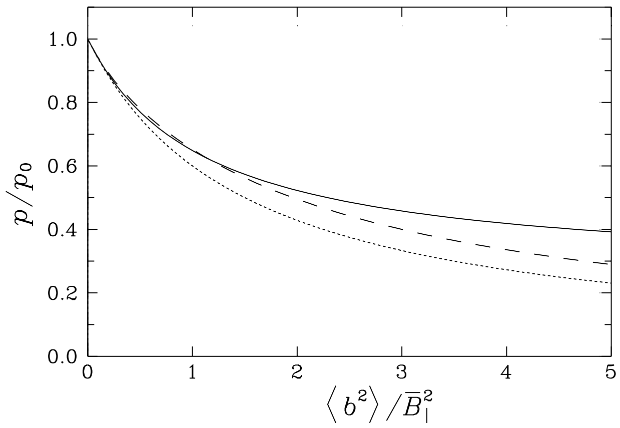

Fig. 2. The fractional polarization with anisotropic random magnetic field as given by Eq. (20) with $a = 1.3$ (solid), under the local equipartition between cosmic rays and magnetic field (dashed) according to Eq. (21), and from the standard relation (2) (dotted) as functions of the ratio $\langle b^2\rangle/\overline{B}_\perp^2$. The standard relation (2) gives significantly overestimated $\overline{B}_\perp$ for a given fractional polarization p and random field strength $\langle b^2\rangle^{1/2}$ if $\langle b^2\rangle/\overline{B}_\perp^2 > 2$.

Anisotropy of both types has the largest component of the random magnetic field roughly elongated with the large-scale Galactic magnetic field $b_\phi \gtrsim b_r \simeq b_z$. This is similar to the anisotropy found by Brown & Taylor (2001) in the fluctuations in the Galactic *RM*.

The degree of polarization produced by the above anisotropic random magnetic field when observed in directions close to the Galactic centre (where $\boldsymbol{b}_\perp$ consists mainly of b_ϕ and b_z with $b_z \simeq b_r$) is given by (see Eq. (19) in Sokoloff et al. 1998)

$$p \simeq p_0 \frac{\overline{B}_\perp^2 + \frac{1}{3}\langle b^2\rangle(a^2-1)}{\overline{B}_\perp^2 + \frac{1}{3}\langle b^2\rangle(a^2+1)}, \qquad (20)$$

where $a = b_\phi/b_r$ is a measure of anisotropy. For $a = 1$, this equation reduces to Eq. (2). This estimate also applies to directions perpendicular to the spiral arms as long as their pitch angle is small.

The solid line in Fig. 2 shows the fractional polarization as given by Eq. (20); it is significantly different from that given by Eq. (2) for realistic values of parameters. The same value of p in fact corresponds to a larger value of $\langle b^2\rangle/\overline{B}_\perp^2$ when the anisotropy has been taken into account. Since the energy density in the random magnetic field is limited from above by the kinetic energy of interstellar turbulence, this implies that values of the regular magnetic field strength obtained from Eq. (2) are *overestimated*. For example, the degree of polarization of $p/p_0 = 0.4$ is obtained for $\langle b^2\rangle/\overline{B}_\perp^2 \simeq 3.0$ according to Eq. (20) and for $\langle b^2\rangle/\overline{B}_\perp^2 \simeq 2.2$ according to Eq. (2). For a fixed $\langle b^2\rangle$, the difference in $\overline{B}_\perp$ is about 1.2.

Thus, because of the anisotropy of the turbulent magnetic field with $b_\phi \gtrsim b_r \simeq b_z$, the regular magnetic field obtained from polarized intensity can be significantly overestimated (whereas estimates of the *total* field remain unaffected). In the Milky Way, this effect is strongest in, roughly, the azimuthal direction where $\boldsymbol{b}_\perp$ is dominated by b_ϕ and b_z and weakest toward the Galactic centre (nearly across the spiral arms) where the main contribution to $\boldsymbol{b}_\perp$ comes from b_r and b_z. The effect of the anisotropy is undoubtedly significant in external galaxies.

4.2. Local equipartition between cosmic rays and magnetic fields

Arguments similar to those of Sect. 3 can be applied to cosmic ray density that may exhibit similar local dependence on the magnetic field strength, with similar consequences for the magnetic field estimates from the total and polarized synchrotron intensities. If the energy equipartition or pressure balance between cosmic rays and magnetic fields were maintained locally (i.e., at any given position), one would expect a strong positive correlation between $n_{\rm cr}$ and B (given that $n_{\rm cr}$ is proportional to the total cosmic ray energy density). Sarkar (1982) argues that such a correlation, implied by cosmic ray confinement ideas, can be enhanced by acceleration of relativistic electrons in compressed supernova shells.

In fact, Eq. (2) widely used in interpretations of polarized radio emission from the Milky Way and external galaxies is inconsistent with energy equipartition or pressure balance between cosmic rays and magnetic fields because it is based on the assumption that the number density of cosmic rays is independent of magnetic fields. This also applies to Eq. (20). If the local equipartition is maintained between cosmic rays and magnetic fields, so that $n_{\rm cr} \propto B^2$ at any position, the total intensity of synchrotron emission will strongly depend on the magnetic field since synchrotron emissivity is proportional to $n_{\rm cr}B_\perp^{1+\alpha} \propto B^2 B_\perp^{1+\alpha}$, where the synchrotron spectral index α is close to unity.

As a result, the equipartition estimate will be biased towards regions with stronger field and the degree of polarization will be larger than predicted by Eq. (2). Sokoloff et al. (1998, their Eq. (28)) have shown that for $n_{\rm cr} \propto B^2$ Eq. (2) is replaced by

$$p = p_0 \frac{\overline{B}_\perp^2 + \frac{7}{3}\langle b^2\rangle}{\overline{B}_\perp^2 + 3\langle b^2\rangle + \frac{10}{9}\langle b^2\rangle^2}, \qquad (21)$$

where it is assumed for simplicity that the regular magnetic field lies in the sky plane, $\overline{B} = \overline{B}_\perp$, and the random magnetic field is isotropic. The resulting degree of polarization is shown in Fig. 2. As expected, Eq. (21) yields significantly stronger fractional polarization for a given ratio $\langle b^2\rangle/\overline{B}_\perp^2$. For example, for $p/p_0 = 0.4$, Eq. (2) predicts that $\langle b^2\rangle/\overline{B}_\perp^2 \simeq 2.2$, whereas Eq. (21) yields $\langle b^2\rangle/\overline{B}_\perp^2 \simeq 4.7$. For a fixed strength of the random magnetic field, this leads to an overestimate by a factor of 1.4 in the regular field strength. This difference is very close to that between the estimates obtained from synchrotron intensities and *RM*. The equipartition value of the *total* field is also an overestimate.

A tight, point-wise correlation between $n_{\rm cr}$ and B can be an oversimplification. Cosmic ray diffusion can smooth variations in $n_{\rm cr}$ at fairly large scales. Then Eq. (21) provides an upper estimate of p. Otherwise, one can still use Eq. (21), but with $\langle b^2 \rangle$ understood as an average over the diffusion length of cosmic-ray electrons.

There are other uncertainties in the equipartition estimates of field strengths. For example, the standard estimate relies on the assumption that relativistic protons and electrons have the same spectral index in the relevant part of the energy spectrum (i.e. around a few GeV for electrons responsible for radio synchrotron emission). This is true in the Solar neighbourhood (Chapter 9 in Longair 1994). However, this is no longer valid if the electrons suffer significant synchrotron or inverse Compton losses, so that their energy spectrum is steeper than that of the protons. In this case the equipartition or minimum-energy method should be applied with care and requires a correction of the electron spectrum derived from the synchrotron spectrum.

5. Discussion

In order to verify our main results, Eqs. (13) and (19), one would need to study the relation between RM, P and p because Eqs. (13) and (19) can be rewritten as

$$RM \simeq RM^{(0)} \left[1 + \frac{2}{3}\kappa \left(1 - \frac{2}{3}\frac{p}{p_0} \right) \right], \tag{22}$$

where Eq. (13) corresponds to $\kappa = -1$, and Eq. (2) has been used with $\overline{B}_\perp^2 = \frac{1}{2}\overline{B}^2$ and $p \ll p_0$. In other words, statistical relation between magnetic field and electron density fluctuations leads to a correlation of the observable Faraday rotation measure with the degree of polarization. In order to rewrite Eq. (22) in terms of observable quantities, we note that $\overline{B}_\parallel$ can be related to the observed polarized intensity given in Eq. (1) since $\overline{B}_\parallel$ and $\overline{B}_\perp$ are related via a geometric factor. Then $RM^{(0)}$ can be expressed in terms of polarized intensity P, $RM^{(0)} \propto P^{1/2}$, and Eq. (22) written as

$$\frac{|RM|}{\sqrt{P}} \propto 1 - \frac{2}{3}\frac{1}{1 + 3/(2\kappa)}\frac{p}{p_0}, \tag{23}$$

where the coefficient of proportionality depends on cosmic ray number density, mean thermal electron density, geometric factors, path length and other variables. Regions with pressure balance yield $1 + 3/(2\kappa) < 0$, and so a positive correlation between $|RM|/\sqrt{P}$ and p, whereas overpressured regions should be detectable via an anticorrelation between these quantities. Without any interdependence between magnetic field and electron density, $|RM|/\sqrt{P}$ would be uncorrelated with the degree of polarization p.

In order to detect the correlation implied by Eq. (23), the observational data have to be selected carefully. Regions in which synchrotron emission and Faraday rotation occur together, so that they probe the same magnetic field, have to be found. The regular magnetic field must be almost uniformly directed throughout the region to reduce the variation in the ratio $\overline{B}_\parallel/\overline{B}_\perp$. The positive and negative correlations from regions in pressure equilibrium and overpressured regions can compensate each other. Thus, to detect the correlation produced by pressure balance, the region must not be affected by any violent activity that could produce overpressure, e.g., expanding supernova shells and H II regions (Jenkins & Tripp 2001). The region must be very well explored in several wavelength ranges in order to have reliable Faraday rotation measures, polarized and total synchrotron intensities, and ionized and neutral gas densities. Finally, Faraday depolarization must not be strong to ensure that P is not affected by variations in Faraday depth.

There are several surveys of the polarized radio background in the Milky Way (at λ = 21–74 cm – Spoelstra 1984; λ = 21 cm – Uyanıker et al. 1999; λ = 80–88 cm – Haverkorn et al. 2000, 2003; and λ = 21 cm – Gaensler et al. 2001), but they cover regions and/or wavelengths where Faraday depolarization is strong and so Faraday rotation and synchrotron emission occur at different depths; this makes it difficult to obtain the value of p required. Polarization intensity data at smaller wavelengths are available (e.g., Duncan et al. 1997), but not the Faraday rotation. Further away from the Galactic plane, a few regions have been observed in radio polarization with high resolution, but Faraday rotation data are available only in small regions (Reich & Uyanıker, in prep.). Further radio polarization studies of carefully selected regions in our Galaxy are required; Galactic polarization surveys at λ6 cm are in planning.

Polarization maps at $\lambda \leq$ 6 cm are available for several external galaxies (Beck 2000). However, the main obstacle is the integration along the long line of sight (several kpc) and the large beam (several 100 pc) that mixes a range of heights above the disc and/or positions within and outside the arms. The available resolution of observations is insufficient to isolate relatively small regions in the plane of the sky where pressure balance can be expected to occur.

An alternative can be the verification of the effect on the standard deviation of RM, described by Eq. (15). A confirmation of the diminishing effect of pressure equilibrium on fluctuations in the Faraday rotation measure of pulsars may have been provided by the results of Mitra et al. (2003) who have shown that the deviations of pulsar Faraday rotation measures from a large-scale model distribution are systematically smaller than predicted by Eq. (14) (we note that the "estimated $\sigma_{\rm rm}$" in their Fig. 8 can be misleading as this quantity is obtained by averaging residuals squared, and so dominated by a few strong deviations). The difference appears to be close to a factor of two, which is consistent with $\langle b^2 \rangle / \overline{B}^2 \simeq 3$ (see Fig. 1).

The apparent disagreement between the values of the regular magnetic field in the Milky Way, obtained from Faraday rotation measures and synchrotron intensity, can be resolved by allowing for the effects of turbulent magnetic fields, and their correlations with thermal and relativistic electron densities. A detailed comparison of the various methods to obtain field estimates (radio polarization, starlight polarization, extragalactic and pulsar rotation measures) in a selected field of the Milky Way is required to derive reliable estimates of the regular magnetic field in the Milky Way. Regarding external galaxies, estimates of the regular magnetic field should be reconsidered with allowance for the anisotropy of the random magnetic field and with more consistent implementation of the cosmic ray equipartition models.

An improved estimate of the strength of the regular magnetic field in the Milky Way can be important, among many

Table 1. Summary of the various effects of random magnetic fields on Faraday rotation measure RM and the degree of synchrotron polarization p discussed in this paper. Consequences for the "standard" estimates of $\overline{B}$ from the observed RM or p are indicated in Col. 2. Reference to relevant equations is given in the third column; the text should be consulted for conditions under which the equations have been obtained; they are briefly summarised in Col. 4, but the effects themselves are more general. Here n_e and n_{cr} are the number densities of thermal and cosmic-ray electrons, respectively, with $N = \langle n_e \rangle$, and $n = n_e - N$; $\boldsymbol{B}$, $\overline{\boldsymbol{B}}$ and $\boldsymbol{b}$ are the total, regular and random magnetic fields, respectively.

Physical effect	Consequences	Equation reference	Assumptions
n_e–B anticorrelation	RM reduced, $\overline{B}$ underestimated,	(11)	pressure balance, isotropic $\boldsymbol{b}$
		(13)	+ magnetic and turbulent energies in equipartition
	σ_{RM} reduced	(15)	as for (13)
n_e–B correlation	RM enhanced, $\overline{B}$ overestimated	(19)	isotropic $\boldsymbol{b}$, $n/N \ll 1$
Anisotropy of $\boldsymbol{b}$	p enhanced, $\overline{B}$ overestimated	(20)	n_{cr} independent of $\overline{B}$
n_{cr}–B correlation	p enhanced, $\overline{B}$ and B overestimated	(21)	isotropic $\boldsymbol{b}$

other topics, for the studies of the origin of high-energy cosmic rays (Kalmykov & Khristiansen 1995).

6. Conclusions

Our main results are summarized in Table 1. We have shown that an anticorrelation between the number density of thermal electrons n_e and magnetic field strength (to which both random and regular magnetic fields contribute) reduces the Faraday rotation measure RM. We have quantified this effect by assuming that the ISM is under pressure equilibrium and shown that values of the regular magnetic field strength $\overline{B}$ obtained from relations similar to Eq. (3) can be underestimated by a factor of about 2. This effect alone can explain the systematic difference between the values of $\overline{B}$ in the Milky Way obtained from Faraday rotation measures and the degree of synchrotron polarization. Another consequence of the anticorrelation is a suppression of fluctuations in RM.

On the contrary, a positive correlation between n_e and magnetic field strength results in enhanced RM and overestimated $\overline{B}$. This effect can be important along selected lines of sight in the Milky Way where the contribution of overpressured regions is significant.

We have also discussed effects that might affect the value of the regular magnetic field inferred from the degree of polarization of synchrotron emission p. The anisotropy of the random magnetic field in the ISM can result from the stretching of turbulent cells by Galactic differential rotation, in addition to a similar anisotropy inherent to magnetohydrodynamic turbulence. The anisotropy of $\boldsymbol{b}$ enhances p and can thereby result in $\overline{B}$ overestimated by a few tens of percent. A correlation between the energy density of cosmic ray electrons and magnetic field strength also enhances the degree of polarization. We have shown that equipartition between cosmic rays and magnetic fields can lead to an overestimate of $\overline{B}$ by roughly the same amount. This implies that the ratio of turbulent to regular magnetic field strengths obtained from the observed degrees of synchrotron polarization should be revised towards larger values; $\langle b^2 \rangle / \overline{B}^2 \simeq 3$–4 seems to be a plausible estimate.

There are further effects on the equipartition estimates of magnetic field strengths; we briefly discussed the role of energy losses of cosmic-ray electrons in Sect. 4.2.

Altogether, the effects discussed in this paper, especially the bias in the estimates of $\overline{B}$ from RM, can reconcile the estimates of the regular magnetic field from Faraday rotation measures and polarized synchrotron emission. However, this requires a more careful interpretation of observational data, especially recent observations of diffuse synchrotron emission in the Milky Way and new determinations of Faraday rotation measures of extragalactic radio sources and pulsars.

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Appendix A: The standard deviation of *RM*

Here we derive Eq. (15) for the standard deviation of the Faraday rotation measure, using the relation $\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2$ for the standard deviation of a random variable X. Thus, apart from Eq. (9), we have to calculate $\langle (n\boldsymbol{b}\cdot\boldsymbol{s})^2 \rangle$. From Eq. (7), we have

$$4\pi^2 F^2 (n\boldsymbol{b}\cdot\boldsymbol{s})^2 = \left[\frac{1}{4}\left(b^2 - \langle b^2 \rangle\right)^2 + \left(b^2 - \langle b^2 \rangle\right)\overline{\boldsymbol{B}}\cdot\boldsymbol{b} + (\overline{\boldsymbol{B}}\cdot\boldsymbol{b})^2\right](\boldsymbol{b}\cdot\boldsymbol{s})^2.$$

The mean values of those terms that contain odd powers of b vanish because (i) the components of $\boldsymbol{b}$ are statistically pointwise independent (so that, say, $\langle b_x^3 b_y^2 \rangle = \langle b_x^3 \rangle \langle b_y^2 \rangle$) and (ii) each of the components has a symmetric probability distribution (so, e.g., $\langle b_x^3 \rangle = 0$). Therefore,

$$\left\langle \left(b^2 - \langle b^2 \rangle\right)(\overline{\boldsymbol{B}}\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{s})^2 \right\rangle = 0.$$

The remaining averages are calculated similarly to the following:

$$\langle b^4 (\boldsymbol{b}\cdot\boldsymbol{s})^2 \rangle = \sum_i \langle b^4 b_i^2 \rangle s_i^2 = \langle b^4 b_x^2 \rangle,$$

where we have taken advantage of the isotropy of $\boldsymbol{b}$ and the fact that $\boldsymbol{s}$ is a unit vector. Now we represent b^4 in terms of its Cartesian components to obtain

$$\langle b^4(\boldsymbol{b}\cdot\boldsymbol{s})^2\rangle = \langle b_x^6\rangle + \frac{16}{27}\langle b^2\rangle^3 + \frac{50}{3}\langle b^2\rangle\langle b_x^4\rangle,$$

where we used the isotropy of $\boldsymbol{b}$, i.e., $\langle b_x^2\rangle = \frac{1}{3}\langle b^2\rangle$. The higher moments of b_i are calculated assuming that b_i is a Gaussian random variable, e.g., $\langle b_i^6\rangle = \frac{5}{9}\langle b^2\rangle^3$ and $\langle b_i^4\rangle = \frac{1}{3}\langle b^2\rangle^2$.

Then Eq. (15) follows after we note that $\overline{B}_\parallel = \overline{B}s_x$, $\overline{B}^2 = \overline{B}_\perp^2 + \overline{B}_\parallel^2$, and

$$\sigma_{RM}^2 = \frac{2K_1Ld}{4\pi^2F^2}\left[\left\langle (n\boldsymbol{b}\cdot\boldsymbol{s})^2\right\rangle - \frac{1}{9}\overline{B}_\parallel^2\langle b^2\rangle\right],$$

where we have used Eqs. (10) and (12), which yields $F^2 = \langle \mathcal{P}_{\rm gas}{}^2\rangle/N^2 = (\overline{B}^2 + \langle b^2\rangle)/(4\pi N^2)$. We have also introduced the volume filling factor of thermal electrons, $f = N^2/\langle n_{\rm e}^2\rangle = N^2/(N^2+\langle n^2\rangle)$.

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