A&A 404, 1051–1065 (2003)
DOI: 10.1051/0004-6361:20030501

Astronomy & Astrophysics

Tidal perturbations of linear, isentropic oscillations in components of circular-orbit close binaries

I. Synchronously rotating components

K. Reyniers and P. Smeyers

Instituut voor Sterrenkunde, Katholieke Universiteit Leuven, Belgium

Received 8 January 2003 / Accepted 27 March 2003

Abstract. The effects of the tidal force exerted by a companion on linear, isentropic oscillations of a uniformly rotating star that is a component of a circular-orbit close binary are studied. In contrast to an earlier perturbation method, which is almost only applicable to polytropic models, the procedure starts from an arbitrary physical model of a spherically symmetric equilibrium star. The tidal field and the nonspherical tidally perturbed star are supposed to be determined by means of the theory of dynamic tides, in which the tides are treated as forced, linear, isentropic oscillations of a nonrotating spherically symmetric star. The equations governing linear, isentropic oscillations of a tidally perturbed star are established in the domain instantaneously occupied by the star and are transformed into equations defined in the domain of the spherically symmetric star, so that usual perturbation methods can be applied. The procedure is developed for the general case in which the star's rotation is not necessarily synchronous with the orbital motion of the companion. The second part of the paper is devoted to the case in which the star rotates synchronously and is subject to an equilibrium tide. The eigenfrequencies of radial modes are shown to remain unaffected by the tidal perturbation at the lowest order of approximation. For the lowest degrees $\ell = 1, 2, 3$, the degeneracy of the eigenvalue problem of the linear, isentropic oscillations of a spherically symmetric star is lift partially, so that a $(2\,\ell + 1)$-fold eigenfrequency is split up into $(\ell + 1)$ eigenfrequencies. A main result is that the eigenfrequencies of the modes belonging to a given degree ℓ are shown to be all split up according to the same pattern. Attention is paid to the linear combinations of eigenfunctions that have to be adopted at order zero when the polar axis of the spherical harmonics of the angular coordinates coincides with the star's rotation axis perpendicular to the orbital plane. The solutions of order zero are also considered in terms of spherical harmonics of angular coordinates for which the polar axis coincides with the tidal axis.

Key words. stars: binaries: close – stars: oscillations – methods: analytical

1. Introduction

The effects of an equilibrium tide on oscillation modes of incompressible liquid or compressible gaseous equilibrium configurations have been studied in a way largely parallel to the effects of an axial rotation. For both phenomena, one is faced with the distortion of the equilibrium configuration from the state of spherical symmetry.

Analytical solutions of linear, isentropic oscillations of uniformly rotating equilibrium configurations have been determined for the MacLaurin spheroids. The case of the incompressible MacLaurin spheroids was treated by Bryan (1889). Later, Chandrasekhar and Lebovitz used the virial method in a series of papers in order to determine low-degree linear, isentropic oscillations of the compressible as well as the incompressible MacLaurin spheroids (Lebovitz 1961; Chandrasekhar & Lebovitz 1962a,b, 1963; Chandrasekhar 1968; see also Chandrasekhar 1969). The eigenvalue problem of the linear, isentropic oscillations of the incompressible MacLaurin spheroids was solved in a more direct way by Smeyers (1986): for modes associated with spherical harmonics of a given azimuthal number up to a certain degree, the author integrated a finite set of equations, expressed in terms of spherical coordinates, from the centre of the spheroid and passed on to the use of oblate spheroidal coordinates in order to impose the boundary conditions at the spheroid's surface. Subsequently, De Boeck (1997) adopted this method for the determination of several low-degree, linear, isentropic oscillations of the compressible MacLaurin spheroids and confirmed earlier results of Tassoul & Tassoul (1967) derived by means of the virial method.

As far as equilibrium configurations distorted by an equilibrium tide are concerned, analytical solutions have been established for the second-harmonic oscillations of the homogeneous masses with prolate spheroidal forms that were considered by Jeans (1917), and reconsidered by Chandrasekhar & Lebovitz (1963) by means of the virial method. Analytical solutions have been determined for both the incompressible

Send offprint requests to: P. Smeyers,
e-mail: paul.smeyers@ster.kuleuven.ac.be

and the compressible Jeans spheroids (see also Chandrasekhar 1969).

For the study of linear, isentropic oscillations of gaseous stars distorted by an axial rotation and/or by an equilibrium tide, one has recourse to methods leading to adequate approximations because of the absence of any general analytical treatment. One of these methods is perturbation theory.

Ledoux (1951) for the first time used a perturbation method in order to determine the effects of a slow uniform rotation on the eigenfrequencies of linear, isentropic oscillations of a star to the first order in the rotational angular velocity.

Ledoux' perturbation method was extended to the second order in the rotational angular velocity by Simon (1969) for the determination of the effects of a slow uniform rotation on radial oscillation modes of a star. The second-order effects stemming from the star's distortion by the centrifugal force were incorporated by a mapping of the various points in the domain of the rotating star on the points in the domain of the nonrotating spherical star. The mapping was performed purely along the radius and was followed correspondingly by a transformation of the governing equations. The perturbation related to the change of the surface of the rotating star is of a type that is referred to as a surface perturbation (see, e.g., Brillouin 1937).

Simon's perturbation procedure was generalized by Smeyers & Denis (1971) for a determination of the second-order rotational effects on nonradial oscillations of a star. These authors used curvilinear coordinates and observed that, from a geometrical point of view, one of the steps involved in the mapping procedure is a parallel transport of the Lagrangian displacements of the mass elements of the rotating star. The necessity of the parallel transport was later emphasized by Smeyers & Martens (1983). Denis (1972) adapted the perturbation procedure for the determination of the lowest-order effects of an equilibrium tide on linear, isentropic oscillation modes of a stellar component in a close binary.

Although its validity has been checked only for distorted equilibrium configurations with a uniform mass density, the perturbation procedure of Smeyers & Denis (1971) and Denis (1972) applies to any stellar model whose nonspherical perturbed equilibrium state can be determined. The latter condition however restricts the applicability of the perturbation procedure in practice almost to polytropic models as those constructed by Chandrasekhar's method (1933).

Later, Saio (1981) adopted the perturbation procedure of Smeyers & Denis (1971) and Denis (1972) to treat the rotational and tidal effects on nonradial oscillations of a polytropic model up to the second order in the angular velocity of rotation Ω. In the introduction of his paper, he noted:

> One of the difficulties in including the terms of order Ω^2 is that we must have a nonspherical model distorted by a tidal force and/or the centrifugal force. Since there are some unresolved problems … in the nonspherical models, we apply our analysis to a tidally and/or rotationally distorted rotating polytrope … of index $n = 3$.

With regard to Saio's investigation, Smeyers & Martens (1983) observed that the parallel transport of the Lagrangian displacement had been omitted but that this omission did not affect the corrections to the eigenfrequencies.

In this investigation, we concentrate on the determination of the effects of the tidal force exerted by a companion on linear, isentropic oscillation modes of a uniformly rotating star that is a component of a close binary with a circular orbit. We consider both the case in which the star rotates synchronously with the companion's orbital motion and the case in which it does not. In the first case, the tide generated by the companion is an equilibrium tide, in the second case, a dynamic tide.

For both cases, we develop a perturbation procedure that applies not only to polytropic models but even to physically realistic models of unperturbed spherically symmetric stars. To this end, we determine the perturbed nonspherical model of the star by using the theory of dynamic tides, in which the tides are considered as forced, linear, isentropic oscillations of a nonrotating spherically symmetric star.

The present paper is the first in a series of three papers. Here, we first establish the equations governing linear, isentropic oscillations in a uniformly rotating star that is subject to the tidal force of a companion moving in a circular orbit, whether or not the star rotates synchronously. The governing equations are derived in the domain instantaneously occupied by the tidally distorted star. Next, by means of an adequate mapping, we transform the governing equations into equations that are defined in the domain of the unperturbed spherically symmetric star.

In the second part of the paper, we consider the case in which the star's rotation is synchronous with the companion's orbital motion. We present a time-independent perturbation procedure which allows one to determine the effects of the equilibrium tide generated by the companion on the star's oscillation modes. For modes belonging to the lower degrees $\ell = 1, 2, 3$, we show that the corrections to the eigenfrequencies obey some general rules and determine the lowest-order approximations of the eigenfunctions.

The validity of the perturbation method will be verified in the second paper. This verification will be done by a comparison with analytical expressions which we derive for the eigenfrequencies of the compressible Jeans spheroids and with eigenfrequencies of a tidally perturbed polytropic model with index $n = 3$ obtained by Saio (1981). The third paper will be devoted to the tidal effects on linear, isentropic oscillation modes of a star that does not rotate synchronously with the orbital motion of its companion.

The plan of the paper is as follows. In Sect. 2, the basic equations are presented. In Sect. 3, we briefly recall the determination of the tidal field and the structure of the nonspherical tidally perturbed star. In Sect. 4, we present the equations that govern linear, isentropic oscillations in the tidally perturbed star and transform them into equations that are defined in the domain of the unperturbed spherically symmetric star. From Sect. 5 on, we concentrate on linear, isentropic oscillations of a component of a close binary with a circular orbit that rotates synchronously with the orbital motion of its companion. In Sect. 5, we present a time-independent perturbation method in order to determine the effects of an equilibrium tide on a linear, isentropic oscillation mode of the component. In Sect. 6, we apply the perturbation method to an arbitrary oscillation mode belonging to one of the lowest degrees $\ell = 0, 1, 2, 3$. The final section is devoted to concluding remarks.

2. Basic equations

Consider a uniformly rotating star with mass M_1 that is a component of a close binary and is subject to the gravitational force of its companion with mass M_2. We assume that the star rotates uniformly with an angular velocity Ω around an axis perpendicular to the orbital plane. The companion is considered to be a point mass and to move in a circular orbit around the star.

We use a frame of reference that is corotating with the star. Its origin coincides with the star's mass centre and its z-axis is perpendicular to the orbital plane. With respect to this frame of reference, we introduce a system of spherical coordinates r, θ, ϕ, which are also denoted as generalized coordinates q^1, q^2, q^3.

Let R_1 be the mean radius of the tidally distorted star, a the radius of the companion's relative orbit, and $\varepsilon_{\rm T}$ a small dimensionless parameter defined as

$$\varepsilon_{\rm T} = \left(\frac{R_1}{a}\right)^3 \frac{M_2}{M_1}. \tag{1}$$

Furthermore, let P be the pressure, ρ the mass density, Φ the potential of self-gravitation, and $\varepsilon_{\rm T}\, W$ the tide-generating potential.

We neglect the effects of the centrifugal force and of the force of Coriolis. The motions of the star's mass elements are then governed by the equations

$$\frac{\partial \dot{q}^j}{\partial t} + \dot{q}^k\, \nabla_k \dot{q}^j = -g^{jk} \left(\nabla_k \Phi + \varepsilon_{\rm T}\, \nabla_k W + \frac{1}{\rho}\, \nabla_k P \right),$$
$$j = 1, 2, 3. \tag{2}$$

The operator ∇_k stands for the operator of partial differentiation with respect to the generalized coordinate q^k as it applies to a scalar, and for the operator of covariant differentiation with respect to that generalized coordinate as it applies to a vector or a tensor component. The g^{jk} are the contravariant components of the metric tensor. Einstein's summation convention is used.

The second-degree tide-generating potential, at time t at the point with spherical coordinates r, θ, ϕ, can be expressed as

$$\varepsilon_{\rm T}\, W\,(\boldsymbol{r}, t) = \varepsilon_{\rm T}\, \frac{G\, M_1}{R_1} \left(\frac{r}{R_1}\right)^2 \frac{1}{2}$$
$$\times \left\{ P_2(\cos\theta) - \frac{1}{2}\, P_2^2(\cos\theta)\, \cos\left\{2\left[\phi + (\Omega - n)\, t\right]\right\} \right\}, \tag{3}$$

where G is the Newtonian gravitation constant, n the mean motion, and where $P_2(\cos\theta)$ and $P_2^2(\cos\theta)$ are respectively the second-degree Legendre polynomial and the associated second-degree Legendre polynomial with azimuthal number 2. The first term gives rise to a static tide, and the second term to a dynamic tide.

3. The tidal field and the nonspherical star

We determine the time-dependent structure of a star that is subject to the tidal action of a companion moving in a circular Keplerian orbit by starting from a physical model of a nonrotating spherically symmetric star in hydrostatic equilibrium, in which the mass elements have no velocities. Next, we introduce the tidal force of the companion. Let $\varepsilon_{\rm T}\, \left(\delta q^j\right)_{\rm T}$, with $j = 1, 2, 3$, be the components of the tidal displacement of a mass element with respect to the local coordinate basis. In accordance with the theory of dynamic tides in which the tides are considered as forced, linear, isentropic oscillations of a spherically symmetric star, the equations governing the linear tidal motions inside the star are obtained by linear perturbation of Eqs. (2) and take the form

$$g_{ij}\, \frac{\partial^2 \left(\delta q^j\right)_{\rm T}}{\partial t^2} + U_{ij} \left(\delta q^j\right)_{\rm T} = -\nabla_i W, \quad i = 1, 2, 3. \tag{4}$$

The operators U_{ij} are similar to the components of the operator applying to the free, linear, isentropic oscillations of a spherically symmetric star and are defined by

$$U_{ij} \left(\delta q^j\right)_{\rm T} = \nabla_i \Phi'_{\rm T} - \frac{\rho'_{\rm T}}{\rho^2}\, \nabla_i P + \frac{1}{\rho}\, \nabla_i P'_{\rm T}. \tag{5}$$

A prime on a quantity denotes the Eulerian perturbation of that quantity.

Equations (4) are completed by the equation expressing the mass conservation of the moving elements

$$\rho'_{\rm T} = -\nabla_i \left[\rho\, \left(\delta q^i\right)_{\rm T}\right] = -\rho\, \alpha_{\rm T} - \left(\delta q^i\right)_{\rm T} \nabla_i \rho, \tag{6}$$

the energy equation expressing that the tidal motions are isentropic

$$P'_{\rm T} = -\left(\delta q^i\right)_{\rm T}\, \nabla_i P - \rho\, c^2\, \alpha_{\rm T}, \tag{7}$$

and Poisson's differential equation

$$\nabla^2 \Phi'_{\rm T} = 4\, \pi\, G\, \rho'_{\rm T}. \tag{8}$$

In these equations, c is the isentropic sound velocity, and $\alpha_{\rm T}$ the divergence of the tidal displacement; $(\delta\rho)_{\rm T}$ and $(\delta P)_{\rm T}$ are the Lagrangian perturbations of the mass density and the pressure related to the tidal displacement.

The solutions of Eqs. (4) and (6)–(8) must satisfy boundary conditions: at the star's centre, the tidal displacement remains finite; at the star's distorted surface, the pressure vanishes, and the gravitational potential and its gradient are continuous.

The second-degree tidal displacement has components with respect to the local coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$ of the form

$$\left.\begin{aligned}
&\varepsilon_{\rm T}\ (\delta r)_{\rm T}\, (r, \theta, \phi, t) = \varepsilon_{\rm T} \Big\{ \xi_{\rm st}(r)\, P_2(\cos\theta) \\
&\quad + 2\, \xi_{\rm dyn}(r)\, P_2^2(\cos\theta)\, \cos\{2\left[\phi + (\Omega - n)\, t\right]\} \Big\}, \\
&\varepsilon_{\rm T}\ (\delta\theta)_{\rm T}\, (r, \theta, \phi, t) = \varepsilon_{\rm T} \Big\{ \frac{\eta_{\rm st}(r)}{r^2}\, \frac{{\rm d}P_2(\cos\theta)}{{\rm d}\theta} \\
&\quad + 2\, \frac{\eta_{\rm dyn}(r)}{r^2}\, \frac{{\rm d}P_2^2(\cos\theta)}{{\rm d}\theta}\, \cos\{2\left[\phi + (\Omega - n)\, t\right]\} \Big\}, \\
&\varepsilon_{\rm T}\ (\delta\phi)_{\rm T}\, (r, \theta, \phi, t) = \varepsilon_{\rm T}\, 2\, \frac{\eta_{\rm dyn}(r)}{r^2} \\
&\quad \times \frac{P_2^2(\cos\theta)}{\sin^2\theta}\, \frac{\partial}{\partial\phi}\, \cos\{2\left[\phi + (\Omega - n)\, t\right]\}.
\end{aligned}\right\} \tag{9}$$

The subscript "st" refers to the static tide, and the subscript "dyn" to the dynamic tide. The function $\xi_{\rm st}(r)$ is the solution of the second-order differential equation

$$\frac{{\rm d}^2\xi_{\rm st}}{{\rm d}r^2} + 2\left[\frac{1}{m(r)}\,\frac{{\rm d}m(r)}{{\rm d}r} - \frac{1}{r}\right]\frac{{\rm d}\xi_{\rm st}}{{\rm d}r} - \frac{4}{r^2}\,\xi_{\rm st} = 0 \tag{10}$$

that remains finite at $r = 0$ and satisfies the boundary condition at $r = R_1$

$$\left(\frac{{\rm d}\,(\xi_{\rm st}/r)}{{\rm d}r}\right)_{R_1} + \frac{2}{R_1}\left(\frac{\xi_{\rm st}}{r}\right)_{R_1} = -\frac{5}{2}\,\frac{1}{R_1}\cdot \tag{11}$$

In Eq. (10), $m(r)$ is the mass contained in the sphere with radius r. Since the static tide is divergence-free, the function $\eta_{\rm st}(r)$ is related to the function $\xi_{\rm st}(r)$ as

$$\eta_{\rm st}(r) = \frac{1}{6}\,\frac{\rm d}{{\rm d}r}\left[r^2\,\xi_{\rm st}(r)\right], \tag{12}$$

and the divergence of the tidal displacement is determined purely by the dynamic tide as

$$\begin{aligned}\varepsilon_{\rm T}\,\alpha_{\rm T}(r,\theta,\phi,t) \;&=\; 2\,\varepsilon_{\rm T}\left[\frac{1}{r^2}\,\frac{\rm d}{{\rm d}r}\left[r^2\,\xi_{\rm dyn}(r)\right] - \frac{6}{r^2}\,\eta_{\rm dyn}(r)\right]\\ &\times P_2^2(\cos\theta)\,\cos\{2\,[\phi + (\Omega - n)\,t]\}.\end{aligned} \tag{13}$$

In the particular case in which the star rotates synchronously with the companion's circular motion, $\Omega = n$, and the tide-generating potential (3) is time-independent and generates a divergence-free equilibrium tide in the star. The relations then hold

$$\xi_{\rm dyn}(r) = -\frac{1}{4}\,\xi_{\rm st}(r), \quad \eta_{\rm dyn}(r) = -\frac{1}{24}\,\frac{\rm d}{{\rm d}r}\left[r^2\,\xi_{\rm st}(r)\right]. \tag{14}$$

The companion is considered to be situated in the direction of the x-axis of the corotating frame of reference.

The spherical coordinates r, θ, ϕ of a mass element in the tidally distorted star are related to the spherical coordinates r_0, θ_0, ϕ_0 of that mass element in the spherically symmetric equilibrium star. In the linear approximation, the relations take the form

$$q^i\left(q_0^1, q_0^2, q_0^3; t\right) = q_0^i + \varepsilon_{\rm T}\left(\delta q^i\right)_{\rm T}\left(q_0^1, q_0^2, q_0^3; t\right),\quad i = 1, 2, 3. \tag{15}$$

The coordinates r_0, θ_0, ϕ_0 are considered as Lagrangian parameters which characterize the moving mass element.

The velocity components of a moving mass element with respect to the local coordinate basis are given by

$$\dot q^i = \varepsilon_{\rm T}\left(\frac{\partial\left(\delta q^i\right)_{\rm T}}{\partial t}\right)_{\boldsymbol{r}_0}, \quad i = 1, 2, 3. \tag{16}$$

Correspondingly, the various physical quantities at the point with spherical coordinates r, θ, ϕ in the tidally distorted star can be related to their values at the point with spherical coordinates r_0, θ_0, ϕ_0 in the spherically symmetric equilibrium star by means of their Lagrangian perturbations:

$$f\,(r,\theta,\phi;t) = f_0\,(r_0) + \varepsilon_{\rm T}\,(\delta f)_{\rm T}\,(r_0,\theta_0,\phi_0;t)\,. \tag{17}$$

4. The equations governing linear, isentropic oscillations of a tidally distorted star

4.1. The initial equations

We now consider the tidally distorted star to be subject to a linear, isentropic oscillation. We derive the governing equations by perturbing linearly Eqs. (2) in the domain instantaneously occupied by the tidally distorted star. The resulting equations are

$$g_{kj}\left[\frac{\partial\left(\dot q^j\right)'}{\partial t} + \dot q^i\,\nabla_i\left(\dot q^j\right)' + \left(\nabla_i\,\dot q^j\right)\left(\dot q^i\right)'\right] = -U_{ki}\,\delta q^i,\quad j = 1, 2, 3, \tag{18}$$

with

$$U_{ki}\,\delta q^i = \nabla_k\Phi' - \frac{\rho'}{\rho^2}\,\nabla_k P + \frac{1}{\rho}\,\nabla_k P'. \tag{19}$$

The δq^i and the $\left(\dot q^j\right)'$ are, respectively, the components of the Lagrangian displacement and the components of the Eulerian perturbation of the velocity of a mass element with respect to the local coordinate basis. The latter components are related to the first ones and to the components of the tidal velocity of the mass element as

$$\left(\dot q^j\right)' = \frac{\partial\left(\delta q^j\right)}{\partial t} + \dot q^i\,\frac{\partial\left(\delta q^j\right)}{\partial q^i} - \delta q^i\,\frac{\partial \dot q^j}{\partial q^i}, \quad j = 1, 2, 3. \tag{20}$$

To Eqs. (18), we add the equation expressing the mass conservation of the moving mass elements

$$\rho' = -\nabla_i\left(\rho\,\delta q^i\right), \tag{21}$$

the energy equation expressing that the oscillations are isentropic

$$P' = -\delta q^i\,\nabla_i P - \rho\,c^2\,\alpha, \tag{22}$$

and the perturbed intregal formula of Poisson

$$\Phi'\,(\boldsymbol{r}) = -G\int_V \rho\,(\boldsymbol{r}')\left[\delta q^j\,\nabla_{\prime j}\left|\boldsymbol{r}' - \boldsymbol{r}\right|^{-1}\right](\boldsymbol{r}')\,{\rm d}V\,(\boldsymbol{r}'), \tag{23}$$

where α is the divergence of the Lagrangian displacement field defined as $\alpha = \nabla_i\,\delta q^i$, and V the volume instantaneously occupied by the tidally distorted star.

Our aim is to solve Eqs. (18) and (21)–(23) by means of a perturbation procedure in which $\varepsilon_{\rm T}$ is the small expansion parameter, and the approximation of the Lagrangian displacement at order zero is a free linear, isentropic oscillation of the spherically symmetric equilibrium star. It should be noticed that the domain instantaneously occupied by the tidally distorted star differs from the domain occupied by the spherically symmetric equilibrium star, so that we have to deal with a surface perturbation. This perturbation has previously been taken into consideration by Simon (1969), Smeyers & Denis (1971), and Denis (1972).

Following the procedure developed by the latter authors, we transform the governing equations determined in the domain of the tidally distorted star into equations determined in the domain of the spherically symmetric equilibrium star. For this purpose, it may be recalled that each point P, with spherical coordinates r, θ, ϕ, in the domain of the tidally distorted star is related to a point P_0, with spherical coordinates r_0, θ_0, ϕ_0, in the domain of the spherically symmetric equilibrium star by means of the relations given by Eqs. (15).

The transformation is performed in two steps.

First, the operators of partial differentiation with respect to the spherical coordinates r, θ, ϕ are transformed into operators of partial differentiation with respect to the spherical coordinates r_0, θ_0, ϕ_0 as

$$\frac{\partial}{\partial q^j} = \left(\delta^i_j - \varepsilon_{\rm T}\, \frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial q^j_0}\right) \frac{\partial}{\partial q^i_0}\cdot \tag{24}$$

The operator of partial differentiation with respect to time, with constant spherical coordinates r, θ, ϕ, is related to the operator of partial differentiation with respect to time, with constant spherical coordinates r_0, θ_0, ϕ_0, as

$$\left(\frac{\partial}{\partial t}\right)_{\boldsymbol r} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol r_0} - \varepsilon_{\rm T} \left(\frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial t}\right)_{\boldsymbol r_0} \frac{\partial}{\partial q^i_0}\cdot \tag{25}$$

Next, the Lagrangian displacement at each point P in the domain of the tidally distorted star is subject to a parallel transport to the associated point P_0 in the domain of the spherically symmetric equilibrium star. The components $\widetilde{\delta q}^j$ of the vector displaced to the point P_0 are related to the components δq^j of the Lagrangian displacement at the point P as

$$\left(\widetilde{\delta q}^j\right)_{P_0} = \left(\delta q^j\right)_P + \varepsilon_{\rm T}\, \Gamma^j_{ik} \left(\delta q^k\right)_{\rm T} \left(\delta q^i\right)_P,\ \ j = 1,2,3, \tag{26}$$

where the Γ^j_{ik} are the Christoffel symbols of the second kind.

4.2. The governing equations defined in the domain of the spherically symmetric star

In the left-hand members of Eqs. (18), the use of the equalities given by Eqs. (20) and (25) yields

$$\begin{aligned}\left[\frac{\partial \left(\dot q^j\right)'}{\partial t}\right]_{\boldsymbol r} &= \left\{ \frac{\partial^2}{\partial t^2}\left[\widetilde{\delta q}^j - \varepsilon_{\rm T}\, \Gamma^j_{is} \left(\delta q^s\right)_{\rm T}\, \widetilde{\delta q}^i\right]\right.\\ &\left.\quad -\varepsilon_{\rm T}\, \frac{\partial}{\partial t}\left(\frac{\partial^2 \left(\delta q^j\right)_{\rm T}}{\partial t\,\partial q^i}\, \widetilde{\delta q}^i\right) - \varepsilon_{\rm T}\, \frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial t}\, \frac{\partial^2 \left(\widetilde{\delta q}^j\right)}{\partial t\,\partial q^i}\right\}_{\boldsymbol r_0} \cdot\end{aligned} \tag{27}$$

Furthermore, one has

$$\left[\dot q^i\, \nabla_i \left(\dot q^j\right)'\right]_{\boldsymbol r} = \left[\varepsilon_{\rm T}\, \frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial t}\, \nabla_i\, \frac{\partial \left(\widetilde{\delta q}^j\right)}{\partial t}\right]_{\boldsymbol r_0}, \tag{28}$$

$$\left[\left(\nabla_i\, \dot q^j\right)\left(\dot q^i\right)'\right]_{\boldsymbol r} = \left[\varepsilon_{\rm T} \left(\nabla_i\, \frac{\partial \left(\delta q^j\right)_{\rm T}}{\partial t}\right) \frac{\partial \left(\widetilde{\delta q}^i\right)}{\partial t}\right]_{\boldsymbol r_0} \cdot \tag{29}$$

The covariant components of the metric tensor are expanded as

$$\left(g_{kj}\right)_{\boldsymbol r} = \left(g_{kj} + \varepsilon_{\rm T}\, \frac{\partial g_{kj}}{\partial q^s}\, \left(\delta q^s\right)_{\rm T}\right)_{\boldsymbol r_0} . \tag{30}$$

For the transformation of the right-hand members of Eqs. (18), the divergence of the Lagrangian displacement is developed as

$$\alpha\left(\boldsymbol r\right) = \widetilde\alpha_0\left(\boldsymbol r_0\right) + \varepsilon_{\rm T}\, \widetilde\alpha_1\left(\boldsymbol r_0\right) \tag{31}$$

with

$$\widetilde\alpha_0\left(\boldsymbol r_0\right) = \left[\frac{1}{\sqrt g}\, \frac{\partial}{\partial q^j}\left(\sqrt g\, \widetilde{\delta q}^j\right)\right]_{\boldsymbol r_0}, \tag{32}$$

$$\begin{aligned}\widetilde\alpha_1\left(\boldsymbol r_0\right) \;=\; &\left\{\left[\frac{\partial}{\partial q^j}\left(\frac{\left(\delta q^i\right)_{\rm T}}{\sqrt g}\, \frac{\partial \sqrt g}{\partial q^i}\right)\right] \widetilde{\delta q}^j\right.\\ &-\frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial q^j}\, \frac{1}{\sqrt g}\, \frac{\partial}{\partial q^i}\left(\sqrt g\, \widetilde{\delta q}^j\right)\\ &\left.-\frac{1}{\sqrt g}\, \frac{\partial}{\partial q^j}\left[\sqrt g\, \Gamma^j_{ik}\left(\delta q^i\right)_{\rm T}\, \widetilde{\delta q}^k\right]\right\}_{\boldsymbol r_0} \cdot\end{aligned} \tag{33}$$

The Eulerian perturbation of the mass density is developed by means of Eq. (21) as

$$\rho'\left(\boldsymbol r\right) = \widetilde\rho'_0\left(\boldsymbol r_0\right) + \varepsilon_{\rm T}\, \widetilde\rho'_1\left(\boldsymbol r_0\right) \tag{34}$$

with

$$\widetilde\rho'_0\left(\boldsymbol r_0\right) = \left(-\rho_0\, \widetilde\alpha_0 - \frac{\partial \rho_0}{\partial q^j}\, \widetilde{\delta q}^j\right)_{\boldsymbol r_0}, \tag{35}$$

$$\begin{aligned}\widetilde\rho'_1\left(\boldsymbol r_0\right) \;=\; &\left[-\rho_0\, \widetilde\alpha_1 - \left(\delta\rho\right)_{\rm T}\, \widetilde\alpha_0\right.\\ &\left.+\frac{\partial \rho_0}{\partial q^j}\left[\nabla_i\left(\delta q^j\right)_{\rm T}\right] \widetilde{\delta q}^i - \frac{\partial \left(\delta\rho\right)_{\rm T}}{\partial q^j}\, \widetilde{\delta q}^j\right]_{\boldsymbol r_0} .\end{aligned} \tag{36}$$

By using expansions for ρ and P of the form of the expansion given by Eq. (17), substituting the expansion for ρ', and transforming the operator $\partial/\partial q^k$, one derives that

$$\left(-\frac{\rho'}{\rho^2}\, \frac{\partial P}{\partial q^k}\right)_{\boldsymbol r} = \left[-\frac{\widetilde\rho'_0}{\rho_0^2}\left(\delta^i_k - \varepsilon_{\rm T}\, \frac{\partial \left(\delta q^i\right)_{\rm T}}{\partial q^k}\right) \frac{\partial P_0}{\partial q^i} + \varepsilon_{\rm T}\, R_{ki}\, \widetilde{\delta q}^i\right]_{\boldsymbol r_0} \tag{37}$$

with

$$R_{ki}\, \widetilde{\delta q}^i = -\frac{\widetilde\rho'_1}{\rho_0^2}\, \frac{\partial P_0}{\partial q^k} + 2\, \frac{\partial P_0}{\partial q^k}\, \frac{\left(\delta\rho\right)_{\rm T}}{\rho_0^3}\, \rho'_0 - \frac{\widetilde\rho'_0}{\rho_0^2}\, \frac{\partial \left(\delta P\right)_{\rm T}}{\partial q^k}\cdot \tag{38}$$

The Eulerian perturbation of the pressure is developed by means of Eq. (22) as

$$P'\left(\boldsymbol r\right) = \widetilde P'_0\left(\boldsymbol r_0\right) + \varepsilon_{\rm T}\, \widetilde P'_1\left(\boldsymbol r_0\right) \tag{39}$$

with

$$\widetilde P'_0\left(\boldsymbol r_0\right) = \left[-\frac{\partial P_0}{\partial q^i}\, \widetilde{\delta q}^i - \left(\rho\, c^2\right)_0\, \widetilde\alpha_0\right]_{\boldsymbol r_0}, \tag{40}$$

$$\begin{aligned}\widetilde P'_1\left(\boldsymbol r_0\right) \;=\; &\left\{\frac{\partial P_0}{\partial q^j}\left[\nabla_i\left(\delta q^j\right)_{\rm T}\right] \widetilde{\delta q}^i\right.\\ &\left.-\frac{\partial \left(\delta P\right)_{\rm T}}{\partial q^j}\, \widetilde{\delta q}^j - \left[\delta\left(\rho\, c^2\right)_{\rm T}\right] \widetilde\alpha_0 - \left(\rho\, c^2\right)_0\, \widetilde\alpha_1\right\}_{\boldsymbol r_0} \cdot\end{aligned} \tag{41}$$

By using the expansion for ρ, substituting the expansion for P', and transforming the operator of partial differentiation $\partial/\partial q^k$, one derives that

$$\left(\frac{1}{\rho}\frac{\partial P'}{\partial q^k}\right)_{\boldsymbol{r}} = \left[\frac{1}{\rho_0}\left(\delta^i_k - \varepsilon_{\mathrm{T}}\frac{\partial\left(\delta q^i\right)_{\mathrm{T}}}{\partial q^k}\right)\frac{\partial \widetilde{P}'_0}{\partial q^i} + \varepsilon_{\mathrm{T}}\, P_{ki}\,\widetilde{\delta q}^{\,i}\right]_{\boldsymbol{r}_0} \tag{42}$$

with

$$P_{ki}\,\widetilde{\delta q}^{\,i} = -\frac{(\delta\rho)_{\mathrm{T}}}{\rho_0^2}\frac{\partial \widetilde{P}'_0}{\partial q^k} + \frac{1}{\rho_0}\frac{\partial \widetilde{P}'_1}{\partial q^k}. \tag{43}$$

The Eulerian perturbation of the gravitational potential is developed by means of the perturbed integral formula of Poisson given by Eq. (23), which, in terms of Cartesian coordinates, takes the form

$$\Phi'(\boldsymbol{r}) = -G\int_V \rho(\boldsymbol{r}')\left(\delta x^j\frac{\partial}{\partial x'^j}\left|\boldsymbol{r}'-\boldsymbol{r}\right|^{-1}\right)(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}'). \tag{44}$$

The integral over the domain V of the tidally distorted star is transformed into an integral over the domain V_0 of the spherically symmetric equilibrium star by means of the mass conservation of the moving elements

$$\rho(\boldsymbol{r}')\,\mathrm{d}V(\boldsymbol{r}') = \rho_0\left(\boldsymbol{r}'_0\right)\mathrm{d}V\left(\boldsymbol{r}'_0\right), \tag{45}$$

the Taylor expansion

$$\left|\boldsymbol{r}'-\boldsymbol{r}\right|^{-1} = \left[1+\varepsilon_{\mathrm{T}}\left(\delta x'^k\right)_{\mathrm{T}}\frac{\partial}{\partial x_0'^k}\right]\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|^{-1}, \tag{46}$$

and the equalities given by Eqs. (24) and (26). It follows that

$$\begin{aligned}\Phi'(\boldsymbol{r}) \;=\; & -G\int_{V_0}\rho_0\left(\boldsymbol{r}'_0\right)\left(\widetilde{\delta x}^{\,j}\frac{\partial}{\partial x_0'^j}\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|^{-1}\right)\left(\boldsymbol{r}'_0\right)\mathrm{d}V\left(\boldsymbol{r}'_0\right)\\ & -\varepsilon_{\mathrm{T}}\,G\int_{V_0}\rho_0\left(\boldsymbol{r}'_0\right)\left(\left(\delta x'^k\right)_{\mathrm{T}}\,\widetilde{\delta x}^{\,j}\frac{\partial^2}{\partial x_0'^k\,\partial x_0'^j}\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|^{-1}\right)\\ & \times\left(\boldsymbol{r}'_0\right)\mathrm{d}V\left(\boldsymbol{r}'_0\right).\end{aligned} \tag{47}$$

Partial integration, and use of Gauss' integral theorem and the definition of $\widetilde{\rho}'_0(\boldsymbol{r}_0)$ yield

$$\begin{aligned}\Phi'(\boldsymbol{r}) = & -G\int_{V_0}\frac{\widetilde{\rho}'_0\left(\boldsymbol{r}'_0\right)}{\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|}\mathrm{d}V\left(\boldsymbol{r}'_0\right) - G\int_{S_0}\frac{\rho_0\left(\boldsymbol{r}'_0\right)\widetilde{\delta r}\left(\boldsymbol{r}'_0\right)}{\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|}\mathrm{d}S\left(\boldsymbol{r}'_0\right)\\ & +\varepsilon_{\mathrm{T}}\,G\int_{V_0}\left(\frac{\partial\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|^{-1}}{\partial x_0'^i}\right)\left(\boldsymbol{r}'_0\right)\\ & \times\left[\frac{\partial}{\partial x_0'^j}\left(\rho_0\left(\delta x^i\right)_{\mathrm{T}}\,\widetilde{\delta x}^{\,j}\right)\right]\left(\boldsymbol{r}'_0\right)\mathrm{d}V\left(\boldsymbol{r}'_0\right)\\ & -\varepsilon_{\mathrm{T}}\,G\int_{S_0}\rho_0\left(\boldsymbol{r}'_0\right)\left(\frac{\partial\left|\boldsymbol{r}'_0-\boldsymbol{r}\right|^{-1}}{\partial x_0'^i}\right)\left(\boldsymbol{r}'_0\right)\\ & \times\left(\delta x^i\right)_{\mathrm{T}}\left(\boldsymbol{r}'_0\right)\widetilde{\delta r}\left(\boldsymbol{r}'_0\right)\mathrm{d}S_0\left(\boldsymbol{r}'_0\right),\end{aligned} \tag{48}$$

where S_0 is the surface of the spherically symmetric equilibrium star.

The Eulerian perturbation of the gravitational potential at a point P with position vector $\boldsymbol{r}$ can then be expanded about the associated point P_0 with position vector $\boldsymbol{r}_0$. In terms of generalized coordinates, one has

$$\Phi'(\boldsymbol{r}) = \widetilde{\Phi}'_0(\boldsymbol{r}_0) + \varepsilon_{\mathrm{T}}\,\widetilde{\Phi}'_1(\boldsymbol{r}_0) \tag{49}$$

with

$$\begin{aligned}\widetilde{\Phi}'_0(\boldsymbol{r}_0) \;=\; & -G\int_{V_0}\frac{\widetilde{\rho}'_0\left(\boldsymbol{r}'_0\right)}{\left|\boldsymbol{r}'_0-\boldsymbol{r}_0\right|}\mathrm{d}V\left(\boldsymbol{r}'_0\right)\\ & -G\int_{S_0}\frac{\rho_0\left(\boldsymbol{r}'_0\right)\widetilde{\delta r}\left(\boldsymbol{r}'_0\right)}{\left|\boldsymbol{r}'_0-\boldsymbol{r}_0\right|}\mathrm{d}S\left(\boldsymbol{r}'_0\right),\end{aligned} \tag{50}$$

$$\begin{aligned}\widetilde{\Phi}'_1(\boldsymbol{r}_0) \;=\; & \Bigg[\left(\delta q^i\right)_{\mathrm{T}}\frac{\partial\widetilde{\Phi}'_0}{\partial q^i} + G\int_{V_0}\left(\frac{\partial\left|\boldsymbol{r}'_0-\boldsymbol{r}_0\right|^{-1}}{\partial q_0'^i}\right)\left(\boldsymbol{r}'_0\right)\\ & \times\left[\left(\delta q^i\right)_{\mathrm{T}}\nabla_{,j_0}\left(\rho_0\,\widetilde{\delta q}^{\,j}\right) + \rho_0\,\widetilde{\delta q}^{\,j}\,\nabla_{,j_0}\left(\delta q^i\right)_{\mathrm{T}}\right]\left(\boldsymbol{r}'_0\right)\mathrm{d}V\left(\boldsymbol{r}'_0\right)\\ & -G\int_{S_0}\rho_0\left(\boldsymbol{r}'_0\right)\left(\frac{\partial\left|\boldsymbol{r}'_0-\boldsymbol{r}_0\right|^{-1}}{\partial q_0'^i}\right)\left(\boldsymbol{r}'_0\right)\\ & \times\left(\delta q^i\right)_{\mathrm{T}}\left(\boldsymbol{r}'_0\right)\widetilde{\delta r}\left(\boldsymbol{r}'_0\right)\mathrm{d}S_0\left(\boldsymbol{r}'_0\right)\Bigg]_{\boldsymbol{r}_0}.\end{aligned} \tag{51}$$

One furthermore has

$$\left(\frac{\partial\Phi'}{\partial q^k}\right)_{\boldsymbol{r}} = \left[\left(\delta^i_k - \varepsilon_{\mathrm{T}}\frac{\partial\left(\delta q^i\right)_{\mathrm{T}}}{\partial q^k}\right)\frac{\partial\widetilde{\Phi}'_0}{\partial q^i} + \varepsilon_{\mathrm{T}}\,F_{ki}\,\widetilde{\delta q}^{\,i}\right]_{\boldsymbol{r}_0} \tag{52}$$

with

$$F_{ki}\,\widetilde{\delta q}^{\,i} = \frac{\partial\widetilde{\Phi}'_1}{\partial q^k}. \tag{53}$$

At this point, the terms of the equations are all considered at the geometrical point P_0, with coordinates r_0, θ_0, ϕ_0, in the domain of the spherically symmetric equilibrium star. From here on, we drop the subscript 0 on the coordinates r_0, θ_0, ϕ_0 for the sake of simplification of the notations.

Finally, by introduction of the linear operators $U^{(0)}_{kj}$ and $V_{kj}(t)$ as

$$U^{(0)}_{kj}\,\widetilde{\delta q}^{\,j} = \frac{\partial\widetilde{\Phi}'_0}{\partial q^k} - \frac{\widetilde{\rho}'_0}{\rho_0^2}\frac{\partial P_0}{\partial q^k} + \frac{1}{\rho_0}\frac{\partial\widetilde{P}'_0}{\partial q^k}, \tag{54}$$

$$V_{kj}(t)\,\widetilde{\delta q}^{\,j} = \left[R_{kj}(t) + P_{kj}(t) + F_{kj}(t)\right]\widetilde{\delta q}^{\,j}, \tag{55}$$

and use of the property that the covariant derivative of the metric tensor is identically zero, the transformed equations can be written as

$$\begin{aligned} & g_{kj}\frac{\partial^2\left(\widetilde{\delta q}^{\,j}\right)}{\partial t^2} - \varepsilon_{\mathrm{T}}\,g_{kj}\left[\nabla_i\frac{\partial^2\left(\delta q^j\right)_{\mathrm{T}}}{\partial t^2}\right]\widetilde{\delta q}^{\,i}\\ & +\varepsilon_{\mathrm{T}}\,g_{ij}\,\Gamma^i_{sk}\left(\delta q^s\right)_{\mathrm{T}}\frac{\partial^2\left(\widetilde{\delta q}^{\,j}\right)}{\partial t^2} =\\ & -\left[\left(\delta^i_k - \varepsilon_{\mathrm{T}}\frac{\partial\left(\delta q^i\right)_{\mathrm{T}}}{\partial q^k}\right)U^{(0)}_{ij} + \varepsilon_{\mathrm{T}}\,V_{kj}(t)\right]\widetilde{\delta q}^{\,j},\\ & k = 1, 2, 3.\end{aligned} \tag{56}$$

These transformed equations, which are valid in the domain of the spherically symmetric equilibrium star, allow one to determine the effects of a tide on the linear, isentropic oscillations of a component of a close binary. The tide is considered to be generated by a companion moving in a circular orbit. The equations apply to both the case in which the star rotates synchronously with the companion's orbital motion and the case in which the star rotates asynchronously. In the remaining part of this paper, we concentrate on the first case.

5. Tidal perturbation of linear, isentropic oscillations of a synchronously rotating stellar component

When the star rotates synchronously with the companion's orbital motion, the tide generated by the companion is an equilibrium tide. Therefore, the coefficients of Eqs. (56) are time-independent, and solutions can be sought that depend on time by a factor $\exp(\mathrm{i}\,\sigma\, t)$, where σ is the angular frequency with respect to the frame of reference corotating with the star. Consequently, the equations reduce to

$$\sigma^2\left[g_{kj}\,\widetilde{\delta q}^{\,j}+\varepsilon_{\rm T}\,g_{ij}\,\Gamma^{i}_{sk}\,(\delta q^s)_{\rm T}\;\widetilde{\delta q}^{\,j}\right]=$$
$$\left[\left(\delta^i_k-\varepsilon_{\rm T}\,\frac{\partial\left(\delta q^i\right)_{\rm T}}{\partial q^k}\right)U^{(0)}_{ij}+\varepsilon_{\rm T}\,V_{kj}\right]\widetilde{\delta q}^{\,j},$$
$$k=1,2,3. \tag{57}$$

In order to determine the effect of a second-degree equilibrium tide on an oscillation mode n of the star, we use a time-independent perturbation method.

5.1. The perturbation method

Adopting $\varepsilon_{\rm T}$ as the expansion parameter, we introduce the following expansions for the eigenfrequency and the components of the vector resulting from the parallel transport of the Lagrangian displacement:

$$\sigma_n=\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma_{n,1}+O\left(\varepsilon_{\rm T}^2\right), \tag{58}$$

$$\left(\widetilde{\delta q}^{\,j}\right)_n(\boldsymbol{r})=\left(\widetilde{\delta q}^{\,j}\right)_{n,0}(\boldsymbol{r})+\varepsilon_{\rm T}\left(\widetilde{\delta q}^{\,j}\right)_{n,1}(\boldsymbol{r})+O\left(\varepsilon_{\rm T}^2\right),$$
$$j=1,2,3. \tag{59}$$

Let the mode n in the spherically symmetric equilibrium star be a spheroidal mode (p-, g-, or f-mode) belonging to a degree ℓ, which has a given radial order in the cases of a p-mode and of a g-mode. Because of the degeneracy of the eigenvalue problem of the linear, isentropic oscillation modes of a spherically symmetric star with respect to the azimuthal number m, the approximation of the eigenfunctions at order zero consists of a linear combination of the Lagrangian displacements of the spheroidal modes of the spherically symmetric equilibrium star that are associated with the same eigenfrequency $\sigma_{n,0}$ and with the spherical harmonics $Y_\ell^{m'}(\theta,\phi)$ of the degree ℓ and the admissible azimuthal numbers $m'=-\ell,\ldots,\ell$. The linear combination can be expressed as

$$\left(\widetilde{\delta q}^{\,j}\right)_{n,0}(\boldsymbol{r})=\sum_{m'=-\ell}^{\ell}a_{n,m'}\left(\delta q^j\right)_{n,0;m'}(\boldsymbol{r}),\ \ j=1,2,3, \tag{60}$$

where the coefficients $a_{n,m'}$ are undetermined constants.

For the first-order perturbations of the eigenfunctions, we use expansions in terms of the corresponding components of the Lagrangian displacements of the spheroidal modes that exist in the spherically symmetric equilibrium star

$$\left(\widetilde{\delta q}^{\,j}\right)_{n,1}(\boldsymbol{r})=\sum_{\lambda,\mu}b_{\lambda,\mu}\left(\delta q^j\right)_{\lambda,0;\mu}(\boldsymbol{r}),\quad j=1,2,3, \tag{61}$$

where the coefficients $b_{\lambda,\mu}$ are also undetermined constants.

A spheroidal mode λ of the spherically symmetric equilibrium star obeys the wave equations

$$\sigma^2_{\lambda,0}\,g_{kj}\left(\delta q^j\right)_{\lambda,0;\mu}-U^{(0)}_{kj}\left(\delta q^j\right)_{\lambda,0;\mu}=0,\quad k=1,2,3. \tag{62}$$

Substitution of the expansions into Eqs. (57) yields, at order $\varepsilon_{\rm T}^0$,

$$\left(\sigma^2_{n,0}\,g_{kj}-U^{(0)}_{kj}\right)\left(\widetilde{\delta q}^{\,j}\right)_{n,0}=0,\quad k=1,2,3, \tag{63}$$

and, at order $\varepsilon_{\rm T}$,

$$\Bigg(2\,\sigma_{n,0}\,\sigma_{n,1}\,g_{kj}+\sigma^2_{n,0}\,g_{\ell j}\,\Gamma^{\ell}_{sk}\,(\delta q^s)_{\rm T}+\frac{\partial\left(\delta q^i\right)_{\rm T}}{\partial q^k}\,U^{(0)}_{ij}$$
$$-V_{kj}\Bigg)\left(\widetilde{\delta q}^{\,j}\right)_{n,0}+\left(\sigma^2_{n,0}\,g_{kj}-U^{(0)}_{kj}\right)\left(\widetilde{\delta q}^{\,j}\right)_{n,1}=0,$$
$$k=1,2,3\ldots \tag{64}$$

Equations (63) are transformed by substitution of the expansions for the components $\left(\widetilde{\delta q}^{\,j}\right)_{n,0}$ and by use of wave Eqs. (62). Multiplication by $\rho_0\,\overline{(\delta q^k)_{n,0;m}}$, where the bar denotes the complex conjugate, integration over the domain V_0 of the spherically symmetric equilibrium star, and use of the orthogonality property between spheroidal modes lead to identities, so that the constants $a_{n,m'}$, with $m'=-\ell,\ldots,\ell$, remain undetermined at order $\varepsilon_{\rm T}^0$.

By the use of wave Eqs. (62), Eqs. (64) reduce to

$$\left[2\,\sigma_{n,0}\,\sigma_{n,1}\,g_{kj}+\sigma^2_{n,0}\,g_{ij}\,\nabla_k\left(\delta q^i\right)_{\rm T}-V_{kj}\right]\left(\widetilde{\delta q}^{\,j}\right)_{n,0}$$
$$+\left(\sigma^2_{n,0}\,g_{kj}-U^{(0)}_{kj}\right)\left(\widetilde{\delta q}^{\,j}\right)_{n,1}=0. \tag{65}$$

First, by substitution of the expansions for the components $\left(\widetilde{\delta q}^{\,j}\right)_{n,0}$ and $\left(\widetilde{\delta q}^{\,j}\right)_{n,1}$, multiplication by $\rho_0\,\overline{(\delta q^k)_{n,0;m}}$, integration over the domain V_0 of the spherically symmetric equilibrium star, and use of the orthogonality property between spheroidal modes, one derives the system of equations

$$2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{n,m}=\sum_{m'=-\ell}^{\ell}a_{n,m'}\,H_{n,m;n,m'},\quad m=-\ell,\ldots,\ell, \tag{66}$$

where the coefficients $H_{n,m;n',m'}$ are defined as

$$H_{n,m;n',m'} = \frac{1}{N_{n,0;m}}\left[\frac{1}{\sigma^2_{n',0}}\int_{V_0}\rho_0\,\overline{(\delta q^k)_{n,0;m}}\,V_{kj}\left(\delta q^j\right)_{n',0;m'}\,\mathrm{d}V\right.$$
$$\left.-\int_{V_0}\rho_0\,g_{ij}\left[\nabla_k\left(\delta q^i\right)_{\mathrm{T}}\right]\overline{(\delta q^k)_{n,0;m}}\left(\delta q^j\right)_{n',0;m'}\,\mathrm{d}V\right]. \quad (67)$$

The factor $N_{n,0;m}$ is the square of the norm of the eigenmode n of order zero that is associated with the spherical harmonic of degree ℓ and azimuthal number m and is determined by

$$N_{n,0;m} = \int_{V_0}\rho_0\,g_{kj}\,\overline{(\delta q^k)_{n,0;m}}\left(\delta q^j\right)_{n,0;m}\,\mathrm{d}V. \quad (68)$$

For a mode n of a degree ℓ, Eqs. (66) form a linear, homogeneous system of $(2\,\ell+1)$ equations for the $(2\,\ell+1)$ unknown constants $a_{n,m}$. The condition for the system of equations to admit of a non-trivial solution yields an equation for the first-order correction $\sigma_{n,1}$ to the angular eigenfrequency. Once this correction is determined, the constants $a_{n,m}$ can be fixed.

Secondly, after substitution of the expansions for the components $\left(\widetilde{\delta q}^j\right)_{n,0}$ and $\left(\widetilde{\delta q}^j\right)_{n,1}$ into Eqs. (65), we multiply by $\rho_0\,\overline{(\delta q^k)_{\lambda,0;\mu}}$, where λ is any mode different from the mode n. By integrating over the domain V_0 of the spherically symmetric equilibrium star and taking into account the orthogonality property between spheroidal modes that are associated with different eigenfrequencies and with spherical harmonics of different degrees, one derives that

$$b_{\lambda,\mu} = \frac{\sigma^2_{n,0}}{\sigma^2_{n,0}-\sigma^2_{\lambda,0}}\sum_{m'=-\ell}^{\ell} a_{n,m'}\,H_{\lambda,\mu;n,m'}. \quad (69)$$

The expansion for the components of the vector resulting from the parallel transport of the Lagrangian displacement then takes the form

$$\left(\widetilde{\delta q}^j\right)_n(\boldsymbol{r}) = \sum_{m'=-\ell}^{\ell} a_{n,m'}\left[\left(\delta q^j\right)_{n,0;m'}(\boldsymbol{r})\right.$$
$$\left.+\varepsilon_{\mathrm{T}}\sum_{\lambda\neq n;\mu}\frac{\sigma^2_{n,0}}{\sigma^2_{n,0}-\sigma^2_{\lambda,0}}\,H_{\lambda,\mu;n,m'}\left(\delta q^j\right)_{\lambda,0;\mu}(\boldsymbol{r})\right]. \quad (70)$$

For the resolution of the homogeneous system of Eqs. (66), explicit expressions for the coefficients $H_{n,m;n,m'}$ are required.

5.2. Form of the eigenfunctions at order zero

The components of the Lagrangian displacement at order zero that are associated with a spherical harmonic $Y_\ell^m(\theta,\phi)$ can be expressed as

$$\left.\begin{aligned}
&(\delta r)_{n,0;m}(r,\theta,\phi) = \xi_{n,\ell}(r)\,Y_\ell^m(\theta,\phi),\\
&(\delta\theta)_{n,0;m}(r,\theta,\phi) = \frac{\eta_{n,\ell}(r)}{r^2}\,\frac{\partial Y_\ell^m(\theta,\phi)}{\partial\theta},\\
&(\delta\phi)_{n,0;m}(r,\theta,\phi) = \frac{\eta_{n,\ell}(r)}{r^2}\,\frac{1}{\sin^2\theta}\,\frac{\partial Y_\ell^m(\theta,\phi)}{\partial\phi}.
\end{aligned}\right\} \quad (71)$$

Correspondingly, the divergence of the Lagrangian displacement and the Eulerian perturbations of the mass density and the pressure at order zero can be expressed as

$$\alpha_{n,0;m}(r,\theta,\phi) = \left\{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\,\xi_{n,\ell}(r)\right]-\frac{\ell(\ell+1)}{r^2}\,\eta_{n,\ell}(r)\right\}$$
$$Y_\ell^m(\theta,\phi) \equiv \alpha_{n,\ell}(r)\,Y_\ell^m(\theta,\phi), \quad (72)$$

$$\rho'_{n,0;m}(r,\theta,\phi) = -\rho_0\left[\alpha_{n,\ell}(r)+\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\,\xi_{n,\ell}(r)\right]Y_\ell^m(\theta,\phi)$$
$$\equiv \rho'_{n,\ell}(r)\,Y_\ell^m(\theta,\phi), \quad (73)$$

$$P'_{n,0;m}(r,\theta,\phi) = -\left[\frac{\mathrm{d}P_0}{\mathrm{d}r}\,\xi_{n,\ell}(r)+\rho_0\,c_0^2\,\alpha_{n,\ell}(r)\right]Y_\ell^m(\theta,\phi)$$
$$\equiv P'_{n,\ell}(r)\,Y_\ell^m(\theta,\phi). \quad (74)$$

Furthermore, the Eulerian perturbation of the gravitational potential at order zero can be written as

$$\Phi'_{n,0;m}(r,\theta,\phi) = -G\int_{V_0}\frac{\rho'_{n,0;m}(\boldsymbol{r}')}{|\boldsymbol{r}'-\boldsymbol{r}|}\,\mathrm{d}V(\boldsymbol{r}')$$
$$-G\int_{S_0}\frac{\rho_0(\boldsymbol{r}')\,(\delta r)_{n,0;m}(\boldsymbol{r}')}{|\boldsymbol{r}'-\boldsymbol{r}|}\,\mathrm{d}S(\boldsymbol{r}'). \quad (75)$$

By expansion of the reciprocal of the distance between a point with position vector $\boldsymbol{r}'$ and the point with position vector $\boldsymbol{r}$ in terms of spherical harmonics and use of the orthogonality property between the spherical harmonics, the Eulerian perturbation of the gravitational potential becomes

$$\Phi'_{n,0;m}(r,\theta,\phi) = -\frac{4\pi G}{2\ell+1}\left[r^{-(\ell+1)}\int_0^r\rho'_{n,\ell}(r')\,r'^{(\ell+2)}\,\mathrm{d}r'\right.$$
$$\left.+r^\ell\int_r^R\rho'_{n,\ell}(r')\,r'^{-(\ell-1)}\,\mathrm{d}r'+\frac{\rho_0(R)\,\xi_{n,\ell}(R)}{R^{\ell-1}}\,r^\ell\right]$$
$$Y_\ell^m(\theta,\phi)\equiv\Phi'_{n,\ell}(r)\,Y_\ell^m(\theta,\phi). \quad (76)$$

By means of the expressions for the components of the Lagrangian displacement, the square of the norm of the eigenmode n can be expressed as

$$N_{n,0;m} = \frac{4\pi}{2\ell+1}\,\frac{(\ell+|m|)!}{(\ell-|m|)!}$$
$$\times\int_0^R\rho(r)\left[\xi^2_{n,\ell}(r)+\frac{\ell(\ell+1)}{r^2}\,\eta^2_{n,\ell}(r)\right]r^2\,\mathrm{d}r$$
$$\equiv \frac{4\pi}{2\ell+1}\,\frac{(\ell+|m|)!}{(\ell-|m|)!}\,\mathcal{N}_{n,0}. \quad (77)$$

5.3. The coefficients $H_{n,m;n,m'}$

The derivation of the explicit expressions for the coefficients $H_{n,m;n,m'}$ is too long to be reproduced here. Therefore, we restrict ourselves to a presentation of the final expressions. A derivation is given to some extent in Reyniers (2002). It follows that

$$H_{n,m;n,m'} = A_{\ell,m';\ell,m}\,\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}\,\frac{1}{\mathcal{N}_{n,0}}$$
$$\times\left\{\frac{1}{\sigma^2_{n,0}}\left[(F_1)_{n,\ell}+(F_2)_{n,\ell}+(F_3)_{n,\ell}\right]+(F_4)_{n,\ell}\right\}\cdot \quad (78)$$

The coefficient $A_{\ell,m';\ell,m}$ results from the integrations over the angular coordinates θ and ϕ and is given by

$$A_{\ell,m';\ell,m} = \int_{4\pi} Y_\ell^{m'}(\theta,\phi)\,\overline{Y_\ell^m(\theta,\phi)} \times\left[Y_2(\theta) - \frac{1}{4}\,Y_2^2(\theta,\phi) - \frac{1}{4}\,Y_2^{-2}(\theta,\phi)\right]\mathrm{d}\omega, \tag{79}$$

where $\mathrm{d}\omega$ is the infinitesimal solid angle. It depends on the degree ℓ of the mode n considered and on the azimuthal numbers m and m'. For the various degrees ℓ, the constants $A_{\ell,m';\ell,m}$ different from zero obey the relations

$$\left.\begin{aligned} A_{\ell,m';\ell,m} &= A_{\ell,m;\ell,m'},\\ A_{\ell,m';\ell,m} &= A_{\ell,-m';\ell,-m}. \end{aligned}\right\} \tag{80}$$

All non-zero coefficients $A_{\ell,m';\ell,m}$ with $m \neq 0$ and/or $m' \neq 0$ can be related to the coefficients $A_{\ell,0;\ell,0}$. Consequently, for a given mode n of a degree ℓ, the coefficients $H_{n,m;n,m'}$ can also be related to the coefficient $H_{n,0;n,0}$, so that only the latter coefficient needs to be determined.

The four terms inside the braces in the right-hand member of Eq. (78) are defined as follows.

5.3.1. The first term

The term $(F_1)_{n,\ell}$ is defined as

$$(F_1)_{n,\ell} = \frac{1}{A_{\ell,m';\ell,m}} \int_{V_0} \rho_0\,\overline{(\delta q^k)_{n,0;m}}\,R_{kj}\left(\delta q^j\right)_{n,0;m'}\,\mathrm{d}V, \tag{81}$$

and it results that

$$(F_1)_{n,\ell} = -\int_0^R \xi_{n,\ell}\,\frac{\mathrm{d}P_0}{\mathrm{d}r}\left[(h_1)_{n,\ell} + \frac{1}{\rho_0}\,\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\,(h_2)_{n,\ell}\right](r)\,\mathrm{d}r \tag{82}$$

with

$$\begin{aligned}[h_1(r)]_{n,\ell} = {}& \left(2\,\xi_{\rm st}(r) - 9\,\frac{\eta_{\rm st}(r)}{r} + 3\,\frac{\mathrm{d}\eta_{\rm st}(r)}{\mathrm{d}r}\right)\xi_{n,\ell}(r)\\ &+3\left[4\,\frac{\eta_{\rm st}(r)}{r} - \left(1 + \frac{\ell(\ell+1)}{3}\right)\xi_{\rm st}(r) - \frac{\mathrm{d}\eta_{\rm st}(r)}{\mathrm{d}r}\right]\frac{\eta_{n,\ell}(r)}{r}\\ &+3\left(\xi_{\rm st}(r) - \frac{\eta_{\rm st}(r)}{r}\right)\frac{\mathrm{d}\eta_{n,\ell}(r)}{\mathrm{d}r} + r^2\,\frac{\mathrm{d}\xi_{\rm st}(r)}{\mathrm{d}r}\,\frac{\mathrm{d}\xi_{n,\ell}(r)}{\mathrm{d}r}, \end{aligned} \tag{83}$$

$$[h_2(r)]_{n,\ell} = \left[r^2\,\frac{\mathrm{d}\xi_{\rm st}(r)}{\mathrm{d}r}\,\xi_{n,\ell}(r) + 3\left(\xi_{\rm st}(r) - \frac{\eta_{\rm st}(r)}{r}\right)\eta_{n,\ell}(r)\right]. \tag{84}$$

5.3.2. The second term

The term $(F_2)_{n,\ell}$ is defined as

$$(F_2)_{n,\ell} = \frac{1}{A_{\ell,m';\ell,m}} \int_{V_0} \rho_0\,\overline{(\delta q^k)_{n,0;m}}\,P_{kj}\left(\delta q^j\right)_{n,0;m'}\,\mathrm{d}V, \tag{85}$$

and it results that

$$\begin{aligned}(F_2)_{n,\ell} = {}& -\int_0^R \alpha_{n,\ell}(r)\left[\rho_0\,c_0^2\,(h_1)_{n,\ell} + \frac{\mathrm{d}P_0}{\mathrm{d}r}\,(h_2)_{n,\ell}\right](r)\,\mathrm{d}r\\ &+\left\{\xi_{n,\ell}\left[\rho_0\,c_0^2\,(h_1)_{n,\ell} + \frac{\mathrm{d}P_0}{\mathrm{d}r}\,(h_2)_{n,\ell}\right]\right\}(R). \end{aligned} \tag{86}$$

5.3.3. The third term

The term $(F_3)_{n,\ell}$ is defined as

$$(F_3)_{n,\ell} = \frac{1}{A_{\ell,m';\ell,m}} \int_{V_0} \rho_0\,(\boldsymbol{r})\,\overline{(\delta q^k)_{n,0;m}}\,F_{ki}\left(\delta q^i\right)_{n,0;m'}\,\mathrm{d}V\,(\boldsymbol{r}), \tag{87}$$

and it results that

$$(F_3)_{n,\ell} = \left\{\rho_0\,r^2\,\xi_{n,\ell}\,(f_3)_{n,\ell}\right\}(R) + \int_0^R \rho'_{n,\ell}(r)\,r^2\,\left[f_3(r)\right]_{n,\ell}\,\mathrm{d}r, \tag{88}$$

where the functions $[f_3(r)]_{n,\ell}$ are given by

$$\begin{aligned}[f_3(r)]_{n,\ell} = \Bigg\{&\left(\xi_{\rm st}\,\frac{\mathrm{d}\Phi'_{n,\ell}}{\mathrm{d}r} + 3\,\frac{\eta_{\rm st}}{r^2}\,\Phi'_{n,\ell}\right)\\ &-\frac{4\,\pi\,G}{2\ell+1}\left[(g_1)_{n,\ell} - (g_2)_{n,\ell} - (g_3)_{n,\ell} + (g_4)_{n,\ell}\right]\Bigg\}(r) \end{aligned} \tag{89}$$

with

$$\begin{aligned}[g_1(r)]_{n,\ell} = {}& r^{-(\ell+1)}\int_0^r \rho'_{n,\ell}(r')\,r'^{\ell+1}\left[\ell\,\xi_{\rm st}(r') + 3\,\frac{\eta_{\rm st}(r')}{r'}\right]\mathrm{d}r'\\ &+r^\ell\int_r^R \rho'_{n,\ell}(r')\,r'^{-\ell}\left[-(\ell+1)\,\xi_{\rm st}(r') + 3\,\frac{\eta_{\rm st}(r')}{r'}\right]\mathrm{d}r', \end{aligned} \tag{90}$$

$$\begin{aligned}[g_2(r)]_{n,\ell} = {}& \ell\,r^{-(\ell+1)}\int_0^r \rho_0(r')\,r'^{\ell-1}\,\left[h_2(r')\right]_{n,\ell}\,\mathrm{d}r'\\ &-(\ell+1)\,r^\ell\int_r^R \rho_0(r')\,r'^{-(\ell+2)}\,\left[h_2(r')\right]_{n,\ell}\,\mathrm{d}r', \end{aligned} \tag{91}$$

$$\begin{aligned}[g_3(r)]_{n,\ell} = {}& r^{-(\ell+1)}\int_0^r \rho_0(r')\,r'^{\ell+1}\,\left[h_3(r')\right]_{n,\ell}\,\mathrm{d}r'\\ &+r^\ell\int_r^R \rho_0(r')\,r'^{-\ell}\,\left[h_3(r')\right]_{n,\ell}\,\mathrm{d}r', \end{aligned} \tag{92}$$

$$[g_4(r)]_{n,\ell} = -\left[(\ell+1)\,\xi_{\rm st}(R) - 3\,\frac{\eta_{\rm st}(R)}{R}\right]\frac{\rho_0(R)\,\xi_{n,\ell}(R)}{R^\ell}\,r^\ell, \tag{93}$$

and

$$\begin{aligned}[h_3(r)]_{n,\ell} = \Bigg\{&3\left(\frac{\mathrm{d}}{\mathrm{d}r}\,\frac{\eta_{\rm st}}{r}\right)\xi_{n,\ell}\\ &+\left[[\ell(\ell+1) - 3]\,\xi_{\rm st} + 9\,\frac{\eta_{\rm st}}{r}\right]\frac{\eta_{n,\ell}}{r^2}\Bigg\}(r). \end{aligned} \tag{94}$$

5.3.4. The fourth term

The term $(F_4)_{n,\ell}$ is defined as

$$\begin{aligned}(F_4)_{n,\ell} = {}& -\frac{1}{A_{\ell,m';\ell,m}}\\ &\times\int_{V_0} \rho_0\,g_{ij}\,\overline{(\delta q^k)_{n,0;m}}\,\left[\nabla_k\left(\delta q^i\right)_{\rm T}\right]\left(\delta q^j\right)_{n,0;m'}\,\mathrm{d}V, \end{aligned} \tag{95}$$

and it results that

$$(F_4)_{n,\ell} = -\int_0^R \rho_0(r)\left[\xi_{n,\ell}\,(h_2)_{n,\ell} + r\,\eta_{n,\ell}\,(h_3)_{n,\ell}\right](r)\,\mathrm{d}r. \tag{96}$$

6. Low-degree oscillation modes of a star perturbed by an equilibrium tide

In this section, we present the solutions of the homogeneous system of Eqs. (66) for oscillation modes n of the lowest degrees $\ell = 0, 1, 2, 3$ in an arbitrary component of a close binary that is subject to an equilibrium tide.

6.1. Oscillation modes belonging to $\ell = 0$

For $\ell = 0$, the zero-order modes n are radial oscillation modes of the spherically symmetric equilibrium star. From the definition of the coefficients $A_{\ell,m';\ell,m}$, it follows that

$$A_{0,0;0,0} = 0, \tag{97}$$

so that, according to Eq. (78),

$$H_{n,0;n,0} = 0 \tag{98}$$

for any radial oscillation mode n. The system of Eqs. (66) then reduces to the single equation

$$2\,\frac{\sigma_{n,1}}{\sigma_{n,0}} = 0. \tag{99}$$

Hence, for any radial mode n of a star, the first-order correction to the eigenfrequency due to a perturbing equilibrium tide is equal to zero.

The components $\left(\widetilde{\delta q}^{\,j}\right)_n$, $j = 1, 2, 3$, of the transported vector remain unchanged at the lowest order of approximation but are affected at order $\varepsilon_{\rm T}$ by the contributions stemming from the nonradial spheroidal modes, as it follows from the expansion given by Eq. (70). After parallel transport to the point in the domain of the tidally perturbed star, one obtains components of the Lagrangian displacement of the form

$$\left(\delta q^j\right)_n(\boldsymbol{r}) = a_{n,0}\Bigg[\delta^j_1\,(\delta r)_{n,0}(\boldsymbol{r}) + \varepsilon_{\rm T}\sum_{\lambda\neq n;\mu}\frac{\sigma^2_{n,0}}{\sigma^2_{n,0}-\sigma^2_{\lambda,0}}\;H_{\lambda,\mu;n,0}\;\left(\delta q^j\right)_{\lambda,0;\mu}(\boldsymbol{r})\Bigg]. \tag{100}$$

6.2. Oscillation modes belonging to $\ell = 1$

For $\ell = 1$, the homogeneous system of Eqs. (66) takes the form

$$2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{n,m} = \sum_{m'=-1}^{1} a_{n,m'}\,H_{n,m;n,m'},\quad m = -1,\ldots,1. \tag{101}$$

Because of the relations between the non-zero coefficients $A_{1,m';1,m}$ and the coefficient $A_{1,0;1,0}$, and the resulting relations between the coefficients $H_{n,m;n,m'}$ and the coefficient $H_{n,0;n,0}$, the system of equations reduces to

$$\left.\begin{aligned}
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{1,-1} &= -\frac{1}{2}\left(a_{1,-1}+3\,a_{1,1}\right)\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{1,0} &= a_{1,0}\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{1,1} &= -\frac{1}{2}\left(3\,a_{1,-1}+a_{1,1}\right)\,H_{n,0;n,0}.
\end{aligned}\right\} \tag{102}$$

The condition for this system to admit of non-trivial solutions for the constants $a_{n,m}$ yields three roots for the first-order relative corrections to the eigenfrequency:

$$2\,\frac{\sigma^{(1)}_{n,1}}{\sigma_{n,0}} = -2\,H_{n,0;n,0},\qquad 2\,\frac{\sigma^{(2)}_{n,1}}{\sigma_{n,0}} = H_{n,0;n,0}. \tag{103}$$

The first root is a single root, and the second one a double root.

Hence, between the corrections to the eigenfrequency of a first-degree mode, the relation holds

$$\sigma^{(2)}_{n,1} = -\frac{1}{2}\,\sigma^{(1)}_{n,1}. \tag{104}$$

By solving the homogeneous system of Eqs. (102) for the correction $\sigma^{(1)}_{n,1}$ to the angular eigenfrequency, one derives

$$a_{n,0} = 0,\quad a_{n,1} = a_{n,-1}, \tag{105}$$

and for the correction $\sigma^{(2)}_{n,1}$ to the angular eigenfrequency,

$$a_{n,1} = -a_{n,-1}, \tag{106}$$

with the coefficient $a_{n,0}$ remaining undetermined. Hence, one obtains a set of three independent solutions for the transported Lagrangian displacement.

In order to determine the components δq^j of the Lagrangian displacement in the domain of the tidally perturbed star, it suffices to use the relations given by Eqs. (26). The resulting lowest-order approximations for these components with respect to the local coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$ are

$$\left.\begin{aligned}
&(\delta r)^{(k)}_n(\boldsymbol{r};t) =\\
&\quad \xi_{n,0}(r)\,V^{(k)}_1(\theta,\phi)\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma^{(k)}_{n,1}\right)t\right],\\
&(\delta\theta)^{(k)}_n(\boldsymbol{r};t) =\\
&\quad \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial V^{(k)}_1(\theta,\phi)}{\partial\theta}\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma^{(k)}_{n,1}\right)t\right],\\
&(\delta\phi)^{(k)}_n(\boldsymbol{r};t) = \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta}\,\frac{\partial V^{(k)}_1(\theta,\phi)}{\partial\phi}\\
&\quad \times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma^{(k)}_{n,1}\right)t\right],\qquad k = 1,2,
\end{aligned}\right\} \tag{107}$$

where the functions $V^{(k)}_1(\theta,\phi)$ are defined as

$$\left.\begin{aligned}
V^{(1)}_1(\theta,\phi) &= A_1\left[Y^1_1(\theta,\phi)+Y^{-1}_1(\theta,\phi)\right],\\
V^{(2)}_1(\theta,\phi) &= B_1\,Y_1(\theta)+C_1\left[Y^1_1(\theta,\phi)-Y^{-1}_1(\theta,\phi)\right],
\end{aligned}\right\} \tag{108}$$

and A_1, B_1, C_1 are undetermined constants. With the single eigenfrequency $\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma^{(1)}_{n,1}$, one eigenfunction is associated, which involves the constant A_1, while, with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma^{(2)}_{n,1}$, two linearly independent eigenfunctions are associated, which involve the constants B_1 and C_1.

Since the axis that joins the mass centre of the tidally perturbed star to the companion is an axis of symmetry of the star, it is interesting to pass also on to a system of spherical coordinates r, θ^*, ϕ^* whose polar axis coincides with this axis and to transform the components δr, $\delta\theta$, $\delta\phi$ of the Lagrangian displacement into components δr, $\delta\theta^*$, $\delta\phi^*$ expressed in terms of spherical harmonics $Y_1^{\widetilde{m}}(\theta^*, \phi^*)$. According to the transformation equations for contravariant vector components, the horizontal components $\delta\theta^*$ and $\delta\phi^*$ are given by

$$\left.\begin{aligned}\delta\theta^* &= \frac{\partial\theta^*}{\partial\theta}\,\delta\theta + \frac{\partial\theta^*}{\partial\phi}\,\delta\phi,\\ \delta\phi^* &= \frac{\partial\phi^*}{\partial\theta}\,\delta\theta + \frac{\partial\phi^*}{\partial\phi}\,\delta\phi.\end{aligned}\right\} \tag{109}$$

The angular coordinates θ and ϕ are related to the angular coordinates θ^* and ϕ^* as

$$\left.\begin{aligned}&\cos\theta = \sin\theta^*\,\sin\phi^*,\\ &\sin\theta\,\sin\phi = \sin\theta^*\,\cos\phi^*,\\ &\sin\theta\,\cos\phi = \cos\theta^*.\end{aligned}\right\} \tag{110}$$

Then, the solution with the single eigenfrequency $\sigma_{n,0} + \varepsilon_{\rm T}\,\sigma_{n,1}^{(1)}$ is associated with the first-degree spherical harmonic $Y_1(\theta^*, \phi^*)$, and the two solutions with the double eigenfrequency $\sigma_{n,0} + \varepsilon_{\rm T}\,\sigma_{n,1}^{(2)}$ are associated with the first-degree spherical harmonics $Y_1^1(\theta^*, \phi^*)$ and $Y_1^{-1}(\theta^*, \phi^*)$:

$$\left.\begin{aligned}(\delta r)_n^{(k)}(\boldsymbol{r};t) &= \xi_{n,0}(r)\,W_1^{(k)}(\theta^*,\phi^*)\\ &\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma_{n,1}^{(k)}\right)t\right],\\ (\delta\theta^*)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial W_1^{(k)}(\theta^*,\phi^*)}{\partial\theta^*}\\ &\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma_{n,1}^{(k)}\right)t\right],\\ (\delta\phi^*)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta^*}\,\frac{\partial W_1^{(k)}(\theta^*,\phi^*)}{\partial\phi^*}\\ &\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\rm T}\,\sigma_{n,1}^{(k)}\right)t\right],\quad k=1,2,\end{aligned}\right\} \tag{111}$$

where the functions $W_1^{(1)}(\theta^*, \phi^*)$ and $W_1^{(2)}(\theta^*, \phi^*)$ are defined as

$$\left.\begin{aligned}&W_1^{(1)}(\theta^*,\phi^*) = -2\,A_1\,Y_1(\theta^*),\\ &W_1^{(2)}(\theta^*,\phi^*) = B_1^*\,Y_1^1(\theta^*,\phi^*) + C_1^*\,Y_1^{-1}(\theta^*,\phi^*)\end{aligned}\right\} \tag{112}$$

and the constants B_1^* and C_1^* are related to the constants B_1 and C_1 as

$$B_1^* = \frac{\mathrm{i}}{2}\,(B_1 + 2\,C_1)\,,\qquad C_1^* = \frac{\mathrm{i}}{2}\,(-B_1 + 2\,C_1)\,. \tag{113}$$

The ratios between the perturbed eigenfrequencies of a first-degree mode n are schematically represented in Fig. 1 in terms of the absolute value of the azimuthal number $\widetilde{m}$ they are associated with.

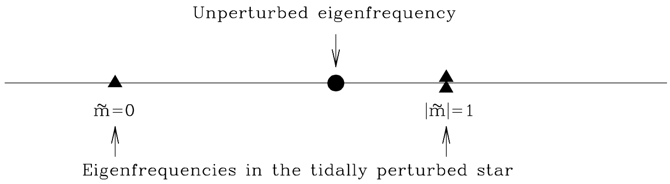

Fig. 1. Schematic representation of the ratios between the perturbed eigenfrequencies of a first-degree mode n.

6.3. Oscillation modes belonging to $\ell = 2$

For $\ell = 2$, the homogeneous system of Eqs. (66) takes the form

$$2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{n,m} = \sum_{m'=-2}^{2} a_{n,m'}\,H_{n,m;n,m'},\quad m=-2,\ldots,2 \tag{114}$$

and reduces to

$$\left.\begin{aligned}2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{2,-2} &= \left(-a_{2,-2} + \frac{1}{4}\,a_{2,0}\right)H_{n,0;n,0},\\ 2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{2,-1} &= \frac{1}{2}\,(a_{2,-1} - 3\,a_{2,1})\;H_{n,0;n,0},\\ 2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{2,0} &= (6\,a_{2,-2} + a_{2,0} + 6\,a_{2,2})\;H_{n,0;n,0},\\ 2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{2,1} &= \frac{1}{2}\,(-3\,a_{2,-1} + a_{2,1})\;H_{n,0;n,0},\\ 2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{2,2} &= \left(\frac{1}{4}\,a_{2,0} - a_{2,2}\right)H_{n,0;n,0}.\end{aligned}\right\} \tag{115}$$

The condition for this system to admit of non-trivial solutions for the constants $a_{n,m}$ yields five roots for the first-order relative corrections to the eigenfrequency:

$$\left.\begin{aligned}&2\,\frac{\sigma_{n,1}^{(1)}}{\sigma_{n,0}} = -2\,H_{n,0;n,0},\quad 2\,\frac{\sigma_{n,1}^{(2)}}{\sigma_{n,0}} = -H_{n,0;n,0},\\ &2\,\frac{\sigma_{n,1}^{(3)}}{\sigma_{n,0}} = 2\,H_{n,0;n,0}.\end{aligned}\right\} \tag{116}$$

The first root is a single root, the two other roots are double roots.

Hence, between the corrections to the eigenfrequency of a second-degree mode, the relations hold

$$\sigma_{n,1}^{(2)} = \frac{1}{2}\,\sigma_{n,1}^{(1)},\qquad \sigma_{n,1}^{(3)} = -\sigma_{n,1}^{(1)}. \tag{117}$$

For the correction $\sigma_{n,1}^{(1)}$, the relations hold

$$\left.\begin{aligned}&a_{n,2} = -\frac{1}{4}\,a_{n,0},\quad a_{n,1}=0,\quad a_{n,-1}=0,\\ &a_{n,-2} = a_{n,2};\end{aligned}\right\} \tag{118}$$

for the correction $\sigma_{n,1}^{(2)}$, the relations

$$a_{n,0} = 0,\quad a_{n,1} = a_{n,-1},\quad a_{n,2} = -a_{n,-2}; \tag{119}$$

and for the correction $\sigma_{n,1}^{(3)}$, the relations

$$a_{n,1} = -a_{n,-1},\quad a_{n,2} = \frac{1}{12}\,a_{n,0},\quad a_{n,2} = a_{n,-2}. \tag{120}$$

One thus obtains a set of five independent solutions for the transported Lagrangian displacement.

One passes on to the components δq^j of the Lagrangian displacement in the domain of the tidally perturbed star by means of the relations given by Eqs. (26). The resulting lowest-order approximations for these components with respect to the local coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$ are

$$\left.\begin{aligned}
(\delta r)_n^{(k)}(\boldsymbol{r};t) &= \xi_{n,0}(r)\, V_2^{(k)}(\theta,\phi)\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\\
(\delta\theta)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial V_2^{(k)}(\theta,\phi)}{\partial\theta}\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\\
(\delta\phi)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta}\,\frac{\partial V_2^{(k)}(\theta,\phi)}{\partial\phi}\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\quad k=1,2,3,
\end{aligned}\right\}\tag{121}$$

where the functions $V_2^{(k)}(\theta,\phi)$ are defined as

$$\left.\begin{aligned}
V_2^{(1)}(\theta,\phi) &= A_2\left[Y_2(\theta)-\frac{1}{4}\,Y_2^2(\theta,\phi)-\frac{1}{4}\,Y_2^{-2}(\theta,\phi)\right],\\
V_2^{(2)}(\theta,\phi) &= B_2\left[Y_2^1(\theta,\phi)+Y_2^{-1}(\theta,\phi)\right]\\
&\quad+C_2\left[Y_2^2(\theta,\phi)-Y_2^{-2}(\theta,\phi)\right],\\
V_2^{(3)}(\theta,\phi) &= D_2\left[Y_2^1(\theta,\phi)-Y_2^{-1}(\theta,\phi)\right]\\
&\quad+E_2\left[Y_2(\theta)+\frac{1}{12}\,Y_2^2(\theta,\phi)+\frac{1}{12}\,Y_2^{-2}(\theta,\phi)\right],
\end{aligned}\right\}\tag{122}$$

and A_2, B_2, C_2, D_2, E_2 are undetermined constants. With the single eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(1)}$, one eigenfunction is associated, while, with each of the double eigenfrequencies $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(2)}$ and $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(3)}$, two linearly independent eigenfunctions are associated. The first eigenfunction involves the constant A_2, and the other two eigenfunctions involve respectively the constants B_2 and C_2 and the constants D_2 and E_2.

In terms of the spherical coordinates r, θ^*, ϕ^*, the solution with the single eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(1)}$ is associated with the second-degree spherical harmonic $Y_2\,(\theta^*,\phi^*)$, the two solutions with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(2)}$ are associated with the second-degree spherical harmonics $Y_2^1\,(\theta^*,\phi^*)$ and $Y_2^{-1}\,(\theta^*,\phi^*)$, and the two solutions with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(3)}$ are associated with the second-degree spherical harmonics $Y_2^2\,(\theta^*,\phi^*)$ and $Y_2^{-2}\,(\theta^*,\phi^*)$:

$$\left.\begin{aligned}
(\delta r)_n^{(k)}(\boldsymbol{r};t) &= \xi_{n,0}(r)\, W_2^{(k)}(\theta^*,\phi^*)\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\\
(\delta\theta^*)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial W_2^{(k)}(\theta^*,\phi^*)}{\partial\theta^*}\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\\
(\delta\phi^*)_n^{(k)}(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta^*}\,\frac{\partial W_2^{(k)}(\theta^*,\phi^*)}{\partial\phi^*}\\
&\quad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma_{n,1}^{(k)}\right)t\right],\quad k=1,2,3,
\end{aligned}\right\}\tag{123}$$

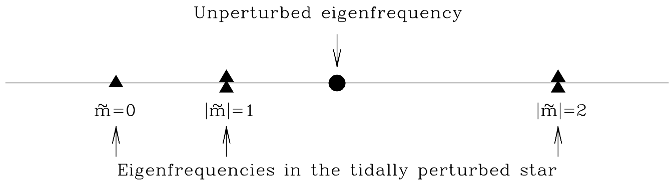

Fig. 2. Schematic representation of the ratios between the perturbed eigenfrequencies of a second-degree mode n.

where the functions $W_2^{(1)}(\theta^*,\phi^*)$, $W_2^{(2)}(\theta^*,\phi^*)$, and $W_2^{(3)}(\theta^*,\phi^*)$ are defined as

$$\left.\begin{aligned}
W_2^{(1)}(\theta^*,\phi^*) &= -2\,A_2\,Y_2(\theta^*),\\
W_2^{(2)}(\theta^*,\phi^*) &= B_2^*\,Y_2^1(\theta^*,\phi^*)+C_2^*\,Y_2^{-1}(\theta^*,\phi^*),\\
W_2^{(3)}(\theta^*,\phi^*) &= D_2^*\,Y_2^2(\theta^*,\phi^*)+E_2^*\,Y_2^{-2}(\theta^*,\phi^*)
\end{aligned}\right\}\tag{124}$$

and the constants B_2^*, C_2^*, D_2^*, E_2^* are related to the constants B_2, C_2, D_2, E_2 as

$$\left.\begin{aligned}
B_2^* &= -\mathrm{i}\,(B_2+2\,C_2), & C_2^* &= \mathrm{i}\,(B_2-2\,C_2),\\
D_2^* &= -\frac{1}{2}\left(D_2+\frac{E_2}{3}\right), & E_2^* &= \frac{1}{2}\left(D_2-\frac{E_2}{3}\right)\cdot
\end{aligned}\right\}\tag{125}$$

The ratios between the perturbed eigenfrequencies of a second-degree mode n are schematically represented in Fig. 2 in terms of the absolute value of the azimuthal number $\widetilde{m}$ they are associated with.

6.4. Oscillation modes belonging to $\ell = 3$

For $\ell=3$, the homogeneous system of Eqs. (66) takes the form

$$2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{n,m}=\sum_{m'=-3}^{3}a_{n,m'}\,H_{n,m;n,m'},\quad m=-3,\ldots,3\tag{126}$$

and reduces to

$$\left.\begin{aligned}
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,-3} &= \frac{1}{8}\,(-10\,a_{3,-3}+a_{3,-1})\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,-2} &= \frac{1}{8}\,a_{3,0}\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,-1} &= \frac{3}{4}\,(10\,a_{3,-3}+a_{3,-1}-2\,a_{3,1})\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,0} &= (15\,a_{3,-2}+a_{3,0}+15\,a_{3,2})\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,1} &= \frac{3}{4}\,(-2\,a_{3,-1}+a_{3,1}+10\,a_{3,3})\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,2} &= \frac{1}{8}\,a_{3,0}\,H_{n,0;n,0},\\
2\,\frac{\sigma_{n,1}}{\sigma_{n,0}}\,a_{3,3} &= \frac{1}{8}\,(a_{3,1}-10\,a_{3,3})\,H_{n,0;n,0}.
\end{aligned}\right\}\tag{127}$$

The condition for this system of equations to admit of non-trivial solutions for the constants $a_{n,m}$ yields seven first-order relative corrections to the eigenfrequency:

$$\left.\begin{aligned} &2\,\frac{\sigma^{(1)}_{n,1}}{\sigma_{n,0}} = -2\,H_{n,0;n,0}, \quad 2\,\frac{\sigma^{(2)}_{n,1}}{\sigma_{n,0}} = -\frac{3}{2}\,H_{n,0;n,0},\\ &2\,\frac{\sigma^{(3)}_{n,1}}{\sigma_{n,0}} = 0, \quad 2\,\frac{\sigma^{(4)}_{n,1}}{\sigma_{n,0}} = \frac{5}{2}\,H_{n,0;n,0}. \end{aligned}\right\} \tag{128}$$

The first root is a single root, the three other roots are double roots.

Hence, between the corrections to the eigenfrequency of a third-degree mode, the relations hold

$$\sigma^{(2)}_{n,1} = \frac{3}{4}\,\sigma^{(1)}_{n,1}, \qquad \sigma^{(3)}_{n,1} = 0, \qquad \sigma^{(4)}_{n,1} = -\frac{5}{4}\,\sigma^{(1)}_{n,1}. \tag{129}$$

For the correction $\sigma^{(1)}_{n,1}$, the relations hold

$$\left.\begin{aligned} &a_{n,0} = 0, \quad a_{n,2} = 0, \quad a_{n,-2} = 0,\\ &a_{n,1} = a_{n,-1}, \quad a_{n,1} = -6\,a_{n,3}, \quad a_{n,3} = a_{n,-3}; \end{aligned}\right\} \tag{130}$$

for the correction $\sigma^{(2)}_{n,1}$, the relations

$$\left.\begin{aligned} &a_{n,0} = -12\,a_{n,2}, \quad a_{n,1} = -a_{n,-1}, \quad a_{n,1} = -2\,a_{n,3},\\ &a_{n,2} = -a_{n,-2}, \quad a_{n,3} = a_{n,-3}; \end{aligned}\right\} \tag{131}$$

for the correction $\sigma^{(3)}_{n,1}$, the relations

$$\left.\begin{aligned} &a_{n,0} = 0, \quad a_{n,1} = -a_{n,-1}, \quad a_{n,1} = 10\,a_{n,3},\\ &a_{n,2} = -a_{n,-2}, \quad a_{n,3} = a_{n,-3}; \end{aligned}\right\} \tag{132}$$

and, for the correction $\sigma^{(4)}_{n,1}$, the relations

$$\left.\begin{aligned} &a_{n,0} = 20\,a_{n,2}, \quad a_{n,1} = -a_{n,-1}, \quad a_{n,1} = 30\,a_{n,3},\\ &a_{n,2} = a_{n,-2}, \quad a_{n,3} = -a_{n,-3}. \end{aligned}\right\} \tag{133}$$

One thus obtains a set of seven independent solutions for the transported Lagrangian displacement.

One passes on to the components δq^j of the Lagrangian displacement in the domain of the tidally perturbed star by means of the relations given by Eqs. (26). The resulting lowest-order approximations for these components with respect to the local coordinate basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$ are

$$\left.\begin{aligned} (\delta r)^{(k)}_n(\boldsymbol{r};t) &= \xi_{n,0}(r)\,V^{(k)}_3(\theta,\phi)\\ &\qquad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right],\\ (\delta\theta)^{(k)}_n(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial V^{(k)}_3(\theta,\phi)}{\partial\theta}\\ &\qquad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right],\\ (\delta\phi)^{(k)}_n(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta}\,\frac{\partial V^{(k)}_3(\theta,\phi)}{\partial\phi}\\ &\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right], \qquad k=1,2,3,4, \end{aligned}\right\} \tag{134}$$

where the functions $V^{(k)}_3(\theta,\phi)$ are defined as

$$\left.\begin{aligned} V^{(1)}_3(\theta,\phi) &= A_3\left\{6\left[Y^1_3(\theta,\phi)+Y^{-1}_3(\theta,\phi)\right]\right.\\ &\qquad\left.-\left[Y^3_3(\theta,\phi)+Y^{-3}_3(\theta,\phi)\right]\right\},\\ V^{(2)}_3(\theta,\phi) &= B_3\left\{2\left[Y^1_3(\theta,\phi)-Y^{-1}_3(\theta,\phi)\right]\right.\\ &\qquad\left.-\left[Y^3_3(\theta,\phi)-Y^{-3}_3(\theta,\phi)\right]\right\}\\ &\qquad+C_3\left\{12\,Y_3(\theta)-\left[Y^2_3(\theta,\phi)+Y^{-2}_3(\theta,\phi)\right]\right\},\\ V^{(3)}_3(\theta,\phi) &= D_3\left\{10\left[Y^1_3(\theta,\phi)+Y^{-1}_3(\theta,\phi)\right]\right.\\ &\qquad\left.+\left[Y^3_3(\theta,\phi)+Y^{-3}_3(\theta,\phi)\right]\right\}\\ &\qquad+E_3\left[Y^2_3(\theta,\phi)-Y^{-2}_3(\theta,\phi)\right],\\ V^{(4)}_3(\theta,\phi) &= F_3\left\{30\left[Y^1_3(\theta,\phi)-Y^{-1}_3(\theta,\phi)\right]\right.\\ &\qquad\left.+\left[Y^3_3(\theta,\phi)-Y^{-3}_3(\theta,\phi)\right]\right\}\\ &\qquad+G_3\left\{20\,Y_3(\theta)+\left[Y^2_3(\theta,\phi)+Y^{-2}_3(\theta,\phi)\right]\right\}, \end{aligned}\right\} \tag{135}$$

and A_3, B_3, C_3, D_3, E_3, F_3, G_3 are undetermined constants. With the single eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(1)}_{n,1}$, one eigenfunction is associated, while, with each of the double eigenfrequencies $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(2)}_{n,1}$, $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(3)}_{n,1}$, and $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(4)}_{n,1}$, two linearly independent eigenfunctions are associated. The first eigenfunction involves the constant A_3, and the other three eigenfunctions involve respectively the constants B_3 and C_3, the constants D_3 and E_3, and the constants F_3 and G_3.

In terms of the spherical coordinates r, θ^*, ϕ^*, the solution with the single eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(1)}_{n,1}$ is associated with the third-degree spherical harmonic $Y_3(\theta^*,\phi^*)$, the two solutions with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(2)}_{n,1}$ are associated with the third-degree spherical harmonics $Y^1_3(\theta^*,\phi^*)$ and $Y^{-1}_3(\theta^*,\phi^*)$, the two solutions with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(3)}_{n,1}$ are associated with the third-degree spherical harmonics $Y^2_3(\theta^*,\phi^*)$ and $Y^{-2}_3(\theta^*,\phi^*)$, and the two solutions with the double eigenfrequency $\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(4)}_{n,1}$ are associated with the third-degree spherical harmonics $Y^3_3(\theta^*,\phi^*)$ and $Y^{-3}_3(\theta^*,\phi^*)$:

$$\left.\begin{aligned} (\delta r)^{(k)}_n(\boldsymbol{r};t) &= \xi_{n,0}(r)\,W^{(k)}_3(\theta^*,\phi^*)\\ &\qquad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right],\\ (\delta\theta^*)^{(k)}_n(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{\partial W^{(k)}_3(\theta^*,\phi^*)}{\partial\theta^*}\\ &\qquad\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right],\\ (\delta\phi^*)^{(k)}_n(\boldsymbol{r};t) &= \frac{\eta_{n,0}(r)}{r^2}\,\frac{1}{\sin^2\theta^*}\,\frac{\partial W^{(k)}_3(\theta^*,\phi^*)}{\partial\phi^*}\\ &\times\exp\left[\mathrm{i}\left(\sigma_{n,0}+\varepsilon_{\mathrm{T}}\,\sigma^{(k)}_{n,1}\right)t\right], \qquad k=1,2,3,4, \end{aligned}\right\} \tag{136}$$

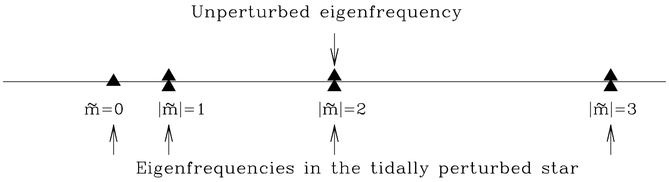

Fig. 3. Schematic representation of the ratios between the perturbed eigenfrequencies of a third-degree mode *n*.

where the functions $W_3^{(1)}(\theta^*,\phi^*)$, $W_3^{(2)}(\theta^*,\phi^*)$, $W_3^{(3)}(\theta^*,\phi^*)$, and $W_3^{(4)}(\theta^*,\phi^*)$ are defined as

$$\left.\begin{aligned}
W_3^{(1)}(\theta^*,\phi^*) &= 48\,A_3\,Y_3(\theta^*),\\
W_3^{(2)}(\theta^*,\phi^*) &= B_3^*\,Y_3^1(\theta^*,\phi^*) + C_3^*\,Y_3^{-1}(\theta^*,\phi^*),\\
W_3^{(3)}(\theta^*,\phi^*) &= D_3^*\,Y_3^2(\theta^*,\phi^*) + E_3^*\,Y_3^{-2}(\theta^*,\phi^*),\\
W_3^{(4)}(\theta^*,\phi^*) &= F_3^*\,Y_3^3(\theta^*,\phi^*) + G_3^*\,Y_3^{-3}(\theta^*,\phi^*)
\end{aligned}\right\} \tag{137}$$

and the constants B_3^*, C_3^*, D_3^*, E_3^*, F_3^*, G_3^* are related to the constants B_3, C_3, D_3, E_3, F_3, G_3 as

$$\left.\begin{aligned}
&B_3^* = -4\,\mathrm{i}\,(2\,B_3 + C_3)\,, && C_3^* = -4\,\mathrm{i}\,(2\,B_3 - C_3)\,,\\
&D_3^* = 4\,D_3 + E_3, && E_3^* = 4\,D_3 - E_3,\\
&F_3^* = -\mathrm{i}\left(4\,F_3 + \frac{2}{3}\,G_3\right), && G_3^* = -\mathrm{i}\left(4\,F_3 - \frac{2}{3}\,G_3\right).
\end{aligned}\right\} \tag{138}$$

The ratios between the perturbed eigenfrequencies of a third-degree mode *n* are schematically represented in Fig. 3 in terms of the absolute value of the azimuthal number $\widetilde{m}$ they are associated with.

7. Concluding remarks

In this paper, we have established the equations that govern linear, isentropic oscillations in a star that is a component of a close binary and is subject to the tidal action of a companion moving in a circular orbit. The star is supposed to rotate uniformly around an axis perpendicular to the orbital plane. We have adopted the rotation axis as the *z*-axis of a corotating frame of reference and as the polar axis for a system of spherical coordinates r, θ, ϕ with associated spherical harmonics $Y_\ell^m(\theta,\phi)$.

The equations are derived in the assumption that the tides generated by the companion are determined by the theory of dynamic tides in which they are considered as forced, linear, isentropic oscillations of a nonrotating spherically symmetric star. An important limitation of our investigation is that the effects of rotation are neglected. We have left it open whether or not the star rotates synchronously with the companion's orbital motion.

In view of the subsequent use of perturbation methods, we have transformed the governing equations into equations that are defined in the domain of the unperturbed spherically symmetric star by using the inverse of the tidal field. The resulting equations are Eqs. (56).

In the second part of the paper, we have concentrated on the case in which the star rotates synchronously, so that the tide generated by the companion is an equilibrium tide. We have presented a time-independent perturbation method in Sect. 5.1, in which $\varepsilon_{\rm T}$ is the small expansion parameter. The perturbation method applies to any physical model of a spherically symmetric star.

Attention is paid to the degeneracy of the eigenvalue problem of the linear, isentropic oscillations of a spherically symmetric star with respect to the azimuthal number *m*. For a mode *n* belonging to a degree ℓ, the $(2\ell+1)$ unknown constants $a_{n,m}$ which are involved in the approximation of the eigenfunction at order zero are determined by means of the linear, homogeneous system of Eqs. (66). The condition for the system of equations to admit of non-trivial solutions leads to an equation for the first-order correction $\sigma_{n,1}$ to the eigenfrequency.

The system of Eqs. (66) contains coefficients $H_{n,m;n,m'}$, with $m, m' = -\ell, \ldots, \ell$. Detailed expressions for these coefficients are presented in Sect. 5.3. It results that coefficients different from zero are related to the coefficient $H_{n,0;n,0}$ in a simple way, so that from a computational point of view only the latter coefficient needs to be determined.

We have solved the system of Eqs. (66) for an arbitrary mode *n* of the lowest degrees $\ell = 0, 1, 2, 3$.

It follows that the eigenfrequency of any radial mode *n* remains unaffected by the tidal perturbation at the lowest order of approximation.

For the degrees $\ell = 1, 2, 3$, the degeneracy of the eigenvalue problem with respect to the azimuthal number is lift partially by the tidal perturbation, in the sense that an eigenfrequency of the unperturbed star is split into $\ell + 1$ eigenfrequencies, which are not equidistant. The partial removal of the degeneracy is due to the introduction of the preferential direction of the tidal axis into the equilibrium configuration. A main result of our investigation is that the eigenfrequencies of the various modes belonging to a given degree ℓ are split according to a common pattern with a scale depending only on the value the coefficient $H_{n,0;n,0}$ has for the mode *n* considered. These patterns are represented schematically in Figs. 1, 2, and 3 and are most conveniently described when the modes are considered to be associated with spherical harmonics of angular coordinates θ^* and ϕ^*, for which the polar axis coincides with the tidal axis of the stellar component. We have denoted the azimuthal number of the latter spherical harmonics by $\widetilde{m}$. With the azimuthal number $\widetilde{m} = 0$, the single eigenfrequency

$$\sigma_{n,0}\,(1 - \varepsilon_{\rm T}\,H_{n,0;n,0})$$

is associated, while with the absolute values of the azimuthal number different from zero, double eigenfrequencies are associated. As $|\widetilde{m}|$ increases, the perturbed eigenfrequencies increase or decrease according as the coefficient $H_{n,0;n,0}$ is positive or negative.

The first-order corrections to the eigenfrequency of a mode n appear to obey the rule

$$\sum_{j=1}^{\ell+1} \alpha_j \, \sigma_{n,1}^{(j)} = 0,$$

where α_j is the multiplicity of the correction $\sigma_{n,1}^{(j)}$.

We have the intention to verify the validity of the perturbation method in a subsequent paper by making comparisons with exact analytical solutions of the eigenfrequency equations which we derive for the second-harmonic oscillations of the compressible Jeans spheroids, and with the eigenfrequencies of a tidally perturbed polytropic model with index $n = 3$ as determined by Saio (1981).

Acknowledgements. The authors acknowledge the careful reading of the manuscript by the referee Dr. A. Claret and the referee's comments which allowed them to improve the presentation of the paper.

References

Brillouin, L. 1937, Comp. Rend., 204, 1863
Bryan, G. H. 1889, Acta Math., 7, 187
Chandrasekhar, S. 1933, MNRAS, 93, 462
Chandrasekhar, S. 1968, ApJ, 152, 293
Chandrasekhar, S. 1969, Ellipsoidal Figures of Equilibrium, The Silliman Foudation Lectures (New Haven: Yale University Press)
Chandrasekhar, S., & Lebovitz, N. R. 1962a, ApJ, 135, 248
Chandrasekhar, S., & Lebovitz, N. R. 1962b, ApJ, 136, 1069
Chandrasekhar, S., & Lebovitz, N. R. 1963, ApJ, 137, 1172
De Boeck, I. 1997, A&A, 317, 832
Denis, J. 1972, A&A, 20, 151
Jeans, J. 1917, MmRAS, 62, 1
Ledoux, P. 1951, ApJ, 114, 373
Lebovitz, N. R. 1961, ApJ, 134, 500
Reyniers, K. 2002, Doctoral Dissertation, Katholieke Universiteit Leuven
Saio, H. 1981, ApJ, 244, 299
Simon, R. 1969, A&A, 2, 390
Smeyers, P. 1986, A&A, 160, 385
Smeyers, P., & Denis, J. 1971, A&A, 14, 311
Smeyers, P., & Martens, L. 1983, A&A, 125, 193
Tassoul, M., & Tassoul, J. L. 1967, ApJ, 150, 1031