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A more powerful evolution model for rotating stars

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Abstract. A new one-dimensional rotating stellar evolution model was developed which, distinct from its previous counterparts based on a conservative rotation case or a "shellular rotation" case, made no special assumptions on the distribution of angular velocity inside a rotating star. The results of the evolutionary calculations are presented and compared with the results of previous studies.

Key words. stars: evolution – stars: rotation – stars: Hertzsprung-Russell (HR) and C-M diagrams

1. Introduction

The study of rotating stars has attracted much attention during the last decades as a consequence of the observational findings such as the He- enrichments in the fast rotators among O-stars (Herrero et al. 1992), the N/C and ^{13} C/^{12}C enrichments on the Red Giant Branch (see e.g. Charbonnel 1994, 1995), the He- and N-enrichments in OBA supergiants and in SN 1987A (Walborn 1976; Fransson et al. 1989; Venn 1995A,b; Venn et al. 1996) and the problem of the blue to red supergiants ratio in galaxies (Langer & Maeder 1995) etc.

In a rotating star, the centrifugal forces reduce the effective gravity according to the latitude and introduce deviations from sphericity. Thus, the structure equations of a rotating star should be two dimensional.The original method devised by Kippenhahn, Meyer-Hofmeister & Thomas (1970) (subsequently referred to as KMT) and applied in most subsequent works (Meyer-Hofmeister 1972; Endal & Sofia 1976 (hereafter ES), 1978; Pinsonneault et al. 1989, 1990; Chaboyer et al. 1995a,b; Sills et al. 2000; Heger et al. 2000) to keep the problem one dimensional is based on a so-called conservative rotation case, i.e., the angular velocity is constant on cylinders. In such a rotation case, the total potential is conservative and the structural variables P, T, ρ, ... are constant on a equipotential surface. Meynet & Maeder (1997) (subsequently referred to as MM) argue that not a conservative rotation case but a "shellular rotation" case, i.e., the angular velocity is constant on an isobar (an isobar is a constant pressure surface) is the real rotation case in an evolved star. In a "shellular rotation" case, they find a

$$\overline{\rho} = \frac{\rho\left(1 - r^2 \sin^2\theta\omega\alpha\right)\left\langle g^{-1}\right\rangle}{\langle g^{-1}\rangle - \left\langle g^{-1} r^2 \sin^2\theta\right\rangle \omega\alpha} \qquad (1)$$

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where $\alpha = \frac{d\omega}{d\psi}$, instead of ρ, is constant on an isobar and it can denote the property of ρ on the whole isobar. Furthermore, the temperature T and the chemical composition μ are supposed to be constant on an isobar. Thus, if the structural equations are written on isobars, the problem can be kept one dimensional.

The distribution of angular velocity inside a real star is still a mystery, therefore, it is important to develop an one dimensional method which is not based on any special assumptions on the distribution of angular velocity. I propose a method based on Meynet & Maeder (1997) to deal with this problem.

The new structural model is introduced in Sect. 2, and the results of evolution calculations are presented in Sect. 3.

2. A new rotating stellar evolution model

In a Roche model, a constant total potential surface is given by the equation:

$$\Psi = \Phi + \frac{1}{2}\omega^2 r^2 \sin^2\theta = \text{constant} \qquad (2)$$

where $\Phi = -V$, and V is the gravitational potential, r is the radius, ω the angular velocity and θ the colatitude.

The constant Ψ_P surfaces, the constant ω surfaces, the constant ρ surfaces, the constant T surfaces, and the isobars do not correspond to each other if no special assumption about the distribution of angular velocity is made. However, some parameters f_Ψ, f_ω, f_d and f_T can always be found to make $f_\Psi\Psi$, $f_\omega\omega$, $f_d\rho$ and $f_T T$ constant on an isobar, furthermore, if ρ_P and T_P denote the value of density and temperature of a point P (r_P θ_P) on an isobar, $\langle f_d\rangle\rho_P$ and $\langle f_T\rangle T_P$ denote the properties of density and temperature on the whole isobar, different values of $\langle f_d\rangle$ and $\langle f_T\rangle$ will be given for different points P. Thus, the problem can be kept one dimensional, if the structural equations are written on isobars.

The four parameters f_Ψ, f_ω, f_d and f_T, indicating the relationship between the constant Ψ_P, ω, ρ, T surfaces and the isobars, accordingly, are not independent of one another. Once f_ω,

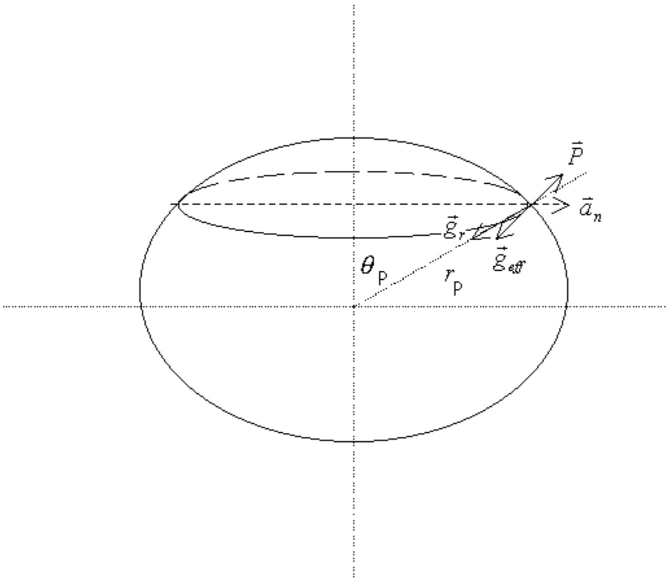

Fig. 1. The isobaric surface.

the distribution of angular velocity, is decided, the other three parameters are decided, too. Consequently, there is only one independent parameter need to be determined by advanced theories or by comparing the results of evolution model with the observational data. The relationships of f_Ψ and f_d with f_ω will be shown in the Appendix and how these parameters are decided in the "shellular rotation" case will be shown in Sect. 3.

An isobar in a rotating star is assumed to be an ellipsoid of resolution with the major axis a and the minor axis b, and the volume inside an isobar is given by

$$V_P = \frac{4\pi}{3}a^2 b. \tag{3}$$

For every isobar, an assumed equivalent sphere with the radius r_P can be defined by:

$$r_p = \left(\frac{3V_p}{4\pi}\right)^{\frac{1}{3}}. \tag{4}$$

The structural equations are written at the point P (r_P θ_P) where an isobar intersects its equivalent sphere (see Fig. 1).

2.1. The equations for the stellar interior

(1) Conservation of mass

As from Eq. (4), the mass between the two isobaric surfaces is given by

$$\mathrm{d}M_P = \langle f_d \rangle \rho_p \mathrm{d}V_p = 4\pi r_P^2 \langle f_d \rangle \rho_P \mathrm{d}r_P \tag{5}$$

which gives the equation of conservation of mass as

$$\frac{\mathrm{d}r_P}{\mathrm{d}M_P} = \frac{1}{4\pi r_P^2 \langle f_d \rangle \rho_P}. \tag{6}$$

(2) The hydrostatic equilibrium

The hydrostatic equilibrium implies that

$$\vec{\nabla} P = -\langle f_d \rangle \rho_P \overrightarrow{g_{\mathrm{eff}}} \tag{7}$$

where $\overrightarrow{g_{\mathrm{eff}}}$ is the effective gravitational acceleration. In the spherical coordinates, its components are given by

$$g_{\mathrm{eff,r}} = -g_r + a_n \sin\theta_P \tag{8}$$

$$g_{\mathrm{eff},\theta} = a_n \cos\theta_P \tag{9}$$

where $\overrightarrow{a_n}$ is the centrifugal acceleration, $\overrightarrow{g_r}$ the gravitational acceleration. The values of $\overrightarrow{a_n}$ and $\overrightarrow{g_r}$ are given by

$$a_n = \omega^2 r_P \sin\theta_P \tag{10}$$

$$g_r = \frac{GM_P}{r_P^2}. \tag{11}$$

The radial component of the vectorial Eq. (7) is

$$\frac{\mathrm{d}P}{\mathrm{d}r_P} = -\langle f_d \rangle \rho_P g_{\mathrm{eff}} \cos\alpha \tag{12}$$

where

$$\cos\alpha = \frac{g_r^2 + g_{\mathrm{eff}}^2 - a_n^2}{2 g_r g_{\mathrm{eff}}}. \tag{13}$$

From Eqs. (11), (12) and (6), the equation of hydrostatic equilibrium is obtained as

$$\frac{\mathrm{d}P}{\mathrm{d}M_P} = -\frac{GM_P}{4\pi r_P^4} f_P \tag{14}$$

where

$$f_P = \frac{1}{2}\frac{g_r^2 + g_{\mathrm{eff}}^2 - a_n^2}{g_r^2} = \frac{1}{2}(1+\xi) \tag{15}$$

and

$$\xi = \frac{g_{\mathrm{eff}}^2 - a_n^2}{g_r^2}. \tag{16}$$

(3) Conservation of the energy

The net energy outflow from a shell between the two isobaric surfaces P and $P + \mathrm{d}P$ is equal to

$$\mathrm{d}L_P = \varepsilon \mathrm{d}M_P \tag{17}$$

where ε is the rate of energy production in the shell. Writing ε in its nuclear, internal and neutrino components, one obtains the equation for conservation of the energy as

$$\frac{\mathrm{d}L_P}{\mathrm{d}M_P} = \varepsilon_n + \varepsilon_g - \varepsilon_\nu. \tag{18}$$

(4) Energy transport in the region of radiative equilibrium

The temperature gradient in the region of radiative equilibrium is defined as

$$\nabla_R = \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_R = \frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P}. \tag{19}$$

Using (12) and (19), one obtains

$$\frac{\mathrm{dT}}{\mathrm{d}n} = -\frac{\rho T g_{\mathrm{eff}}}{P}\nabla_R \tag{20}$$

where n is the unit normal of the isobar at point P. The radiative energy flux at the point P on the isobaric surface is given by

$$\pi F = -\frac{4acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}n}\cdot \tag{21}$$

Hence, one obtains the luminosity as

$$L_\mathrm{P} = -S_\mathrm{P}\frac{4acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}n} \tag{22}$$

where S_P is the area of the isobaric surface. Comparing (22) with (20), one obtains

$$\nabla_\mathrm{R} = \frac{3\kappa L_\mathrm{P} P}{4acT^4 S_\mathrm{P} g_\mathrm{eff}} = \frac{3\kappa L_\mathrm{P} P}{16\pi ac(\langle f_\mathrm{T}\rangle T_\mathrm{P})^4 G M_\mathrm{P}} f_\mathrm{R} \tag{23}$$

where

$$f_\mathrm{R} = \frac{g_\mathrm{r}}{g_\mathrm{eff}}\frac{4\pi r_\mathrm{p}^2}{S_\mathrm{P}}\cdot \tag{24}$$

Using (14) and (19), one obtains the equation of energy transport in the region of radiative equilibrium as

$$\frac{\mathrm{d}(\langle f_\mathrm{T}\rangle T_\mathrm{P})}{\mathrm{d}M_\mathrm{P}} = -\frac{G M_\mathrm{P}(\langle f_\mathrm{T}\rangle T_\mathrm{P}) f_\mathrm{P}}{4\pi r^4 \mathrm{P}}\nabla_\mathrm{R}. \tag{25}$$

(5) Energy transport in the convective region

The equation of energy transported in the convective region of stellar interior can be written as

$$\frac{\mathrm{d}(\langle f_\mathrm{T}\rangle T_\mathrm{P})}{\mathrm{d}M_\mathrm{P}} = -\frac{G M_\mathrm{P}(\langle f_\mathrm{T}\rangle T_\mathrm{P}) f_\mathrm{P}}{4\pi r^4 P}\nabla_\mathrm{ad} \tag{26}$$

where

$$\nabla_\mathrm{ad} = \frac{\delta P}{C_\mathrm{P}(\langle f_\mathrm{T}\rangle T_\mathrm{P})\rho_\mathrm{P}}\cdot \tag{27}$$

2.2. Calculation of the quantities f_P and f_R

In the practical calculations, the non-spherical stratification of rotating stars can be replaced by the spherical stratification of equivalent sphere. Thus, the calculations of the structure Eqs. (6), (14), (18), (25) and (26) are based on the equivalent sphere. However, the two factors f_P and f_R, defined by (15) and (24), which are related to the characteristic of rotation, should be calculated on the isobar.

When the radius of the equivalent sphere r_P and the angular velocity ω are given,the volume V_P inside the isobaric surface can be written from (4) as

$$V_\mathrm{P} = \frac{4\pi}{3} r_\mathrm{P}^3. \tag{28}$$

From Eq. (2), one obtains

$$f_{\Psi a}\left[\frac{GM_\mathrm{p}}{a} + \frac{1}{2}(f_{\omega a}\omega)^2 a^2\right] = f_{\Psi b}\left(\frac{GM_\mathrm{p}}{b}\right) \tag{29}$$

$f_{\omega a}$ can be supposed to be 1.

Then, the major and minor axis of the isobar can be obtained as

$$a = \frac{b}{f-\eta} \tag{30}$$

$$b = r_\mathrm{p}(f-\eta)^{\frac{2}{3}} \tag{31}$$

$$f = \frac{f_{\Psi b}}{f_{\Psi a}} \tag{32}$$

where

$$\eta = \frac{1}{2}\frac{\omega^2 r_\mathrm{p}^3}{G M_\mathrm{p}}\cdot \tag{33}$$

Using the value b and the relation

$$f_\Psi\left[\frac{GM_\mathrm{p}}{r_\mathrm{p}} + \frac{1}{2}(f_\omega\omega)^2 r_\mathrm{p}^2\sin^2\theta_\mathrm{p}\right] = f_{\Psi b}\left(\frac{GM_\mathrm{p}}{b}\right) \tag{34}$$

the value of the colatitude θ_p can be obtained.

The factor f_p can be calculated by using (15) when the values of ω, r_p and θ_p are known.

The factor f_R can be calculated by using (24) when the surface area S_P of the isobar is given by

$$S_\mathrm{p} = \frac{4\pi}{3}(2a^2 + b^2). \tag{35}$$

3. Evolutionary calculations

The calculation of the case without rotation is performed by using a modified version of the stellar structure and evolution program developed by Kippenhahn et al. (1967). This version has included the recent opacities (cf. Iglesias & Rogers 1996, complemented at lower temperature with the low-temperature rosseland opacities by Alexander & Ferguson 1994) and energy generating rates (cf. Maeder 1983; Maeder & Meynet 1987, 1989). The case with rotation is calculated using the rotating evolution code which is modified from the previous nonrotating program, and have included the equations represented in Sect. 2.

To compare the new results with previous ones, the "shellular rotation" case (compared with MM) has been calculated.

MM assumed that the angular velocity ω, the pressure P and the temperature T and a $\overline{\rho}$ are constant on isobars. Therefore, in the new method,

$$f = f_\omega = \langle f_\mathrm{T}\rangle = 1 \tag{36}$$

$$\langle f_\mathrm{d}\rangle = \frac{\rho\left(1 - r^2\sin^2\theta\omega\alpha\right)\left\langle g^{-1}\right\rangle}{\rho_\mathrm{P}(\langle g^{-1}\rangle - \left\langle g^{-1}r^2\sin^2\theta\right\rangle\omega\alpha)} \tag{37}$$

where $\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}\psi}$.

The comparison can only be made on the ZAMS because in the new method, the mixing of chemical species and angular momentum by circulation currents and turbulence has not been included while it has been considered in MM's method, and solid body rotation is assumed on the ZAMS, therefore:

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}\psi} = 1 \tag{38}$$

$$\langle f_\mathrm{d}\rangle = 1. \tag{39}$$

Some characteristics of the models being calculated ($X = 0.7$, $Z = 0.02$.) are given in Table 1 as a function of the initial mass and angular velocity, and Fig. 2 presents the ZAMS obtained in the theoretical HR diagram. These results do not differ from MM's very much.

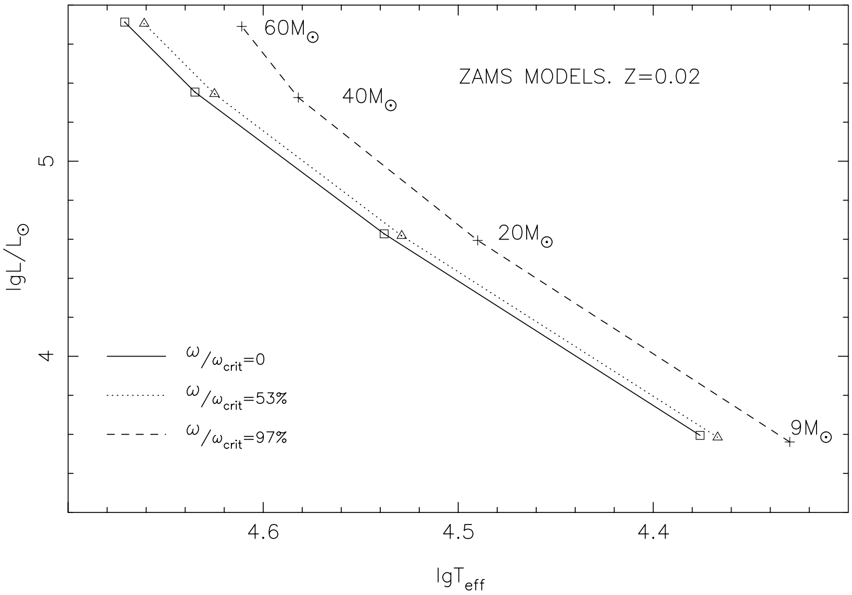

Fig. 2. Upper ZAMS in the theoretical HR diagram for different rotation rates.

Table 1. Effect of rotation on the ZAMS.

Initial mass	ω_0	$\omega_0/\ \omega_{\rm crit}$ [10^{-4} s^{-1}]	$V_{\rm eq}$ [km s^{-1}]	$\log L/L_\odot$	$\log T_{\rm eff}$	$\frac{R_{\rm P}(\omega)}{R_{\rm e}(\omega)}$	$\frac{R_{\rm e}(\omega)}{R_{\rm e}(\omega=0)}$	$E_{\rm h}$ [10^{37} erg]
	9 $M_\odot$	$\omega_{\rm crit} = 0.1581 \times 10^{-3}$ [s^{-1}]						
9 $M_\odot$	0.00	0.00	0.000	3.595	4.376	1.0000	1.000	0.7743
9 $M_\odot$	0.84	53.13%	227.6	3.585	4.367	0.9626	1.029	0.7563
9 $M_\odot$	1.54	97.41%	529.2	3.561	4.330	0.7978	1.189	0.7154
	20 $M_\odot$	$\omega_{\rm crit} = 0.1194 \times 10^{-3}$ [s^{-1}]						
20 $M_\odot$	0.00	0.00	0.000	4.628	4.538	1.0000	1.000	3.763
20 $M_\odot$	0.64	53.60%	270.9	4.618	4.529	0.9628	1.029	3.673
20 $M_\odot$	1.54	97.57%	629.3	4.594	4.490	0.7977	1.196	3.476
	40 $M_\odot$	$\omega_{\rm crit} = 0.8920 \times 10^{-4}$ [s^{-1}]						
40 $M_\odot$	0.00	0.00	0.000	5.354	4.635	1.0000	1.000	10.01
40 $M_\odot$	0.495	55.49%	309.9	5.345	4.625	0.9639	1.032	9.814
40 $M_\odot$	0.88	98.65%	722.6	5.326	4.582	0.7974	1.232	9.382
	60 $M_\odot$	$\omega_{\rm crit} = 0.7050 \times 10^{-4}$ [s^{-1}]						
60 $M_\odot$	0.00	0.00	0.000	5.713	4.671	1.0000	1.000	15.25
60 $M_\odot$	0.413	58.58%	331.4	5.705	4.661	0.9647	1.038	14.99
60 $M_\odot$	0.701	99.43%	765.7	5.691	4.611	0.7981	1.285	14.50

* Symbols: ω_0 and $\omega_{\rm crit}$ denote the initial angular velocity and the breaking equatorial angular velocity at the surface on the ZAMS, respectively; $V_{\rm eq}$ denotes the equatorial velocity at the surface; $\frac{R_{\rm P}(\omega)}{R_{\rm e}(\omega)}$ and $\frac{R_{\rm e}(\omega)}{R_{\rm e}(\omega=0)}$ denote the oblateness (polar radius over the equatorial radius) and the ratios between the equatorial radius obtained with rotation and that obtained without rotation, respectively; and $E_{\rm h}$ denote the energy production of core H-burning.

4. Conclusion

A new one-dimensional rotating stellar evolution model was developed. Distinct from its previous counterparts based on a conservative rotation case or a "shellular rotation" case, it made no special assumptions on the distribution of angular velocity inside a rotating star.

One independent parameter f_ω has been introduced in the new method; to determine its value is another complicated task. This and other improvements, such as including the mixing of chemical species and angular momentum by circulation currents and turbulent, will be presented in the future.

Acknowledgements. I wish to express thanks to R.Q. Huang for his most fruitful advice.

Appendix A:

Following MM,

$$\begin{aligned} \vec{\nabla} P &= -\rho \vec{g}_{\rm eff} \\ &= -\rho\left(\vec{\nabla}\Psi_{\rm P} - r^2 \sin^2\theta\omega\vec{\nabla}\omega\right) \end{aligned} \quad \text{(A1)}$$

and under the condition of no special assumption about ω , one can get:

$$\begin{aligned}\vec{\nabla} P &= -\rho \vec{g}_{\rm eff} \\ &= -\rho\left[\frac{\vec{\nabla}\Psi_{\rm P}}{\vec{\nabla}(f_\Psi\Psi_{\rm P})}\vec{\nabla}(f_\Psi\Psi_{\rm P}) - r^2\sin^2\theta\omega\frac{\vec{\nabla}\omega}{\vec{\nabla}(f_\omega\omega)}\vec{\nabla}(f_\omega\omega)\right] \\ &= -\rho[\frac{1}{\langle f_\Psi\rangle}\vec{\nabla}(f_\Psi\Psi_{\rm P}) - r^2\sin^2\theta\omega\frac{1}{\langle f_\omega\rangle}\vec{\nabla}(f_\omega\omega)] \qquad \text{(A2)}\end{aligned}$$

$\vec{\nabla}(f_\Psi\Psi_{\rm P})$ is parallel to $\vec{\nabla}(f_\omega\omega)$, thus, the effective gravity can be expressed as

$$g = \left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]\frac{{\rm d}(f_\Psi\Psi_{\rm P})}{{\rm d}n} \qquad \text{(A3)}$$

where: $\alpha = \frac{{\rm d}(f_\omega\omega)}{{\rm d}(f_\Psi\Psi_{\rm P})}$, or

$$g = \left[\frac{\beta}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\frac{1}{\langle f_\omega\rangle}\right]\frac{{\rm d}(f_\omega\omega)}{{\rm d}n} \qquad \text{(A4)}$$

where $\beta = \frac{{\rm d}(f_\Psi\Psi_{\rm P})}{{\rm d}(f_\omega\omega)}$, hence:

$$\langle f_\Psi\rangle = \frac{\beta}{g\frac{{\rm d}n}{{\rm d}(f_\omega\omega)} + r^2\sin^2\theta\omega\frac{1}{\langle f_\omega\rangle}} \qquad \text{(A5)}$$

and:

$$\frac{{\rm d}n}{{\rm d}(f_\Psi\Psi_{\rm P})} = \frac{\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}}{g}. \qquad \text{(A6)}$$

The hydrostatic equation then becomes

$$\frac{{\rm d}P}{{\rm d}n} = -\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]\frac{{\rm d}(f_\Psi\Psi_{\rm P})}{{\rm d}n} \qquad \text{(A7)}$$

or

$$\frac{{\rm d}P}{{\rm d}(f_\Psi\Psi_{\rm P})} = -\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]. \qquad \text{(A8)}$$

From this equation, one can deduce that $-\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]$ is constant on an isobar.

Using (A6), one can get

$$\begin{aligned}{\rm d}M_{\rm P} &= \int_{\rm isobar}\rho{\rm d}n{\rm d}\sigma \\ &= {\rm d}(f_\Psi\Psi_{\rm P})\int_{\rm isobar}\rho\frac{{\rm d}n}{{\rm d}(f_\Psi\Psi_{\rm P})}{\rm d}\sigma \\ &= {\rm d}(f_\Psi\Psi_{\rm P})\int_{\rm isobar}\rho\frac{\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}}{g}{\rm d}\sigma. \qquad \text{(A9)}\end{aligned}$$

and

$$\begin{aligned}{\rm d}V_{\rm P} &= \int_{\rm isobar}{\rm d}n{\rm d}\sigma \\ &= {\rm d}(f_\Psi\Psi_{\rm P})\int_{\rm isobar}\frac{{\rm d}n}{{\rm d}(f_\Psi\Psi_{\rm P})}{\rm d}\sigma \\ &= {\rm d}(f_\Psi\Psi_{\rm P})\int_{\rm isobar}\frac{\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}}{g}{\rm d}\sigma. \qquad \text{(A10)}\end{aligned}$$

Thus

$$\frac{{\rm d}(f_\Psi\Psi_{\rm P})}{{\rm d}M_{\rm P}} = \frac{1}{\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]\langle g^{-1}\rangle S_{\rm P}} \qquad \text{(A11)}$$

and

$$\frac{{\rm d}V_{\rm P}}{{\rm d}(f_\Psi\Psi_{\rm P})} = \left[\frac{\langle g^{-1}\rangle}{\langle f_\Psi\rangle} - \left\langle g^{-1}r^2\sin^2\theta\omega\right\rangle\frac{\alpha}{\langle f_\omega\rangle}\right]S_{\rm P}. \qquad \text{(A12)}$$

Hence

$$\overline{\rho} = \frac{\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]\langle g^{-1}\rangle}{\left[\frac{\langle g^{-1}\rangle}{\langle f_\Psi\rangle} - \left\langle g^{-1}r^2\sin^2\theta\omega\right\rangle\frac{\alpha}{\langle f_\omega\rangle}\right]} \qquad \text{(A13)}$$

$$\langle f_{\rm d}\rangle = \frac{\rho\left[\frac{1}{\langle f_\Psi\rangle} - r^2\sin^2\theta\omega\alpha\frac{1}{\langle f_\omega\rangle}\right]\langle g^{-1}\rangle}{\rho_{\rm P}\left[\frac{\langle g^{-1}\rangle}{\langle f_\Psi\rangle} - \left\langle g^{-1}r^2\sin^2\theta\omega\right\rangle\frac{\alpha}{\langle f_\omega\rangle}\right]}. \qquad \text{(A14)}$$

Furthermore, once f_ω is given, $f_{\rm T}$ can be decided from other ways and (A5) and (A14), which decide f_ψ and $f_{\rm d}$ respectively, can be more applicable.

References

Alexander, D. R., & Ferguson, J. W. 1994, ApJ, 437, 879

Chaboyer, B., Demarque, P., & Pinsonneault, M. H. 1995a, ApJ, 441, 865

Chaboyer, B., Demarque, P., & Pinsonneault, M. H. 1995b, ApJ, 441, 876

Charbonnel, C. 1994, A&A, 282, 811

Charbonnel, C. 1995, ApJ, 453, L41

Endal, A. S., & Sofia, S. 1976, ApJ, 210, 184 (ES)

Endal, A. S., & Sofia, S. 1978, ApJ, 220, 279

Fransson, C., Cassatella, A., Gilmozzi, R., et al. 1989, ApJ, 336, 429

Heger, A., Langer, & N., Woosley, S. E. 2000, ApJ, 528, 368

Herrero, A., Kudritzki, R. P., Vilchez, J. M., et al. 1992, A&A, 261, 209

Iglesias, C. A., & Rogers, F. J. 1996, ApJ, 464, 943

Kippenhahn, R., Weigert, A., & Hofmeister, E. 1967, Meth. Comp. Phys., 7, 129

Kippenhahn, R., Meyer-Hofmeister, E., & Thomas, H. C. 1970, A&A, 5, 155 (KMT)

Langer, N., & Maeder, A. 1995, A&A, 295, 685

Maeder, A. 1983, A&A, 120, 113

Maeder, A., & Meynet, G. 1987, A&A, 182, 243

Maeder, A., & Meynet, G. 1989, A&A, 210, 155

Meyer-Hofmeister, E. 1972, A&A, 16, 282

Meynet, G., & Maeder, A. 1997, A&A, 321, 465 (MM)

Pinsonneault, M. H., Kawaler, S. D., Sofia, S., & Demarque, P. 1989, ApJ, 338, 424

Pinsonneault, M. H., Kawaler, S. D., Sofia, S., & Demarque, P. 1990, ApJS, 74, 501

Sills, A., Pinsonneault, M. H., & Terndrup, D. M. 2000, ApJ, 534, 335

Venn, K. A. 1995a, ApJ, 449, 839

Venn, K. A. 1995b, ApJS, 99, 659

Venn, K. A., Lambert, D. L., & Lemke, M. 1996, A&A, 307, 849

Walborn, N. R. 1976, ApJ, 205, 419