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Simple equation of state for partially degenerate semirelativistic electrons

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Abstract. A simple method is given to evaluate the equation of state of a weakly relativistic, partially degenerate electron gas.

Key words. equation of state – stars: interiors – Sun: interior

1. Introduction

The plasma in the cores of the Sun and many other stars, which consists of hydrogen, helium, and a small fraction of heavier elements, is almost a perfect gas. The partially degenerate electrons are described by the generalized Fermi-Dirac integrals. These are complicated functions which have to be evaluated numerically; analytic expressions are only available for certain limiting cases. It is common to use the nonrelativistic approximation of the integrals. In the centre of the Sun ($T \approx 1.5 \times 10^7$ K), e.g., the relativistic effect for electrons is still small. However, the enormous accuracy of helioseismological observations has made feasible the detection of even small relativistic effects. It turned out that the significant discrepancies between the measured solar p-mode frequencies and model calculations are removed by inclusion of relativistic effects in the equation of state (Elliott & Kosovichev 1998; Elliott 1998; Rogers & Nayfonov 2002). Thus it is of advantage to make available techniques to calculate the relativistic equation of state for the electrons. The ionic component of the plasma may still be described by the perfect gas law.

Divine (1965) has presented a method for the numerical evaluation of the well-known equations relating the temperature, pressure, density, and internal energy of an electron obeying Fermi-Dirac statistics. An alternative way to evaluate the equation of state was adopted by Eggleton et al. (1973) who approximated the Fermi-Dirac integrals by a type of formula containing a polynomial in two variables. Their method has the advantage that a single formula approximates the Fermi-Dirac integral over the entire range of its arguments temperature and electron degeneracy. In the nonrelativistic limit Cloutman (1989) has calculated the Fermi-Dirac integrals numerically to 12 digit accuracy. Recently, Aparicio (1998) has developed a method based on a split of the integration domain into four parts, whereby he achieved 15-digit accuracy. In the present paper a simple method is given to evaluate the relativistic corrections to the equation of state of a partially degenerate electron gas by means of series expansions.

2. Equation of state

The number density of electrons is given by (Cox & Giuli 1968)

$$n_{\mathrm{e}} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2}{\mathrm{e}^{-\eta+\epsilon/kT} + 1}\, \mathrm{d}p, \tag{1}$$

where p and ϵ is the electron momentum and kinetic energy, respectively, T is the temperature, h is Planck's constant, k is the Boltzmann constant, and η is the degeneracy parameter. Likewise, the electron pressure has the form

$$P_{\mathrm{e}} = \frac{8\pi}{3h^3 m} \int_0^\infty \frac{p^4 \left[1 + (p/mc)^2\right]^{-1/2}}{\mathrm{e}^{-\eta+\epsilon/kT} + 1}\, \mathrm{d}p, \tag{2}$$

where m is the electron mass and c is the velocity of light. Using the relation $p^2c^2 = \epsilon^2 + 2\epsilon mc^2$ between ϵ and p, the electron number density and pressure may be expressed by means of the generalized Fermi-Dirac integral of index k,

$$F_k(\eta,\beta) = \int_0^\infty \frac{x^k(1+\frac{1}{2}\beta x)^{1/2}}{\mathrm{e}^{-\eta+x} + 1}\, \mathrm{d}x, \tag{3}$$

viz.,

$$n_{\mathrm{e}} = 8\pi\sqrt{2}(mc/h)^3\beta^{3/2}\left[F_{1/2}(\eta,\beta) + \beta\, F_{3/2}(\eta,\beta)\right] \tag{4}$$

and

$$P_{\mathrm{e}} = \frac{16}{3}\pi\sqrt{2}(m^4c^5/h^3)\beta^{5/2}\left[F_{3/2}(\eta,\beta) + \frac{1}{2}\beta\, F_{5/2}(\eta,\beta)\right]. \tag{5}$$

Here $x = \epsilon/kT$ and $\beta = kT/mc^2$. If the dimensionless temperature β is assumed to be small against unity, the function $F_k(\eta,\beta)$ can be expanded into the Taylor series

$$F_k(\eta,\beta) \approx F_k(\eta) + \frac{\beta}{4} F_{k+1}(\eta) - \frac{\beta^2}{32} F_{k+2}(\eta) + \frac{\beta^3}{128} F_{k+3}(\eta) - \frac{5\beta^4}{2048} F_{k+4}(\eta) + \frac{7\beta^5}{8192} F_{k+5}(\eta) \mp \ldots \tag{6}$$

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containing the Fermi-Dirac integrals

$$F_k(\eta) \equiv F_k(\eta,\beta=0) = \int_0^\infty \frac{x^k}{\mathrm{e}^{-\eta+x}+1}\,\mathrm{d}x. \qquad (7)$$

Using Eq. (6) the expressions for n_e and P_e take the approximate form

$$n_\mathrm{e} \approx 8\pi\sqrt{2}\left(mkT/h^2\right)^{3/2}\left\{F_{1/2}(\eta)+\frac{5}{4}\beta F_{3/2}(\eta)+\frac{7}{32}\beta^2F_{5/2}(\eta) -\frac{3}{128}\beta^3F_{7/2}(\eta)+\frac{11}{2\,048}\beta^4F_{9/2}(\eta)-\frac{13}{8192}\beta^5F_{11/2}(\eta)\right\} \qquad (8)$$

and

$$P_\mathrm{e} \approx 8\pi\sqrt{2}\left(mkT/h^2\right)^{3/2}kT\left\{\frac{2}{3}F_{3/2}(\eta)+\frac{1}{2}\beta F_{5/2}(\eta) +\frac{1}{16}\beta^2F_{7/2}(\eta)-\frac{1}{192}\beta^3F_{9/2}(\eta) +\frac{1}{1024}\beta^4F_{11/2}(\eta)-\frac{1}{4096}\beta^5F_{13/2}(\eta)\right\}. \qquad (9)$$

The integral $F_k(\eta)$ can be expanded in the series

$$F_k(\eta) = \Gamma(k+1)\,\mathrm{e}^\eta\sum_{r=0}^{\infty}(-1)^r\frac{\mathrm{e}^{r\eta}}{(r+1)^{k+1}}, \qquad (10)$$

which is valid for $\eta \le 0$ and $k > -1$ (Cox & Giuli 1968). In order to achieve rapid convergence, it is, however, more convenient to substitute the function $F_{1/2}(\eta)$ for the degeneracy parameter η. Employing the expansion (10) and eliminating e^η one gets

$$F_{3/2}(\eta) \approx \frac{3}{2}F_{1/2}(\eta)\left\{1+a_1F_{1/2}+a_2F_{1/2}^2+a_3F_{1/2}^3 +a_4F_{1/2}^4+a_5F_{1/2}^5+a_6F_{1/2}^6\right\}, \qquad (11)$$

$$F_{5/2}(\eta) \approx \frac{15}{4}F_{1/2}\left\{1+b_1F_{1/2}+b_2F_{1/2}^2 +b_3F_{1/2}^3+b_4F_{1/2}^4+b_5F_{1/2}^5\right\}, \qquad (12)$$

$$F_{7/2}(\eta) \approx \frac{105}{8}F_{1/2}(\eta)\left\{1+c_1F_{1/2}+c_2F_{1/2}^2 +c_3F_{1/2}^3+c_4F_{1/2}^4+c_5F_{1/2}^5\right\}, \qquad (13)$$

$$F_{9/2}(\eta) \approx \frac{945}{16}F_{1/2}(\eta)\left\{1+d_1F_{1/2}+d_2F_{1/2}^2 +d_3F_{1/2}^3+d_4F_{1/2}^4+d_5F_{1/2}^5\right\}, \qquad (14)$$

$$F_{11/2}(\eta) \approx \frac{10\,395}{32}F_{1/2}(\eta)\left\{1+e_1F_{1/2}+e_2F_{1/2}^2 +e_3F_{1/2}^3+e_4F_{1/2}^4+e_5F_{1/2}^5\right\}. \qquad (15)$$

The coefficients

$$a_1 = 1/\sqrt{8\pi} \approx 0.199\,471\,140\,201,$$

$$a_2 = \pi^{-1}\left(\frac{1}{2}-\frac{8}{27}\sqrt{3}\right) \approx -4.201\,766\,662\times10^{-3},$$

$$a_3 = \pi^{-3/2}\left(\frac{3}{4}+\frac{5}{8}\sqrt{2}-\frac{2}{3}\sqrt{6}\right) \approx 1.598\,890\,418\times10^{-4},$$

$$a_4 = \pi^{-2}\left(\frac{317}{108}+2\sqrt{2}-\frac{8}{3}\sqrt{3}-\frac{64}{125}\sqrt{5}\right) \approx -5.739\,649\,1\times10^{-6},$$

$$a_5 = \pi^{-5/2}\left(\frac{35}{4}+\frac{1\,687}{216}\sqrt{2}-\frac{20}{9}\sqrt{3}-\frac{40}{9}\sqrt{6}-\frac{8}{5}\sqrt{10}\right) \approx 1.534\,080\times10^{-7},$$

$$a_6 = \pi^{-3}\left(\frac{173}{4}+18\sqrt{2}-\frac{4\,076}{243}\sqrt{3}-\frac{192}{25}\sqrt{5}-\frac{32}{3}\sqrt{6} -\frac{384}{343}\sqrt{7}+\frac{128}{75}\sqrt{15}\right) \approx -7.5588\times10^{-10},$$

$$b_1 = (3/4)/\sqrt{2\pi} \approx 0.299\,206\,710,$$

$$b_2 = \pi^{-1}\left(\frac{3}{4}-\frac{32}{81}\sqrt{3}\right) \approx 0.020\,923\,468\,3,$$

$$b_3 = \pi^{-3/2}\left(\frac{15}{16}+\frac{15}{16}\sqrt{2}-\frac{25}{27}\sqrt{6}\right) \approx -8.478\,026\,816\times10^{-4},$$

$$b_4 = \pi^{-2}\left(\frac{2\,725}{648}+\frac{21}{8}\sqrt{2}-\frac{34}{9}\sqrt{3}-\frac{384}{625}\sqrt{5}\right) \approx 4.197\,722\,6\times10^{-5},$$

$$b_5 = \pi^{-5/2}\left(\frac{189}{16}+\frac{14\,287}{1\,296}\sqrt{2}-\frac{77}{27}\sqrt{3} -\frac{532}{81}\sqrt{6}-\frac{252}{125}\sqrt{10}\right) \approx -1.940\,519\,9\times10^{-6},$$

$$c_1 = (7/8)/\sqrt{2\pi} \approx 0.349\,074\,495,$$

$$c_2 = \pi^{-1}\left(\frac{7}{8}-\frac{104}{243}\sqrt{3}\right) \approx 0.042\,561\,458\,54,$$

$$c_3 = \pi^{-3/2}\left(\frac{63}{64}+\frac{35}{32}\sqrt{2}-\frac{167}{162}\sqrt{6}\right) \approx 1.091\,854\,825\times10^{-3},$$

$$c_4 = \pi^{-2}\left(\frac{18\,563}{3\,888}+\frac{91}{32}\sqrt{2}-\frac{115}{27}\sqrt{3}-\frac{1\,984}{3\,125}\sqrt{5}\right) \approx -7.946\,649\times10^{-5},$$

$$c_5 = \pi^{-5/2}\left(\frac{833}{64}+\frac{96\,425}{7\,776}\sqrt{2}-\frac{983}{324}\sqrt{3} -\frac{68}{9}\sqrt{6}-\frac{1\,342}{625}\sqrt{10}\right) \approx 5.201\,375\times10^{-6},$$

$$d_1 = (15/16)/\sqrt{2\pi} \approx 0.374\,008\,388,$$

$$d_2 = \pi^{-1}\left(\frac{15}{16}-\frac{320}{729}\sqrt{3}\right) \approx 0.056\,405\,578,$$

$$d_3 = \pi^{-3/2}\left(\frac{255}{256}+\frac{75}{64}\sqrt{2}-\frac{1\,045}{972}\sqrt{6}\right) \approx 3.577\,696\,9\times10^{-3},$$

$$d_4 = \pi^{-2}\left(\frac{117\,505}{23\,328}+\frac{375}{128}\sqrt{2}-\frac{2\,175}{486}\sqrt{3}-\frac{9\,984}{15\,625}\sqrt{5}\right) \approx 2.835\,022\,3\times10^{-6},$$

$$d_5 = \pi^{-5/2}\left(\frac{3\,465}{256} + \frac{607\,075}{46\,656}\sqrt{2} - \frac{12\,005}{3\,888}\sqrt{3} - \frac{5\,845}{729}\sqrt{6} + \frac{6\,867}{3\,125}\sqrt{10}\right) \approx -3.690\,209\,8\times 10^{-6},$$

$$e_1 = (31/32)/\sqrt{2\pi} \approx 0.386\,475\,334,$$

$$e_2 = \pi^{-1}\left(\frac{31}{32} - \frac{968}{2\,187}\sqrt{3}\right) \approx 6.433\,601\,236\times 10^{-2},$$

$$e_3 = \pi^{-3/2}\left(\frac{1\,023}{1\,024} + \frac{155}{128}\sqrt{2} - \frac{6\,383}{5\,832}\sqrt{6}\right) \approx 5.501\,333\,76\times 10^{-3},$$

$$e_4 = \pi^{-2}\left(\frac{722\,387}{139\,968} + \frac{1\,519}{512}\sqrt{2} - \frac{4\,447}{972}\sqrt{3} - \frac{49\,984}{78\,125}\sqrt{5}\right) \approx 1.859\,243\times 10^{-4},$$

$$e_5 = \pi^{-5/2}\left(\frac{14\,105}{1\,024} + \frac{3\,720\,857}{279\,936}\sqrt{2} - \frac{144\,815}{46\,656}\sqrt{3} - \frac{6\,005}{729}\sqrt{6} - \frac{69\,359}{31\,250}\sqrt{10}\right) \approx -3.565\,924\times 10^{-6}$$

decrease much more rapidly than the coefficients of the series (10) so that the series (11) to (15) are rapidly convergent and can even be used for small positive values of η. The relative error of these formulae at $\eta = 2$, corresponding to $F_{1/2}(\eta) \approx 2.5$ (note that inside the Sun $\eta < -1.5$), is 8.2×10^{-8} for $F_{3/2}(\eta)$, 9.3×10^{-6} for $F_{5/2}(\eta)$, 3.0×10^{-5} for $F_{7/2}(\eta)$, 3.2×10^{-5} for $F_{9/2}(\eta)$, and 8.4×10^{-7} for $F_{11/2}(\eta)$. The series (11) for $F_{3/2}(\eta)$ is expanded up to the 6th order in $F_{1/2}(\eta)$ because the leading term of $P_{\rm e}$ is given by this function. Since $F_{5/2}$ to $F_{11/2}$ are multiplied by powers of the small quantity β, their accuracy is sufficient.

By means of Eqs. (7) and (11) to (15) the electron density and pressure are given by

$$n_{\rm e} \approx 4\pi\left(2mkT/h^2\right)^{3/2} F_{1/2}(\eta)\left\{1 + \frac{15}{8}\beta\left(1 + \sum_{i=1}^{6} a_i F_{1/2}^i\right) + \frac{105}{128}\beta^2\left(1 + \sum_{i=1}^{5} b_i F_{1/2}^i\right) - \frac{315}{1024}\beta^3\left(1 + \sum_{i=1}^{5} c_i F_{1/2}^i\right) + \frac{10\,395}{32\,768}\beta^4\left(1 + \sum_{i=1}^{5} d_i F_{1/2}^i\right) - \frac{135\,135}{262\,144}\beta^5\left(1 + \sum_{i=1}^{5} e_i F_{1/2}^i\right)\right\} \quad (16)$$

and

$$P_{\rm e} \approx 4\pi(2mkT/h^2)^{3/2} kT F_{1/2}(\eta)\left\{1 + \sum_{i=1}^{6} a_i F_{1/2}^i + \frac{15}{8}\beta\left(1 + \sum_{i=1}^{5} b_i F_{1/2}^i\right) + \frac{105}{128}\beta^2\left(1 + \sum_{i=1}^{5} c_i F_{1/2}^i\right) - \frac{315}{1\,024}\beta^3\left(1 + \sum_{i=1}^{5} d_i F_{1/2}^i\right) + \frac{10\,395}{32\,768}\beta^4\left(1 + \sum_{i=1}^{5} e_i F_{1/2}^i\right)\right\}. \quad (17)$$

Likewise, the internal energy per unit volume,

$$u_{\rm e} = 8\pi\sqrt{2}\left(m^4c^5/h^3\right)\beta^{5/2}\left[F_{3/2}(\eta,\beta) + \beta F_{5/2}(\eta,\beta)\right], \quad (18)$$

(Cox & Giuli 1968) can be written as

$$u_{\rm e} \approx 6\pi\left(2mkT/h^2\right)^{3/2} kT F_{1/2}(\eta)\left\{1 + \sum_{i=1}^{6} a_i F_{1/2}^i + \frac{25}{8}\beta\left(1 + \sum_{i=1}^{5} b_i F_{1/2}^i\right) + \frac{245}{128}\beta^2\left(1 + \sum_{i=1}^{5} c_i F_{1/2}^i\right) - \frac{945}{1\,024}\beta^3\left(1 + \sum_{i=1}^{5} d_i F_{1/2}^i\right) + \frac{38\,115}{32\,768}\beta^4\left(1 + \sum_{i=1}^{5} e_i F_{1/2}^i\right)\right\}. \quad (19)$$

Equations (16) and (17) represent the equation of state of the electrons in parametric form. For given values of the temperature T and of the electron density $n_{\rm e}$ (or the mass density $\rho = (\mu_{\rm e}/N_{\rm A})n_{\rm e}$, where $\mu_{\rm e}$ is the mean molecular weight per free electron and $N_{\rm A}$ is the Avogadro constant), Eq. (16) is readily solved for $F_{1/2}(\eta)$ by iteration, the first approximation being the nonrelativistic limit of $F_{1/2}(\eta)$ for weak degeneracy,

$$F_{1/2}(\eta) \approx \frac{n_{\rm e}}{4\pi}\left(\frac{h^2}{2mkT}\right)^{3/2}. \quad (20)$$

Once $F_{1/2}$ is known the electron pressure $P_{\rm e}$ is computed from (17). These formulae are very accurate for $\eta \le 4$ or $F_{1/2}(\eta) \le 5.8$ and $\beta \le 0.1$ ($T \le 5.9\times 10^8$ K), i.e., for weakly degenerate and semirelativistic electrons. In the worst case, $\beta = 0.1$ and $\eta = 4$, the magnitude of the last term in the curly brackets of Eq. (16) is $\approx 3.4\times 10^{-5}$. Since the signs of the terms are alternating, the error of $F_{1/2}$ will be less than 10^{-5}. Likewise, the last term in the curly brackets of Eq. (17) is $\approx 2.2\times 10^{-4}$. Due to the small factor of the term proportional to β^5 in Eq. (9) the error of the missing terms in the β expansion (17) is less than 1.5×10^{-5}. In the range of validity the Eqs. (16), (17), and (19) are superior to the approximations of Eggleton et al. (1973). The above conditions are satisfied in the solar centre as well as in the hotter interior of more massive stars. The effects of degeneracy and relativity are opposite: At high temperatures, when relativistic corrections are becoming significant, the degeneracy parameter decreases, and vice versa.

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