

Images for an isothermal ellipsoidal gravitational lens from a single real algebraic equation

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Abstract. We present explicit expressions for the lens equation for a cored isothermal ellipsoidal gravitational lens as a single real sixth-order algebraic equation in two approaches; 2-dimensional Cartesian coordinates and 3-dimensional polar ones. We find a condition for physical solutions which correspond to at most five images. For a singular isothermal ellipsoid, the sixth-order equation is reduced to fourth-order one for which analytic solutions are well-known. Furthermore, we derive analytic criteria for determining the number of images for the singular lens, which give us simple expressions for the caustics and critical curves. The present formulation offers a useful way for studying galaxy lenses frequently modeled as isothermal ellipsoids.

Key words. gravitational lensing – galaxies: general – cosmology: theory

1. Introduction

Gravitational lensing due to a galaxy is important for probing mass distributions and determining cosmological parameters. Galaxy lenses are often modeled as cored isothermal ellipsoids. Although the ellipsoidal model is quite simple, it enables us to understand a number of physical properties of the galactic lens. Furthermore, it fits well with mass profiles implied by observations (For instance, Binney & Tremaine 1987).

Until now, the lens equation for the cored isothermal ellipsoid has been solved numerically as a *nonlinearly coupled* system. For a binary gravitational lens, it has recently been shown that the lens equation is reduced to a single real fifth-order algebraic equation (Asada 2002). Also for a singular isothermal ellipsoidal lens, furthermore, the apparently coupled lens equations can be reduced to a single equation (Schneider et al. 1992), though explicit expressions were not given there. Along this course, we reexamine the coupled lens equations for the cored isothermal ellipsoid. The main purpose of the present paper is to show that they are reduced to a single equation with a condition for physical solutions and to give analytic criteria for determining the number of images for isothermal ellipsoidal lenses.

2. Lens equation for a singular isothermal ellipsoid

First, let us consider the singular isothermal ellipsoidal lens with ellipticity $0 \leq \epsilon < 1/5$. A condition that the surface mass

density projected onto the lens plane must be non-negative everywhere puts a constraint on the ellipticity as $\epsilon < 1$. A tighter constraint $\epsilon < 1/5$ comes from that the density contours must be convex, which is reasonable for an isolated relaxed system. The lens equation is expressed as

$$\beta_1 = \theta_1 - \frac{(1 - \epsilon)\theta_1}{\sqrt{(1 - \epsilon)\theta_1^2 + (1 + \epsilon)\theta_2^2}}, \quad (1)$$

$$\beta_2 = \theta_2 - \frac{(1 + \epsilon)\theta_2}{\sqrt{(1 - \epsilon)\theta_1^2 + (1 + \epsilon)\theta_2^2}}, \quad (2)$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2)$ denote the positions of the source and images, respectively.

For simplicity, we introduce variables as $x \equiv \sqrt{1 - \epsilon}\theta_1$, $y \equiv \sqrt{1 + \epsilon}\theta_2$, $a \equiv \sqrt{1 - \epsilon}\beta_1$ and $b \equiv \sqrt{1 + \epsilon}\beta_2$, so that the lens equation can be rewritten as

$$a = x \left(1 - \frac{1 - \epsilon}{\sqrt{x^2 + y^2}} \right), \quad (3)$$

$$b = y \left(1 - \frac{1 + \epsilon}{\sqrt{x^2 + y^2}} \right). \quad (4)$$

2.1. Sources on the symmetry axes

There are two symmetry axes in the ellipsoid, $a = 0$ and $b = 0$. We consider a source on the axis $a = 0$. In this case, we can find analytic solutions for the lens equation as follows. We can make a replacement as $a \leftrightarrow b$, $x \leftrightarrow y$ and $1 + \epsilon \leftrightarrow 1 - \epsilon$ to obtain solutions for $b = 0$.

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For $a = 0$, Eq. (3) becomes

$$x \left(1 - \frac{1 - \epsilon}{\sqrt{x^2 + y^2}} \right) = 0, \quad (5)$$

which means

$$x = 0,$$

or

$$\frac{1 - \epsilon}{\sqrt{x^2 + y^2}} = 1.$$

In the case of $x = 0$, Eq. (4) becomes

$$b - y = -(1 + \epsilon) \frac{y}{|y|}, \quad (6)$$

which is solved as

$$y = \begin{cases} b - (1 + \epsilon) & \text{for } b < -(1 + \epsilon), \\ b \pm (1 + \epsilon) & \text{for } -(1 + \epsilon) \leq b \leq 1 + \epsilon, \\ b + (1 + \epsilon) & \text{for } 1 + \epsilon < b. \end{cases} \quad (7)$$

Here, we should pay attention to the cases of $b = \pm(1 + \epsilon)$, since they produce a solution at the singularity $(x, y) = (0, 0)$ of the potential.

Next, we consider the case of Eq. (7). Let us investigate the two cases, $\epsilon = 0$ and $\epsilon \neq 0$ separately. For $\epsilon = 0$, Eq. (4) means $b = 0$, so that images become a ring as

$$x^2 + y^2 = 1. \quad (8)$$

Below, we assume $\epsilon \neq 0$. Eliminating $\sqrt{x^2 + y^2}$ from Eqs. (4) and (7), we obtain

$$y = -\frac{(1 - \epsilon)b}{2\epsilon}. \quad (9)$$

Substituting this into y in Eq. (7), we obtain

$$x^2 = \left(\frac{1 - \epsilon}{2\epsilon} \right)^2 (4\epsilon^2 - b^2), \quad (10)$$

which has real solutions if and only if $|b| \leq 2\epsilon$,

$$x = \pm \frac{1 - \epsilon}{2\epsilon} \sqrt{4\epsilon^2 - b^2}. \quad (11)$$

Consequently, Eqs. (9) and (13) show that four, two or one images occur for $|b| < 2\epsilon$, $2\epsilon < |b| < 1 + \epsilon$ or $1 + \epsilon < |b|$, respectively.

In the similar manner, we obtain the solutions for $b = 0$. A point is that $2\epsilon < |a| < 1 - \epsilon$ can hold only for $\epsilon < 1/3$.

2.2. Off-axis sources

Here, we consider off-axis sources ($a \neq 0$ and $b \neq 0$). In this case, Eqs. (3) and (4) show $x \neq 0$ and $y \neq 0$. Eliminating $\sqrt{x^2 + y^2}$ from Eqs. (3) and (4), we obtain

$$y = \frac{(1 - \epsilon)bx}{(1 + \epsilon)a - 2\epsilon x}, \quad (12)$$

which determines y uniquely for any given x . For finite b , Eq. (4) means that y is also finite, so that $x \neq (1 + \epsilon)a/2\epsilon$

from Eq. (14). Substituting Eq. (14) into Eq. (3), we obtain the fourth-order polynomial for x as

$$D(x) \equiv [(a - x)^2 - (1 - \epsilon)^2] [(1 + \epsilon)a - 2\epsilon x]^2 + (1 - \epsilon)^2 b^2 (a - x)^2 = 0, \quad (13)$$

where we used $x \neq 0$.

The number of real roots for a fourth-order equation is discussed by the discriminant D_4 (e.g. van der Waerden 1966), which becomes for Eq. (15)

$$D_4 = -64a^2b^2\epsilon^2(1 - \epsilon)^{12} \times \left[(a^2 + b^2 - 4\epsilon^2)^3 + 108a^2b^2\epsilon^2 \right]. \quad (14)$$

Namely, if

$$\left(\frac{a^2}{4\epsilon^2} + \frac{b^2}{4\epsilon^2} - 1 \right)^3 + 27 \left(\frac{a^2}{4\epsilon^2} \right) \left(\frac{b^2}{4\epsilon^2} \right) < 0, \quad (15)$$

the number of real roots is either four or zero, which is determined as four by explicit solutions for on-axis sources. Otherwise, it is two. However, the number does not necessarily indicate that of images as shown below.

Since $x \neq 0$ for off-axis sources, Eq. (3) is rewritten as

$$\frac{a - x}{x} = -\frac{1 - \epsilon}{\sqrt{x^2 + y^2}}, \quad (16)$$

whose right-hand side is necessarily negative since $1 - \epsilon > 0$. As a result, we find that any solution of the lens equation must satisfy

$$\frac{a - x}{x} < 0. \quad (17)$$

This implies that $x < 0$ or $a < x$ for positive a , while $x < a$ or $0 < x$ for negative a . It should be noted that Eq. (19) always holds in the limit of $a \rightarrow 0$.

Let us investigate the number of roots in $(a - x)/x > 0$, namely an interval between 0 and a . We find out

$$D(a) = -a^2(1 - \epsilon)^4 < 0, \quad (18)$$

$$D(0) = a^2(1 - \epsilon^2)^2 \left(\frac{a^2}{(1 - \epsilon)^2} + \frac{b^2}{(1 + \epsilon)^2} - 1 \right). \quad (19)$$

If a and b satisfy

$$\frac{a^2}{(1 - \epsilon)^2} + \frac{b^2}{(1 + \epsilon)^2} \geq 1, \quad (20)$$

$D(0)$ is not negative, so that $D(x) = 0$ has at least one root between 0 and a . For $\epsilon < 1/3$, Eq. (22) implies $a^2/4\epsilon^2 + b^2/4\epsilon^2 > 1$, so that the left-hand side of Eq. (17) becomes positive. Hence, D_4 is negative, so that $D(x) = 0$ has two roots. As a result, it has only one root in the interval, which means only one image appears. Unless Eq. (22) holds, $D(0)$ is negative, so that the number of roots for $D(x) = 0$ for $(a - x)/x < 0$, namely that of images are four or two, respectively for $D_4 > 0$ or < 0 .

The inner and outer caustics (Fig. 1) are given by

$$\left(\frac{a^2}{4\epsilon^2} + \frac{b^2}{4\epsilon^2} - 1 \right)^3 + 27 \left(\frac{a^2}{4\epsilon^2} \right) \left(\frac{b^2}{4\epsilon^2} \right) = 0, \quad (21)$$

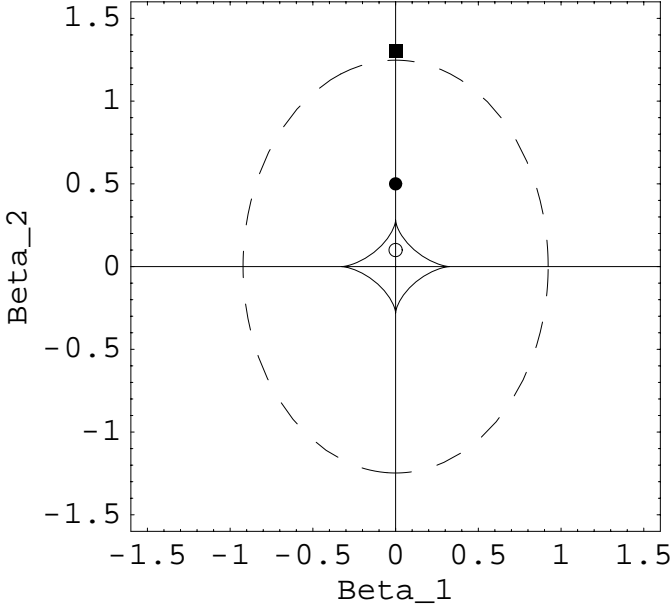


Fig. 1. Caustics for a singular isothermal ellipsoidal lens in the physical coordinates (β_1, β_2) . For $\epsilon = 0.15$, the solid curve denotes the inner caustic, and the dashed one for the outer caustic. Sources locate at $(0, 0.1)$, $(0, 0.5)$ and $(0, 1.3)$, denoted by the circle, filled disk and square, respectively.

$$\frac{a^2}{(1-\epsilon)^2} + \frac{b^2}{(1+\epsilon)^2} = 1. \quad (24)$$

The inner caustic given by Eq. (23) is an asteroid which is parametrized as

$$a = 2\epsilon \cos^3 t, \quad (25)$$

$$b = 2\epsilon \sin^3 t, \quad (26)$$

where $t \in [0, 2\pi)$.

The critical curves on the lens plane correspond to the caustics on the source plane (Schneider et al. 1992). We introduce the polar coordinates as $(x, y) \equiv (\rho \cos \xi, \rho \sin \xi)$. By substituting Eqs. (25) and (26) with $t = -\xi$ into the lens Eqs. (3) and (4), we obtain the parametric representation of the critical curve as

$$\rho = 1 + \epsilon \cos 2\xi. \quad (27)$$

In the similar manner to the outer caustic given by Eq. (24), we find

$$\rho = 0, \quad (28)$$

which is the origin in (x, y) (Fig. 2).

3. Lens equation for a cored isothermal ellipsoid

Let us consider a cored isothermal ellipsoidal lens with the angular core radius c . The lens equation is expressed as

$$a = x \left(1 - \frac{1-\epsilon}{\sqrt{x^2 + y^2 + c^2}} \right), \quad (29)$$

$$b = y \left(1 - \frac{1+\epsilon}{\sqrt{x^2 + y^2 + c^2}} \right), \quad (30)$$

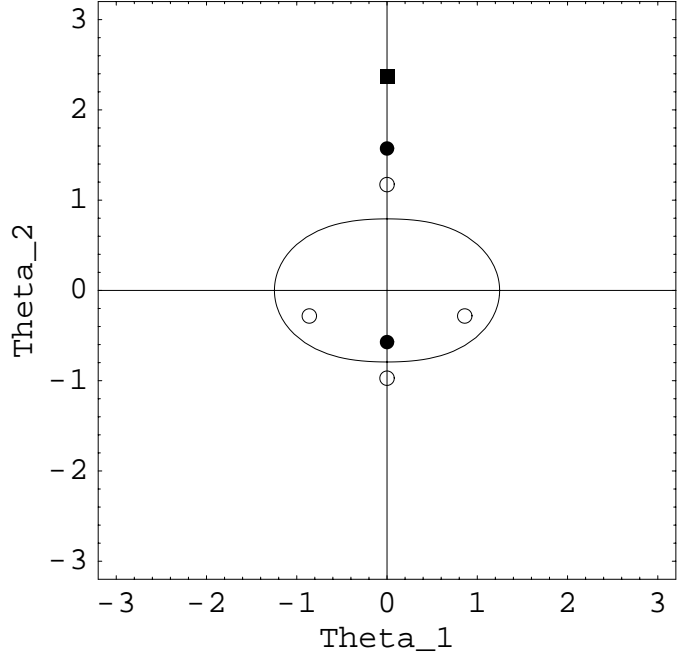


Fig. 2. Critical curves for a singular isothermal ellipsoidal lens in the physical coordinates (θ_1, θ_2) . For $\epsilon = 0.15$, the solid curve denotes the outer critical curve which corresponds to the inner caustic. The origin corresponds to the outer caustic. The images correspond to the sources in Fig. 1.

which are apparently similar to the set of Eqs. (3) and (4). However, there are differences in their algebraic properties as shown below.

3.1. Sources on the symmetry axes

We consider a source on the axis $a = 0$. In this case, we can find analytic solutions for the lens equation as follows. To consider the case of $b = 0$, it is enough to make a replacement as $a \leftrightarrow b$, $x \leftrightarrow y$ and $1 + \epsilon \leftrightarrow 1 - \epsilon$.

For $a = 0$, Eq. (29) becomes

$$x \left(1 - \frac{1-\epsilon}{\sqrt{x^2 + y^2 + c^2}} \right) = 0, \quad (31)$$

which means

$$x = 0, \quad (32)$$

or

$$\frac{1-\epsilon}{\sqrt{x^2 + y^2 + c^2}} = 1. \quad (33)$$

In the case of $x = 0$, Eq. (30) becomes the fourth-order polynomial for y as

$$E(y) \equiv y^4 - 2by^3 + [b^2 + c^2 - (1+\epsilon)^2]y^2 - 2bc^2y + b^2c^2 = 0. \quad (34)$$

Explicit solutions for a fourth-order equation take a lengthy form (e.g. van der Waerden 1966). Now, Eq. (30) implies

$$\frac{b-y}{y} < 0. \quad (35)$$

Using

$$E(0) = b^2c^2 > 0, \tag{36}$$

$$E(b) = -b^2(1 + \epsilon)^2 < 0, \tag{37}$$

we find that $E(y)$ has at least one zero point between 0 and b . In other words, $E(y) = 0$ has at most three roots for $(b - y)/y < 0$.

Next, we consider the case of Eq. (33). For $\epsilon = 0$, Eq. (30) means $b = 0$, so that a ring image appears at

$$x^2 + y^2 + c^2 = 1. \tag{38}$$

We assume $\epsilon \neq 0$ in the following. Eliminating $\sqrt{x^2 + y^2 + c^2}$ from Eqs. (30) and (33), we obtain

$$y = -\frac{(1 - \epsilon)b}{2\epsilon}. \tag{39}$$

Substituting this into y in Eq. (33), we obtain

$$x^2 = \left(\frac{1 - \epsilon}{2\epsilon}\right)^2 (4\epsilon^2 - b^2) - c^2, \tag{40}$$

which has the real solutions

$$x = \pm \frac{1 - \epsilon}{2\epsilon} \sqrt{4\epsilon^2 \left[1 - \left(\frac{c}{1 - \epsilon}\right)^2\right] - b^2}, \tag{41}$$

if and only if

$$b^2 \leq 4\epsilon^2 \left[1 - \left(\frac{c}{1 - \epsilon}\right)^2\right]. \tag{42}$$

3.2. Off-axis sources

Here, we consider off-axis sources ($a \neq 0$ and $b \neq 0$). In this case, Eqs. (29) and (30) imply $x \neq 0$ and $y \neq 0$. Eliminating $\sqrt{x^2 + y^2 + c^2}$ from Eqs. (29) and (30), we obtain

$$y = \frac{(1 - \epsilon)bx}{(1 + \epsilon)a - 2\epsilon x}. \tag{43}$$

Equation (30) shows that y is finite for finite b , so that $x \neq (1 + \epsilon)a/2\epsilon$ from Eq. (43). Substituting Eq. (43) into Eq. (29), we obtain the sixth-order polynomial for x as

$$\begin{aligned} F(x) &\equiv (a - x)^2 \\ &\quad \times \left[(x^2 + c^2) [(1 + \epsilon)a - 2\epsilon x]^2 + (1 - \epsilon)^2 b^2 x^2 \right] \\ &\quad - (1 - \epsilon)^2 x^2 [(1 + \epsilon)a - 2\epsilon x]^2 \\ &= 0. \end{aligned} \tag{44}$$

This equation has at most six real solutions whose analytic expressions can not be given by algebraic manners (e.g. van der Waerden 1966). As shown below, however, six real roots *never* mean six images.

In the same manner as for the singular isothermal ellipsoid, we obtain a condition for x as

$$\frac{a - x}{x} < 0. \tag{45}$$

Let us prove that there exists a root between 0 and a . We can find out

$$F(0) = a^4 c^2 (1 + \epsilon)^2 > 0, \tag{46}$$

$$F(a) = -a^4 (1 - \epsilon)^4 < 0. \tag{47}$$

Since $F(x)$ is a continuous function, $F(x) = 0$ has at least one root in the interval. For $(a - x)/x < 0$, consequently, $F(x) = 0$ has at most five solutions. Since the polynomial is sixth-order, the discriminant is not sufficient to determine the exact number of roots. Hence, the determination is beyond the scope of our paper.

3.3. Polar coordinates

Up to this point, we have used 2-dimensional Cartesian coordinates: We must solve Eq. (44) and choose appropriate roots which satisfy the inequality by Eq. (45). Here, we adopt 3-dimensional polar coordinates to simplify the inequality, as shown below. By taking c as a fictitious third dimension, we define

$$r = \sqrt{x^2 + y^2 + c^2} \geq c, \tag{48}$$

$$\cos \Psi = \frac{c}{r}, \tag{49}$$

$$x = r \sin \Psi \cos \phi, \tag{50}$$

$$y = r \sin \Psi \sin \phi, \tag{51}$$

where we can assume $\sin \Psi \geq 0$. Then, Eqs. (29) and (30) are rewritten as

$$a = r \left(1 - \frac{1 - \epsilon}{r}\right) \sin \Psi \cos \phi, \tag{52}$$

$$b = r \left(1 - \frac{1 + \epsilon}{r}\right) \sin \Psi \sin \phi. \tag{53}$$

We concentrate on off-axis sources ($a \neq 0$ and $b \neq 0$).

Eliminating $\sin \Psi$ from Eqs. (52) and (53), we obtain

$$\tan \phi = \frac{b[r - (1 - \epsilon)]}{a[r - (1 + \epsilon)]}, \tag{54}$$

which determines $\tan \phi$ uniquely for any given r . Equations (52) and (53) show that $r \neq 1 \pm \epsilon$ for nonvanishing a and b , since $x \neq 0$ and $y \neq 0$ mean $\cos \phi \neq 0$ and $\sin \phi \neq 0$. Hence, Eqs. (52) and (53) can be rewritten as

$$x = \frac{ar}{r - (1 - \epsilon)}, \tag{55}$$

$$y = \frac{br}{r - (1 + \epsilon)}. \tag{56}$$

Substituting these into $r^2 = x^2 + y^2 + c^2$, we obtain the sixth-order equation for r as

$$\begin{aligned} G(r) &\equiv (r^2 - c^2) [r - (1 - \epsilon)]^2 [r - (1 + \epsilon)]^2 \\ &\quad - a^2 r^2 [r - (1 + \epsilon)]^2 - b^2 r^2 [r - (1 + \epsilon)]^2 \\ &= 0, \end{aligned} \tag{57}$$

which has at most six real solutions. Let us show that there are at most five roots compatible with $r \geq c$. Using $G(c) = -a^2 c^2 [c - (1 + \epsilon)]^2 - b^2 c^2 [c - (1 - \epsilon)]^2 < 0$ and $G(-\infty) = +\infty > 0$ for a continuous function $G(r)$, we find that $G(r) = 0$ has at least one root for $r < c$. Consequently, it has at most five roots for $r \geq c$.

We should note that r and $\tan \phi$ are not enough to determine the location of images. A strategy for determining the location is as follows: First, we solve Eq. (57) for $r \geq c$. Next, we substitute r into Eqs. (55) and (56) to obtain the image position as (x, y) .

4. Conclusion

We have carefully reexamined the lens equation for a cored isothermal ellipsoid both in 2-dimensional Cartesian and 3-dimensional polar coordinates. We have shown that the nonlinearly coupled equations are reduced to a single real sixth-order polynomial Eq. (44) or (57), which coincides with the fourth-order equation for a singular isothermal ellipsoid as the core radius approaches zero. For the singular case, explicit expressions of image positions for sources on the symmetry axis are given by Eqs. (6), (9), (11) and (13). Furthermore, we have presented analytic criteria for determining the number of images, which correspond to the caustics given by Eqs. (23) and (24). Consequently, analytic expressions for the critical curves are given by Eqs. (27) and (28). We have shown for the cored case that a condition Eq. (45) or (48) gives us physical solutions of the sixth-order polynomial, which are at most five images.

The present formulation based on the one-dimensional Eq. (44) or (57) enables us to study a cored isothermal ellipsoidal lens with considerable efficiency and accuracy, in comparison with previous two-dimensional treatments for which

there are no well-established numerical methods (Press et al. 1988). Particularly for a source close to the caustics, the image position is unstable so that careful computations are needed. The amount of computations can be reduced by our approach. As a result, it must be powerful in rapid and accurate parameter fittings to observational data.

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